A LIOUVILLE TYPE RESULT FOR BOUNDED, ENTIRE SOLUTIONS TO A CLASS OF VARIATIONAL SEMILINEAR ELLIPTIC SYSTEMS

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Abstract. We prove a Liouville type result for bounded, entire solutions to a class of variational semilinear elliptic systems, based on the growth of their potential energy over balls with growing radius. Important special cases to which our result applies are the Ginzburg-Landau system and systems that arise in the study of multi-phase transitions.

We consider the semilinear elliptic system

$$\Delta u = W_u(u) \quad \text{in} \quad \mathbb{R}^n, \quad n \geq 2, \tag{1}$$

where $W : \mathbb{R}^m \to \mathbb{R}$, $m \geq 1$, is sufficiently smooth and

$$W \geq 0.$$

Solutions that are defined in all of $\mathbb{R}^n$ are appropriately called entire. Under these general assumptions, it is well known (see for instance [1, 4]) that solutions of (1) satisfy the following monotonicity property:

$$\frac{d}{dr}\left(\frac{1}{r^{n-2}}\int_{B_r}\left\{\frac{1}{2}|\nabla u|^2 + W(u)\right\}dx\right) \geq 0, \quad r > 0, \tag{2}$$

where B_r stands for the n-dimensional ball of radius r and center at the origin (keep in mind that (1) is translation invariant). As a direct consequence, we have the following Liouville type theorem:

Liouville property A *The only solutions of (1) with finite energy, that is*

$$\int_{\mathbb{R}^n}\left\{\frac{1}{2}|\nabla u|^2 + W(u)\right\}dx < \infty, \tag{3}$$

are the constant ones.

(The case $n = 2$ requires some extra work, see [1]).

If $m = 1$, bounded solutions of (1) satisfy the gradient bound:

$$\frac{1}{2}|\nabla u|^2 \leq W(u) \quad \text{in} \quad \mathbb{R}^n, \tag{4}$$

(see [7, 13, 17]), and thus a stronger Liouville property holds:

Liouville propertly B *The only bounded solutions of (1) with finite potential energy, that is*

$$\int_{\mathbb{R}^n} W(u)\,dx < \infty, \tag{5}$$

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are the constant ones.

In fact, the gradient estimate (4) implies that a stronger monotonicity formula holds with $n-1$ in place of $n-2$ in (2) (see [1, 7, 18]).

If $m \geq 2$, the analog of the gradient bound (4) does not hold in general, as is indicated by the counterexamples in [11, 19]. Despite of this, there are such systems for which the Liouville property B is in effect. Indeed, let us consider the class of systems, arising in the study of multi-phase transitions (see [3]), where W has a finite number of global minima which, in addition, are non-degenerate. Let u be a bounded solution of (1) such that (5) holds. Using that $|\nabla u|$ is also bounded (by standard elliptic estimates [15]) in the same way as in [6], and exploiting that the global minimizers of W are isolated, we find that there exists a global minimizer $a \in \mathbb{R}^m$ of W such that

$$|u(x) - a| \to 0 \quad \text{as} \quad |x| \to \infty. \tag{6}$$

In turn, the non-degeneracy of a yields that

$$|u(x) - a| + |\nabla u| \leq C_0 e^{-C_1|x|}, \quad x \in \mathbb{R}^n,$$

for some constants $C_0, C_1 > 0$ (see [16]). The above relation clearly implies that (3) holds and, thanks to the Liouville property A, we conclude that $u \equiv a$.

In the case where the global minima of W are not isolated, the previous simple argument fails. Nevertheless, it was shown in [11] that the Liouville property B holds for the Ginzburg-Landau system

$$\Delta u = \left(|u|^2 - 1\right) u, \text{ (here } W(u) = \frac{\left(1-|u|^2\right)^2}{4} \text{ vanishes on } \mathbb{S}^{m-1}\text{)}, \tag{7}$$

arising in superconductivity, provided that $n \geq 4$. On the other hand, this property does not hold when $n = m = 2$ (see [6]), whereas it was shown to hold if $n = 3$ and $m = 2$ in [12]. The approach in [11] is based on standard elliptic estimates and the monotonicity formula (2). On the other side, let us mention that for this system one can control the growth of the potential energy (over growing balls) in terms of the corresponding growth of the kinetic energy (see [6, 10]).

In this note, we will prove that a stronger assertion than the Liouville property B holds for a broad class of systems, which includes both the phase transition and the Ginzburg-Landau system that we mentioned above. As will be apparent, another advantage of our proof is its simplicity.

Our main result is the following.

Theorem 1. Assume that $W \in C^1(\mathbb{R}^m; \mathbb{R})$ is nonnegative and satisfies

$$-\left(u - Q\right) \cdot W_u(u) \leq C_2 \left(W(u)\right)^{\frac{p-1}{p}}, \quad u \in \mathbb{R}^m, \tag{8}$$

for some $Q \in \mathbb{R}^m$, $p \geq 2$ and a constant $C_2 > 0$ (here $\cdot$ stands for the Euclidean inner product of $\mathbb{R}^m$). Let $u \in C^2(\mathbb{R}^n; \mathbb{R}^m)$ be a bounded solution of (1).

I: If $n \geq 4$, the following Liouville property holds:

$$\int_{B_r} W(u) dx = o(r^q) \quad \textit{as} \quad r \to \infty, \tag{9}$$

with

$$q = \frac{p}{p-1}\left(n - 2 - \frac{n}{p}\right) = n - 2 - \frac{2}{p-1}, \tag{10}$$

implies that u is a constant, (where $o(\cdot)$ is the standard Landau symbol).

II: If $n = 4$ and $p = 2$, the Liouville property B holds, that is (5) implies that u is a constant.

Proof. We consider first the Case I. Without loss of generality, we may assume that $Q = 0$. The main effort will be placed in estimating the corresponding growth of the kinetic energy. To this end, we will adapt an argument from [22]. We take the inner product of (1) with u and integrate by parts over B_r, $r > 0$, to arrive at the relation

$$\int_{B_r} |\nabla u|^2 dx = -\int_{B_r} u \cdot W_u(u) dx + \int_{\partial B_r} u \cdot u_r dS, \tag{11}$$

where u_r denotes the radial derivative of u. The first term in the righthand side will be estimated initially by (8). Concerning the second term, letting

$$I(r) = \frac{1}{r^{n-1}} \int_{\partial B_r} |u|^2 dS,$$

we note that

$$I'(r) = \frac{2}{r^{n-1}} \int_{\partial B_r} u \cdot u_r dS,$$

(see also [8, Ch. 6]). Now, from equation (11), we obtain

$$\begin{aligned} \int_{B_r} |\nabla u|^2 dx &\leq C_2 \int_{B_r} (W(u))^{\frac{p-1}{p}} dx + \tfrac{1}{2} r^{n-1} I'(r) \\ &\leq C_3 r^{\frac{n}{p}} \left(\int_{B_r} W(u) dx \right)^{\frac{p-1}{p}} + \tfrac{1}{2} r^{n-1} I'(r) \\ \text{using (9)}: &\leq o(r^{n-2}) + \tfrac{1}{2} r^{n-1} I'(r), \end{aligned} \tag{12}$$

for some constant $C_3 > 0$, as $r \to \infty$.

Next, we claim that there exists a sequence $r_j \to \infty$ such that

$$r_j I'(r_j) \to 0.$$

If not, without loss of generality, there would exist positive constants δ, R_0 such that

$$I'(r) \geq \frac{\delta}{r}, \quad r \geq R_0,$$

which implies that

$$I(r) \geq I(R_0) + \delta \ln \left(\frac{r}{R_0} \right), \quad r \geq R_0.$$

However, this is not possible since I is a bounded function (from the assumption that u is bounded). It is worth mentioning that this argument can improve slightly the related estimates below relations (2.1) and (2.5) in [9].

To conclude, we take $r = r_j$ in (12) to find that

$$\int_{B_{r_j}} |\nabla u|^2 dx \leq o(r_j^{n-2}) \;\text{ as } j \to \infty.$$

Clearly, the monotonicity formula (2), the assumption (9) and the above relation yield that u is a constant, as desired.

The proof in the Case II requires some minor modifications. Again we may assume that $Q = 0$. Motivated from [11], we now integrate the relation $u \cdot \Delta u = u \cdot W_u(u)$ over the annulus $B_t \setminus B_s$ with $t > s$. Then, instead of (12), we have

$$\int_{B_t \setminus B_s} |\nabla u|^2 dx \leq C_3 t^2 \left(\int_{\mathbb{R}^n \setminus B_s} W(u) dx \right)^{\frac{1}{2}} + O(s^3) + \frac{1}{2} t^3 I'(t),$$

for all $t > s > 0$, where we also used that $|\nabla u|$ is bounded in $\mathbb{R}^n$ (by standard elliptic estimates [15]). In turn, this implies that

$$\int_{B_t} |\nabla u|^2 dx \leq C_3 t^2 \left(\int_{\mathbb{R}^n \setminus B_s} W(u) dx \right)^{\frac{1}{2}} + O(s^4) + \frac{1}{2} t^3 I'(t),$$

for all $t > s > 1$. As before, for any $s > 1$, there exists a sequence $\{t_j\}$ such that $t_j \to \infty$ and

$$\int_{B_{t_j}} |\nabla u|^2 dx \leq C_3 t_j^2 \left(\int_{\mathbb{R}^n \setminus B_s} W(u) dx \right)^{\frac{1}{2}} + O(s^4) + o(t_j^2), \tag{13}$$

as $j \to \infty$. Let $r > 0$ (independent of s, j). By the monotonicity formula (2), we obviously have that

$$\frac{1}{r^2} \int_{B_r} \left\{ \frac{1}{2} |\nabla u|^2 + W(u) \right\} dx \leq \frac{1}{t_j^2} \int_{B_{t_j}} \left\{ \frac{1}{2} |\nabla u|^2 + W(u) \right\} dx,$$

provided that j is sufficiently large (depending on r, s). Letting $j \to \infty$, via (13), we obtain that

$$\frac{1}{r^2} \int_{B_r} \left\{ \frac{1}{2} |\nabla u|^2 + W(u) \right\} dx \leq C_3 \left(\int_{\mathbb{R}^n \setminus B_s} W(u) dx \right)^{\frac{1}{2}},$$

for all $r > 0$ and $s > 1$. Finally, letting $s \to \infty$ and recalling (5), we deduce that u is a constant.

The proof of the theorem is complete. □

Remark 1. Condition (8) holds, with $p = 2$ (and for any $Q \in \mathbb{R}^m$), if $W \geq 0$ is C^2 and vanishes on finitely many smooth, compact manifolds which are non-degenerate, in the sense that at each point u of such a manifold M we have $T_u M = \mathrm{Ker}\,[W_{uu}(u)]$ (where $T_u M$ denotes the tangent space of M at the point u).

Remark 2. Let us revisit the phase transition systems that we mentioned previously in relation to (6). Condition (8) with $p = 2$ is clearly satisfied near the global minima of W (for any $Q \in \mathbb{R}^m$), and thus also in bounded subsets of $\mathbb{R}^m$. In fact, it is easy to see that the same is true (with $p \geq 2$) in the more general case where near each zero a of W the following holds: There exist $c > 0$ and $p \geq 2$ such that

$$W(u) \geq c|u - a|^p \quad \text{and} \quad |W_u(u)| \leq \frac{1}{c} |u - a|^{p-1} \tag{14}$$

(this is certainly satisfied if W vanishes up to a finite order at a). In passing, let us note that this class of systems appears in the recent study [5]. For such W, we showed very recently in [20] that, for any $k > 0$, bounded, nonconstant solutions of (1) satisfy:

$$\frac{1}{(\ln r)^k} \frac{1}{r^{n-2}} \int_{B_r} \left\{ \frac{1}{2} |\nabla u|^2 + W(u) \right\} dx \to \infty \quad \text{as} \quad r \to \infty, \tag{15}$$

(actually, this was shown in the case of nondegenerate minima but the proof carries over straightforwardly to this degenerate case). Armed with the above information, and modifying the proof of Theorem 1 accordingly, we obtain the following Liouville property for bounded solutions to such systems in $n \geq 4$ dimensions:

$$\int_{B_r} W(u)dx = O\left((\ln r)^k r^q\right), \quad \text{for some } k > 0, \quad \text{as} \quad r \to \infty \implies u \text{ is a constant,}$$

where the exponent q is as in (10).

We cannot resist to make the following observation. If W has only one such global minimum, say at the origin, it holds that

$$|u \cdot W_u(u)| \leq C_4 W(u)$$

for some constant $C_4 > 0$ (at least in the range of the considered bounded solution). Therefore, using this to estimate directly the first term in the righthand side of (11), and modifying slightly the rest of the proof, we infer that a stronger Liouville property holds for any solution in $n \geq 2$ dimensions:

$$\int_{B_r} W(u)dx = o\left(r^{n-2}\right) \quad \text{as} \quad r \to \infty \implies u \text{ is a constant.} \tag{16}$$

It is natural to conjecture that this property continues to hold when there are arbitrary many such minima. In this direction, see Remark 4 below.

Remark 3. Assume that W is nonnegative, sufficiently smooth ($C^{2,\alpha}$ with $0 < \alpha < 1$ suffices), and there exists an $M > 0$ such that

$$u \cdot W_u(u) > 0, \quad |u| > M.$$

Under these assumptions, it was shown recently in [19], in the spirit of [7], that there exists a constant $C_5 > 0$ such that all bounded solutions of (1) satisfy the gradient bound:

$$|\nabla u|^2 \leq C_5 \left(M^2 - |u|^2\right) \quad \text{in} \quad \mathbb{R}^n.$$

Assume further that there exists a constant $C_6 > 0$ such that

$$W(u) \geq C_6 \left(M^2 - |u|^2\right)^2, \quad |u| \leq M.$$

The above assumptions are clearly satisfied by the Ginzburg-Landau potential (with $M = 1$). Furthermore, as was observed in [19], they are also satisfied by a class of symmetric phase transition potentials.

It follows that there is a constant $C_7 > 0$ such that

$$\int_{B_r} |\nabla u|^2 dx \leq C_7 r^{\frac{n}{2}} \left(\int_{B_r} W(u)dx\right)^{\frac{1}{2}}, \quad r > 0,$$

(compare with (12) for $p = 2$). Now, we can argue by contradiction and use the monotonicity formula (2) to show the following: If u is a bounded, nonconstant solution of (1) in $n \geq 5$ dimensions, there exists a constant $C_8 > 0$ such that

$$\int_{B_r} W(u)dx \geq C_8 r^{n-4}, \quad r \geq 1.$$

Notice that the above lower bound represents a slight improvement over Theorem 1 for this particular class of potentials.

Suppose that, in addition, the potential W is of phase transition type and satisfies (14). Then, we can exploit (15) to obtain the stronger property: Given $k>0$, nonconstant, bounded solutions in $n\geq 4$ dimensions satisfy

$$\frac{1}{(\ln r)^k}\frac{1}{r^{n-4}}\int_{B_r} W\left(u\right)dx \to \infty \quad \text{as} \quad r\to\infty.$$

Remark 4. In the case of phase transition potentials with non-degenerate global minima, it was shown recently in [2] that each nonconstant, bounded and local minimizing solution (in the sense of Morse) satisfies

$$\int_{B_r}\left\{\frac{1}{2}|\nabla u|^2+W\left(u\right)\right\}dx\geq C_9 r^{n-1}, \quad r\geq 1,$$

for some constant $C_9>0$. Using this instead of the monotonicity formula in the proof of Theorem 1 (with $p=2$), we deduce that such solutions satisfy the Liouville property (16). Interestingly enough, if $n=2$ and under slightly more general assumptions, it was shown recently in [21] that such solutions satisfy

$$\int_{B_r} W\left(u\right)dx\geq C_{10} r, \quad r\geq 1,$$

for some constant $C_{10}>0$. For related Liouville type results for minimizing solutions, we refer the interested reader to [14].

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