ON THE SOLVABILITY OF A LINEAR INHOMOGENEOUS PROBLEM ARISING IN THE BLOW-UP ANALYSIS OF THE PHASE SEPARATION IN BOSE-EINSTEIN CONDENSATES

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ABSTRACT. We study the inhomogeneous linear system which arises in the higher order asymptotic expansion of the wave functions of multi-component Bose-Einstein condensates, across the regular part of the interface, in the case of segregation.

1. Introduction

The system

$$\Delta V_i = V_i \sum_{j=1, j\neq i}^{m} V_j^2, \quad V_i \geq 0 \text{ in } \mathbb{R}^n, \quad i = 1, \cdots, m,$$

arises in the blow-up analysis of the phase separation phenomenon in multi-component Bose-Einstein condensates (see for instance [8]). In fact, near the regular part of the interface, the phase separation should be governed by the corresponding one-dimensional system with just two nontrivial components, see [9], for which there holds the following result.

Proposition 1. [3, 4] *Given $\psi_0 > 0$, there exists a unique solution (V_1, V_2) with positive components to the system*

$$\begin{cases} \ddot{V}_1 = V_2^2 V_1, \\ \ddot{V}_2 = V_1^2 V_2, \end{cases} \tag{1}$$

such that

$$\frac{V_1}{x} \to \psi_0 \ \ \textit{and} \ \ V_2 \to 0 \ \textit{as} \ x \to \infty, \tag{2}$$

and

$$V_1(-x) = V_2(x), \ x \in \mathbb{R}. \tag{3}$$

Furthermore, every other entire solution of (1) with positive components is given by

$$\left(\mu V_1\left(\mu(x-h)\right), \mu V_2\left(\mu(x-h)\right)\right) \tag{4}$$

for some $\mu > 0$ and $h \in \mathbb{R}$.

Going further in the blow-up analysis, as is frequently needed in the study of problems related to the appearance of interfaces (see for instance [2, 5, 6] and the references therein), requires a good understanding of the invertibility properties of the associated linearized operator. In this regard, the following *injectivity* result was proven in [3].

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Proposition 2. *If* $\Phi_1, \Phi_2 \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *satisfy*

$$L\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix},$$

where

$$L\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix} = \begin{pmatrix}-\ddot{\Phi}_1 + V_2^2\Phi_1 + 2V_1V_2\Phi_2\\ -\ddot{\Phi}_2 + V_1^2\Phi_2 + 2V_1V_2\Phi_1\end{pmatrix}, \tag{5}$$

then

$$(\Phi_1, \Phi_2) \equiv \lambda(\dot{V}_1, \dot{V}_2)$$

for some $\lambda \in \mathbb{R}$.

The purpose of this note is to show how the above proposition can be used to establish the existence of solutions with *linear growth* to the inhomogeneous linear problem:

$$L\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix} = \begin{pmatrix}F_1\\ F_2\end{pmatrix}, \quad x \in \mathbb{R}, \tag{6}$$

with F_1, F_2 being smooth and decaying exponentially fast, which seems to be what is needed for the aforementioned purpose (see [1]). We remark that it is not clear to us how to use tools from functional analysis to achieve this because, as we expect from the general theory in [7], the continuous spectrum of L (when defined in the natural Hilbert space) should be the interval $[0, \infty)$ (it was also shown in [3] that the whole spectrum of L is nonnegative). Another obstruction is that, even though we are aware of two elements in the (formal) kernel of L, namely

$$(\dot{V}_1, \dot{V}_2) \text{ and } (x\dot{V}_1 + V_1, x\dot{V}_2 + V_2) \text{ (generated by the invariances in (4))},$$

the remaining two elements or their asymptotic behaviour are not known to us (in fact, we suspect that the latter should involve a super-exponential growth). Therefore, in contrast to related scalar second order problems (see for example [6, Lem. 4.1]), it is not clear how to derive conclusions from the corresponding variations of constants formula. Lastly, in relation to the cooperative character of L (see [4]), let us mention that we have not been able to construct appropriate upper and lower solution pairs to (6).

2. Main results

In view of (3), it is natural to decompose the space into the following two subspaces:

$$\mathcal{S} = \{(\varphi_1, \varphi_2) \ : \ \varphi_1(-x) \equiv \varphi_2(x)\}, \quad \mathcal{A} = \{(\varphi_1, \varphi_2) \ : \ \varphi_1(-x) \equiv -\varphi_2(x)\},$$

where $\mathcal{S}$ stands for (mirror) symmetric while $\mathcal{A}$ for anti-symmetric.

The important point is that Proposition 2 implies that the kernel of L does not have nontrivial bounded elements in $\mathcal{S}$. Based on this, the following *surjectivity* property was established recently in [1].

Proposition 3. *Given* $(F_1, F_2) \in \mathcal{S} \cap C(\mathbb{R})$ *satisfying*

$$|F_i(x)| \le Ce^{-c|x|}, \quad x \in \mathbb{R}, \ i = 1, 2, \tag{7}$$

for some constants $c, C > 0$, *there exists a unique solution* $(\Phi_1, \Phi_2) \in \mathcal{S} \cap C^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *to (6). Moreover, it holds*

$$\Phi_1(x) = a + \mathcal{O}(e^{-c'x}), \quad \Phi_2(x) = \mathcal{O}(e^{-c'x}) \quad \text{as} \quad x \to \infty,$$

for some $a \in \mathbb{R}$ and any $c' \in (0, c)$.

The main step in the proof was to obtain uniform estimates for solutions to the problem on finite intervals of the form $(-M, M)$ with Neumann boundary conditions that are *independent of M*. One can then easily pass to the limit uniformly over compacts (along a sequence $M_n \to \infty$), using a standard diagonal-compactness argument, to get a *bounded* solution in $\mathcal{S}$ of (6). The asymptotic behaviour follows at once from the following simple lemma (also keep in mind that V_2 decays super-exponentially fast as $x \to \infty$).

Lemma 1. *Suppose that u, q are smooth and satisfy*

$$-\ddot{u} + q(x)u = \mathcal{O}(e^{-c_0 x}) \quad \text{as} \quad x \to \infty,$$

for some constant $c_0 > 0$. Then, the following properties hold.

- $\liminf_{x\to\infty} q(x) = \infty$ *and u has at most algebraic growth* $\Longrightarrow$ $u = \mathcal{O}(e^{-c_0 x})$ *as $x \to \infty$,*
- $q = \mathcal{O}(e^{-c_0 x})$ *as $x \to \infty$* $\Longrightarrow$ $u = a_1 + b_1 x + \mathcal{O}(e^{-c_1 x})$ *as $x \to \infty$ for some $a_1, b_1 \in \mathbb{R}$ and any $c_1 \in (0, c_0)$.*

Proof. The first property can be shown as in [5, Lem. 7.3], while the second as in [2, Lem. 3.2]. □

In light of the above proposition, we may restrict ourselves to the subspace $\mathcal{A}$. The next lemma is a sort of *Fredholm alternative* for L.

Lemma 2. *Given $(F_1, F_2) \in \mathcal{A} \cap C(\mathbb{R})$ satisfying the exponential decay (7) and the orthogonality condition*

$$\int_{-\infty}^{\infty} \left(F_1 \dot{V}_1 + F_2 \dot{V}_2\right) dx = 0, \tag{8}$$

there exists a unique solution $(\Phi_1, \Phi_2) \in \mathcal{A} \cap C^2(\mathbb{R})$ to (6) such that

$$\Phi_1(x) = \mathcal{O}(e^{-c'x}), \quad \Phi_2(x) = \mathcal{O}(e^{-c'x}) \quad \text{as} \quad x \to \infty,$$

for any $c' \in (0, c)$.

Proof. Motivated by the comments after Proposition 3, we will construct the desired solution through a limiting process. We consider the sequence of approximate problems:

$$L\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} F_{1,n} \\ F_{2,n} \end{pmatrix}, \quad x \in (-n, n); \quad \Phi_i(\pm n) = 0, \ i = 1, 2, \tag{9}$$

where

$$\begin{pmatrix} F_{1,n} \\ F_{2,n} \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} - d_n \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-c|x|},$$

where

$$d_n = \frac{\int_{-n}^{n} \dot{V}_1 F_1 dx}{\int_{-n}^{n} \dot{V}_1 e^{-c|x|} dx} \qquad \text{(keep in mind that } \dot{V}_1 > 0\text{)}$$

is chosen so that

$$\int_{-n}^{n} \left(F_{1,n} \dot{V}_1 + F_{2,n} \dot{V}_2\right) dx = 0. \tag{10}$$

We note that (7), (8) and Lebesgue's dominated convergence theorem yield that

$$d_n \to 0.$$

Problem (9) has a unique solution $(\Phi_{1,n}, \Phi_{2,n})$ for large n, as is guaranteed by [10, Prop. 2.3]. In particular, this solution belongs to the subspace $\mathcal{A}$. We stress that the orthogonality relation (10) is not used at all for this. However, without the orthogonality relation, the latter proposition just gives us the estimate

$$\|\Phi_{i,n}\|_{L^\infty(-n,n)} \le C_0 n, \quad i = 1, 2,$$

for some fixed constant $C_0 > 0$. Even though this points in the right direction as to what we eventually wish to show in this note (see the next proposition), it is not helpful in passing to the limit in (9). The key observation is that by incorporating the orthogonality condition (10) at the end of the proof of the above estimate, *the resulting estimate does not degenerate as* $n \to \infty$. This allows us to pass to the limit (along a subsequence), in the standard way, and produce a *bounded* solution in $\mathcal{A}$ of (6). The desired asymptotic behaviour follows at once from Lemma 1 and the fact that $(\dot{V}_1, \dot{V}_2) \in \mathcal{A}$ is in the kernel of L. Finally, the uniqueness of such a solution is a direct consequence of Proposition 2. □

Remark 1. *In the above proof, one may be tempted to achieve the desired orthogonality of the righthand side by a term of the form* $d_n(\dot{V}_1, \dot{V}_2)$. *However, we stress that the argument in* [10, Prop. 2.3] *requires that the resulting righthand side has a* uniform decay in both directions.

We can now establish our main result:

Proposition 4. *Given* $(F_1, F_2) \in \mathcal{A} \cap C(\mathbb{R})$ *satisfying (7), there exists a unique solution* $(\Phi_1, \Phi_2) \in \mathcal{A} \cap C^2(\mathbb{R})$ *to (6) such that*

$$\Phi_1(x) = bx + \mathcal{O}(e^{-c'x}), \quad \Phi_2(x) = \mathcal{O}(e^{-c'x}) \quad \text{as} \quad x \to \infty,$$

for some $b \in \mathbb{R}$ *and any* $c' \in (0, c)$.

Proof. The main idea is to search for a solution in the form

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = B \begin{pmatrix} V_1 \\ -V_2 \end{pmatrix} + \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$$

with $B \in \mathbb{R}$ and $(\Psi_1, \Psi_2) \in \mathcal{A}$. The new equation that now needs to be satisfied is

$$L \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} + B \begin{pmatrix} 2V_1V_2^2 \\ -2V_2V_1^2 \end{pmatrix}.$$

To this end, we can apply Lemma 2 by choosing

$$B = -\frac{\int_{-\infty}^{\infty} F_1 \dot{V}_1 dx}{\int_{-\infty}^{\infty} 2V_1V_2^2 \dot{V}_1 dx} \overset{(1),(2)}{=} -\frac{1}{\psi_0^2} \int_{-\infty}^{\infty} F_1 \dot{V}_1 dx.$$

To complete the proof, it remains to add to the obtained solution the pair

$$-\frac{B}{\psi_0} \left[\lim_{x \to \infty} (V_1 - \psi_0 x) \right] (\dot{V}_1, \dot{V}_2).$$

□

Remark 2. *The constants* a, b *in Propositions 3, 4 may be determined in terms of* (F_1, F_2) *by testing (6) with* $(\dot{V}_1, \dot{V}_2)$ *and* $(x\dot{V}_1 + V_1, x\dot{V}_2 + V_2)$ *respectively.*

Remark 3. *The above results allow us to define the pair* $(\hat{\Phi}_1, \hat{\Phi}_2)$ *of* [10, Sec. 3] *in its natural domain, the entire real line that is. As a consequence, after some obvious modifications, we can dispense of the slightly annoying logarithms appearing in the corresponding estimates of the latter reference and not in* [1].

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