ON THE EVEN POSITIVE SOLUTION WITH LINEAR GROWTH
OF \( u'' = u^2 - x^2, \ x \in \mathbb{R} \)

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From [1], [2], we know that there exists a unique solution \( U \) to the problem
\[
\begin{cases}
  u'' = u^2 - x^2, \ x \in \mathbb{R}, \\
  u(x) - |x| \to 0 \quad \text{as} \ |x| \to \infty, \\
  u(x) > |x|, \ x \in \mathbb{R}.
\end{cases}
\] (1)

We have that \( U \) is even, \( U' > 0 \), \( U'' > 0 \), \( x > 0 \), and
\[
\frac{1}{C(|x| + 1)^{-\frac{1}{4}} e^{-\frac{2\sqrt{2}}{3}|x|^\frac{3}{2}}} \leq U(x) - |x| \leq C(|x| + 1)^{-\frac{1}{4}} e^{-\frac{2\sqrt{2}}{3}|x|^\frac{3}{2}}, \ x \in \mathbb{R}.
\]

We present a hopefully new proof that yields an explicit upper bound for \( U \).

**Proposition 1.** If \( \delta > 0 \), there exists a solution \( u_\delta \) of
\[
\begin{cases}
  u'' = u^2 - x^2 - \delta, \ x \in \mathbb{R}, \\
  u(x) - (x^2 + \delta)^{\frac{1}{2}} \to 0 \quad \text{as} \ |x| \to \infty, \\
  u(x) > (x^2 + \delta)^{\frac{1}{2}}, \ x \in \mathbb{R}.
\end{cases}
\]

**Proof.** Let
\[
u(x) = (x^2 + \delta)^{\frac{1}{2}}.
\]
We have
\[
-\nu'' + \nu^2 - x^2 - \delta = -\delta(x^2 + \delta)^{-\frac{3}{2}}, \ x \in \mathbb{R}.
\]
Hence \( \nu \) is a subsolution of the equation.

There exists a unique \( \phi \in L^2(\mathbb{R}), \ \phi > 0, \) such that
\[
-\phi'' + 2(x^2 + \delta)^{\frac{1}{2}} \phi = -\delta(x^2 + \delta)^{-\frac{3}{2}}, \ x \in \mathbb{R}.
\]

Let
\[
u_\delta(x) = (x^2 + \delta)^{\frac{1}{2}} + \phi.
\]
We have
\[
-\nu'' + \nu^2 - x^2 - \delta = -\nu'' + \left( \nu - (x^2 + \delta)^{\frac{1}{2}} \right) \left( \nu + (x^2 + \delta)^{\frac{1}{2}} \right)
\]
\[
= -\delta(x^2 + \delta)^{-\frac{3}{2}} - \phi'' + \phi \left( 2(x^2 + \delta)^{\frac{1}{2}} + \phi \right)
\]
\[
= -\delta(x^2 + \delta)^{-\frac{3}{2}} - \phi'' + 2(x^2 + \delta)^{\frac{1}{2}} \phi + \phi^2 = \phi^2.
\]
Hence \( \nu \) is an ordered supersolution of the equation.
It follows that there exists a solution $u_\delta$ to the equation such that
\[(x^2 + \delta)^{\frac{1}{2}} < u_\delta < (x^2 + \delta)^{\frac{1}{2}} + \phi.\]
The proof is concluded. \qed

Note that $u_\delta$ is even, and satisfies
\[
\begin{aligned}
&-\bar{u}_\delta'' + 2(x^2 + \delta)^{\frac{1}{2}} \bar{u}_\delta = 2(x^2 + \delta), \quad x \in \mathbb{R}, \\
&\bar{u}_\delta(x) - (x^2 + \delta)^{\frac{1}{2}} \to 0 \quad \text{as } |x| \to \infty, \\
&\bar{u}_\delta(x) > (x^2 + \delta)^{\frac{1}{2}}, \quad x \in \mathbb{R}.
\end{aligned}
\]

**Proposition 2.** There exists a constant $C > 0$ such that
\[0 < \bar{u}_\delta - |x| \leq C, \quad x \in \mathbb{R} \quad \text{as } \delta \to 0.\]

**Proof.** We just have to show it for $x > 0$. Let
\[v_\delta = \bar{u}_\delta - x > 0, \quad x \geq 0.\]

We have
\[
-w_\delta'' + 2(x^2 + \delta)^{\frac{1}{2}} w_\delta = 2\delta \frac{(x^2 + \delta)^{\frac{1}{2}}}{(x^2 + \delta)^{\frac{1}{2}} + x}, \quad x > 0. \tag{2}
\]

To show that $\|v_\delta\|_{L^\infty(0,\infty)}$ is uniformly bounded as $\delta \to 0$, we argue by contradiction. Suppose that, for a sequence, $\|v_\delta\|_{L^\infty(0,\infty)} \to \infty$ as $\delta \to 0$.

Let
\[w_\delta = \frac{v_\delta}{\|v_\delta\|_{L^\infty(0,\infty)}}, \quad x > 0.
\]

We have
\[
-w_\delta'' + 2(x^2 + \delta)^{\frac{1}{2}} w_\delta = \frac{2\delta}{\|v_\delta\|_{L^\infty(0,\infty)}} \frac{(x^2 + \delta)^{\frac{1}{2}}}{(x^2 + \delta)^{\frac{1}{2}} + x}, \quad x > 0. \tag{3}
\]

Moreover, $\|w_\delta\|_{L^\infty(0,\infty)} = 1$, $w_\delta > 0$, $x \geq 0$, $w_\delta'(0) = -\|v_\delta\|_{L^\infty(0,\infty)}^{-1}$, and $w_\delta(x) \to 0$ as $x \to \infty$. There exist $x_\delta \geq 0$ such that
\[w_\delta(x_\delta) = 1.
\]

If $x_\delta > 0$ then
\[w_\delta'(x_\delta) = 0, \quad w_\delta''(x_\delta) \leq 0.
\]

Hence by (3),
\[0 \leq x_\delta < \delta.
\]

Using (3), Arzela-Ascoli’s theorem, and the standard diagonal argument, we obtain that, for a subsequence,
\[w_\delta \to w_0 \quad \text{in } C^2_{\text{loc}}(\mathbb{R}) \quad \text{as } \delta \to 0,
\]
where
\[-w_0'' + 2xw_0 = 0, \quad x > 0, \quad w_0(0) = 1, \quad w_0'(0) = 0, \quad w_0 \geq 0, \quad x \geq 0, \quad \|w_0\|_{L^\infty(0,\infty)} = 1.
\]

We thus obtain an even, smooth nontrivial bounded solution of
\[-w'' + 2|x|w = 0, \quad x \in \mathbb{R}.
\]

A contradiction, and the proof is concluded. \qed
Proposition 3. There exists a constant $D > 0$ such that

$$0 < \bar{u}_\delta - |x| \leq D|x|^{-1}, \quad |x| > 1 \quad \text{as} \quad \delta \to 0.$$ 

Proof. Again we show it for $x > 1$. The function

$$\bar{v}(x) = Dx^{-1}, \quad x > 1,$$

satisfies

$$-\bar{v}'' + 2(x^2 + \delta)^{\frac{1}{2}} \bar{v} - 2\delta \frac{(x^2 + \delta)^{\frac{1}{2}}}{(x^2 + \delta)^{\frac{1}{2}} + x} =$$

$$-2Dx^{-3} + 2D(1 + \delta x^{-2})^{\frac{1}{2}} - 2\delta \frac{(x^2 + \delta)^{\frac{1}{2}}}{(x^2 + \delta)^{\frac{1}{2}} + x} > 0, \quad x > 1,$$

(if $D$ is large and $\delta$ small). Hence by Proposition 2, (2), and the comparison principle,

$$\bar{u}_\delta - x = v_\delta \leq \bar{v} = Dx^{-1}, \quad x > 1.$$

The proof is concluded. \hfill \Box

Now we can pass to the limit $\delta \to 0$ and summarize everything in

Theorem 1. There exists an even solution $U$ of (1). Moreover,

$$U(x) \leq \bar{U}(x), \quad x \in \mathbb{R},$$

where $\bar{U}$ is the unique solution of

$$\begin{cases}
- u'' + 2|x|u = 2x^2, \quad x \in \mathbb{R}, \\
u(x) - |x| \to 0 \quad \text{as} \quad |x| \to \infty.
\end{cases}$$

$(\bar{U} > |x|)$ is the limit in $C^{2}_{\text{loc}}$ of $\bar{u}_\delta$.

REFERENCES


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