

ON THE CONFINEMENT OF BOUNDED ENTIRE SOLUTIONS TO A CLASS OF SEMILINEAR ELLIPTIC SYSTEMS

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ABSTRACT. Under appropriate assumptions, we show that all bounded entire solutions to a class of semilinear elliptic systems are confined in a convex domain. Moreover, we prove a Liouville type theorem in the case where the domain is strictly convex. Our result represents an extension, under less regularity assumptions, of a recent result in [6]. We also provide several applications.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

The following result is contained in the very recent paper of P. Smyrnelis [6]:

Theorem 1.1. Let $W \in C^{2,\alpha}(\mathbb{R}^m, \mathbb{R})$, $\alpha \in (0, 1)$, be such that

$$u \cdot \nabla W(u) > 0 \text{ for } u \in \mathbb{R}^m \text{ with } |u| > R,$$

where $R > 0$ is some constant. If $u \in C^2(\mathbb{R}^n; \mathbb{R}^m) \cap L^\infty(\mathbb{R}^n; \mathbb{R}^m)$ is an entire solution to the equation

$$\Delta u = \nabla W(u), \quad x \in \mathbb{R}^n, \tag{1.1}$$

we have that $|u(x)| \leq R$, $x \in \mathbb{R}^n$. In addition, if u is not constant, then $|u(x)| < R$, $x \in \mathbb{R}^n$.

The proof is based on the P -function technique, see [7] for the case of scalar equations. Essentially, this technique consists in applying the maximum principle to a second order elliptic equation that is satisfied by a convenient scalar function $P(u; x)$ where u solves (1.1). The choice made in [6] was

$$P(u; x) = \frac{1}{2} |\nabla u(x)|^2 + C (|u(x)|^2 - R^2),$$

for some large constant $C > 0$. In fact, the gradient structure of the righthand side of (1.1) did not play any role in the proof of the above theorem. We point out that the reason for assuming that $W \in C^{2,\alpha}$ was to justify taking the Laplacian of the above function P .

The purpose of this note is to prove the following extension and improvement (as far as regularity is concerned) of the above result, and present some applications.

Theorem 1.2. Let \mathcal{D} be a smooth convex domain of \mathbb{R}^m (at least C^2). We assume that $F \in C^{0,1}(\mathbb{R}^m; \mathbb{R})$ and

$$(u - u_0) \cdot F(u) > 0 \quad \forall u \in \mathbb{R}^m \setminus \bar{\mathcal{D}}, \tag{1.2}$$

where $u_0 \in \partial\mathcal{D}$ is such that $|u - u_0| = \text{dist}(u, \partial\mathcal{D})$.

If $u \in C^2(\mathbb{R}^n; \mathbb{R}^m) \cap L^\infty(\mathbb{R}^n; \mathbb{R}^m)$ is an entire solution of

$$\Delta u = F(u) \text{ in } \mathbb{R}^n, \tag{1.3}$$

then

$$u(x) \in \bar{\mathcal{D}}, \quad x \in \mathbb{R}^n. \tag{1.4}$$

In addition, if \mathcal{D} is strictly convex, and u is non constant, we have that

$$u(x) \in \mathcal{D}, \quad x \in \mathbb{R}^n.$$

2. PROOF OF THE MAIN RESULT

The first assertion of Theorem 1.2 will follow from the following lemma which is of independent interest.

Lemma 2.1. Let $F \in C^{0,1}(\mathbb{R}^m; \mathbb{R}^m)$ satisfy

$$F(u_1, u_2, \dots, u_m) \cdot (u_1, 0, \dots, 0) > 0 \quad \text{if } u_1 > L, \quad u_i \in \mathbb{R}, \quad i = 2, \dots, m, \quad (2.1)$$

for some $L \geq 0$.

If $u = (u_1, \dots, u_m) \in C^2(\mathbb{R}^n; \mathbb{R}^m)$ is an entire bounded solution to the elliptic system (1.3), we have that

$$u_1(x) \leq L, \quad x \in \mathbb{R}^n.$$

Proof. We will argue by contradiction. For this purpose, suppose that

$$M = \sup_{x \in \mathbb{R}^n} u_1(x) > L, \quad (2.2)$$

(clearly $M < \infty$). There exist $x_j \in \mathbb{R}^n$ such that

$$u_1(x_j) \rightarrow M.$$

Let

$$v_j(x) = u(x + x_j).$$

We have that

$$\Delta v_j = F(v_j), \quad |v_j| \leq C_1, \quad x \in \mathbb{R}^n, \quad j \geq 1,$$

for some $C_1 > 0$. Moreover, the first component of v_j satisfies

$$(v_j)_1(0) = u_1(x_j) \rightarrow M \quad \text{and} \quad (v_j)_1(x) \leq M, \quad x \in \mathbb{R}^n.$$

By standard interior elliptic regularity estimates [1, 3], we deduce that

$$\|v_j\|_{C^{2,\alpha}(\mathbb{R}^n; \mathbb{R}^m)} \leq C_2, \quad j \geq 1, \quad (2.3)$$

where $0 < \alpha < 1$ is fixed, for some $C_2 > 0$. Hence, by well known compactness imbeddings (see [1, 3]), and the standard diagonal Cantor type argument, passing to a subsequence if necessary, we find that

$$v_j \rightarrow V \quad \text{in } C_{loc}^2(\mathbb{R}^n; \mathbb{R}^m),$$

for some $V = (V_1, \dots, V_m)$ which satisfies

$$\Delta V = F(V) \quad \text{in } \mathbb{R}^n, \quad \text{and} \quad V_1(0) = \sup_{x \in \mathbb{R}^n} V_1(x) = M. \quad (2.4)$$

In view of (2.1) and (2.2), we may assume that

$$\Delta V_1 > 0, \quad |x| < \delta,$$

for some small $\delta > 0$. On the other hand, the second relation in (2.4) contradicts the strong maximum principle (see [1, 3]). \square

We can now proceed to the proof of our main result.

Proof of Theorem 1.2. Let $p \in \partial D$ and T_p denote the tangent plane to ∂D at p . That tangent plane separates \mathbb{R}^m to two open connected components. The one component contains \mathcal{D} and the other one contains $\mathbb{R}^m \setminus \bar{\mathcal{D}}$. The first assertion of the theorem will follow if we show that the points $u(x)$ belong to the closure of the component that contains \mathcal{D} for every $x \in \mathbb{R}^n$. Since the equation (1.3) is invariant under translations and rotations, we may assume that p is the origin and that T_p is the hyperplane $\{u_1 = 0\}$ with $\mathcal{D} \subset \{u_1 < 0\}$. Clearly, assumption (2.1) is satisfied with $L = 0$. It then follows from Lemma 2.1 that the first component of u satisfies is nonnegative, as desired.

The second assertion of the theorem follows directly from [9]. For the sake of completeness, we will give a self-contained proof in the spirit of [8]. Let $u \in C^2(\mathbb{R}^n; \mathbb{R}^m)$ be a solution to (1.3) such that (1.4) holds and $u(x_0) \in \partial \mathcal{D}$ for some $x_0 \in \mathbb{R}^n$, where \mathcal{D} is additionally assumed to be strictly convex. We denote the signed distance of a point $u \in \mathbb{R}^m$ from $\partial \mathcal{D}$ by $d(u)$, that is $d(u) < 0$ if $u \in \mathcal{D}$ and $d(u) > 0$ if $u \in \mathbb{R}^m \setminus \bar{\mathcal{D}}$. It is well known that the function d is convex in \mathbb{R}^m , and smooth in a tubular neighborhood of $\partial \mathcal{D}$ (see [3]). In particular, by the strict convexity of $\partial \mathcal{D}$, we have that the Hessian

$$\partial^2 d(u) \text{ is positive definite for } u \in \partial \mathcal{D}. \quad (2.5)$$

The function

$$U(x) = d(u(x))$$

is smooth in a neighborhood of x_0 , say if $|x - x_0| < \epsilon$ for some small $\epsilon > 0$. For such x , using (1.2), (1.3) and (2.5), we find that

$$\begin{aligned} \Delta U(x) &= \operatorname{tr} \left\{ (\partial^2 d(u(x))) (\nabla u(x)) (\nabla u(x))^T \right\} + [(\nabla d)(u(x))] \cdot F(u(x)). \\ &\geq c |\nabla u(x)|^2 + [(\nabla d)(u(x))] \cdot F(u(x)), \end{aligned} \quad (2.6)$$

for some $c > 0$, having decreased $\epsilon > 0$ if needed. For $|x - x_0| < \epsilon$, let

$$Q(x) = \begin{cases} \frac{[(\nabla d)(u(x))] \cdot F(u(x))}{U(x)}, & \text{if } U(x) < 0, \\ 0, & \text{otherwise.} \end{cases}$$

If $u(x) \in \mathcal{D}$ with $|x - x_0| < \epsilon$, let $\tilde{u} \in \partial \mathcal{D}$ be such that $U(x) = -|u(x) - \tilde{u}|$. Note that $\nabla d(\tilde{u}) = \nu_{\tilde{u}}$, where $\nu_{\tilde{u}}$ denotes the outward unit normal vector to $\partial \mathcal{D}$ at \tilde{u} . So, from (1.2), we have that

$$Q(x) \leq \frac{[(\nabla d)(u(x))] \cdot F(u(x)) - \nabla d(\tilde{u}) \cdot F(\tilde{u})}{-|u(x) - \tilde{u}|} \leq C_3,$$

for some constant $C_3 > 0$, where we used the Lipschitz continuity of F and the smoothness of $\partial \mathcal{D}$. Since

$$\Delta U \geq Q(x)U \text{ if } |x - x_0| < \epsilon, \text{ and } U(x) \leq 0 = U(x_0),$$

a refinement of Hopf's boundary point lemma (see [1, Ch. 9]) yields that $\nabla U(x_0) \neq 0$ or U is constant, namely zero, for $|x - x_0| < \epsilon$ (apply the aforementioned lemma in [1] for $v = -U \geq 0$ and $c = Q$, noting that c bounded from above suffices for the proof to go through). On the other hand, since $U(x) \leq 0 = U(x_0)$ for $|x - x_0| < \epsilon$, we have that $\nabla U(x_0) = 0$. Thus, only the second scenario is possible. Hence, we infer that

$$U(x) = 0 \text{ if } |x - x_0| < \epsilon.$$

By a simple continuity argument, we have that

$$U(x) = 0, \quad x \in \mathbb{R}^n, \quad \text{that is } u(x) \in \partial\mathcal{D}, \quad x \in \mathbb{R}^n.$$

Observe that this holds for \mathcal{D} smooth and merely convex. Now, we will make use of the strict inequality in (2.5). By differentiating the above relation, and making use of (2.6), we infer that

$$\nabla u(x) = 0, \quad x \in \mathbb{R}^n,$$

that is

$$u(x) = u(x_0), \quad x \in \mathbb{R}^n,$$

as desired. □

3. APPLICATIONS

Below, we will present some applications of our main result.

3.1. The Ginzburg-Landau system. Consider the Ginzburg-Landau system which arises in superconductivity:

$$A\Delta u = (|u|^2 - 1)u, \quad x \in \mathbb{R}^n,$$

where u takes values in \mathbb{R}^m and A is a diagonal matrix with positive entries in the diagonal (see for example [5]). In the case where A is the identity, it was shown in [6], as a corollary of Theorem 1.1, that every entire bounded solution satisfies $|u| \leq 1$ in \mathbb{R}^n , and $|u| < 1$ in \mathbb{R}^n if u is nonconstant (actually, it was already shown in [2] that every entire solution is bounded and satisfies $|u| \leq 1$ in \mathbb{R}^n). In the general case, where A is not a positive constant multiple of the identity, it follows readily from Theorem 1.2 that the same properties continue to hold. Indeed, firstly observe that the function $v = Au$ satisfies

$$\Delta v = (|A^{-1}v|^2 - 1)A^{-1}v \quad \text{in } \mathbb{R}^n.$$

Let \mathcal{D} be the smooth and strictly convex domain $\{v \in \mathbb{R}^m : |A^{-1}v| < 1\}$. Let $v \in \mathbb{R}^m \setminus \bar{\mathcal{D}}$, that is $|A^{-1}v| > 1$, and $v_0 \in \partial\mathcal{D}$ be such that $|v - v_0| = \text{dist}(v, \partial\mathcal{D})$. Since the outer unit normal vector to $\partial\mathcal{D}$ at v_0 is $\frac{A^{-2}v_0}{|A^{-2}v_0|}$, we have that

$$v = v_0 + \frac{|v - v_0|}{|A^{-2}v_0|}A^{-2}v_0.$$

Using this, we find readily that

$$\begin{aligned} (|A^{-1}v|^2 - 1)A^{-1}v \cdot (v - v_0) &= (|A^{-1}v|^2 - 1) \frac{|v - v_0|}{|A^{-2}v_0|} (A^{-1}v) \cdot \left(A + \frac{|v - v_0|}{|A^{-2}v_0|} A^{-1} \right)^{-1} (A^{-1}v) \\ &\geq c|A^{-1}v|^2 > c, \end{aligned}$$

for some positive c . Theorem 1.2 then implies that $|A^{-1}v| \leq 1$ in \mathbb{R}^n , and $|A^{-1}v| < 1$ in \mathbb{R}^n if v is nonconstant. The corresponding assertions for $u = A^{-1}v$ follow at once.

3.2. The vectorial Allen-Cahn equation. Let $W : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a smooth function with three global nondegenerate minima at $a, b, c \in \mathbb{R}^2$ (not contained in the same line). Some special bounded solutions $u \in C^2(\mathbb{R}^2; \mathbb{R}^2)$ of the elliptic system (1.1) with $n = 2$, taking values close to a, b or c away from three half-lines (domain walls) that meet at the origin, are related to the study of some models of three-boundary motion in material science (see [4] and the references therein). The most natural choice is

$$W(u) = |u - a|^2 |u - b|^2 |u - c|^2.$$

Let u be a bounded entire solution to (1.1) for this W . By translating and rotating this solution, we may assume that the resulting function \tilde{u} solves (1.1) with W as above but with $a = (0, a_2)$, $b = (0, -a_2)$, $c = (c_1, c_2)$ such that $a_2 > 0$ and $c_1 < 0$. It is easy to show that (2.1) is satisfied. Hence, by Lemma 2.1, we see that the first component of \tilde{u} is non-positive. In turn, reversing the Euclidean motions, this implies that the values of u are on the same side of the line joining a and b as the triangle \widehat{abc} . Analogously, we can show that the range of u is contained in the closed \widehat{abc} triangle. In fact, from the proof of the second assertion of this theorem, we find that if a bounded entire solution touches one of the sides of the triangle, then it must be contained in this side for all $x \in \mathbb{R}^n$; clearly, this cannot happen for the solutions constructed in [4] which “take” all three phases.

3.3. Symmetry of components of a semilinear elliptic system. Our Lemma 2.1 also implies the following interesting property: If $F \in C^{0,1}(\mathbb{R}^2; \mathbb{R}^2)$ satisfies

$$(-u_2, u_1) \cdot F(u_1, u_2) > 0 \text{ for } u_1 \neq u_2,$$

then every bounded entire solution $u = (u_1, u_2)$ of (1.3) satisfies

$$u_1(x) = u_2(x), \quad x \in \mathbb{R}^n.$$

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