

ANALYSIS OF AN IRREGULAR BOUNDARY LAYER BEHAVIOR FOR THE STEADY STATE FLOW OF A BOUSSINESQ FLUID

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Dedicated to the memory of Professor Paul C. Fife

ABSTRACT. Using a perturbation approach, we make rigorous the formal boundary layer asymptotic analysis of Turcotte, Spence and Bau from the early eighties for the vertical flow of an internally heated Boussinesq fluid in a vertical channel with viscous dissipation and pressure work. A key point in our proof is to establish the non-degeneracy of a special solution of the Painlevé-I transcendent. To this end, we relate this problem to recent studies for the ground states of the focusing nonlinear Schrödinger equation in an annulus. We also relate our result to a particular case of the well known Lazer-McKenna conjecture from nonlinear analysis.

1. INTRODUCTION

1.1. The problem. In [30], Turcotte, Spence and Bau considered the vertical flow of an internally heated Boussinesq fluid in a vertical channel with viscous dissipation and pressure work. Starting from the basic equations for conservation of mass, momentum and energy in a compressible fluid, and after making various appropriate assumptions, they were led to the study of the following boundary value problem for the steady state flow:

$$\begin{cases} 2u'' = u^2 - A(1 - x^2), & x \in (-1, 1), \\ u(-1) = u(1) = 0, \end{cases} \quad (1.1)$$

where $A \geq 0$ is a parameter. More specifically, the parameter A represents the (non-dimensional) heat addition, u is the velocity, x is the scaled position and $[-1, 1]$ is the horizontal cross section of the vertical channel (we refer to [30] for more details).

1.2. Formal asymptotic analysis as $A \rightarrow \infty$. A formal asymptotic analysis carried out in [30] predicts the existence of solutions to (1.1) which, as $A \rightarrow \infty$, behave roughly in the following way. They converge uniformly to $\sqrt{A(1-x^2)}$ over fixed compacts of $(-1, 1)$; they converge (in some sense) to

$$(2A)^{\frac{2}{5}} Y \left((2A)^{\frac{1}{5}} (x+1) \right) \quad \text{and} \quad (2A)^{\frac{2}{5}} Z \left((2A)^{\frac{1}{5}} (1-x) \right) \quad (1.2)$$

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near $x = -1$ and $x = 1$ respectively, where Y and Z should satisfy the following boundary value problem:

$$\begin{cases} 2y'' = y^2 - s, & s > 0, \\ y(0) = 0, & y - s^{\frac{1}{2}} \rightarrow 0 \text{ as } s \rightarrow \infty. \end{cases} \quad (1.3)$$

To convince the skeptical reader, let us note that, letting

$$\varepsilon = \sqrt{\frac{2}{A}} \text{ and } v = \frac{1}{\sqrt{A}}u,$$

problem (1.1) is equivalent to the singular perturbation problem:

$$\begin{cases} \varepsilon^2 v'' = v^2 - (1 - x^2), & x \in (-1, 1), \\ v(-1) = v(1) = 0, \end{cases} \quad (1.4)$$

to which one can apply standard, but non-rigorous, matching asymptotic techniques (see for instance [23]). In these terms, $\sqrt{1 - x^2}$ serves as an outer solution which, however, has an *irregular boundary layer*, thus creating the need for the inner solutions in (1.2).

The familiar reader may have already observed that, after a simple normalization, the differential equation in (1.3) is non other than the Painlevé-I transcendent (see for example [9, Ch. 5]). Despite of this fact, the study of the limit problem (1.3) is nontrivial and, in fact, has quite a history (see Proposition 2.1 herein for more details). Combining the results of [11, 16], we know that problem (1.3) has exactly two solutions: Y_+ which is strictly increasing; Y_- which has negative slope at the origin and exactly one local minimum.

Actually, analogous formal considerations leave open the possibility of existence of solutions converging to $-\sqrt{A(1 - x^2)}$, in some sense, as $A \rightarrow \infty$, but this case lies beyond the scope of the present article.

1.3. Rigorous known results. To the best of our knowledge, there are no studies of the problem (1.1) that link it rigorously to the limit problem (1.3). On the other hand, we were truly surprised when we realized the striking similarities that the former problem shares with a class of extensively studied superlinear elliptic problems of Ambrosetti-Prodi type and the famous Lazer-McKenna conjecture that accompanies them. Interestingly enough, however, both problems date to the early 1980's. In particular, for the simplified problem:

$$\begin{cases} -\Delta u = |u|^p - A\varphi_1(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.5)$$

where Ω is a smooth, bounded domain of \mathbb{R}^N , $p \in \left(1, \frac{N+2}{N-2}\right)$ if $N \geq 3$, $p \in (1, \infty)$ if $N = 1, 2$, and $\varphi_1 > 0$ is the principal eigenfunction of $-\Delta$ in Ω with Dirichlet boundary conditions, the Lazer-McKenna conjecture asserts roughly that the number of solutions diverges as $A \rightarrow \infty$ (see [4, 5] and the references therein for more details). Remarkably, the same was also conjectured in [30] for problem (1.1) and was subsequently verified by Hastings and McLeod in [11] via a shooting argument. Let us point out that solutions to (1.5) should also develop an irregular boundary layer, as $A \rightarrow \infty$, since the gradient of φ_1 on $\partial\Omega$ is nonzero (by Hopf's boundary point lemma). In [4], Dancer and Yan proved the Lazer-McKenna conjecture for

(1.5) by constructing solutions with an arbitrary number of sharp downward spikes, located near the maximum points of φ_1 and superimposed on a positive minimizing solution, provided that A is sufficiently large. They also studied the asymptotic behavior, as $A \rightarrow \infty$, of the mountain pass solutions to (1.5) and showed that they have a small steep peak near the boundary (combined with the irregular boundary layer). In connection with this, let us note that the aforementioned increasing solution Y_+ of (1.3) is a minimizer of the natural associated energy, while the other solution Y_- , which has a negative peak, is a mountain pass. Even though the irregular boundary layer of the problem (1.5) was treated mostly as a tangential issue in [4], the author still had to study the elliptic analog of (1.3) (with exponent p in the nonlinearity).

It follows readily from the analysis in [4], which was variational in nature, that problem (1.1) has two even solutions u_{\pm} such that

$$u_{\pm}(x) = \sqrt{A(1-x^2)} - (1-x^2)^{-2} + A^{-\frac{1}{2}}\mathcal{O}(1), \quad (1.6)$$

uniformly over fixed compacts of $(-1, 1)$, as $A \rightarrow \infty$ (here, and throughout this paper, Landau's symbol $\mathcal{O}(1)$ denotes a quantity which is bounded independently of large A);

$$(2A)^{-\frac{2}{5}}u_{\pm}\left(-1 + (2A)^{-\frac{1}{5}}s\right) \rightarrow Y_{\pm}(s) \text{ in } C_{loc}[0, \infty), \text{ as } A \rightarrow \infty. \quad (1.7)$$

Actually, only even solutions were considered in [30]. As may be expected, the minus case in the above result is considerably harder to establish and, for this purpose, the authors had to adapt some ideas from [6]. Let us emphasize that u_- has to be constructed as a mountain pass in the class of even functions. Indeed, the associated linearization on u_- , for large A , has similarities with a semiclassical Schrödinger operator with a double-well potential; thus, it is expected to have at least two unstable eigenvalues in the non-symmetric class (by the instability of u_- in the symmetric class and tunnelling phenomena, see [13] and Remark 3.1 below), contradicting [14] should u_- was a mountain pass solution in the non-symmetric class. The mountain pass solutions, in the general class, are expected to have the (re-scaled) profile of Y_- at one boundary point and that of Y_+ at the other, for large A . However, this does not seem to follow directly from the analysis in [4]. Let us also point out that the variational approach, used for showing the above, does not require any knowledge of the non-degeneracy of the solutions Y_{\pm} of (1.3), that is the absence of bounded elements in the kernel of the associated linearizations. On the other hand, it is essentially the non-degeneracy of the corresponding Y_+ that allowed the authors of [4] to add sharp downward spikes on top of the corresponding minimal solution u_+ by means of a finite dimensional variational reduction procedure.

1.4. The main result. In this article, using a perturbation argument, we will give optimal estimates for the convergence in (1.7) and also provide the missing estimates in the intermediate zones that are not covered by (1.6) and (1.7). In the process, we will prove the non-degeneracy of the “blow-up” profile Y_- , and at the same time provide a new proof of the fact that (1.3) has only Y_+ and Y_- as solutions, which was originally shown in [11].

The following is our main result.

Theorem 1.1. *Let Y and Z be either one of the two solutions Y_+ and Y_- of (1.3). There exists a solution $u = u_{YZ}$ of (1.1) such that*

$$\begin{cases} u = (2A)^{\frac{2}{5}}Y \left((2A)^{\frac{1}{5}}(x+1) \right) + \mathcal{O}(A^{\frac{2}{5}})(x+1), & 0 \leq x+1 \leq (2A)^{-\frac{1}{5}}D, \\ u = (2A)^{\frac{2}{5}}Y \left((2A)^{\frac{1}{5}}(x+1) \right) + \mathcal{O}(A^{\frac{1}{2}})(x+1)^{\frac{3}{2}}, & (2A)^{-\frac{1}{5}}D \leq x+1 \leq \delta, \\ u = (2A)^{\frac{2}{5}}Z \left((2A)^{\frac{1}{5}}(1-x) \right) + \mathcal{O}(A^{\frac{2}{5}})(1-x), & 0 \leq 1-x \leq (2A)^{-\frac{1}{5}}D, \\ u = (2A)^{\frac{2}{5}}Z \left((2A)^{\frac{1}{5}}(1-x) \right) + \mathcal{O}(A^{\frac{1}{2}})(1-x)^{\frac{3}{2}}, & (2A)^{-\frac{1}{5}}D \leq 1-x \leq \delta, \end{cases}$$

and

$$u = \sqrt{A(1-x^2)} + \mathcal{O}(1)(1-x^2)^{-2}, \quad x \in \left[-1 + (2A)^{-\frac{1}{5}}D, 1 - (2A)^{-\frac{1}{5}}D \right],$$

for some constants $0 < \delta \ll D$, uniformly as $A \rightarrow \infty$ (for the above notation, see Subsection 1.9 below).

Moreover, the following a-priori estimate holds for the associated linearized operator: There exist constants $A_1, C > 0$ such that if $\varphi \in C^2[-1, 1]$ and $f \in C[-1, 1]$ satisfy

$$\begin{cases} -\varphi'' + u_{YZ}\varphi = f, & x \in (-1, 1), \\ \varphi(-1) = 0 = \varphi(1), \end{cases}$$

for $A \in (A_1, \infty)$, then

$$\|\varphi\|_{L^\infty(-1,1)} \leq CA^{-\frac{2}{5}}\|f\|_{L^\infty(-1,1)}.$$

Although we do not show it, the above estimates are optimal as can be easily verified by simple scaling arguments.

In the case of even solutions, it turns out that $u_{Y_+Y_+}$ is asymptotically stable while $u_{Y_-Y_-}$ is unstable with Morse index equal to two (see Remarks 3.1, 3.3 and 4.1 below). In the nonsymmetric case, we can tell that $u_{Y_-Y_+}$ and $u_{Y_+Y_-}$ have Morse index one (see Remark 3.2 below).

We expect that the above a-priori estimate for the linearized operator can allow to extend the usual variational reduction procedure, similarly to [4], in order to construct new solutions to (1.1), having an arbitrary number of downward spikes near the origin (each of scale $A^{-\frac{1}{2}}$ and at an $\mathcal{O}\left((\ln A)A^{-\frac{1}{2}}\right)$ distance from the others) that are superimposed on the profile of u_{YZ} , for large $A > 0$.

The proof of Theorem 1.1 carries over directly to the case of the singular perturbation problem

$$\begin{cases} \varepsilon^2 u'' = F(u, x), & x \in (a, b), \\ u(a) = 0 = u(b), \end{cases}$$

provided that the following assumptions are met: $F \in C^3(\mathbb{R} \times [a, b])$ and there exists a $u_0 \in C^2(a, b) \cap C[a, b]$ such that $u_0(a) = 0 = u_0(b)$,

$$F(u_0(x), x) = 0, \quad x \in [a, b], \quad F_u(u_0(x), x) > 0, \quad x \in (a, b),$$

$$F_u = 0, \quad F_x < 0, \quad F_{uu} > 0, \quad F_{ux} = 0 \quad \text{at } (0, a),$$

$$F_u = 0, \quad F_x > 0, \quad F_{uu} > 0, \quad F_{ux} = 0 \quad \text{at } (0, b).$$

1.5. Method of proof. Our strategy is to apply a perturbation argument that has been used in many papers in the last years. This type of argument consists of three main steps: Firstly, one constructs a sufficiently good approximate solution to the problem, then studies the invertibility properties of the associated linearization about this approximation, and finally captures a true solution that is close, in some sense, to the approximate one by some type of fixed point argument. This approach, however, relies heavily on the good understanding of the corresponding limit problems, something which is not the case here since the non-degeneracy of Y_- does not seem to be known. In addition, the solutions that we expect to find are not localized in the conventional sense, as they should develop irregular boundary layers. In this regard, let us point out that an extra difficulty is that the convergence of Y_{\pm} to the square root profile is algebraically slow (see (2.4) below).

We are able to prove the non-degeneracy of Y_- by reducing (1.3) to the ground state problem for a nonlinear Schrödinger equation in the half-line with zero boundary conditions (see (2.2) below), and take advantage of the many studies that have been conducted on uniqueness and non-degeneracy issues for the latter problem (see [3, 8, 17, 21]). In fact, this also allows us to give a new proof of the uniqueness of Y_{\pm} , as was originally conjectured in [16] and proven by completely different techniques in [11]. Armed with the knowledge of the non-degeneracy of the blow-up profiles Y_{\pm} , we can deal with the difficulties related to the irregular boundary layer behavior by adapting the perturbative approach that was developed in the recent papers [20, 28], where the corresponding blow-up problem featured the Painlevé-II transcendent.

Let us point out that, in contrast to the problems in the aforementioned references, the instability of Y_- (recall our discussion in Subsection 1.3) suggests that the solutions of (1.1) with this blow-up profile in one of the boundaries should also be unstable. Therefore, the well known method of upper and lower solutions (barriers), see for example [26], should not be applicable to capture such solutions of (1.1).

1.6. Relations with geometric singular perturbation theory. The ordinary differential equation in (1.4) can be written as a three-dimensional, slow-fast system (see [29]), having a one-dimensional slow manifold which undergoes saddle-node bifurcations at the values ± 1 of the slow variable x . In light of the non-degeneracy of Y_- that we will prove, it seems plausible that our main result can also be proven by the blow-up approach to geometric singular perturbation theory (see [27] for a related problem with one turning point that involves the Painlevé-II transcendent).

1.7. Extensions. We expect that an analogous result to Theorem 1.1 holds for positive solutions to (1.5) (at least for $p \geq 2$), having the profile of the corresponding to Y_+ one-dimensional stable solution of the blow-up problem

$$\begin{cases} y_{ss} + \Delta_{\mathbb{R}^{N-1}} = |y|^p - s, & (s, \theta) \in (0, \infty) \times \mathbb{R}^{N-1}, \\ y(0, \theta) = 0, \quad y(s, \theta) - s^{\frac{1}{p}} \rightarrow 0 \quad \text{as } s \rightarrow \infty, & \text{uniformly in } \theta \in \mathbb{R}^{N-1}. \end{cases}$$

orthogonal to the boundary.

In view of the preceding discussion and the results of the current article, the only other solution of the above blow-up problem for which we have some non-degeneracy information is $Y_-(s)$ for the case $p = 2$. However, note that this solution has infinite Morse index as a solution of the above problem, and thus a corresponding

perturbation result should involve resonance phenomena (see [7, 22], and especially [19] where solutions exhibiting similar irregular layered behavior were studied).

1.8. Outline of the paper. In Section 2, we will construct sufficiently good approximate solutions to (1.1). This is the main section of the paper and it is where we will prove the non-degeneracy of Y_- (the full details will be postponed to Appendix A). In Section 3, we will study the invertibility properties of the linearization of (1.1) on the constructed approximate solutions, relying heavily on the non-degeneracy of Y_{\pm} . In Section 4, we will use the obtained linear estimates to perturb the approximate solutions to genuine ones, and also obtain related estimates for their difference by various comparison arguments. Finally, in Section 5, we will combine everything together to prove Theorem 1.1. We will close the paper with an appendix, providing the full details of the proof of the non-degeneracy of Y_- .

1.9. Notation. In the sequel, we will often suppress the obvious dependence on A of various functions and quantities. Furthermore, by c/C we will denote small/large generic constants, independent of A , whose value will change from line to line. The value of A will constantly increase so that all previous relations hold. The Landau symbol $\mathcal{O}(1)$, $A \rightarrow \infty$, will denote quantities that remain uniformly bounded as $A \rightarrow \infty$, whereas $o(1)$ will denote quantities that approach zero as $A \rightarrow \infty$.

2. CONSTRUCTION OF AN APPROXIMATE SOLUTION u_{ap}

In this section, we will construct sufficiently good approximate solutions to the problem (1.1) with the same type of behavior as the solutions that we are looking for.

2.1. The inner (boundary layer) solution u_{in} . In this subsection, motivated from the aforementioned formal analysis in [30], we will use the solutions Y_{\pm} of the blow-up problem (1.3) to construct approximate solutions to (1.1) which, however, are effective only near the boundary of the interval.

The properties of the blow-up profiles Y_{\pm} that we will need for the purposes of this paper are contained in the following proposition.

Proposition 2.1. *The boundary value problem (1.3) has exactly two solutions Y_+ and Y_- . We have that $(Y_+)' > 0$ in $[0, \infty)$, while $(Y_-)'(0) < 0$ and Y_- has a unique minimum at some point in $(0, \infty)$. Moreover, the solutions Y_{\pm} are non-degenerate, in the sense that there do not exist nontrivial bounded smooth solutions of*

$$\psi'' - Y_{\pm}\psi = 0 \quad \text{in } [0, \infty), \quad \varphi(0) = 0. \quad (2.1)$$

Proof. Existence of two solutions, satisfying the monotonicity properties described in the assertion of the proposition, has been established by Holmes and Spence [16] by a shooting argument (and in [4], via the method of upper/lower solutions and variational arguments, perhaps unaware of [16]). The authors of [16] also conjectured that these solutions were indeed the only ones. Their conjecture was settled, to the affirmative, by Hastings and Troy [11]. However, their proof was, as we discover (*almost 25 years later!*), much more complicated than necessary, and relied on some four decimal point numerical calculations. A truly simple proof of the uniqueness result of [11], which in the process implies the desired non-degeneracy of solutions can be given, based on a previous remark of ours from [20] (see Remark 35 therein), as follows. It is easy to see that Y_+ is non-degenerate and the unique increasing solution of (1.3) (see for example [4]). Let \tilde{Y} be any other solution of

(1.3), and let $u = Y_+ - \tilde{Y}$. By an easy calculation, and the maximum principle, we find that u has to be a solution of

$$2u'' - 2Y_+(s)u + u^2 = 0, \quad s > 0, \quad u(s) > 0, \quad s > 0, \quad u(0) = 0, \quad u(s) \rightarrow 0 \text{ as } s \rightarrow \infty. \quad (2.2)$$

It has been shown recently in [8] that the general problem

$$\begin{cases} u'' + \frac{\nu}{s}u' - V(s)u + u^p = 0, & s > a, \\ u(s) > 0, & s > a, \quad u(a) = 0, \quad u(s) \rightarrow 0 \text{ as } s \rightarrow \infty, \end{cases} \quad (2.3)$$

where $a > 0$, $\nu \geq 0$, $p \in (1, \infty)$ and $V \in C^1([a, \infty))$, has at most one solution provided that the auxiliary function

$$U(s) = V'(s)s^3 + \beta V(s)s^2 + (\beta - 2)L,$$

with

$$\alpha = \frac{2\nu}{p+3}, \quad \beta = (p-1)\alpha, \quad \text{and} \quad L = \alpha(\nu - 1 - \alpha),$$

satisfies

$$\liminf_{s \rightarrow \infty} U(s) > 0,$$

and one of the following conditions:

- (i): U is positive in (a, ∞) ,
- (ii): $U(a) < 0$ and U changes sign only once in (a, ∞) .

Moreover, if we further assume that $\nu > 0$, it has been shown in the same reference that the unique solution of (2.3) is non-degenerate (if such solution exists).

We would like to adapt the proof of the aforementioned result in order to establish that the solution $Y_+ - Y_-$ of (2.2) is unique and non-degenerate. Comparing with (2.3), in the problem at hand (2.2) we have $a = 0$, $p = 2$, $\nu = 0$, $\alpha = 0$, $V(s) = Y_+(s)$, $s > 0$, and $U(s) = s^3 Y_+'(s)$, $s > 0$. We note that, by scaling, the result of [8] continues to hold when a positive constant multiplies the power nonlinearity. Observe that the corresponding case (i) above holds. However, our potential $V = Y_+$ loses its positivity at $s = 0$ and also becomes unbounded as $s \rightarrow \infty$. On top of that, in our case $\nu = 0$ and not positive as required in [8] for showing the non-degeneracy of the solution. Nevertheless, as we will see, the proof of [8] can be easily adapted to establish uniqueness for the problem (2.2), and with some care the same can be done for showing that the solution $Y_+ - Y_-$ is non-degenerate. This implies at once that (1.3) has exactly the two solutions Y_+ and Y_- , while the desired non-degeneracy property of the solution Y_- follows readily. In Proposition A.1 of Appendix A we will indicate how the arguments of [8] can be adapted to provide uniqueness and non-degeneracy for (2.2).

The proof of the proposition is complete. □

Remark 2.1. *In [18, 19] we had previously applied the same idea, used in the proof of Proposition 2.1 for the study of Y_- , to the problem*

$$y'' = y^2 - s^2, \quad s \in \mathbb{R}; \quad y(s) - |s| \rightarrow 0 \text{ as } |s| \rightarrow \infty,$$

and showed that it has exactly two solutions, one which is stable and another one which is unstable. Interestingly enough, during the preparation of the current article, we came across the paper [15] where the same result was previously obtained by different techniques (which are similar to those that were subsequently used in [16]). Actually, the non-degeneracy of the unstable solution to the above problem, which

we also proved in [18] and enabled us to carry out the corresponding perturbation analysis, does not seem to be contained in [15].

Remark 2.2. In [16] it was also shown that there are solutions to (1.3) that approach $-\sqrt{s}$, instead of \sqrt{s} , as $s \rightarrow \infty$. For a thorough analysis of such solutions and more up to date references, we refer the interested reader to [12].

It is easy to show that

$$Y(s) - s^{\frac{1}{2}} = \mathcal{O}(s^{-2}) \text{ as } s \rightarrow \infty, \quad (2.4)$$

and

$$Y'(s) = \frac{1}{2}s^{-\frac{1}{2}} + \mathcal{O}(s^{-3}), \quad Y''(s) = \mathcal{O}(s^{-\frac{3}{2}}) \text{ as } s \rightarrow \infty, \quad (2.5)$$

(see also [28, App. A]).

Let Y , Z denote either one of Y_{\pm} , we define the inner solution of (1.1) near $x = -1$ as

$$u_{in}(x) = (2A)^{\frac{2}{5}}Y \left((2A)^{\frac{1}{5}}(x+1) \right) \text{ for } 0 \leq s \equiv (2A)^{\frac{1}{5}}(x+1) \leq \delta(2A)^{\frac{1}{5}}, \quad (2.6)$$

where $\delta > 0$ is a small constant independent of A . Similarly, close to $x = 1$, we define

$$u_{in}(x) = (2A)^{\frac{2}{5}}Z \left((2A)^{\frac{1}{5}}(1-x) \right) \text{ for } 0 \leq t \equiv (2A)^{\frac{1}{5}}(1-x) \leq \delta(2A)^{\frac{1}{5}}. \quad (2.7)$$

The effectiveness of u_{in} as an approximate solution can be mainly measured from the estimate in the following proposition.

Proposition 2.2. *We have*

$$2u_{in}'' - u_{in}^2 + A(1-x^2) = \mathcal{O}(A)(1-x^2)^2 \text{ as } A \rightarrow \infty,$$

uniformly on $[-1, -1+\delta] \cup [1-\delta, 1]$.

Proof. We will sketch the proof in the case where $x \in [-1, -1+\delta]$, the other case can be treated identically. The desired estimate follows readily by linearizing $1-x^2$ at $x = -1$, which reads as

$$1-x^2 = 2(x+1) - (x+1)^2,$$

and using (1.3).

The proof of the proposition is complete. \square

2.2. The modified outer solution \tilde{u}_{out} . Instead of using the outer solution

$$u_{out} = \sqrt{A(1-x^2)},$$

we will use a more sophisticated approximation

$$\tilde{u}_{out} = \left\{ A(1-x^2) - (2A)^{\frac{4}{5}} [s - Y^2(s)] n_{\delta}(1+x) - (2A)^{\frac{4}{5}} [t - Z^2(t)] n_{\delta}(1-x) \right\}^{\frac{1}{2}}, \quad (2.8)$$

$x \in [-1, 1]$, where s, t as in (2.6), (2.7) respectively (but now defined on $[0, \infty)$), and n_{δ} is a smooth cutoff function such that

$$n_d(r) = \begin{cases} 1 & \text{if } |r| \leq d, \\ 0 & \text{if } |r| \geq 2d. \end{cases} \quad (2.9)$$

Our motivation for the definition of \tilde{u}_{out} comes from [19, 20]. However, let us note that formulas of related nature can be found (at the formal level) in some books of asymptotic analysis (see [23, Ch. 8]).

The main result concerning \tilde{u}_{out} is the following.

Proposition 2.3. *We have*

$$2\tilde{u}_{out}'' - \tilde{u}_{out}^2 + A(1 - x^2) = \mathcal{O}(A^{\frac{1}{2}})(1 - x^2)^{-\frac{1}{2}}, \quad (2.10)$$

uniformly on $[-1 + \delta^{-1}(2A)^{-\frac{1}{5}}, 1 - \delta^{-1}(2A)^{-\frac{1}{5}}]$, as $A \rightarrow \infty$. Moreover, we find that

$$\tilde{u}_{out} - u_{in} = \mathcal{O}(A^{\frac{1}{2}})(1 - x^2)^{\frac{3}{2}}, \quad (\tilde{u}_{out} - u_{in})' = \mathcal{O}(A^{\frac{1}{2}})(1 - x^2)^{\frac{1}{2}}, \quad (2.11)$$

$$(\tilde{u}_{out} - u_{in})'' = \mathcal{O}(A^{\frac{1}{2}})(1 - x^2)^{-\frac{1}{2}},$$

uniformly on $[-1 + \delta^{-1}(2A)^{-\frac{1}{5}}, -1 + \delta] \cup [1 - \delta, 1 - \delta^{-1}(2A)^{-\frac{1}{5}}]$, as $A \rightarrow \infty$.

Proof. If $x \in [-1 + \delta^{-1}(2A)^{-\frac{1}{5}}, -1 + \delta]$, recalling (2.8) and (2.9), we obtain that

$$\tilde{u}_{out}^2 - A(1 - x^2) = (2A)^{\frac{4}{5}} [Y^2(s) - s] = u_{in}''. \quad (2.12)$$

In the same interval, we can write

$$\tilde{u}_{out}(x) = (2A)^{\frac{2}{5}} \left[-\frac{1}{2}(2A)^{\frac{1}{5}}(x+1)^2 + Y^2 \left((2A)^{\frac{1}{5}}(x+1) \right) \right]^{\frac{1}{2}}.$$

Hence, by (2.4) and (2.6), we get

$$\tilde{u}_{out} = (2A)^{\frac{2}{5}} Y(s) \left[1 - \frac{1}{2}(2A)^{-\frac{1}{5}} s^2 Y^{-2} \right]^{\frac{1}{2}} = (2A)^{\frac{2}{5}} Y(s) \left[1 + \mathcal{O}(A^{-\frac{1}{5}})s \right] = u_{in} + \mathcal{O}(A^{\frac{1}{5}})s^{\frac{3}{2}},$$

uniformly for $s \in [\delta^{-1}, \delta(2A)^{\frac{1}{5}}]$, as $A \rightarrow \infty$. Moreover, direct differentiation yields that

$$\begin{aligned} \tilde{u}'_{out} &= \frac{(2A)^{\frac{2}{5}}}{2} \left[-(2A)^{\frac{1}{5}}(x+1) + 2(2A)^{\frac{1}{5}} Y Y' \right] \left[-\frac{1}{2}(2A)^{\frac{1}{5}}(x+1)^2 + Y^2 \right]^{-\frac{1}{2}} \\ &= \frac{(2A)^{\frac{2}{5}}}{2} \left[-s + 2(2A)^{\frac{1}{5}} Y Y'(s) \right] \left[-\frac{1}{2}(2A)^{-\frac{1}{5}} s^2 + Y^2(s) \right]^{-\frac{1}{2}}, \end{aligned}$$

and

$$\begin{aligned} \tilde{u}''_{out} &= \frac{(2A)^{\frac{2}{5}}}{2} \left[-(2A)^{\frac{1}{5}} + 2(2A)^{\frac{2}{5}} (Y')^2(s) + 2(2A)^{\frac{2}{5}} Y'' Y(s) \right] \\ &\quad \times \left[-\frac{1}{2}(2A)^{-\frac{1}{5}} s^2 + Y^2(s) \right]^{-\frac{1}{2}} \\ &\quad - \frac{(2A)^{\frac{2}{5}}}{4} \left[-s + 2(2A)^{\frac{1}{5}} Y Y'(s) \right]^2 \left[-\frac{1}{2}(2A)^{-\frac{1}{5}} s^2 + Y^2(s) \right]^{-\frac{3}{2}}. \end{aligned}$$

By (2.4), (2.5), and (2.6), we have that

$$\begin{aligned} \tilde{u}'_{out} &= \left[u'_{in} - \frac{(2A)^{\frac{2}{5}}}{2} s Y^{-1} \right] \left[1 - \frac{1}{2}(2A)^{-\frac{1}{5}} s^2 Y^{-2} \right]^{-\frac{1}{2}} \\ &= \left[(2A)^{\frac{2}{5}} Y'(s) - \frac{(2A)^{\frac{2}{5}}}{2} \mathcal{O}(s^{\frac{1}{2}}) \right] \left[1 - \frac{1}{2}(2A)^{-\frac{1}{5}} \mathcal{O}(s) \right] \\ &= u'_{in} + \mathcal{O}(A^{\frac{2}{5}})s^{\frac{1}{2}} \end{aligned}$$

uniformly for $s \in [\delta^{-1}, \delta(2A)^{\frac{1}{5}}]$, as $A \rightarrow \infty$. In the same fashion, we can show that

$$\tilde{u}''_{out} = u''_{in} + \mathcal{O}(A^{\frac{3}{5}})s^{-\frac{1}{2}},$$

uniformly for $s \in [\delta^{-1}, \delta(2A)^{\frac{1}{5}}]$, as $A \rightarrow \infty$. We point out that $\tilde{u}''_{out} = (\tilde{u}''_{out})_1 - (\tilde{u}''_{out})_2$, with the obvious notation, and

$$(\tilde{u}''_{out})_1 = (2A)^{\frac{4}{5}}Y'' + (2A)^{\frac{4}{5}}(Y')^2Y^{-1} + \mathcal{O}(A^{\frac{3}{5}})s^{-\frac{1}{2}},$$

$$(\tilde{u}''_{out})_2 = (2A)^{\frac{4}{5}}(Y')^2Y^{-1} + \mathcal{O}(A^{\frac{3}{5}})s^{-\frac{1}{2}},$$

uniformly as $A \rightarrow \infty$.

In $[-1 + \delta, -1 + 2\delta]$, it follows readily from (2.4), (2.5), and (2.8) that

$$\tilde{u}_{out}^2 - A(1 - x^2) = \mathcal{O}(A^{\frac{1}{2}}), \quad \tilde{u}'_{out} = \mathcal{O}(A^{\frac{1}{2}}), \quad \tilde{u}''_{out} = \mathcal{O}(A^{\frac{1}{2}}),$$

uniformly as $A \rightarrow \infty$ (An easy way to see these is to note that we have $\tilde{u}_{out} \geq cA^{\frac{1}{2}}$ and then differentiate twice (2.12) with righthand side multiplied by the cutoff). Similar estimates hold for the remaining regions of $[-1, 1]$.

The desired assertions of the proposition follow readily from the above relations. \square

The following estimates will be useful in the sequel.

Lemma 2.1. *We have*

$$\tilde{u}_{out} = \sqrt{A(1 - x^2)} + \mathcal{O}((1 - x^2)^{-2}) \quad (2.13)$$

uniformly on $[-1 + \delta^{-1}(2A)^{-\frac{1}{5}}, -1 + 2\delta] \cup [1 - 2\delta, 1 - \delta^{-1}(2A)^{-\frac{1}{5}}]$, and

$$\tilde{u}_{out} = \sqrt{A(1 - x^2)}, \quad x \in [-1 + 2\delta, 1 - 2\delta]. \quad (2.14)$$

Proof. If $x \in [-1 + \delta^{-1}(2A)^{-\frac{1}{5}}, -1 + 2\delta]$, which implies that $s = (2A)^{\frac{1}{5}}(x+1) \geq \delta^{-1}$, from (2.8), via (2.4), we obtain that

$$\tilde{u}_{out} = A^{\frac{1}{2}}(1 - x^2)^{\frac{1}{2}} \left[1 + \mathcal{O}(s^{-\frac{5}{2}}) \right]^{\frac{1}{2}} = A^{\frac{1}{2}}(1 - x^2)^{\frac{1}{2}} \left[1 + \mathcal{O}(s^{-\frac{5}{2}}) \right],$$

and (2.13) follows readily. Analogously we treat the case where $x \in [1 - 2\delta, 1 - \delta^{-1}(2A)^{-\frac{1}{5}}]$. Relation (2.14) follows immediately from the definitions (2.8) and (2.9).

The proof of the lemma is complete. \square

2.3. Gluing the inner and outer approximations in order to create the global approximation u_{ap} . We define our global approximate solution to (1.1) to be the smooth function

$$u_{ap} = \begin{cases} u_{in}, & x \in [-1, -1 + \delta^{-1}(2A)^{-\frac{1}{5}}] \cup [1 - \delta^{-1}(2A)^{-\frac{1}{5}}, 1], \\ \tilde{u}_{out} + (\chi_- + \chi_+)(u_{in} - \tilde{u}_{out}), & x \in [-1 + \delta^{-1}(2A)^{-\frac{1}{5}}, 1 - \delta^{-1}(2A)^{-\frac{1}{5}}], \end{cases} \quad (2.15)$$

with u_{in} , \tilde{u}_{out} as in (2.6)–(2.7), (2.8) respectively, and

$$\chi_{\mp}(x) = n_{\delta^{-1}} \left((2A)^{\frac{1}{5}}(1 \pm x) \right), \quad (2.16)$$

where $n_{\delta^{-1}}$ defined through (2.9).

The main result concerning u_{ap} is the following.

Proposition 2.4. *Letting*

$$\mathcal{E} \equiv 2u''_{ap} - u_{ap}^2 + A(1 - x^2),$$

we have

$$\mathcal{E} = \begin{cases} \mathcal{O}(A)(1-x^2)^2, & x \in [-1, -1 + \delta^{-1}(2A)^{-\frac{1}{5}}] \cup [1 - \delta^{-1}(2A)^{-\frac{1}{5}}, 1], \\ \mathcal{O}(A^{\frac{1}{2}})(1-x^2)^{-\frac{1}{2}}, & x \in [-1 + \delta^{-1}(2A)^{-\frac{1}{5}}, 1 - \delta^{-1}(2A)^{-\frac{1}{5}}], \end{cases} \quad (2.17)$$

uniformly as $A \rightarrow \infty$.

Proof. Outside of the interpolation region $[-1 + \delta^{-1}(2A)^{-\frac{1}{5}}, -1 + 2\delta^{-1}(2A)^{-\frac{1}{5}}] \cup [1 - 2\delta^{-1}(2A)^{-\frac{1}{5}}, 1 - \delta^{-1}(2A)^{-\frac{1}{5}}]$, relation (2.17) follows at once from the assertions of Propositions 2.2 and 2.3. In $[-1 + \delta^{-1}(2A)^{-\frac{1}{5}}, -1 + 2\delta^{-1}(2A)^{-\frac{1}{5}}]$, by the estimates of Proposition 2.2 and 2.3, we have that

$$\begin{aligned} 2u_{ap}'' - u_{ap}^2 + A(1-x^2) &= 2\tilde{u}_{out}'' - \tilde{u}_{out}^2 + A(1-x^2) \\ &\quad + 2\chi_-''(u_{in} - \tilde{u}_{out}) + 4\chi_-'(u_{in} - \tilde{u}_{out})' + 2\chi_-(u_{in} - \tilde{u}_{out})'' \\ &\quad - 2\tilde{u}_{out}\chi_-(u_{in} - \tilde{u}_{out}) - \chi_-^2(u_{in} - \tilde{u}_{out})^2 \\ &= \mathcal{O}(A^{\frac{1}{2}})(1+x)^{-\frac{1}{2}} \\ &\quad + \mathcal{O}\left(A^{\frac{1}{2}}(1+x)^{\frac{3}{2}}A^{\frac{2}{5}} + A^{\frac{1}{5}}A^{\frac{1}{2}}(1+x)^{\frac{1}{2}} + A^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}\right) \\ &\quad + \mathcal{O}\left(A^{\frac{2}{5}}A^{\frac{1}{2}}(1+x)^{\frac{3}{2}} + A(1+x)^3\right) \\ &= \mathcal{O}(A^{\frac{1}{2}})(1+x)^{-\frac{1}{2}}, \end{aligned} \quad (2.18)$$

uniformly as $A \rightarrow \infty$. Analogous estimates hold true in the interpolation region $[1 - 2\delta^{-1}(2A)^{-\frac{1}{5}}, 1 - \delta^{-1}(2A)^{-\frac{1}{5}}]$.

The proof of the proposition is complete. \square

We have the following two easy corollaries.

Corrolary 2.1. *We have*

$$\|\mathcal{E}\|_{L^\infty(-1,1)} = \|2u_{ap}'' - u_{ap}^2 + A(1-x^2)\|_{L^\infty(-1,1)} \leq CA^{\frac{3}{5}}. \quad (2.19)$$

Proof. It follows directly from (2.17). \square

Corrolary 2.2. *We have*

$$u_{ap} = \sqrt{A(1-x^2)} + \mathcal{O}((1-x^2)^{-2}) \quad (2.20)$$

uniformly on $[-1 + \delta^{-1}(2A)^{-\frac{1}{5}}, -1 + 2\delta] \cup [1 - 2\delta, 1 - \delta^{-1}(2A)^{-\frac{1}{5}}]$, and

$$u_{ap} = \sqrt{A(1-x^2)}, \quad x \in [-1 + 2\delta, 1 - 2\delta]. \quad (2.21)$$

Proof. In view of (2.15), if x is not in the interpolation intervals $[-1 + \delta^{-1}(2A)^{-\frac{1}{5}}, -1 + 2\delta^{-1}(2A)^{-\frac{1}{5}}]$ and $[1 - 2\delta^{-1}(2A)^{-\frac{1}{5}}, 1 - \delta^{-1}(2A)^{-\frac{1}{5}}]$, the assertions of the corollary follow directly from the corresponding ones of Lemma 2.1. For x in the interpolation intervals, we also have to use (2.11).

The proof of the corollary is complete. \square

3. LINEAR ANALYSIS

In this section, we will study the linearization of (1.1) about the approximate solution u_{ap} , namely the linear Schrödinger operator

$$\mathcal{L}(\varphi) = -\varphi'' + u_{ap}\varphi, \quad \varphi \in C^2(-1,1) \cap C[-1,1], \quad \varphi(\pm 1) = 0, \quad (3.1)$$

(for convenience, we have divided by two).

3.1. Properties of the potential of the Schrödinger operator. In view of (2.6), we have

$$(2A)^{-\frac{2}{5}} u_{in} \left(-1 + (2A)^{-\frac{1}{5}} s \right) = Y(s), \quad 0 \leq s \leq 2(2A)^{\frac{1}{5}}.$$

Moreover, it follows from (2.11) that

$$(2A)^{-\frac{2}{5}} (\tilde{u}_{out} - u_{in}) \left(-1 + (2A)^{-\frac{1}{5}} s \right) = \mathcal{O}(A^{-\frac{1}{5}}) s^{\frac{3}{2}},$$

uniformly on $[\delta^{-1}, \delta(2A)^{\frac{1}{5}}]$, as $A \rightarrow \infty$. Hence, via (2.15), we find that

$$(2A)^{-\frac{2}{5}} u_{ap} \left(-1 + (2A)^{-\frac{1}{5}} s \right) \rightarrow Y(s) \text{ in } C_{loc}[0, \infty) \text{ as } A \rightarrow \infty. \quad (3.2)$$

Analogously, we find that

$$(2A)^{-\frac{2}{5}} u_{ap} \left(1 - (2A)^{-\frac{1}{5}} t \right) \rightarrow Z(t) \text{ in } C_{loc}[0, \infty) \text{ as } A \rightarrow \infty. \quad (3.3)$$

The asymptotic behavior of Y_{\pm} (recall (1.3)) and the definitions (2.6)–(2.7) imply that there exist constants $c, D > 0$, independent of A, δ , such that

$$u_{in} \geq cA^{\frac{1}{2}}(1 - x^2)^{\frac{1}{2}}$$

for $x \in \left[-1 + D(2A)^{-\frac{1}{5}}, -1 + 2\delta^{-1}(2A)^{-\frac{1}{5}} \right] \cup \left[1 - 2\delta^{-1}(2A)^{-\frac{1}{5}}, 1 - D(2A)^{-\frac{1}{5}} \right]$. Observe also that if $x \in \left[-1 + \delta^{-1}(2A)^{-\frac{1}{5}}, 1 - \delta^{-1}(2A)^{-\frac{1}{5}} \right]$ (which implies that $s, t \geq \delta^{-1}$), thanks to (2.4) and (2.8), we have

$$\tilde{u}_{out}^2 \geq A(1 - x^2) - CA^{\frac{4}{5}}\delta^{\frac{3}{2}} \geq A(1 - x^2) - CA\delta^{\frac{5}{2}}(1 - x^2) \geq \frac{A}{2}(1 - x^2),$$

for some constant $C > 0$ independent of both A and δ , where we have decreased the value of δ if necessary. Combining the above two relations with (2.11) and (2.15), we arrive at

$$u_{ap} \geq c\sqrt{\frac{A}{2}}\sqrt{1 - x^2}, \quad x \in \left[-1 + D(2A)^{-\frac{1}{5}}, 1 - D(2A)^{-\frac{1}{5}} \right]. \quad (3.4)$$

3.2. Uniform a-priori estimates. Let

$$\|\varphi\|_0 \equiv \|\varphi\|_{L^\infty(-1,1)} \equiv \sup_{(-1,1)} |\varphi(x)|. \quad (3.5)$$

The main result of this section is the following.

Proposition 3.1. *There exist constants $A_0, C > 0$ such that, given $f \in C[-1, 1]$, there exists a unique classical solution to the boundary value problem*

$$\mathcal{L}(\varphi) = f \text{ in } (-1, 1), \quad \varphi(\pm 1) = 0, \quad (3.6)$$

and this solution satisfies

$$\|\varphi\|_0 \leq CA^{-\frac{2}{5}} \|f\|_0, \quad (3.7)$$

provided that $A \geq A_0$.

Proof. To establish existence and uniqueness for (3.6), it suffices to show the a-priori estimate (3.7) which implies that the kernel of \mathcal{L} is empty (see for example [31]). Suppose that the latter estimate does not hold. Then, there would exist sequences $A_n > 0$, $\varphi_n \in C^2[-1, 1]$, $f_n \in C[-1, 1]$ such that

$$\mathcal{L}(\varphi_n) = f_n \text{ in } (-1, 1), \quad \varphi_n(\pm 1) = 0, \quad (3.8)$$

$$A_n \rightarrow \infty, \quad \|\varphi_n\|_0 = 1, \quad \text{and} \quad A_n^{-\frac{2}{5}} \|f_n\|_0 \rightarrow 0. \quad (3.9)$$

Without loss of generality, we may assume that there are $x_n \in (-1, 1)$ such that

$$\varphi_n(x_n) = \|\varphi_n\|_0 = 1, \quad \varphi_n'(x_n) = 0, \quad \text{and} \quad \varphi_n''(x_n) \leq 0,$$

(otherwise we can consider $-\varphi_n$). Equation (3.8), for $x = x_n$, gives us that

$$u_{ap}(x_n) \leq f_n(x_n).$$

In view of (3.4) and (3.9), we find that

$$\text{the } x_n \text{'s cannot be in } \left(-1 + 2\delta^{-1}(2A_n)^{-\frac{1}{5}}, 1 - 2\delta^{-1}(2A_n)^{-\frac{1}{5}}\right) \text{ for large } n.$$

Consequently, there are infinitely many n 's such that

$$x_n \in \left(-1, -1 + 2\delta^{-1}(2A_n)^{-\frac{1}{5}}\right] \quad \text{or} \quad x_n \in \left[1 - 2\delta^{-1}(2A_n)^{-\frac{1}{5}}, 1\right).$$

We may assume, without loss of generality, that the former case occurs. Therefore, abusing notation, we can choose a subsequence so that

$$x_n \in \left(-1, -1 + 2\delta^{-1}(2A_n)^{-\frac{1}{5}}\right], \quad n \geq 1. \quad (3.10)$$

Let

$$\Phi_n(s) \equiv \varphi_n(x), \quad F_n(s) \equiv f_n(x), \quad x = -1 + (2A_n)^{-\frac{1}{5}}s.$$

Then, relations (3.8) and (3.9) become

$$\Phi_n'' - (2A_n)^{-\frac{2}{5}}u_{ap} \left(-1 + (2A_n)^{-\frac{1}{5}}s\right) \Phi_n = (2A_n)^{-\frac{2}{5}}F_n \quad (3.11)$$

in $I_n \equiv \left[0, 2(2A_n)^{\frac{1}{5}}\right]$, $\Phi_n = 0$ on the boundary of I_n , and

$$\|\Phi_n\|_{L^\infty(I_n)} = 1, \quad A_n^{-\frac{2}{5}}\|F_n\|_{L^\infty(I_n)} \rightarrow 0, \quad (3.12)$$

respectively. Furthermore, recalling (3.10), we have that

$$\Phi_n(s_n) = 1, \quad \text{where } s_n \equiv (2A_n)^{\frac{1}{5}}(x_n + 1) \in (0, 2\delta^{-1}]. \quad (3.13)$$

Making use of (3.2), (3.11), (3.12), (3.13), and a standard diagonal compactness argument, passing to a further subsequence, we find that

$$\Phi_n \rightarrow \Phi_* \quad \text{in } C_{loc}^2[0, \infty), \quad s_n \rightarrow s_* \in [0, 2\delta^{-1}],$$

where

$$\Phi_*'' - Y(s)\Phi_* = 0 \quad \text{in } (0, \infty), \quad \Phi_*(0) = 0, \quad \|\Phi_*\|_{L^\infty(0, \infty)} \leq 1, \quad \text{and} \quad \Phi_*(s_*) = 1.$$

On the other hand, by the non-degeneracy of Y_\pm (recall Proposition 2.1), we arrive at a contradiction. We have thus established the validity of (3.7).

The proof of the proposition is complete. □

Remark 3.1. Let $\mu_1^\pm < \mu_2^\pm < \dots$, with $\mu_i^\pm \rightarrow \infty$ as $i \rightarrow \infty$, denote the eigenvalues of the linear operators

$$\mathcal{M}_\pm(\psi) = -\psi'' + Y_\pm(s)\psi$$

with domain $\{\psi \in H^2(0, \infty), \sqrt{s}\psi \in L^2(0, \infty), \psi(0) = 0\}$, which are self-adjoint in $L^2(0, \infty)$ and have only simple eigenvalues in their spectrum since $Y_\pm(s) \rightarrow \infty$ as $s \rightarrow \infty$ (see [13] for more details). It follows from Propositions 2.1 and A.1 that

$$\mu_1^- < 0, \quad \mu_2^- > 0, \quad \text{while} \quad \mu_1^+ > 0.$$

In the case where u_{ap} is even, using the obvious notation, we denote the eigenvalues of the linear operators \mathcal{L}_\pm in (3.1) by $\lambda_1^\pm < \lambda_2^\pm < \dots$. Arguing as in [10, 18], it follows readily that

$$\lambda_{i+1}^\pm - \lambda_i^\pm = \mathcal{O}(A^{-k}) \quad \text{and} \quad \lambda_i^\pm = \mu_i^\pm A^{-\frac{2}{5}} + \mathcal{O}\left(A^{-\frac{4}{5}}\right), \quad i = 1, 3, 5, \dots, 2\left[\frac{n}{2}\right] + 1,$$

with $k, n \in \mathbb{N}$ fixed, as $A \rightarrow \infty$. The main observation is that, because of the simplicity of the eigenvalues, the associated (normalized) eigenfunction to λ_{2m-1} is even whereas that associated to λ_{2m} is odd, for $m \geq 1$. Thus, the eigenvalue problem for \mathcal{L} in $(-1, 1)$ reduces to two eigenvalue problems in $(-1, 0)$ with boundary condition $\varphi(-1) = 0$, $\varphi(0) = 0$ and $\varphi(-1) = 0$, $\varphi'(0) = 0$ respectively. The main point being that the reduced eigenvalue problems have only one turning point (at $x = -1$) and the proof of [18, Prop. 3.25] applies directly. We expect that, as in [24], the difference between two clustering eigenvalues is actually exponentially small.

Remark 3.2. In the case where u_{ap} is nonsymmetric, one can adapt the proof of [18, Prop. 3.25] to show that the corresponding linear operator \mathcal{L} in (3.1) has only one negative eigenvalue, which satisfies $\lambda_1 = \mu_1^- A^{-\frac{2}{5}} + \mathcal{O}\left(A^{-\frac{4}{5}}\right)$ as $A \rightarrow \infty$ (where μ_1^- as in Remark 3.1). However, it is not clear to us how to obtain asymptotic expansions for the rest of the eigenvalues. Certainly this has to depend on the ordering between $\{\mu_i^-\}$ and $\{\mu_i^+\}$.

Remark 3.3. It is easy to see that relation (3.7) as well as the assertions of Remarks 3.1 and 3.2 continue to hold if the potential of \mathcal{L} was $u_{ap} + \phi$ with $A^{-\frac{2}{5}}\|\phi\|_0 \rightarrow 0$ as $A \rightarrow \infty$ (clearly ϕ has to be even for the latter remark to hold).

4. EXISTENCE OF SOLUTIONS AND ESTIMATES

We seek solutions of (1.1) in the form

$$u = u_{ap} + \phi. \tag{4.1}$$

Substituting this ansatz in (1.1), and rearranging terms, we see that ϕ solves

$$-2\phi'' + 2u_{ap}\phi = -\phi^2 + 2u_{ap}'' - u_{ap}^2 + A(1 - x^2), \quad x \in (-1, 1); \quad \phi(\pm 1) = 0. \tag{4.2}$$

The next proposition is the main result of this section.

Proposition 4.1. *If A is sufficiently large, there exists a constant $C > 0$ and a unique solution of (4.2) such that*

$$\|\phi\|_0 \leq CA^{\frac{1}{5}}. \tag{4.3}$$

Proof. Let us write (4.2) in the abstract form

$$2\mathcal{L}(\phi) = \mathcal{N}(\phi) + \mathcal{E}; \quad \phi(\pm 1) = 0, \tag{4.4}$$

where \mathcal{L} was studied in Section 3,

$$\mathcal{N}(\phi) \equiv -\phi^2,$$

and \mathcal{E} was defined in (2.17).

For $M > 0$, consider the closed ball of $C[-1, 1]$ that is defined by

$$\mathcal{B}_M = \left\{ \phi \in C[-1, 1] : \|\phi\|_0 \leq MA^{\frac{1}{5}} \right\}.$$

We will show that, if M is chosen sufficiently large, the mapping $\mathcal{T} : \mathcal{B}_M \rightarrow C^2[-1, 1]$, defined by

$$\mathcal{L}(\mathcal{T}(\phi)) = \mathcal{N}(\phi) + \mathcal{E}; \quad \mathcal{T}(\phi)(\pm 1) = 0,$$

(recall Proposition 3.1), maps \mathcal{B}_M into itself and is a contraction with respect to the $\|\cdot\|_0$ norm, provided that A is sufficiently large. Let $\phi \in \mathcal{B}_M$, via (2.19) and (3.7), we have

$$\begin{aligned} \|\mathcal{T}(\phi)\|_0 &\leq CA^{-\frac{2}{5}} (\|\mathcal{N}(\phi)\|_0 + \|\mathcal{E}\|_0) \\ &\leq CA^{-\frac{2}{5}} M^2 A^{\frac{2}{5}} + CA^{-\frac{2}{5}} A^{\frac{3}{5}} \\ &\leq CA^{\frac{1}{5}} (M^2 A^{-\frac{1}{5}} + 1), \end{aligned}$$

where $C > 0$ is independent of both large A and M . By virtue of the above relation, we can choose a large $M > 0$ such that \mathcal{T} maps \mathcal{B}_M into itself, for all sufficiently large A . From now on, we fix such an M . Similarly, for $\phi_1, \phi_2 \in \mathcal{B}_M$, we have

$$\|\mathcal{T}(\phi_1) - \mathcal{T}(\phi_2)\|_0 \leq CA^{-\frac{2}{5}} A^{\frac{1}{5}} \|\phi_1 - \phi_2\|_0 = CA^{-\frac{1}{5}} \|\phi_1 - \phi_2\|_0,$$

which implies that, for large A , the mapping $\mathcal{T} : \mathcal{B}_M \rightarrow \mathcal{B}_M$ is a contraction. Hence, by Banach's fixed point theorem, we infer that \mathcal{T} has a unique fixed point in the closed set \mathcal{B}_M . In turn, this furnishes a solution of (4.2) which satisfies the uniform estimate (4.3).

The proof of the proposition is complete. □

Remark 4.1. *If u_{ap} is even, we can of course restrict ourselves to even fluctuations ϕ in (4.1).*

In the next two lemmas we will show that estimate (4.3) can be improved away from the boundary points.

Lemma 4.1. *Let ϕ be as in Proposition 4.1. Given $L \geq 1$, there exists a constant $C_L > 0$ such that*

$$|\phi'(x)| \leq C_L A^{\frac{2}{5}}, \quad x \in \left[-1, -1 + (2A)^{-\frac{1}{5}} L\right] \cup \left[1 - (2A)^{-\frac{1}{5}} L, 1\right], \quad (4.5)$$

for all A sufficiently large.

Proof. Let

$$\Psi(s) = (2A)^{-\frac{1}{5}} \phi \left(-1 + (2A)^{-\frac{1}{5}} s\right), \quad s \in \left[0, 2(2A)^{\frac{1}{5}}\right]. \quad (4.6)$$

From (4.2), we find that

$$-\Psi'' + (2A)^{-\frac{2}{5}} u_{ap} \left(-1 + (2A)^{-\frac{1}{5}} s\right) \Psi = -\frac{1}{2} (2A)^{-\frac{1}{5}} \Psi^2 + \frac{1}{2} (2A)^{-\frac{3}{5}} \mathcal{E} \left(-1 + (2A)^{-\frac{1}{5}} s\right), \quad (4.7)$$

for $s \in \left(0, 2(2A)^{\frac{1}{5}}\right)$, and $\Psi(0) = 0$. Furthermore, from (4.3), and (4.6), it follows that

$$\|\Psi\|_{L^\infty(0, 2(2A)^{\frac{1}{5}})} \leq C. \quad (4.8)$$

In turn, relations (2.19), (3.2), (4.7) and (4.8) imply that, given $L \geq 1$, there exists a constant $C_L > 0$ such that

$$|\Psi''(s)| \leq C_L \quad \text{on } [0, L],$$

provided that A is sufficiently large. Consequently, it follows from (4.8), the above relation, and the elementary interpolation inequality

$$\|\Psi'\|_{L^\infty(0,L)} \leq 2\|\Psi\|_{L^\infty(0,L)} + \|\Psi''\|_{L^\infty(0,L)}$$

(keep in mind that $L \geq 1$), that

$$|\Psi'(s)| \leq C_L \text{ on } [0, L],$$

provided that A is sufficiently large (for some possibly larger constant C_L). Now, the validity of estimate (4.5) on the interval $[-1, -1 + (2A)^{-\frac{1}{5}}L]$ follows directly via (4.6). Analogously we can show its validity on the interval $[1 - (2A)^{-\frac{1}{5}}L, 1]$.

The proof of the lemma is complete. \square

Lemma 4.2. *Let ϕ be as in Proposition 4.1. There exist positive constants C, D such that*

$$|\phi(x)| \leq C(1-x^2)^{-1}, \quad x \in \left[-1 + (2A)^{-\frac{1}{5}}D, 1 - (2A)^{-\frac{1}{5}}D\right], \quad (4.9)$$

provided that A is sufficiently large.

Proof. Let

$$\underline{\psi}(x) = -K(1-x^2)^{-1}, \quad x \in \left[-1 + (2A)^{-\frac{1}{5}}D, 1 - (2A)^{-\frac{1}{5}}D\right],$$

with constant $K > 0$ to be determined, and $D > 0$ to be chosen larger than that in (3.4), such that $\underline{\psi}$ is a lower solution to (4.2) on the above interval. Differentiating twice gives us

$$\underline{\psi}'' = -K(6x^2 + 2)(1-x^2)^{-3} \geq -8K(1-x^2)^{-3}.$$

Recalling (2.17), (3.4), and (4.3), we find that

$$\begin{aligned} -2\underline{\psi}'' + 2u_{ap}\underline{\psi} + \underline{\psi}^2 - \mathcal{E} &\leq 16K(1-x^2)^{-3} - cA^{\frac{1}{2}}K(1-x^2)^{-\frac{1}{2}} + CA^{\frac{2}{5}} \\ &\quad + cA^{\frac{1}{2}}(1-x^2)^{-\frac{1}{2}} \\ &\leq (1-x^2)^{-\frac{1}{2}} \left[CK(1-x^2)^{-\frac{5}{2}} - cKA^{\frac{1}{2}} + CA^{\frac{1}{2}} \right] \\ &\leq (1-x^2)^{-\frac{1}{2}} \left[CKD^{-\frac{5}{2}}A^{\frac{1}{2}} - cKA^{\frac{1}{2}} + CA^{\frac{1}{2}} \right] \\ &\leq (1-x^2)^{-\frac{1}{2}} \left[-\frac{1}{2}cKA^{\frac{1}{2}} + CA^{\frac{1}{2}} \right], \end{aligned}$$

where the constants c, C are independent of both A and D , having increased the value of D if necessary. Hence, we can chose a large $K > 0$ such that

$$-2\underline{\psi}'' + 2u_{ap}\underline{\psi} + \underline{\psi}^2 - \mathcal{E} \leq 0, \quad x \in \left[-1 + (2A)^{-\frac{1}{5}}D, 1 - (2A)^{-\frac{1}{5}}D\right],$$

provided that A is sufficiently large. By virtue of (4.2), (4.3), the above equation, and making use of the maximum principle, we deduce that

$$-C(1-x^2)^{-1} \leq \phi(x), \quad x \in \left[-1 + (2A)^{-\frac{1}{5}}D, 1 - (2A)^{-\frac{1}{5}}D\right],$$

for some large constant $C > 0$ and all large A . Analogously we can establish the other side of the desired estimate (4.9).

The proof of the lemma is complete. \square

In summary, we have the following.

Proposition 4.2. *There exists a solution of (1.1) such that*

$$u - u_{ap} = \mathcal{O}(A^{\frac{2}{5}})(1 - x^2), \quad x \in \left[-1, -1 + (2A)^{-\frac{1}{5}}D\right] \cup \left[1 - (2A)^{-\frac{1}{5}}D, 1\right],$$

$$u - u_{ap} = \mathcal{O}(1)(1 - x^2)^{-1}, \quad x \in \left[-1 + (2A)^{-\frac{1}{5}}D, 1 - (2A)^{-\frac{1}{5}}D\right],$$

for some constant $D \gg 1$, uniformly as $A \rightarrow \infty$.

5. PROOF OF THE MAIN RESULT

From Propositions 2.3 and 4.2, relation (2.15), Corollary 2.2 and Remark 3.3, we can infer the validity of Theorem 1.1.

APPENDIX A. UNIQUENESS AND NON-DEGENERACY OF SOLUTIONS FOR PROBLEM (2.2)

Proposition A.1. *Problem (2.2) has $Y_+ - Y_-$ as its unique solution. Moreover, this solution is non-degenerate, namely there are no nontrivial bounded solutions to (2.1) $_-$.*

Proof. To show uniqueness, we argue by contradiction and assume that there exist two distinct solutions u_1 and u_2 of (2.2). As in [3, 8, 17], the solutions u_1 and u_2 can be chosen such that $u'_1(0) < u'_2(0)$ and such that they intersect *at most once* in $(0, \infty)$ (this is achieved by a shooting argument, making use only of the smooth dependence on the initial data of solutions to the ordinary differential equation in (2.2)). Under this assumption, as in [3, 8, 17], we have that

$$\frac{d}{ds} \left(\frac{u_1(s)}{u_2(s)} \right) > 0, \quad s > 0. \quad (\text{A.1})$$

We point out that, in the above calculation, the terms involving Y_+ cancel each other, and thus the form of Y_+ is irrelevant for this part of the proof. Furthermore, if we define

$$E(s; u) = [u'(s)]^2 - Y_+(s)u^2(s) + \frac{1}{3}u^3(s), \quad s > 0, \quad u \in C^2([0, \infty)),$$

a direct calculation yields that

$$\frac{d}{ds} E(s; u_i) = -Y'_+ u_i^2 < 0, \quad s > 0, \quad i = 1, 2. \quad (\text{A.2})$$

Therefore, using the standard fact that any solution of (2.2) decays super-exponentially as $s \rightarrow \infty$, we obtain that

$$E(s; u_i) > \lim_{s \rightarrow \infty} E(s; u_i) = 0, \quad s > 0, \quad i = 1, 2. \quad (\text{A.3})$$

Next, as in [21], we set

$$F(s) = E(s; u_2) - \left(\frac{u_2}{u_1} \right)^2 E(s; u_1), \quad s > 0.$$

Note that, thanks to l'hospital's rule, we have $F(0) = 0$. A direct calculation, making use of (A.2), yields that

$$F'(s) = -\frac{d}{ds} \left\{ \left(\frac{u_2}{u_1} \right)^2 \right\} E(s; u_1), \quad s > 0.$$

So, in view of (A.1) and (A.3), we get that

$$F'(s) > 0, \quad s > 0.$$

Consequently, noting that (A.1) implies that

$$0 < \frac{u_2(s)}{u_1(s)} < \frac{u_2'(0)}{u_1'(0)}, \quad s > 0,$$

and making once more use of the super-exponential decay of u_1 and u_2 , we arrive at the *strict* inequality

$$0 = F(0) < \lim_{s \rightarrow \infty} F(s) = 0,$$

which is a contradiction. Hence, problem (2.2) has $Y_+ - Y_-$ as its only solution.

With some care, the non-degeneracy of φ can also be shown as in [8] (see also [3], [17]). The fact that $V(s) = Y_+(s) \rightarrow \infty$, as $s \rightarrow \infty$, poses an obstruction in adapting some proofs of [8] to our setting (especially the second part of the proof of Proposition 3.1 therein). Nevertheless, the fact that Y_+' is positive on $[0, \infty)$ and decays to zero at an algebraic rate, see (2.5), will allow us to bypass some of the arguments in [8], and in fact provide a more direct proof as follows. Firstly, motivated from [3], we define

$$\|\phi\| = \left(\int_0^\infty [(\phi')^2 + Y_+(s)\phi^2] ds \right)^{\frac{1}{2}},$$

and let X be the completion of $C_0^\infty(0, \infty)$ with respect to $\|\cdot\|$. We note that

$$\|\phi\|^2 \geq \mu_1 \int_0^\infty \phi^2 ds,$$

where $\mu_1 > 0$ is the principal eigenvalue of

$$-\psi'' + Y_+(s)\psi = \mu\psi, \quad s > 0, \quad \psi(0) = 0, \quad \psi \in L^2(\mathbb{R}).$$

Let $\varphi = Y_+ - Y_-$ be the unique solution of (2.2), then φ is a critical point of the functional

$$I(u) = \int_0^\infty \left(|u'|^2 + Y_+(s)u^2 - \frac{1}{3}u_+^3 \right) ds,$$

where $I : X \rightarrow \mathbb{R}$ is of class C^2 (here $u_+ = \max\{u, 0\}$). This functional has the mountain pass structure (see for instance [1]), and the unique solution φ of (2.2) corresponds to a mountain pass solution. We point out that, even though the interval $(0, \infty)$ is infinite, compactness is restored by the property that $Y_+(s) \rightarrow \infty$ as $s \rightarrow \infty$ (see [25]). We define the Morse index of φ as

$$i(I, \varphi) = \max \{ \dim H : H \subset X \text{ is a subspace such that } I''(\varphi)(h, h) < 0 \text{ for all } h \in H \setminus \{0\} \}.$$

It follows from the general theorem in [14] that

$$i(I, \varphi) \leq 1. \tag{A.4}$$

In fact, for the specific equation, this can be shown in an elementary way (see [2]). As in [3, 8, 17], we introduce a perturbed functional

$$J_\delta(u) = I(u) - \delta \int_0^\infty \left(\frac{1}{3}u_+^3 - \frac{1}{2}\varphi(s)u^2 \right) ds, \quad u \in X,$$

for small $\delta > 0$. By the maximum principle, we see that non-trivial critical points of J_δ are solutions to the problem

$$\begin{cases} 2u'' - (2Y_+(s) + \delta\varphi(s))u + (1 + \delta)u^2 = 0, & s > 0, \quad u(s) > 0, \quad s > 0, \\ u(0) = 0, \quad \lim_{s \rightarrow \infty} u(s) = 0. \end{cases} \quad (\text{A.5})$$

Observe that φ is a solution of (A.5) for all $\delta > 0$. As in [8], our primary objective is to apply the arguments that were used for showing uniqueness for (2.2) in order to infer that φ is the only solution of (A.5) if $\delta > 0$ is sufficiently small. These arguments can be applied almost word for word to (A.5), once we show that the corresponding relation to (A.2) holds. In other words, we have to show that

$$2Y'_+(s) + \delta\varphi'(s) > 0, \quad s > 0, \quad (\text{A.6})$$

for sufficiently small $\delta > 0$ (under the assumptions of [8], recall our discussion following (2.3), this was not possible and the authors had to argue indirectly). To this end, note that we have the following rough estimates:

$$Y'_+(s) \geq \min \left\{ c, \frac{1}{4}s^{-\frac{1}{2}} \right\}, \quad |\varphi'(s)| \leq Ce^{-s}, \quad s \geq 0,$$

for some positive constants c, C (the former estimate holds via (2.5), while the latter from the super-exponential decay of φ and (2.2)). We therefore deduce that (A.6) is valid if $\delta \in (0, m)$, where $m > 0$ is the minimum value of the function

$$2C^{-1} \min \left\{ c, \frac{1}{4}s^{-\frac{1}{2}} \right\} e^s, \quad s \geq 0.$$

Consequently, if $\delta > 0$ is sufficiently small, the function $\varphi = Y_+ - Y_-$ is the only solution to (A.5).

As in [3, 8, 17], in order to show that the unique solution φ of (2.2) is non-degenerate, we will argue by contradiction. So, assume that φ is degenerate. In view of (A.4), this implies that there exists a 2-dimensional subspace $H \subset X$ such that

$$I''(\varphi)(h, h) \leq 0 \quad \text{for all } h \in H.$$

Since

$$J''_\delta(u)(h, h) = I''(u)(h, h) - \delta \int_0^\infty (2u_+ - \varphi(s)) h^2 ds,$$

for any $u \in X$, $h \in H$, we have

$$J''_\delta(\varphi)(h, h) = I''(\varphi)(h, h) - \delta \int_0^\infty \varphi h^2 ds.$$

In particular, we see that

$$J''_\delta(\varphi)(h, h) < 0 \quad \text{for all } h \in H \setminus \{0\},$$

which implies that $i(J_\delta, \varphi) \geq 2$. On the other hand, since J_δ has the mountain pass structure, and (A.5) has φ as its only solution for small $\delta > 0$, we must have $i(J_\delta, \varphi) \leq 1$ for small $\delta > 0$. We have therefore arrived at a contradiction, thus completing the proof of the non-degeneracy of φ .

The proof of the proposition is complete. □

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