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THE HETEROCLINIC CONNECTION PROBLEM FOR GENERAL DOUBLE-WELL POTENTIALS

CHRISTOS SOURDIS

Abstract. By variational methods, we provide a simple proof of existence of a heteroclinic orbit to the Hamiltonian system $u'' = \nabla W(u)$ that connects the two global minima of a double-well potential W. Moreover, we consider several inhomogeneous extensions.

1. Introduction

1.1. **The problem.** In this paper, we will prove existence of solutions $u \in C^2(\mathbb{R}, \mathbb{R}^n)$ to the following problem:

$$u_{xx} = \nabla W(u), \quad x \in \mathbb{R}, \quad \lim_{x\to\pm\infty} u(x) = a_\pm, \tag{1.1}$$

where

$$W \in C^1(\mathbb{R}^n),\ n \geq 1,\ \text{satisfies } W(a_-) = W(a_+) = 0,\ W(u) > 0 \text{ if } u \neq a_\pm, \tag{1.2}$$

for some $a_- \neq a_+$, and the function

$$\omega(s) = \min_{|u|=s} W(u), \quad s \geq 0, \tag{1.3}$$

satisfies

$$\int_0^\infty \sqrt{\omega(s)}ds = \infty. \tag{1.4}$$

In passing, we note that ω is upper semicontinuous. In fact, we will prove a more general result (see Theorem 1.1 below).

Since $a_- \neq a_+$, such a solution is called a *heteroclinic connection*, as opposed to a homoclinic. Motivated from mechanics, in relation with Newton's second law of motion (where x plays the role of time), we will often refer to W as a double-well potential (see also [6, 18] and the references therein). In order to avoid confusion, we point out that $-W$ is what is usually referred to as the potential in classical mechanics.

We note that the quantity

$$\frac{1}{2}|u_x|^2 - W(u)$$

is constant along solutions of the equation, which easily implies that $W(a_-) = W(a_+)$ is a necessary condition for a heteroclinic connection to exist between a_- and a_+.

We will also study the inhomogeneous problem

$$u_{xx} = h(x)\nabla W(u), \quad \lim_{x\to\pm\infty} u(x) = a_\pm, \tag{1.5}$$

under various assumptions on h.

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1.2. **Motivation.** The theory of phase transitions has led to the extensive study of singularly perturbed, non-convex energies of the form

$$J_\varepsilon(u) = \int_\Omega \left\{ \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u) \right\} dx,$$

where W is a nonnegative potential with a finite number of global minima; usually these are assumed to be nondegenerate, and that W is coercive at infinity or at least that

$$\liminf_{|u|\to\infty} W(u) > 0. \tag{1.6}$$

In the scalar case, this problem was studied by Modica [41] using De Giorgi's notion of Γ-convergence (see also [3, 19] and the references therein). In the vectorial case of two global minima, that is when (1.2) and (1.6) hold, the Γ-limit of this energy was studied in [15], [30] (for a thorough discussion around condition (1.6) in this context, we refer to [39]). The case where W has more than two wells was considered in [14] (see also [49]). In this context, the heteroclinic connections determine the interfacial energy.

In parallel, the interest in the heteroclinic connection problem stems also from the study of the vectorial Allen-Cahn equation that models multi-phase transitions (see [1], [3], [5], [6], [17], [22], and the references therein). Loosely speaking, the heteroclinic connections are expected to describe the way in which the solutions to the multi-dimensional parabolic system

$$u_t = \varepsilon^2 \Delta u - \nabla W(u),$$

for small $\varepsilon > 0$, transition from one state to the other (see [21]).

The heteroclinic connection problem also comes up when studying phase coexistence in consolidating porous medium (see [27] and the references therein), crystalline grain boundaries (see [20]), planar transition front solutions to the Cahn-Hilliard system [33], and domain walls in coupled Gross-Pitaevskii equations (see [2, 32] and the references therein).

We emphasize that some of these applications require a triple-well or four-well potential. Nevertheless, under a reflection symmetry assumption on W (which is frequently inherited from the physical model), the problem can easily be reduced to the double-well case (see [2] or [48]). The heteroclinic connection problem for nonsymmetric multi-well potentials is much more complicated to treat (see [51]).

For an application which requires one to consider potentials with degenerate minima, we refer to [13].

Our motivation for the inhomogeneous problems is twofold:

In [42], among other things, by employing singular perturbation techniques, the author constructed heteroclinic connections to the scalar spatially inhomogeneous Allen-Cahn equation

$$u_{xx} = h(\varepsilon x)W'(u) \text{ such that } \lim_{x\to\pm\infty} u(x) = a_\pm, \tag{1.7}$$

provided that $\varepsilon > 0$ is sufficiently small, where W has the same features as in the present paper but assuming non-degeneracy of the global minima; h is strictly positive, bounded, and having at least one non-degenerate local minimum. The result relies on the fact that the $\varepsilon = 0$ limit problem has a unique, asymptotically stable heteroclinic solution. Our results provide existence for all $\varepsilon > 0$ and hold for systems with more general W. Moreover, we believe that, with some more effort, they can provide information about the $\varepsilon \to 0$ asymptotic behavior of the solutions.

Recently, there has been an interest in constructing heteroclinic solutions to semilinear elliptic systems with variational structure (see [8]). In that case, in order to exclude the possibility of constructing the one dimensional heteroclinic, one has to impose some spatial inhomogeneity to the problem. We believe that our approach, a refinement of that of [6, 8], has the advantage of being flexible enough to potentially treat the case of these semilinear elliptic systems.

1.3. **Known results.** The problem (1.1) is completely understood if $n = 1$, see for instance [3], [18]; in fact, assumption (1.6) is not needed in that case.

If $n \geq 2$, under assumptions (1.2) and (1.6), the existence of a heteroclinic orbit was proven in [43] via a variational approach (see also [18, Thm. 2.3]).

Under various additional nondegeneracy or geometric conditions near the global minima of W, this problem has been dealt, mostly as a tangential issue, in several references. Under the assumption that

$$W(a_\pm + \rho\nu) \ \text{ is increasing in } \rho \in [0, \delta],\ \forall\ \nu \in \mathbb{S}^{n-1}, \tag{1.8}$$

(for some small $\delta > 0$), where $\mathbb{S}^{n-1}$ stands for the unit sphere, the existence of a heteroclinic connection was proven recently in [6] (see also [7] and [48]). Their novelty was to employ constraints which are subsequently removed. It is worthwhile noting that they observed that their proof goes through even when (1.6) is replaced with the assumption that the function ω in (1.3) is decreasing as $s \to \infty$ and

$$s^2\omega(s) \to \infty \ \text{ as } \ s \to \infty,$$

(compare with (1.4), which was also found independently in [40]). If $W(a_\pm + \rho\nu) \geq c\rho^\gamma$, $\rho \in [0, \delta]$, for some $c, \gamma, \delta > 0$, and assuming that the level sets of W near $a_\pm$ are strictly convex, the existence of a heteroclinic connection was proven very recently in [36] in the spirit of the concentrated compactness method. If the global minima of W are non-degenerate, that is the Hessian $\partial^2 W(a_\pm)$ is positive definite, the existence of a heteroclinic connection was proven in [49] by using techniques from Γ-convergence theory (an additional growth condition as $|u| \to \infty$ was also assumed). Other variational proofs, which usually require some non-degeneracy of the global minima, can be found in [1], [2], [5], [22], [32] and [44]. In fact, as is pointed out, the proof of [2] carries over to the case where W vanishes to finite order at $a_\pm$.

To the best of our knowledge, there are only a few corresponding results for spatially inhomogeneous systems, which we will refer to in the subsequent remarks. For the state of the art in the case of the scalar problem, we refer the interested reader to [9] and the references therein.

Lastly, we note that homoclinic and periodic orbits for conservative systems as in (1.1) can also be studied variationally (see for instance [24] and [50] respectively); of course there are corresponding spatially inhomogeneous extensions.

1.4. **The main result.** In view of the above discussion, and motivated by related literature (see Remarks 2.1, 3.3, 3.6, 3.7 below), it is natural to embed problem (1.1) into the more general family of inhomogeneous problems (1.5) with h positive and periodic.

Our primary goal is to prove the following theorem.

Theorem 1.1. *Let $h \in C(\mathbb{R}; \mathbb{R})$ be T-periodic ($T > 0$) and satisfy*

$$h(x) \geq h_0 > 0, \quad x \in \mathbb{R}, \tag{1.9}$$

for some constant h_0. Then, under assumptions (1.2) and (1.4), there exists a solution $u \in C^2(\mathbb{R}, \mathbb{R}^n)$ to the problem (1.5).

Subsequently, we adapt this proof to treat in a unified way a broader class of spatially inhomogeneous problems of the form (1.5).

1.5. **Method of proof and outline of the paper.** Our proof is motivated from the constraint variational set up of [6] but, instead of using energy decreasing local replacement arguments as a substitute of the maximum principle, we will use energy controlling local replacements together with a clearing-out argument. In particular, we do not need to employ the polar representation that was used in [6] (see also the introduction in [17]), that is to write a function $u \in W^{1,2}(\mathbb{R}, \mathbb{R}^n)$ as

$$u(x) = a_\pm + \rho_\pm(x)\Theta_\pm(x) \ \text{ whenever } \rho_\pm(x) = |u(x) - a_\pm| \neq 0; \quad u(x) = a_\pm \ \text{ otherwise},$$

which turns out to be a rather cumbersome issue (see, however, the observation below (2.14) herein), especially in the case of the corresponding elliptic problems (see [8]).

In our opinion, besides of rendering the most general result, our proof is the simplest available.

The outline of the paper is the following: In Section 2 we present the proof of Theorem 1.1, and in Section 3 we consider some other extensions to the inhomogeneous case.

2. Proof of the main result

Proof of Theorem 1.1. The main part of the proof will be devoted in showing that there exists a solution $u \in C^2(\mathbb{R}, \mathbb{R}^n)$ to the equation

$$u_{xx} = h(x)\nabla W(u), \tag{2.1}$$

and an $L > 0$, such that

$$|u(x) - a_-| < \delta, \quad x \leq -L; \quad |u(x) - a_+| < \delta, \quad x \geq L, \tag{2.2}$$

for some small

$$\delta < \frac{|a_+ - a_-|}{2}. \tag{2.3}$$

To this end, as in [6], for $L > 2$, let

$$X_L^- = \left\{ u \in W_{loc}^{1,2}(\mathbb{R}, \mathbb{R}^n) \ : \ |u(x) - a_-| \leq \delta, \quad x \leq -L \right\}, \tag{2.4}$$

$$X_L^+ = \left\{ u \in W_{loc}^{1,2}(\mathbb{R}, \mathbb{R}^n) \ : \ |u(x) - a_+| \leq \delta, \quad x \geq +L \right\}. \tag{2.5}$$

It is standard to show that there exists a $u_L \in X_L^- \cap X_L^+$ such that

$$J(u_L) = \inf_{u \in X_L^- \cap X_L^+} J(u) < \infty, \tag{2.6}$$

where $J : W_{loc}^{1,2}(\mathbb{R}, \mathbb{R}^n) \to [0, \infty]$ is the associated energy functional

$$J(u) = \int_{\mathbb{R}} \left\{ \frac{1}{2}|u_x|^2 + h(x)W(u) \right\} dx. \tag{2.7}$$

This was shown in [6] in the case where $h \equiv 1$, but the general case where $h \geq 0$ can be treated completely analogously (no other property of h besides reasonable regularity is needed at this point). Our goal is to show that there exists $L \gg 1$ such that u_L (or some translation of it) satisfies (2.2), since this will imply that u_L is a classical solution to (2.1).

We note that, a-priori, the minimizer u_L is C^2 and satisfies the Euler-Lagrange equation (2.1) *only in* $(-L, L)$ *and wherever it is away from the cylindrical boundary of the constraints.*

By constructing a piecewise linear competitor that is identically equal to a_- for $x \leq -1$ and equal to a_+ for $x \geq 1$, it is easy to show that

$$J(u_L) \leq C_1, \tag{2.8}$$

where the constant $C_1 > 0$ *is independent of* $L > 2$ (an analogous argument also appears in [23] and many other papers).

We claim that, given any $d \in (0, \delta]$, there exists $\varepsilon \in (0, \frac{d}{2})$, *independent of* $L > 2$, such that

$$\text{if} \;\; x_2 - x_1 \geq 3 \;\; \text{and} \;\; |u_L(x_i) - a_\pm| \leq \varepsilon, \; i = 1, 2, \tag{2.9}$$

then

$$|u_L(x) - a_\pm| < d, \quad x \in [x_1, x_2]. \tag{2.10}$$

In passing, we note that an analogous property was established independently in [29, Prop. 8.1] for the corresponding problem with nonlocal diffusion, where it is appropriately called 'stickiness property' (see also [12] for a more involved application of this property in the local setting). It is clear that we only have to verify this claim for the + case. To this end, suppose that $x_1, x_2 \in \mathbb{R}$ and $\varepsilon \in (0, \frac{d}{2})$ are such that the corresponding case of (2.9) holds for u_L. The main observation is that the minimality property of u_L implies that there exists a constant $C_2 > 0$, *independent of* ε, x_1, x_2, L, such that

$$\int_{x_1}^{x_2} \left\{ \frac{1}{2} \left|(u_L)_x\right|^2 + h(x) W(u_L) \right\} dx \leq C_2 \varepsilon. \tag{2.11}$$

In fact, if W was C^2 near a_+, we would have ε^2 instead of ε in the above relation. This follows by comparing the energy of u_L (keep in mind (2.6)) to that of a $\tilde{u}_L \in X_L^- \cap X_L^+$ which agrees with u_L outside of (x_1, x_2), is identically equal to a_+ over $[x_1 + 1, x_2 - 1]$, and is linear in the intermediate interpolation intervals. We point out that the assumption that the distance between x_1 and x_2 is larger than some universal constant (the number 3 here is chosen for convenience purposes only) plays a crucial role in controlling the gradient of $\tilde{u}_L$ in the interpolation zones. One could say that the competitor $\tilde{u}_L$ is obtained by performing *surgery* on u_L; we refer the interested reader to [10, Rem. 2.3] and [26] for related surgery type constructions. The desired claim now follows by applying the clearing-out lemma in [17] (see Lemma 1 therein). For the sake of completeness, and for future purposes, let us present a different argument. Suppose to the contrary that there exists $x_* \in (x_1, x_2)$ such that

$$|u_L(x) - a_+| < d, \quad x \in [x_1, x_*), \quad \text{and} \quad |u_L(x_*) - a_+| = d. \tag{2.12}$$

Note that there exists a $V \in C[0, \delta]$, $V > 0$ on $(0, \delta]$, such that

$$W(a_\pm + \rho\nu) \geq V(\rho) \;\; \forall \, \rho \in [0, \delta], \; \nu \in \mathbb{S}^{n-1}. \tag{2.13}$$

Indeed, plainly set $V(\rho) = \min\{V_-(\rho), V_+(\rho)\}$, where

$$V_\pm(\rho) = \min_{\nu \in \mathbb{S}^{n-1}} W(a_\pm + \rho\nu), \quad \rho \in [0, \delta]. \tag{2.14}$$

In passing, we observe that $u_L(x) \neq a_+$, $x \in [x_1, x_*]$ (if not and $u_L(\bar{x}) = a_+$ for some $\bar{x}$, the function which coincides with u_L for $x < \bar{x}$ and is identically equal to a_+ for $x \geq \bar{x}$

would belong in $X_L^- \cap X_L^+$ while having less energy than the minimizer u_L). Armed with this information, we have

$$\begin{aligned}
\int_{x_1}^{x_*}\left\{\tfrac{1}{2}\left|(u_L)_x\right|^2+h(x)W(u_L)\right\}dx &\overset{(1.9),(2.13)}{\geq} \int_{x_1}^{x_*}\left\{\tfrac{1}{2}\left|(u_L-a_+)_x\right|^2+h_0V\left(|u_L-a_+|\right)\right\}dx\\
\text{via the diamagnetic inequality [31]:} &\geq \int_{x_1}^{x_*}\left\{\tfrac{1}{2}\left|u_L-a_+\right|_x^2+h_0V\left(|u_L-a_+|\right)\right\}dx\\
\text{by Young's inequality:} &\geq \sqrt{2h_0}\int_{x_1}^{x_*}|u_L-a_+|_xV^{\frac{1}{2}}\left(|u_L-a_+|\right)dx\\
\text{by the area formula [38]:} &= \sqrt{2h_0}\int_{\mathbb{R}}V^{\frac{1}{2}}(\rho)\text{card}|u_L-a_+|^{-1}\left(\{\rho\}\right)d\rho\\
\text{from (2.9), (2.12) :} &\geq \sqrt{2h_0}\int_{\frac{d}{2}}^{d}V^{\frac{1}{2}}(\rho)d\rho,
\end{aligned}$$

where *card* stands for the cardinality and

$$|u_L-a_+|^{-1}\left(\{\rho\}\right)=\left\{x\in(x_1,x_*)\ :\ |u_L(x)-a_+|=\rho\right\}.$$

It is worthwhile to mention that the use of Young's inequality in related contexts seems to be due to [41] and is frequently referred to as Modica's trick (recall also the discussion in Subsection 1.2). Therefore, on account of (2.11), we can exclude the possibility (2.12) by choosing

$$\varepsilon\in\left(0,\frac{d}{2}\right)\quad\text{such that}\quad \varepsilon<\frac{\sqrt{h_0}}{C_2}\int_{\frac{d}{2}}^{d}V^{\frac{1}{2}}(\rho)d\rho, \tag{2.15}$$

which proves the claim. It is worth mentioning that, if (1.8) holds, the maximum principle of [6, 8], which follows from a sophisticated surgery type argument, asserts the following: if $|u_L(y_i)-a_+|<d$, $i=1,2$, for some $y_1<y_2$ and $d\in(0,\delta]$, then $|u_L(x)-a_+|<d$, $x\in(y_1,y_2)$. In fact, the main difference of our proof with that in [6] lies in that we use the 'asymptotic maximum principle' in (2.9)-(2.10) instead of the aforementioned maximum principle that was developed and used therein (see also [7, 8] for various extensions).

Next, we claim that, for any $\zeta>0$ sufficiently small, there exists

$$M>3, \tag{2.16}$$

independent of L, and a sequence of positive numbers $x_1^+<x_2^+<\cdots$, with

$$x_1^+\in(0,M),\quad M<x_{i+1}^+-x_i^+<3M,\quad i\geq 1, \tag{2.17}$$

such that

$$W\left(u_L(x_i^+)\right)\leq\zeta,\quad i\geq 1. \tag{2.18}$$

To see this, plainly take

$$M\geq C_1h_0^{-1}\zeta^{-1}, \tag{2.19}$$

where C_1 is as in (2.8) (we may assume that $M>3$), and apply the integral mean value theorem in the intervals $[0,M],[2M,3M],\cdots$. Analogously, given $\zeta>0$ sufficiently small, we can find negative numbers $\cdots<x_2^-<x_1^-$, with $x_1^-\in(-M,0)$, $M<x_i^--x_{i+1}^-<3M$ (increasing the value of M if needed), such that $W\left(u_L(x_i^-)\right)\leq\zeta$, $i\geq 1$.

We also claim that there exists a constant $C_3>0$, independent of $L>2$, such that

$$|u_L(x)|\leq C_3,\quad x\in\mathbb{R}. \tag{2.20}$$

Indeed, in view of the definition (1.3) and the property (2.8), we can easily adapt the previous argument, leading to (2.15), to get that

$$C_1 \geq \sqrt{2h_0} \int_{\min_{x\in\mathbb{R}} |u_L|}^{\max_{x\in\mathbb{R}} |u_L|} \sqrt{\omega(s)} ds, \tag{2.21}$$

(see also [39]). Now, the desired estimate follows at once from (1.4) and the trivial observation that $\min_{x\in\mathbb{R}} |u_L| \leq C_4$ for some constant $C_4 > 0$ that is independent of $L > 2$ (just take $C_4 = |a_-| + \delta$).

Let $\varepsilon > 0$ be as in (2.15) with

$$d = \delta, \tag{2.22}$$

so that property (2.9)-(2.10) is valid. Then, let $\zeta > 0$ be such that the following property holds:

$$W(u) \leq \zeta \ \text{ and } \ |u| \leq C_3 \ \text{ imply that } \ |u - a_-| \leq \varepsilon \text{ or } |u - a_+| \leq \varepsilon, \tag{2.23}$$

which of course is possible thanks to (1.2). We then choose

$$L = 1000M,$$

where $M > 3$ is any fixed number satisfying (2.19). We note that this choice of L will turn out to be much larger than what is actually needed, that is we make it for convenience purposes only. From (2.9), (2.10), (2.17), (2.18), and (2.23), it follows readily that

$$|u_L(x) - a_-| < \delta \ \text{ if } \ x \leq -1010M; \quad |u_L(x) - a_+| < \delta \ \text{ if } \ x \geq 1010M. \tag{2.24}$$

Indeed, we first note that (2.17) certainly implies that there exists $i_0 \in \mathbb{N}$ such that

$$x_{i_0}^+ \in (1000M, 1010M). \tag{2.25}$$

Then, in view of (2.3) and (2.5), we obtain from (2.18) with $i \geq i_0$, (2.20) and property (2.23) that

$$|u_L(x_i^+) - a_+| \leq \varepsilon, \quad i \geq i_0.$$

So, thanks to (2.16), (2.17) and (2.18), we can use the property (2.9)-(2.10) in each interval (x_i^+, x_{i+1}^+), $i \geq i_0$, to deduce that

$$|u_L(x) - a_+| < \delta, \quad x \in [x_{i_0}^+, \infty)$$

(keep in mind (2.22)). The second relation in (2.24) now follows at once if we recall (2.25). Similarly we can show the validity of the first relation.

In the remainder of the proof, we will further restrict M to be an integer multiple of the period T of h.

In view of (2.18) and (2.23), only two possibilities can occur:

(1) $|u_L(x_1^+) - a_+| \leq \varepsilon$. Then, by the property (2.9)–(2.10), the condition (2.22), and the second part of (2.24), we infer that

$$|u_L(x) - a_+| < \delta \ \text{ for } \ x \geq x_1^+ \in (0, M). \tag{2.26}$$

In light of the first relation in (2.24) and the above estimate, if u_L does not touch the cylindrical constraint somewhere on $[-1010M, -1000M]$ we are done. In any case, its translate

$$v_L(\cdot) = u_L(\cdot - 20M)$$

does satisfy the desired relation (2.2). Indeed, the first relation in (2.24) yields that

$$|v_L(x) - a_-| < \delta \ \text{ if } \ x \leq -990M = -L + 10M.$$

On the other side, relation (2.26) gives that

$$|v_L(x) - a_+| < \delta, \quad x \geq 21M = L - 979M.$$

Moreover, since h is T-periodic and M is an integer multiple of T, we have that v_L has the same energy as u_L. We note that the periodicity of h is used only at this point. In other words, v_L is also a minimizer of problem (2.6) which further satisfies (2.2) (i.e. it does not touch the cylindrical constraints).

(2) $|u_L(x_1^+) - a_-| \leq \varepsilon$. Then, we have that $|u_L(x) - a_-| < \delta$ for $x \leq x_1^+ \in (0, M)$ (from (2.9)-(2.10) and the first part of (2.24)). In that case, as before, replacing u_L by the translated minimizer $u_L(\cdot + 20M)$, if necessary, we find that (2.2) holds, as desired.

The above argument of using a suitable translate of u_L is mainly motivated from [6]. Actually, under the monotonicity assumption (1.8), it was shown in the latter reference that u_L can touch the cylindrical constraints at most at one of the points $x = -L$ or $x = L$, provided that L is sufficiently large. So, in contrast to the general case at hand, they could use any sufficiently small translation in the appropriate direction.

We have thus shown that the minimizer u_L satisfies (2.2). In particular, by standard arguments (see [6]), it induces a classical solution to (2.1). To complete the proof of the theorem, we will show that

$$\lim_{x \to \pm\infty} u_L(x) = a_\pm. \tag{2.27}$$

Indeed, for any arbitrarily small $\tilde{d} > 0$, let $\tilde{\varepsilon} \in (0, \frac{\tilde{d}}{2})$ be such that the corresponding property to (2.9)-(2.10) holds. As before, using (2.8) and (2.23), we can find a sequence $\{\tilde{x}_i^+\}$ such that $\tilde{x}_{i+1}^+ - \tilde{x}_i^+ \geq 3$, $\tilde{x}_i^+ \geq L$, $i \geq 1$, satisfying $|u_L(\tilde{x}_i^+) - a_+| \leq \tilde{\varepsilon}$ for $i \geq 1$. Thus, by the aforementioned property (2.9)-(2.10), we deduce that $|u_L(x) - a_+| < \tilde{d}$, $x \geq \tilde{x}_1^+$, which clearly implies the validity of the $+$ case in (2.27). Similarly we can show the $-$ case.

The proof of the theorem is complete. □

Remark 2.1. *In the scalar case ($n = 1$), further assuming that $a_\pm$ are non-degenerate minima of W, this problem was considered in* [4], *and for W as above in* [18]. *It is easy to see that, when $n = 1$, the above theorem as well as Theorem 3.1 below do not need the assumption (1.4). In fact, by appropriately modifying W in the two intervals outside of its global minima (see for example* [11, Ch. 1]*), we can capture a heteroclinic solution to the resulting system with values strictly between them. Of course this is also a heteroclinic to the original problem.*

Remark 2.2. *For a recent application of the above theorem, we refer to* [45].

Remark 2.3. *The proof of Theorem 1.1 carries over without difficulty to the quasi-linear setting:*

$$\left(|u_x|^{p-2}u_x\right)_x = \nabla W(u), \quad \lim_{x \to \pm\infty} u(x) = a_\pm, \quad (p > 2),$$

at least when (1.6) is assumed. This problem was considered in [35], *and the references therein, under assumption (1.8). The only essential difference is that one has to modify slightly the proof of the clearing-out lemma of* [17] *by using the Hölder inequality instead of the Cauchy-Schwarz .*

Remark 2.4. *Recently in* [8], *building on the arguments of* [6] *which rely on assumption (1.8), the authors constructed heteroclinic connections for semilinear elliptic systems of the form $\Delta u = \nabla W(u)$ in singly periodic domains of $\mathbb{R}^m$ with Neumann boundary conditions. In* *fact, as is pointed out, their approach can be extended to construct heteroclinic connections for the problem*

$$\begin{cases} \Delta u = h(x_1, \cdots, x_m)\nabla W(u) \\ \text{in cylindrical domains (in the } x_1 \text{ direction)} \\ \text{with Neumann boundary conditions,} \end{cases} \tag{2.28}$$

where h is positive and periodic in x_1. It would be interesting to see how much our approach can be pushed towards this direction.

3. Further inhomogeneous problems

3.1. The asymptotically constant inhomogeneity.

Theorem 3.1. *Assume that $h \in C(\mathbb{R})$ satisfies (1.9),*

$$\lim_{x\to\pm\infty} h(x) = h_\infty \in (0,\infty) \quad \text{and} \quad h(x) \le h_\infty, \ x \in \mathbb{R}. \tag{3.1}$$

Under assumptions (1.2) and (1.4) on W, there exists a solution to the problem (1.5).

Proof. The main difference of the problem at hand with the previous ones is that there is no translation invariance (continuous or discrete).

As before, for $L > 2$, let

$$m_L = \inf_{u\in X_L^-\cap X_L^+} J(u), \tag{3.2}$$

where $X_L^\pm$ are as in (2.4)–(2.5), and the energy functional J is as in (2.7). As we mentioned in the proof of Theorem 1.1, it is easy to show that the infimum is attained at some $u_L \in X_L^- \cap X_L^+$.

Motivated from [16], where ground states to the nonlinear Schrödinger equation with potential h were considered, we will compare m_L with the 'limiting energy'

$$m_{\infty,L} = \inf_{u\in X_L^-\cap X_L^+} \int_{\mathbb{R}} \left\{ \frac{1}{2}|u_x|^2 + h_\infty W(u) \right\} dx.$$

As we have already shown in Theorem 1.1, the above infimum is attained by a classical solution $u_{\infty,L} \in X_L^- \cap X_L^+$ of the problem

$$u_{xx} = h_\infty \nabla W(u), \quad \lim_{x\to\pm\infty} u(x) = a_\pm,$$

provided that L is sufficiently large. Clearly, since $u_{\infty,L}$ is continuous and (2.3) holds, there exists $x_L \in \mathbb{R}$ such that

$$|u_{\infty,L}(x_L) - a_-| \ge \delta \quad \text{and} \quad |u_{\infty,L}(x_L) - a_+| \ge \delta. \tag{3.3}$$

Observe that all the properties in the proof of Theorem 1.1 up to (2.23) remain true for this u_L as well as for $u_{\infty,L}$ (recall also a related comment in Case (1) therein), with the same constants in fact as those in the aforementioned theorem. In light of this, let us keep the same notation.

We may assume that $h(x) < h_\infty$ somewhere, say that

$$h(x) < h_\infty, \quad x \in (x_-, x_+), \tag{3.4}$$

for some $x_-, x_+ \in \mathbb{R}$. Thanks to (3.3), by translating $u_{\infty,L}$ if necessary, we may assume that

$$|u_{\infty,L}(x_-) - a_-| \ge \delta \quad \text{and} \quad |u_{\infty,L}(x_-) - a_+| \ge \delta. \tag{3.5}$$

Abusing notation, we will keep denoting by $u_{\infty,L}$ the possibly translated solution. The main point is that, by increasing L if needed, the possibly new $u_{\infty,L}$ is still in $X_L^- \cap X_L^+$. Indeed, as in the proof of Theorem 1.1, there exist $z_{1,L} \in (x_- - M, x_-)$, $z_{2,L} \in (x_-, x_- + M)$

$$|u_{\infty,L}(z_{i,L}) - a_-| \leq \varepsilon \ \text{ or } \ |u_{\infty,L}(z_{i,L}) - a_+| \leq \varepsilon, \quad i = 1, 2.$$

Hence, taking into consideration property (2.9)-(2.10) and (3.5), we infer that

$$|u_{\infty,L}(x) - a_-| < \delta, \quad x \leq z_{1,L}; \quad |u_{\infty,L}(x) - a_+| < \delta, \quad x \geq z_{2,L},$$

which clearly implies that $u_{\infty,L} \in X_L^- \cap X_L^+$ provided that $L \geq |x_-| + M$.

By (3.5) and the analog of (2.8), it is easy to see that

$$|u_{\infty,L}(x) - a_-| \geq \frac{\delta}{2} \ \text{ and } \ |u_{\infty,L}(x) - a_+| \geq \frac{\delta}{2} \ \text{ for } \ x \in \left[x_-, x_- + \frac{\delta^2}{8C_1}\right], \tag{3.6}$$

where C_1 is as in (2.8), (the point being that this interval is independent of large L). Indeed, if $x \in \left[x_-, x_- + \frac{\delta^2}{8C_1}\right]$, letting $\rho_\pm(x) = |u_{\infty,L}(x) - a_\pm|$, we have

$$|\rho_\pm(x) - \rho_\pm(x_-)| \leq \int_{x_-}^{x} |u_{\infty,L} - a_\pm|_t dt \leq \int_{x_-}^{x} |(u_{\infty,L})_t| \, dt \leq |x - x_-|^{\frac{1}{2}} (2C_1)^{\frac{1}{2}} \leq \frac{\delta}{2}. \tag{3.7}$$

Then, using $u_{\infty,L}$ as a test function, we find that

$$\begin{aligned} m_L \quad & \leq \quad \int_{\mathbb{R}} \left\{ \tfrac{1}{2} |(u_{\infty,L})_x|^2 + h(x) W(u_{\infty,L}) \right\} dx \\ & = \quad \int_{\mathbb{R}} \left\{ \tfrac{1}{2} |(u_{\infty,L})_x|^2 + h_\infty W(u_{\infty,L}) \right\} dx + \int_{\mathbb{R}} \left(h(x) - h_\infty \right) W(u_{\infty,L}) dx \\ \text{via (3.4), (3.6)} \quad & \leq \quad m_{\infty,L} - c \end{aligned} \tag{3.8}$$

where $c > 0$ is independent of large L.

This time we let

$$L = L_j = jM,$$

with j a sufficiently large integer that is to be determined so that (2.2) holds, which in particular will imply that u_L is a classical solution to

$$u_{xx} = h(x) \nabla W(u). \tag{3.9}$$

Suppose, to the contrary, that there exists a sequence of $L_j \to \infty$ such that (2.2) with $L = L_j$ is violated at some $x_j \leq -L_j$ (the other case is completely analogous). Then, similarly to (2.24), denoting u_{L_j} by u_j, we would have that

$$|u_j(x) - a_-| < \delta \ \text{ if } \ x \leq -(j + 10)M; \quad |u_j(x) - a_+| < \delta \ \text{ if } \ x \geq (j + 10)M. \tag{3.10}$$

In particular, this implies that $x_j \in (-(j + 10)M, -jM]$. Thus, as in the proof of Theorem 1.1, this gives that

$$|u_j(x) - a_+| < \delta \ \text{ if } \ x \geq -(j - 1)M. \tag{3.11}$$

From the above relation (which implies that u_j solves (3.9) for $x \geq -(j - 1)M$), making use of Arczela-Ascoli's theorem and the standard diagonal argument, passing to a subsequence if needed, we find that

$$u_j \to U \ \text{ in } C_{loc}(\mathbb{R}, \mathbb{R}^n), \tag{3.12}$$

where U satisfies

$$U_{xx} - h(x) \nabla W(U) = 0, \quad |U(x) - a_+| \leq \delta, \quad x \in \mathbb{R}. \tag{3.13}$$

Moreover, from the minimality of u_j, and the second part of (3.10), it follows readily that U is a minimizer of the energy subject to its boundary conditions, that is

$$J(U) \leq J(U+\varphi) \quad \forall\, \varphi \in W_0^{1,2}(I,\mathbb{R}^n) \ \text{ and any bounded interval } I \subset \mathbb{R},$$

(this can be proven as in [28]). Similarly to (2.27), we find that

$$\lim_{x\to\pm\infty} U(x) = a_+.$$

Then, by means of the analog of property (2.9)-(2.10), we get that

$$U \equiv a_+. \tag{3.14}$$

Actually, in the case where (1.8) holds, not necessarily with strict monotonicity, this can also be deduced by the weak sub-harmonicity of the function $|U-a_+|$, which follows directly from (3.13). It follows from (3.12) and (3.14) that

$$W(u_j) \to 0 \ \text{ in } C_{loc}(\mathbb{R}). \tag{3.15}$$

On the other hand, we have

$$\begin{aligned} m_{L_j} &= \int_{\mathbb{R}} \left\{ \tfrac{1}{2}|(u_j)_x|^2 + h(x)W(u_j) \right\} dx \\ &= \int_{\mathbb{R}} \left\{ \tfrac{1}{2}|(u_j)_x|^2 + h_\infty W(u_j) \right\} dx + \int_{\mathbb{R}} \left(h(x)-h_\infty\right) W(u_j) dx \\ u_j \in X_{L_j}^- \cap X_{L_j}^+ : \quad &\geq m_{\infty,L_j} + \int_{\mathbb{R}} \left(h(x)-h_\infty\right) W(u_j) dx \\ \text{via (2.8), (3.1), (3.15)}: \quad &\geq m_{\infty,L_j} + o(1), \end{aligned}$$

where $o(1) \to 0$ as $j \to \infty$, which contradicts (3.8). In more detail, to get the last relation, we estimate as follows:

$$\begin{aligned} \int_{\mathbb{R}} \left|\left(h(x)-h_\infty\right) W(u_j)\right| dx &= \int_{|x|<K} \left|\left(h(x)-h_\infty\right) W(u_j)\right| dx + \int_{|x|>K} \left|\left(h(x)-h_\infty\right) W(u_j)\right| dx \\ &\leq 4Kh_\infty \max_{|x|\leq K} W(u_j) + C_1 h_0^{-1} \sup_{|x|\geq K} \left|h(x)-h_\infty\right|, \end{aligned}$$

for any $K>0$. Then, given any $\epsilon>0$, we choose K so that the second term is smaller than $\epsilon/2$ and subsequently j_0 so that the first term is smaller than $\epsilon/2$ for $j \geq j_0$.

Having established that (2.2) holds for sufficiently large L, the rest of the proof proceeds verbatim as that of Theorem 1.1. □

Remark 3.1. *We note that the first condition in (3.1) was used only at the very end of the above proof for showing that*

$$\int_{\mathbb{R}} \left(h(x)-h_\infty\right) W(u_j) dx \to 0 \quad \text{as} \quad j \to \infty.$$

Interestingly enough, in view of (2.20), the above relation can also be deduced from (3.15) and Lebesgue's dominated convergence theorem if we assume instead that

$$h - h_\infty \in L^1(\mathbb{R}).$$

Remark 3.2. *Condition (1.9) was used crucially in obtaining the uniform estimate (2.20) (recall (2.21)), which allowed us to get (3.12). Nevertheless, with some care, we can still obtain a uniform estimate (with respect to L) under the weaker condition:*

$$h(x) \geq h_0, \quad |x| \geq N \ \text{ for some } h_0, N > 0; \quad h(x) \geq 0, \quad x \in \mathbb{R}, \tag{3.16}$$

instead of (1.9). Indeed, the proof of (2.20) gives us first a uniform estimate for $|u_L|$ *on* $|x| \geq N$*. Then, this can be extended in the remaining region by arguing as in (3.7). Clearly, the validity of (3.10) is not affected by weakening the assumption (1.9) to (3.16). The same is also true for (3.11). Indeed, as in Theorem 1.1 (recall especially (2.18)), there exist* $\xi_{1,j} \in (-jM, -(j-1)M)$*,* $\xi_{2,j} \in (-(N+M), -N)$ *such that* $|u_j(\xi_{i,j}) - a_+| \leq \varepsilon$*,* $i = 1, 2$*,* $j \geq 1$*. Now, property (2.9)-(2.10) gives us the desired relation in* $(-(j-1)M, \xi_{1,j})$*. Moreover, by relation (3.7) with fixed end point* $\xi_{1,j}$*, reducing* $\varepsilon > 0$ *if needed, we deduce that the same holds in* $(\xi_{1,j}, \xi_{2,j})$*. Lastly, in the remaining interval* $(\xi_{2,j}, \infty)$ *we just apply property (2.9)-(2.10) between* $\xi_{2,j}$ *and* $(j + 11)M$*. We argue similarly for showing (3.14) under these conditions.* We therefore conclude that the assertion of Theorem 3.1 continues to hold even if (1.9) is replaced by the weaker condition (3.16).

In the case where h *is periodic, we can repeat the above procedure in a periodic fashion, and find that the assertion of Theorem 1.1 still holds even if (1.9) is weakened to* $h \geq 0$*,* h *nontrivial.*

Remark 3.3. *Using a different variational argument, Theorem 3.1 was proven in the scalar case in* [18, Thm. 2.2] *(see also* [47] *for a result which allows* $h_\infty - h(x)$ *to change sign for arbitrarily large values of* $|x|$*).*

Remark 3.4. *It may be plausible that the above theorem generalizes to the case where* h_∞ *is a periodic function (with the obvious interpretation of (3.1)). A related result for the scalar problem can be found in* [9]*. It is worth mentioning that the results in* [25] *do not require the corresponding inhomogeneity to be asymptotically periodic (however they require further assumptions on the corresponding potential which include the nondegeneracy of its global minima).*

Remark 3.5. *In* [37]*, the authors used the method of upper and lower solutions together with an approximation by large finite intervals to show that there is a unique solution to the problem*

$$u'' = h(x)(u^3 - u), \ x \in \mathbb{R}; \quad \lim_{x \to \pm\infty} u(x) = \pm 1,$$

which is an odd and strictly increasing function, under the assumptions that $h \in C^1(\mathbb{R})$ *is even,* $h' < 0$ *for almost all* $x > 0$*, and* $\lim_{x \to +\infty} h(x) > 0$*. Observe that this result is not contained in our Theorem 3.1 because the second assumption in (3.1) is violated. Nevertheless, an inspection of the proof of Theorem 1.1 (see also Remark 2.1) yields that there exists an odd solution to the scalar problem*

$$u'' = h(x)W'(u), \ x \in \mathbb{R}; \quad \lim_{x \to \pm\infty} u(x) = \pm 1,$$

such that $u > 0$ *in* $(0, \infty)$*, provided that the following assumptions are fulfilled:* $h \in C(\mathbb{R})$*,* $W \in C^1(\mathbb{R})$ *are even,* $h \geq 0$*,* $\liminf_{x \to +\infty} h(x) > 0$*, and* $W(u) > 0$ *for* $u \in [0, \infty) \setminus \{1\}$*. The main point is that we can 'pin' the minimizer* u_L *at the origin by restricting ourselves to the class of odd functions.*

3.2. The diverging inhomogeneity.

Theorem 3.2. *Assume that* $h \in C(\mathbb{R})$ *is nonnegative, and*

$$\lim_{x \to \pm\infty} h(x) = \infty. \tag{3.17}$$

Under solely the assumption (1.2) on W*, there exists a solution to the problem (1.5).*

Proof. Our strategy remains the same. We consider the constraint minimization problem (2.2)-(2.7) and show that any minimizer u_L (which exists by standard arguments) satisfies (2.2), provided that L is sufficiently large. Clearly, estimate (2.8) holds (abusing notation).

We claim that, for large L, we have that

$$|u_L(x) - a_+| < \delta, \quad x \geq L.$$

Indeed, suppose to the contrary that there exists $x_+ \geq L$ such that $|u_L(x_+) - a_+| = \delta$ (we have suppressed the obvious dependence of x_+ on L, x_+ is not related to that in the proof of Theorem 3.1). Then, arguing as we did before for showing (3.6), we find that

$$\delta \geq |u_L(x) - a_+| \geq \frac{\delta}{2} \quad \text{for} \quad x \in \left[x_+, x_+ + \frac{\delta^2}{8C_1}\right].$$

In turn, this implies that

$$W\left(u_L(x)\right) \geq c > 0, \quad x \in \left[x_+, x_+ + \frac{\delta^2}{8C_1}\right],$$

where *the constant $c > 0$ is independent of large L.* On the other hand, if L is sufficiently large, the above relation contradicts the fact that

$$\int_L^\infty W\left(u_L(t)\right) dt \to 0 \quad \text{as} \quad L \to \infty,$$

which follows directly from (2.8) and (3.17). Analogously, we can show that

$$|u_L(x) - a_-| < \delta, \quad x \leq -L.$$

Having established that u_L satisfies (2.2) (and as a consequence (3.9)), for sufficiently large L, we can proceed in a similar manner to show that it also satisfies the desired asymptotic behavior at respective infinities. ☐

Remark 3.6. *If $h(x) > 0$, $x \in \mathbb{R}$, the above theorem is contained in* [34].

Remark 3.7. *In* [46], *relying on the oddness of the nonlinearity, we used a shooting argument to show that there exists a unique odd solution to the problem*

$$u_{xx} = |x|^\alpha (u^3 - u), \quad \lim_{x \to \pm\infty} u(x) = \pm 1,$$

where $\alpha > 0$. Moreover, this solution is increasing and asymptotically stable. This heteroclinic connection describes the profile of the transition layer, near $x = 0$, of the singular perturbation problem (1.7) with $h \sim |x|^\alpha$ as $x \to 0$ and $h > 0$ elsewhere (here $W(u) = \frac{(u^2-1)^2}{4}$).

Remark 3.8. *The results in this paper generalize straightforwardly to the more general class of systems $u'' = \nabla_u W(x, u)$.*

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Department of Mathematics, University of Turin, Via Carlo Alberto 10, 20123, Turin, Italy.

E-mail address: christos.sourdis@unito.it