ON THE EVEN POSITIVE SOLUTION WITH LINEAR GROWTH OF $u'' = u^2 - x^2,\ x \in \mathbb{R}$

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From [1], [2], we know that there exists a unique solution U to the problem

$$\begin{cases} u'' = u^2 - x^2, & x \in \mathbb{R}, \\ u(x) - |x| \to 0 & \text{as } |x| \to \infty, \\ u(x) > |x|, & x \in \mathbb{R}. \end{cases} \tag{1}$$

We have that U is even, $U'(x) > 0,\ U''(x) > 0,\ x > 0$, and

$$\frac{1}{C}(|x|+1)^{-\frac{1}{4}} e^{-\frac{2\sqrt{2}}{3}|x|^{\frac{3}{2}}} \le U(x) - |x| \le C(|x|+1)^{-\frac{1}{4}} e^{-\frac{2\sqrt{2}}{3}|x|^{\frac{3}{2}}}, \quad x \in \mathbb{R}.$$

We present a hopefully new proof that yields an explicit upper bound for U.

Proposition 1. *If $\delta > 0$, there exists a solution u_δ of*

$$\begin{cases} u'' = u^2 - x^2 - \delta, & x \in \mathbb{R}, \\ u(x) - (x^2+\delta)^{\frac{1}{2}} \to 0 & \textit{as } |x| \to \infty, \\ u(x) > (x^2+\delta)^{\frac{1}{2}}, & x \in \mathbb{R}. \end{cases}$$

Proof. Let

$$\underline{u}(x) = (x^2+\delta)^{\frac{1}{2}}.$$

We have

$$-\underline{u}'' + \underline{u}^2 - x^2 - \delta = -\delta(x^2+\delta)^{-\frac{3}{2}}, \quad x \in \mathbb{R}.$$

Hence $\underline{u}$ is a subsolution of the equation.

There exists a unique $\phi \in L^2(\mathbb{R}),\ \phi > 0$, such that

$$-\phi'' + 2(x^2+\delta)^{\frac{1}{2}}\phi = \delta(x^2+\delta)^{-\frac{3}{2}}, \quad x \in \mathbb{R}.$$

Let

$$\bar{u}_\delta(x) = (x^2+\delta)^{\frac{1}{2}} + \phi.$$

We have

$$\begin{aligned} -\bar{u}'' + \bar{u}^2 - x^2 - \delta &= -\bar{u}'' + \left(\bar{u} - (x^2+\delta)^{\frac{1}{2}}\right)\left(\bar{u} + (x^2+\delta)^{\frac{1}{2}}\right) \\ &= -\delta(x^2+\delta)^{-\frac{3}{2}} - \phi'' + \phi\left(2(x^2+\delta)^{\frac{1}{2}} + \phi\right) \\ &= -\delta(x^2+\delta)^{-\frac{3}{2}} - \phi'' + 2(x^2+\delta)^{\frac{1}{2}}\phi + \phi^2 = \phi^2. \end{aligned}$$

Hence $\bar{u}$ is an ordered supersolution of the equation.

It follows that there exists a solution u_δ to the equation such that

$$(x^2+\delta)^{\frac{1}{2}} < u_\delta < (x^2+\delta)^{\frac{1}{2}} + \phi.$$

The proof is concluded. □

Note that u_δ is even, and satisfies

$$\begin{cases} -\bar{u}_\delta'' + 2(x^2+\delta)^{\frac{1}{2}}\bar{u}_\delta = 2(x^2+\delta), & x\in\mathbb{R}, \\ \bar{u}_\delta(x) - (x^2+\delta)^{\frac{1}{2}} \to 0 & \text{as } |x|\to\infty, \\ \bar{u}_\delta(x) > (x^2+\delta)^{\frac{1}{2}}, & x\in\mathbb{R}. \end{cases}$$

Proposition 2. *There exists a constant $C>0$ such that*

$$0 < \bar{u}_\delta - |x| \le C, \quad x\in\mathbb{R} \quad \textit{as } \delta\to 0.$$

Proof. We just have to show it for $x>0$. Let

$$v_\delta = \bar{u}_\delta - x > 0, \quad x \ge 0.$$

We have

$$-v_\delta'' + 2(x^2+\delta)^{\frac{1}{2}} v_\delta = 2\delta \frac{(x^2+\delta)^{\frac{1}{2}}}{(x^2+\delta)^{\frac{1}{2}} + x}, \quad x>0. \tag{2}$$

To show that $\|v_\delta\|_{L^\infty(0,\infty)}$ is uniformly bounded as $\delta\to 0$, we argue by contradiction. Suppose that, for a sequence,

$$\|v_\delta\|_{L^\infty(0,\infty)} \to \infty \ \text{ as } \delta\to 0.$$

Let

$$w_\delta = \frac{v_\delta}{\|v_\delta\|_{L^\infty(0,\infty)}}, \quad x>0.$$

We have

$$-w_\delta'' + 2(x^2+\delta)^{\frac{1}{2}} w_\delta = \frac{2\delta}{\|v_\delta\|_{L^\infty(0,\infty)}} \frac{(x^2+\delta)^{\frac{1}{2}}}{(x^2+\delta)^{\frac{1}{2}} + x}, \quad x>0. \tag{3}$$

Moreover, $\|w_\delta\|_{L^\infty(0,\infty)} = 1,\ w_\delta > 0,\ x\ge 0,\ w_\delta'(0) = -\|v_\delta\|^{-1}_{L^\infty(0,\infty)}$, and $w_\delta(x)\to 0$ as $x\to\infty$. There exist $x_\delta \ge 0$ such that

$$w_\delta(x_\delta) = 1.$$

If $x_\delta > 0$ then

$$w_\delta'(x_\delta) = 0, \quad w_\delta''(x_\delta) \le 0.$$

Hence by (3),

$$0 \le x_\delta < \delta.$$

Using (3), Arzcela-Ascoli's theorem, and the standard diagonal argument, we obtain that, for a subsequence,

$$w_\delta \to w_0 \ \text{ in } \ C^2_{loc}(\mathbb{R}) \ \text{ as } \delta\to 0,$$

where

$$-w_0'' + 2xw_0 = 0, \quad x>0,$$

$w_0(0)=1,\ w_0'(0)=0,\ w_0\ge 0,\ x\ge 0,\ \|w_0\|_{L^\infty(0,\infty)} = 1$. We thus obtain an even, smooth nontrivial bounded solution of

$$-w'' + 2|x|w = 0, \quad x\in\mathbb{R}.$$

A contradiction, and the proof is concluded. □

Proposition 3. *There exists a constant $D > 0$ such that*

$$0 < \bar{u}_\delta - |x| \leq D|x|^{-1}, \quad |x| > 1 \quad \text{as} \quad \delta \to 0.$$

Proof. Again we show it for $x > 1$. The function

$$\bar{v}(x) = Dx^{-1}, \quad x > 1,$$

satisfies

$$-\bar{v}'' + 2(x^2+\delta)^{\frac{1}{2}}\bar{v} - 2\delta\frac{(x^2+\delta)^{\frac{1}{2}}}{(x^2+\delta)^{\frac{1}{2}} + x} =$$

$$-2Dx^{-3} + 2D(1+\delta x^{-2})^{\frac{1}{2}} - 2\delta\frac{(x^2+\delta)^{\frac{1}{2}}}{(x^2+\delta)^{\frac{1}{2}} + x} > 0, \quad x > 1,$$

(if D is large and δ small). Hence by Proposition 2, (2), and the comparison principle,

$$\bar{u}_\delta - x = v_\delta \leq \bar{v} = Dx^{-1}, \quad x > 1.$$

The proof is concluded. □

Now we can pass to the limit $\delta \to 0$ and summarize everything in

Theorem 1. *There exists an even solution U of (1). Moreover,*

$$U(x) \leq \bar{U}(x), \quad x \in \mathbb{R},$$

where $\bar{U}$ is the unique solution of

$$\begin{cases} -u'' + 2|x|u = 2x^2, & x \in \mathbb{R}, \\ \\ u(x) - |x| \to 0 & \text{as } |x| \to \infty. \end{cases}$$

($\bar{U} > |x|$ is the limit in C^2_{loc} of $\bar{u}_\delta$).

References

[1] Iida, M., Nakashima, K., Yanagida, E., *On certain one-dimensionar elliptic systems under different growth conditions at respective infinities*, in Asympiotic Analysis and Singularities, Advanced Studies in Pure Mathematics **47-2**, 2007, pp. 565–572.

[2] Schecter, S., and Sourdis, C., *Heteroclinic orbits in slow-fast Hamiltonian systems with slow manifold bifurcations*, preprint (2009).

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