OPTIMAL ENERGY GROWTH LOWER BOUNDS FOR A CLASS OF SOLUTIONS TO THE VECTORIAL ALLEN-CAHN EQUATION

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Abstract. We prove optimal lower bounds for the growth of the energy over balls of minimizers to the vectorial Allen-Cahn energy in two spatial dimensions, as the radius tends to infinity. In the case of radially symmetric solutions, we can prove a stronger result in all dimensions.

Consider the semilinear elliptic system

$$\Delta u = \nabla W(u) \ \text{ in } \ \mathbb{R}^n, \quad n \geq 1, \tag{0.1}$$

where $W : \mathbb{R}^m \to \mathbb{R}$, $m \geq 1$, is sufficiently smooth and nonnegative. This system has variational structure, and solutions in a smooth bounded domain $\Omega \subset \mathbb{R}^n$ are critical points of the energy

$$E(v; \Omega) = \int_\Omega \left\{ \frac{1}{2}|\nabla v|^2 + W(v) \right\} dx$$

(subject to their own boundary conditions). A solution $u \in C^2(\mathbb{R}^n; \mathbb{R}^m)$ is called globally minimizing if

$$E(u; \Omega) \leq E(u + \varphi; \Omega)$$

for every smooth bounded domain $\Omega \subset \mathbb{R}^n$ and for every $\varphi \in W_0^{1,2}(\Omega; \mathbb{R}^m) \cap L^\infty(\Omega; \mathbb{R}^m)$ (see also [13] and the references therein).

In the scalar case, namely $m = 1$, Modica [18] used the maximum principle to show that every bounded solution to (0.1) satisfies the pointwise gradient bound

$$\frac{1}{2}|\nabla u|^2 \leq W(u) \ \text{ in } \ \mathbb{R}^n, \tag{0.2}$$

(see also [6] and [10]). Using this, together with Pohozaev identities, it was shown in [19] that the energies of such solutions satisfy the following monotonicity property:

$$\frac{d}{dR}\left(\frac{1}{R^{n-1}} \int_{B(x_0,R)} \left\{ \frac{1}{2}|\nabla u|^2 + W(u) \right\} dx \right) \geq 0, \quad R > 0, \ x_0 \in \mathbb{R}^n, \tag{0.3}$$

where $B(x_0, R)$ stands for the n-dimensional ball of radius R that is centered at x_0. Combining the above two relations yields that, if $x_0 \in \mathbb{R}^n$, the "potential" energy of each bounded nonconstant solution to the scalar problem satisfies the lower bound:

$$\int_{B(x_0,R)} W(u)\, dx \geq cR^{n-1}, \quad R > 0, \quad \text{for some } c > 0. \tag{0.4}$$

In the scalar case, the most famous representative of this class of equations is the Allen-Cahn equation

$$\Delta u = u^3 - u \ \text{ in } \ \mathbb{R}^n, \quad \text{where } \ W(u) = \frac{(1-u^2)^2}{4}, \tag{0.5}$$

which is used to model phase transitions (see [11] and the references therein).

In the vectorial case, that is when $m \geq 2$, in the absence of the maximum principle, it is not true in general that the gradient bound (0.2) holds (see [22] for a counterexample). Nevertheless, it was shown by Alikakos [1] using a stress energy tensor (see also [21]), and earlier by Caffarelli and Lin [8] via Pohozaev identities, that the energy of every solution to (0.1) (not necessarily bounded) satisfies the following weak monotonicity property:

$$\frac{d}{dR}\left(\frac{1}{R^{n-2}}\int_{B(x_0,R)}\left\{\frac{1}{2}|\nabla u|^2+W\left(u\right)\right\}dx\right)\geq 0,\quad R>0,\ x_0\in\mathbb{R}^n,\ n\geq 2. \tag{0.6}$$

In fact, as was observed in the former reference, if a solution u satisfies Modica's gradient bound (0.2), it follows that its energy satisfies the strong monotonicity property (0.2). Armed with (0.6), and doing some more work in the case $n=2$ (see [1]), it is easy to show that, if $x_0\in\mathbb{R}^n$, the energy of each nonconstant solution to the system (0.1) satisfies:

$$\int_{B(x_0,R)}\left\{\frac{1}{2}|\nabla u|^2+W\left(u\right)\right\}dx\geq\begin{cases} cR^{n-2} & \text{if } n\geq 3,\\ c\ln R & \text{if } n=2,\end{cases} \tag{0.7}$$

for all $R>1$ and some $c>0$.

The above results hold for arbitrary smooth and nonnegative W. If additionally W vanishes at least at one point, it is easy to cook up a suitable competitor for the energy and show that bounded globally minimizing solutions satisfy

$$\int_{B(x_0,R)}\left\{\frac{1}{2}|\nabla u|^2+W\left(u\right)\right\}dx\leq CR^{n-1},\quad R>0,\ x_0\in\mathbb{R}^n,$$

for some $C>0$ (see for example [3, Rem. 2.3]). The system (0.1) with $W\geq 0$ vanishing at a finite number of global minima is used to model multi-phase transitions (see [5] and the references therein). In this case, the system (0.1) is frequently referred to as the vectorial Allen-Cahn equation. Under appropriate assumptions (symmetries or non-degeneracy assumptions), it is possible to construct by variational methods "heteroclinic" solutions that "connect" the global minima of W (see [12, 16, 20] and the references therein); the energy of these solutions over $B(x_0,R)$ is of order R^{n-1} as $R\to\infty$. This observation implies that the estimate (0.7) is far from optimal for this class of W's. On the other side, for the case of the Ginzburg-Landau system

$$\Delta u=\left(|u|^2-1\right)u,\quad u:\mathbb{R}^2\to\mathbb{R}^2,\quad\left(\text{here } W(u)=\frac{\left(1-|u|^2\right)^2}{4}\text{ vanishes on }\mathbb{S}^1\right),$$

there are globally minimizing solutions with energy over $B(x_0,R)$ of order $\ln R$ as $R\to\infty$ (see [4, 21] and the references therein). In other words, the estimate (0.7) captures the optimal growth in the case of globally minimizing solutions to the Ginzburg-Landau system.

In this note, we will establish the corresponding optimal lower bound in the case of the phase transition case when $n=2$. In fact, we will prove the analog of the lower bound (0.4). As will be apparent, our proof does not work if $n\geq 3$. To the best of our knowledge, there is no related published result. Our approach combines ideas from two disciplines:

- We adapt to this setting clearing-out arguments from the study of the Ginzburg-Landau system, see [4].

- We employ variational maximum principles for globally minimizing solutions that have been recently devised and used for the study of the vectorial Allen-Cahn equation in [2].

Our main result is the following.

Theorem 0.1. Assume that $W \in C^1(\mathbb{R}^m;\mathbb{R})$, $m \geq 1$, and that there exist finitely many $N \geq 1$ points $a_i \in \mathbb{R}^m$ such that

$$W(u) > 0 \quad \text{in} \quad \mathbb{R}^m \setminus \{a_1, \cdots, a_N\}, \tag{0.8}$$

and there exists small $r_0 > 0$ such that the functions

$$r \mapsto W(a_i + r\nu), \quad \text{where} \quad \nu \in \mathbb{S}^1, \quad \text{are strictly increasing for } r \in (0, r_0), \quad i = 1, \cdots, N. \tag{0.9}$$

Moreover, we assume that

$$\liminf_{|u| \to \infty} W(u) > 0. \tag{0.10}$$

If $u \in C^2(\mathbb{R}^2;\mathbb{R}^m)$ is a bounded, nonconstant, and globally minimizing solution to the elliptic system

$$\Delta u = \nabla W(u) \quad \text{in} \quad \mathbb{R}^2, \tag{0.11}$$

for any $x_0 \in \mathbb{R}^2$, there exist constants $c_0, R_0 > 0$ such that

$$\int_{B(x_0,R)} W(u(x))\, dx \geq c_0 R \quad \text{for} \quad R \geq R_0.$$

Proof. Since the problem is translation invariant, without loss of generality, we may carry out the proof for $x_0 = 0$.

Suppose, to the contrary, that there exists a bounded, nonconstant, and globally minimizing solution u and radii $R_j \to \infty$ such that

$$\int_{B(0,R_j)} W(u(x))\, dx = o(R_j) \quad \text{as} \quad j \to \infty. \tag{0.12}$$

By the co-area formula (see for instance [9, Ap. C]), the nonnegativity of W, and the mean value theorem, there exist

$$s_j \in \left(\frac{R_j}{2}, R_j\right)$$

such that

$$\int_{\partial B(0,s_j)} W(u(x))\, dS(x) = o(1) \quad \text{as} \quad j \to \infty.$$

From this, as in the clearing-out lemma of [4], it follows that

$$\max_{|x|=s_j} W(u(x)) = o(1) \quad \text{as} \quad j \to \infty. \tag{0.13}$$

Indeed, if not, passing to a subsequence if necessary, there would exist $c_1 > 0$ and $x_j \in \partial B(0, s_j)$ such that

$$W(u(x_j)) \geq c_1 \quad \text{for} \quad j \geq 1.$$

On the other hand, since u is bounded in $\mathbb{R}^2$, by standard interior elliptic regularity estimates (see [9, 14]), the same is true for ∇u. Hence, there exists $r_* > 0$ such that

$$W(u(x)) \geq \frac{c_1}{2}, \quad x \in B(x_j, r_*), \quad \text{for} \quad j \geq 1,$$

which implies that

$$\int_{\partial B(0,s_j)} W\left(u(x)\right) dS(x) \geq \frac{c_1}{2}\mathcal{H}^1\left\{B(x_j,r_*)\cap \partial B(0,s_j)\right\} \geq c_2 \ \text{ for } \ j\geq 1.$$

for some $c_2>0$. Clearly, the above relation contradicts (0.12).

In view of (0.10), relation (0.13) implies that there exist $i_j\in\{1,\cdots,N\}$ such that

$$\max_{|x|=s_j}\left|u(x)-a_{i_j}\right| = o(1) \ \text{ as } \ j\to\infty.$$

By virtue of (0.9), exploiting the fact that u is a globally minimizing solution, we can apply a recent variational maximum principle from [2] to deduce that

$$\max_{|x|\leq s_j}\left|u(x)-a_{i_j}\right| \leq \max_{|x|=s_j}\left|u(x)-a_{i_j}\right| \ \text{ for } \ j\gg 1,$$

(so that the righthand side is smaller than $r_0/2$). The idea is that, if this is violated, one can construct a suitable function which agrees with u on $\partial B(0,s_j)$ but with less energy, thus contradicting the minimality of u. The above two relations imply the existence of an $i_0\in\{1,\cdots,N\}$ such that

$$\max_{|x|\leq s_j}\left|u(x)-a_{i_0}\right| = o(1) \ \text{ as } \ j\to\infty.$$

Now, letting $j\to\infty$ in the above relation yields that $u\equiv a_{i_0}$ which contradicts our assumption that u is nonconstant. □

Remark 0.1. Clearly, the monotonicity condition (0.9) is satisfied if the global minima are nondegenerate (the Hessian matrix $D^2W(a_i)$ is invertible for all $i=1,\cdot,N$).

Remark 0.2. In dimensions $n\geq 2$, a Liouville type theorem of Fusco [13] tells us that, if W is as in Theorem 0.1, nonconstant global minimizing solutions to (0.1) must enter any neighborhood of at least two of the global minima. Intuitively, this suggests that the optimal lower bound for the growth of the energy, that is kinetic (or interfacial) and potential, should be of order R^{n-1} in all dimensions. In this regard, see [7, 18] for the scalar case ($m=1$), provided that the global minima are nondegenerate.

If we restrict our attention to radially symmetric solutions, we have a stronger result which follows at once from the following proposition which is of independent interest.

Proposition 0.1. *Let $W\in C^1(\mathbb{R}^m;\mathbb{R})$, $m\geq 1$, possibly sign-changing. If $u\in C^2(\mathbb{R}^n;\mathbb{R}^m)$, $n\geq 2$, satisfies (0.1), we have that*

$$\frac{1}{2}\left|u'(R)\right|^2 \leq W\left(u(R)\right)-W\left(u(0)\right), \quad R>0, \tag{0.14}$$

and

$$\frac{d}{dR}\left(\frac{1}{R^n}\int_{B(0,R)}\left\{\frac{n-2}{2}|\nabla u|^2+nW(u)\right\}dx\right)\geq 0, \quad R>0. \tag{0.15}$$

Proof. We know that

$$u''+\frac{n-1}{r}u'-\nabla W(u)=0, \quad r>0, \quad u'(0)=0.$$

So, letting

$$e(r)=\frac{1}{2}\left|u'(r)\right|^2-W\left(u(r)\right), \quad r>0,$$

we find that

$$e'(r) = -\frac{n-1}{r} |u'(r)|^2, \quad r > 0. \tag{0.16}$$

Then, estimate (0.14) follows at once by integrating the above relation over $(0, R)$.

By Pohozaev's identity (the idea is to test the equation by $ru'(r)$, see for instance [21, Ch. 5]), for $R > 0$, we have that

$$\frac{1}{R}\int_{B(0,R)} \left\{ \frac{n-2}{2}|\nabla u|^2 + nW(u) \right\} dx = \int_{\partial B(0,R)} \left\{ W\left(u(R)\right) - \frac{1}{2}\left|u'(R)\right|^2 \right\} dS = -\mathcal{H}^1\left\{\mathbb{S}^1\right\} R^{n-1} e(R).$$

Then, we can arrive at (0.15) by dividing both sides by R^{n-1}, differentiating, and using (0.16). □

Remark 0.3. Radial solutions to the Allen-Cahn equation (0.5), decaying to zero in an oscillatory manner, as $r \to \infty$, have been constructed in [15].

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