EXISTENCE OF THE EVEN POSITIVE SOLUTION WITH 
LINEAR GROWTH OF $u'' = u^2 - x^2$ VIA TOPOLOGICAL DEGREE

CHRISTOS SOURDIS

We will show existence of a solution $u$ of

$$
\begin{cases}
  u'' = u^2 - x^2, & x \in \mathbb{R}, \\
  u(x) - |x| \to 0 \text{ as } |x| \to \infty, \\
  u(x) \geq |x|, & x \in \mathbb{R}.
\end{cases}
$$

(1)

Equivalently we study

$$
\begin{cases}
  u'' = u^2 - x^2, & x > 0, \\
  u'(0) = 0 & u(x) - x \to 0 \text{ as } x \to \infty, \\
  u(x) \geq x, & x \in \mathbb{R},
\end{cases}
$$

(2)

or, by setting $u = x + \phi$, $x \geq 0$,

we will study solutions $\phi$ of

$$
\begin{cases}
  -\phi'' + 2x\phi = -\phi^2, & x > 0, \\
  \phi'(0) = -1 & \lim_{x \to \infty} \phi(x) = 0, \\
  \phi(x) \geq 0 & x \geq 0
\end{cases}
$$

(3)

Remark 1. Note that if $\phi$ solves (3) then $\phi(x) > 0$, $\phi'(x) < 0$, $\phi''(x) > 0$, $x \geq 0$.

Proposition 1. There exists a constant $M_1 > 0$ such that if $\phi \in C^2[0, \infty)$, $s \in [0, 1]$ satisfy

$$
\begin{cases}
  -\phi'' + 2x\phi = -s\phi^2, & x > 0, \\
  \phi'(0) = -1 & \lim_{x \to \infty} \phi(x) = 0, \\
  \phi(x) \geq 0 & x \geq 0
\end{cases}
$$

(4)

then

$$0 < \phi(x) \leq \phi(0) \leq M_1, \quad x \geq 0.$$

Proof. We argue by contradiction. Suppose that $\phi_n \in C^2[0, \infty)$, $s_n \in [0, 1]$ satisfy (4), and

$$
\max_{x \geq 0} \phi_n(x) = \phi_n(0) \to \infty \text{ as } n \to \infty,
$$

(5)

(recall Remark 1).

We distinguish three cases:
Using Arzela-Ascoli’s theorem, (5), (A), we obtain that, for a subsequence, moreover

\[ \limsup_n \phi_n(x) = 0, \quad x \geq 0, \quad n \geq 1. \]

In view of Remark 1 we have

\[ 0 \leq \phi_n(x) \leq 1, \quad x \geq 0, \quad \phi_n(0) = 1, \quad \phi'_n(0) = -\phi_n(0)^{-\frac{1}{2}} s_n^{-\frac{1}{2}}. \]

Moreover

\[ -\phi''_n + 2[s_n \phi_n(0)]^{-\frac{1}{2}} x \phi_n = -\phi^n_n, \quad x > 0. \]

Using Arzela-Ascoli’s theorem, (5), (A), we obtain that, for a subsequence,

\[ \phi_n \to \phi_* \quad \text{in } C^2_{loc}(0, \infty) \quad \text{as } n \to \infty, \]

for some \( \phi_* \in C^2[0, \infty) \), satisfying

\[ \phi_0 = \phi_*^2, \quad x > 0, \quad (6) \]

and

\[ \phi_0(0) = 1, \quad \phi'_0(0) = 0, \quad 0 \leq \phi_* \leq 1, \quad x \geq 0. \]

On the other hand one can see that all nontrivial solutions of (6) are unbounded. We arrived at a contradiction.

(B) \( s_n \phi_n(0) \to 0 \) for a subsequence \( n \to \infty \). Let

\[ \tilde{\phi}_n(x) = \phi_n(0)^{-1} \phi_n(x), \quad x \geq 0. \]

We have

\[ 0 \leq \tilde{\phi}_n(x) \leq 1, \quad x \geq 0, \quad \tilde{\phi}_n(0) = 1, \quad \tilde{\phi}'_n(0) = -\phi_n(0)^{-1}. \]

Moreover

\[ -\tilde{\phi}''_n + 2 x \tilde{\phi}_n = -s_n \tilde{\phi}_n(0) \tilde{\phi}^2_n, \quad x > 0. \]

Using Arzela-Ascoli’s theorem, (5), (B), we obtain that, for a subsequence,

\[ \tilde{\phi}_n \to \phi_* \quad \text{in } C^2_{loc}(0, \infty) \quad \text{as } n \to \infty, \]

for some \( \phi_* \in C^2[0, \infty) \), satisfying

\[ -\phi''_0 + 2 x \phi_* = 0, \quad x > 0, \quad (7) \]

\[ \phi_0(0) = 1, \quad \phi'_0(0) = 0, \quad 0 \leq \phi_* \leq 1, \quad x \geq 0. \]

We see that \( \phi_0(x) \to 0 \) as \( x \to \infty \) super-exponentially and by multiplying (7) by \( \phi_* \) and integrating by parts we obtain a contradiction.

(C) \( s_n \phi_n(0) \to c > 0 \) for a subsequence \( n \to \infty \). As in Case (B) we get a \( \phi_* \)

satisfying

\[ -\phi''_0 + 2 x \phi_* = -c \phi^2_* \quad \text{as } x \to \infty \]

\[ \phi_0(0) = 1, \quad \phi'_0(0) = 0, \quad 0 \leq \phi_* \leq 1, \quad x \geq 0. \]

We see that \( \phi_0(x) \to 0 \) as \( x \to \infty \) super-exponentially and by multiplying (8) by \( \phi_* \) and integrating by parts we obtain a contradiction.

The proof of the proposition is complete. \( \square \)

**Corollary 1.** There exists a constant \( M_2 > 0 \) such that if \( \phi \in C^2[0, \infty) \), \( s \in [0,1] \)

satisfy (4), then

\[ 0 < \phi(x) \leq M_2 e^{-x}, \quad x \geq 0. \]
Proof. Note that
\[-\phi'' + (2x + s\phi)\phi = 0, \quad x > 0,\]
where \(2x + s\phi \geq 2x, \quad x \geq 0\). The corollary now follows from a standard barrier argument. \qed

Consider the Banach space
\[X = \{ \phi \in C[0, \infty) : \|\phi\| := \sup_{x \geq 0} e^x|\phi(x)| < \infty \}\]
equipped with the norm \(\|\cdot\|\), and its closed convex subsets
\[Y = \{ \phi \in X : \phi'(0) = -1 \}, \quad Z = \{ \phi \in X : \phi'(0) = -1, \quad \phi(x) \geq 0, \quad x \geq 0 \}.

A standard barrier argument yields

**Proposition 2.** If \(f \in X\), there exists a unique \(\phi = T(f) \in Y\) such that
\[-\phi'' + 2x\phi = f, \quad x > 0.\]
Moreover, \(T : X \to X\) is compact, and there exists a constant \(C > 0\) such that
\[\|T(f_1) - T(f_2)\| \leq C\|f_1 - f_2\|, \quad \forall f_1, f_2 \in X.\]

**Theorem 1.** There exists a solution \(\phi\) of (3).

**Proof.** In view of Proposition 2 we have to show the existence of a solution \(\phi \in Z\) to the operator equation
\[\phi = T(-\phi^2).\]
The nonlinear operator
\[N_s(\phi) = \phi - T(-s\phi^2)\]
is continuous for \((s, \phi) \in [0, 1] \times Z\) and is a compact perturbation of the identity (see Proposition 2). Let \(B = Z \cap \{\|\phi\| \leq 2M_2\}\) where \(M_2\) is as in Corollary 1. The Schauder degree \(d(N_s(\cdot), B, 0)\) is defined and constant provided \(N_s(\phi) \neq 0\) for all \(\phi \in \partial B\) and \(s \in [0, 1]\). In view of Corollary 1 we know this to be the case. Since \(N_0 = I - T(0)\) (I the identity), we have
\[1 = d(I - T(0), B, 0) = d(N_1, B, 0).\]
Hence, there exists \(\phi \in B\) such that \(N_1(\phi) = \phi - T(-\phi^2) = 0\). The proof of the theorem is complete. \qed

**Remark 2.** It is easy to check that the solution of (3) is unique.

**References**


[3] Sourdis, C., On the even positive solution with linear growth of \(u'' = u^2 - x^2, \quad x \in \mathbb{R}\), preprint (2009).

C. Sourdis, Departamento de Ingenieria Matematica, Universidad de Chile, Santiago, Chile

E-mail address: schristos@dim.uchile.cl