EXISTENCE OF THE EVEN POSITIVE SOLUTION WITH LINEAR GROWTH OF $u'' = u^2 - x^2$ VIA TOPOLOGICAL DEGREE

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We will show existence of a solution u of

$$\begin{cases} u'' = u^2 - x^2, & x \in \mathbb{R}, \\ u(x) - |x| \to 0 & \text{as } |x| \to \infty, \\ u(x) \geq |x|, & x \in \mathbb{R}. \end{cases} \tag{1}$$

Equivalently we study

$$\begin{cases} u'' = u^2 - x^2, & x > 0, \\ u'(0) = 0 & u(x) - x \to 0 \ \text{ as } x \to \infty, \\ u(x) \geq x, & x \in \mathbb{R}, \end{cases} \tag{2}$$

or, by setting

$$u = x + \phi, \quad x \geq 0,$$

we will study solutions ϕ of

$$\begin{cases} -\phi'' + 2x\phi = -\phi^2, & x > 0, \\ \phi'(0) = -1 & \lim_{x\to\infty} \phi(x) = 0, \\ \phi(x) \geq 0 & x \geq 0 \end{cases} \tag{3}$$

Remark 1. *Note that if ϕ solves (3) then $\phi(x) > 0,\ \phi'(x) < 0,\ \phi''(x) > 0,\ x \geq 0$.*

Proposition 1. *There exists a constant $M_1 > 0$ such that if $\phi \in C^2[0,\infty),\ s \in [0,1]$ satisfy*

$$\begin{cases} -\phi'' + 2x\phi = -s\phi^2, & x > 0, \\ \phi'(0) = -1 & \lim_{x\to\infty} \phi(x) = 0, \\ \phi(x) \geq 0 & x \geq 0. \end{cases} \tag{4}$$

then

$$0 < \phi(x) \leq \phi(0) \leq M_1, \quad x \geq 0.$$

Proof. We argue by contradiction. Suppose that $\phi_n \in C^2[0,\infty),\ s_n \in [0,1]$ satisfy (4), and

$$\max_{x\geq 0} \phi_n(x) = \phi_n(0) \to \infty \quad \text{as} \quad n \to \infty, \tag{5}$$

(recall Remark 1).

We distinguish three cases:

(A) $s_n\phi_n(0) \to \infty$ for a subsequence $n \to \infty$. Let

$$\tilde{\phi}_n(x) = \phi_n(0)^{-1}\phi_n\left([s_n\phi_n(0)]^{-\frac{1}{2}}x\right), \quad x \geq 0, \quad n \geq 1.$$

In view of Remark 1 we have

$$0 \leq \tilde{\phi}_n(x) \leq 1, \quad x \geq 0, \quad \tilde{\phi}_n(0) = 1, \ \tilde{\phi}_n'(0) = -\phi_n(0)^{-\frac{3}{2}}s_n^{-\frac{1}{2}}.$$

Moreover

$$-\tilde{\phi}_n'' + 2[s_n\phi_n(0)]^{-\frac{3}{2}}x\tilde{\phi}_n = -\tilde{\phi}_n^2, \quad x > 0.$$

Using Arzcela-Ascoli's theorem, (5), (A), we obtain that, for a subsequence,

$$\tilde{\phi}_n \to \phi_* \quad \text{in } C^2_{loc}[0,\infty) \ \text{ as } n \to \infty,$$

for some $\phi_* \in C^2[0,\infty)$, satisfying

$$\phi_*'' = \phi_*^2, \quad x > 0, \tag{6}$$

and

$$\phi_*(0) = 1, \ \phi_*'(0) = 0, \quad 0 \leq \phi_* \leq 1, \ x \geq 0.$$

On the other hand one can see that all nontrivial solutions of (6) are unbounded. We arrived at a contradiction.

(B) $s_n\phi_n(0) \to 0$ for a subsequence $n \to \infty$. Let

$$\tilde{\phi}_n(x) = \phi_n(0)^{-1}\phi_n(x), \quad x \geq 0.$$

We have

$$0 \leq \tilde{\phi}_n(x) \leq 1, \quad x \geq 0, \quad \tilde{\phi}_n(0) = 1, \ \tilde{\phi}_n'(0) = -\phi_n(0)^{-1}.$$

Moreover

$$-\tilde{\phi}_n'' + 2x\tilde{\phi}_n = -s_n\tilde{\phi}_n(0)\tilde{\phi}_n^2, \quad x > 0.$$

Using Arzcela-Ascoli's theorem, (5), (B), we obtain that, for a subsequence,

$$\tilde{\phi}_n \to \phi_* \quad \text{in } C^2_{loc}[0,\infty) \ \text{ as } n \to \infty,$$

for some $\phi_* \in C^2[0,\infty)$, satisfying

$$-\phi_*'' + 2x\phi_* = 0, \quad x > 0, \tag{7}$$

$$\phi_*(0) = 1, \quad \phi_*'(0) = 0, \quad 0 \leq \phi_* \leq 1, \ x \geq 0.$$

We see that $\phi_*(x) \to 0$ as $x \to \infty$ super-exponentially and by multiplying (7) by ϕ_* and integrating by parts we obtain a contradiction.

(C) $s_n\phi_n(0) \to c > 0$ for a subsequence $n \to \infty$. As in Case (B) we get a ϕ_* satisfying

$$-\phi_*'' + 2x\phi_* = -c\phi_*^2, \quad x > 0, \tag{8}$$

and

$$\phi_*(0) = 1, \ \phi_*'(0) = 0, \ 0 \leq \phi_* \leq 1, \ x \geq 0.$$

We see that $\phi_*(x) \to 0$ as $x \to \infty$ super-exponentially and by multiplying (8) by ϕ_* and integrating by parts we obtain a contradiction.

The proof of the proposition is complete. □

Corollary 1. *There exists a constant $M_2 > 0$ such that if $\phi \in C^2[0,\infty)$, $s \in [0,1]$ satisfy (4), then*

$$0 < \phi(x) \leq M_2 e^{-x}, \quad x \geq 0.$$

Proof. Note that
$$-\phi'' + (2x + s\phi)\phi = 0, \quad x > 0,$$
where $2x + s\phi \geq 2x,\ x \geq 0$. The corollary now follows from a standard barrier argument. □

Consider the Banach space
$$\mathbf{X} = \{\phi \in C[0,\infty) \mid \|\phi\| := \sup_{x\geq 0} e^x|\phi(x)| < \infty\}$$
equipped with the norm $\|\cdot\|$, and its closed convex subsets
$$\mathbf{Y} = \{\phi \in \mathbf{X} \mid \phi'(0) = -1\},$$
$$\mathbf{Z} = \{\phi \in \mathbf{X} \mid \phi'(0) = -1,\ \phi(x) \geq 0,\ x \geq 0\}.$$
A standard barrier argument yields

Proposition 2. *If $f \in \boldsymbol{X}$, there exists a unique $\phi = T(f) \in \boldsymbol{Y}$ such that*
$$-\phi'' + 2x\phi = f, \quad x > 0.$$
Moreover, $T : \boldsymbol{X} \to \boldsymbol{X}$ is compact, and there exists a constant $C > 0$ such that
$$\|T(f_1) - T(f_2)\| \leq C\|f_1 - f_2\|, \quad \forall f_1,\ f_2 \in \boldsymbol{X}.$$

Theorem 1. *There exists a solution ϕ of (3).*

Proof. In view of Proposition 2 we have to show the existence of a solution $\phi \in \mathbf{Z}$ to the operator equation
$$\phi = T(-\phi^2).$$
The nonlinear operator
$$N_s(\phi) = \phi - T(-s\phi^2)$$
is continuous for $(s,\phi) \in [0,1] \times \mathbf{Z}$ and is a compact perturbation of the identity (see Proposition 2). Let $B = \mathbf{Z} \cap \{\|\phi\| \leq 2M_2\}$ where M_2 is as in Corollary 1. The Schauder degree $d\,(N_s(\cdot), B, 0)$ is defined and constant provided $N_s(\phi) \neq 0$ for all $\phi \in \partial B$ and $s \in [0,1]$. In view of Corollary 1 we know this to be the case. Since $N_0 = I - T(0)$ (I the identity), we have
$$1 = d(I - T(0), B, 0) = d(N_1, B, 0).$$
Hence, there exists $\phi \in B$ such that $N_1(\phi) = \phi - T(-\phi^2) = 0$. The proof of the theorem is complete. □

Remark 2. *It is easy to check that the solution of (3) is unique.*

References

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