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Optimal potential energy growth lower bounds for minimizing solutions to the vectorial Allen-Cahn equation in two space dimensions

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We prove optimal lower bounds for the growth of the potential energy over balls of minimizers to the vectorial Allen-Cahn energy in two spatial dimensions, as the radius tends to infinity. Copyright © 2009 John Wiley & Sons, Ltd.

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1. Introduction

Consider the semilinear elliptic system

$$\Delta u = W_u(u) \ \text{ in } \ \mathbb{R}^n, \quad n \geq 1, \tag{1}$$

where $W : \mathbb{R}^m \to \mathbb{R}$, $m \geq 1$, is sufficiently smooth and nonnegative. This system has variational structure, and solutions in a smooth bounded domain $\Omega \subset \mathbb{R}^n$ are critical points of the energy

$$E(v;\Omega) = \int_\Omega \left\{ \frac{1}{2}|\nabla v|^2 + W(v) \right\} dx$$

(subject to their own boundary conditions). A solution $u \in C^2(\mathbb{R}^n;\mathbb{R}^m)$ is called *globally minimizing or minimal* if

$$E(u;\Omega) \leq E(u+\varphi;\Omega) \tag{2}$$

for every smooth bounded domain $\Omega \subset \mathbb{R}^n$ and for every $\varphi \in W_0^{1,2}(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$ (see also [22] and the references therein).

In the scalar case, namely $m = 1$, Modica [29] used the maximum principle to show that every bounded solution to (1) satisfies the pointwise gradient bound

$$\frac{1}{2}|\nabla u|^2 \leq W(u) \ \text{ in } \ \mathbb{R}^n, \tag{3}$$

(see also [14] and [19]). Using this, together with Pohozaev identities, it was shown in [30] that the energies of such solutions satisfy the following monotonicity property:

$$\frac{d}{dR}\left(\frac{1}{R^{n-1}} \int_{B(x_0,R)} \left\{ \frac{1}{2}|\nabla u|^2 + W(u) \right\} dx \right) \geq 0, \quad R > 0, \ x_0 \in \mathbb{R}^n, \tag{4}$$

where $B(x_0,R)$ stands for the n-dimensional ball of radius R that is centered at x_0. Combining the above two relations yields that, if $x_0 \in \mathbb{R}^n$, the "potential" energy of each bounded, nonconstant solution to the scalar problem satisfies the lower bound:

$$\int_{B(x_0,R)} W(u)\, dx \geq cR^{n-1}, \quad R > 0, \ \text{ for some } c > 0. \tag{5}$$

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In the scalar case, the most famous representative of this class of equations is the Allen-Cahn equation

$$\Delta u = u^3 - u \quad \text{in} \quad \mathbb{R}^n, \quad \text{where} \quad W(u) = \frac{(1-u^2)^2}{4}, \tag{6}$$

which is used to model phase transitions (see [20] and the references therein).

In the vectorial case, that is when $m \geq 2$, in the absence of the maximum principle, it is not known in general whether the analog of the gradient bound (3) holds; see [23] and [35] for examples and counterexamples respectively of its validity in the case of the phase transition systems that we describe below (9), and [18] for the case of the Ginzburg-Landau system (10). Actually, for potentials $W \geq 0$ that vanish on a codimension one manifold $\mathcal{M}$, easy counterexamples are provided by the one-dimensional periodic solutions in [28] which shadow closed, non-degenerate geodesics of $\mathcal{M}$ (whenever the latter exist). In turn, counterexamples to the Modica estimate for multiple well potentials can trivially be obtained by modifying W in the complement of a tubular neighborhood of such a closed orbit (see also [42] for a related construction). Nevertheless, it was shown in [1] using a stress energy tensor (see also [33, 38]), and earlier by [10, 16] via Pohozaev identities, that the energy of every solution to (1) (not necessarily bounded) satisfies the following weak monotonicity property:

$$\frac{d}{dR}\left(\frac{1}{R^{n-2}}\int_{B(x_0,R)}\left\{\frac{1}{2}|\nabla u|^2 + W(u)\right\}dx\right) \geq 0, \quad R>0,\ x_0 \in \mathbb{R}^n,\ n \geq 2. \tag{7}$$

In fact, as was observed in [1] (see also [38]), if a solution u satisfies Modica's gradient bound (3), it follows that its energy satisfies the strong monotonicity property (4). Armed with (7), and doing some more work in the case $n = 2$ (see [1]), it is easy to show that, if $x_0 \in \mathbb{R}^n$, the energy of each nonconstant solution to the system (1) satisfies:

$$\int_{B(x_0,R)}\left\{\frac{1}{2}|\nabla u|^2 + W(u)\right\}dx \geq \begin{cases} cR^{n-2} & \text{if } n \geq 3, \\ c\ln R & \text{if } n = 2, \end{cases} \tag{8}$$

for all $R > 1$ and some $c > 0$.

The above results hold for arbitrary smooth and nonnegative W. If additionally W vanishes at least at one point, it is easy to cook up a suitable competitor for the energy and show that bounded, globally minimizing solutions satisfy

$$\int_{B(x_0,R)}\left\{\frac{1}{2}|\nabla u|^2 + W(u)\right\}dx \leq CR^{n-1}, \quad R>0,\ x_0 \in \mathbb{R}^n, \tag{9}$$

for some $C > 0$ (see for example [15]). The system (1) with $W \geq 0$ vanishing at a finite number of non-degenerate, global minima is used to model *multi-phase transitions* (see [4, 12, 26] and the references therein). In this case, the system (1) is frequently referred to as the *vectorial Allen-Cahn equation*. Under appropriate assumptions (symmetries or non-degeneracy assumptions), it is possible to construct by variational methods "heteroclinic" solutions that "connect" along certain directions the global minima of W (see [3, 13, 21, 25], and [32] for a case without symmetry assumptions); the energy of these solutions over $B(x_0, R)$ is of order R^{n-1} as $R \to \infty$. This observation implies that the estimate (8) is far from optimal for this class of W's. In passing, we remark that the aforementioned entire equivariant solutions are minimizers within some symmetry class, and may not be minimal in the general sense of (2). On the other side, for the case of the Ginzburg-Landau system

$$\Delta u = \left(|u|^2 - 1\right)u, \quad u : \mathbb{R}^n \to \mathbb{R}^m, \quad \left(\text{here } W(u) = \frac{\left(1-|u|^2\right)^2}{4} \text{ vanishes on } \mathbb{S}^{m-1}\right), \tag{10}$$

when $n = m = 2$ there are globally minimizing solutions with energy over $B(x_0, R)$ of order $\ln R$ as $R \to \infty$ (see [9, 33] and the references therein). Moreover, when $n = m \geq 3$, there are analogous solutions with corresponding energy growth of order R^{n-2} as $R \to \infty$ (see [31]). In other words, the estimate (8) captures the optimal growth in the case of globally minimizing solutions to the Ginzburg-Landau system.

In the rest of this paper, we will restrict ourselves to the class of multi-phase transition systems, where $W \geq 0$ vanishes only at a finite number of global minimizers. If the latter are assumed to be non-degenerate, by adapting a lemma of [34] from the study of the Ginzburg-Landau system, we showed in [36] that, for any $k > 0$ and $x_0 \in \mathbb{R}^n$, bounded, nonconstant solutions of (1) satisfy:

$$\frac{1}{(\ln R)^k}\frac{1}{R^{n-2}}\int_{B(x_0,R)}\left\{\frac{1}{2}|\nabla u|^2 + W(u)\right\}dx \to \infty \quad \text{as} \quad R \to \infty,$$

(see also [41] for related estimates for just the potential energy term, there it was also noted that the above estimate remains valid even if the global minima of W have a finite degree of degeneracy). If in addition the solution is minimal, it was shown in [6], as a consequence of the density estimates proven therein (a natural but nontrivial extension of those in [15] for the scalar problem), that

$$\liminf_{R\to\infty}\frac{1}{R^{n-1}}\int_{B(x_0,R)}\left\{\frac{1}{2}|\nabla u|^2 + W(u)\right\}dx > 0, \quad \forall\ x_0 \in \mathbb{R}^n, \tag{11}$$

which in light of (9) is optimal.

From our perspective, property (11) gains its importance from the fact that its validity for bounded and minimal solutions of (1), in the case when W has a unique global minimum, when combined with our recent asymptotic monotonicity formula [39], implies the following Liouville type result: *If W has a unique global minimizer, then the only bounded and minimal solution of (1) is the aforementioned minimizer.* In turn, this Liouville type result easily implies uniform estimates for bounded, minimal solutions in domains with respect to the distance from the nonempty part of the boundary (when W has a unique global minimum in the closure of the range of such solutions). We emphasize that the latter estimates have played a pivotal role in some of the recent constructions of equivariant solutions that were mentioned previously in relation with (9), see in particular [21]. This Liouville property was shown originally in [22], in the case where W is strictly convex near its global minimum (see also [6] for a more recent proof, which rests upon the density estimates developed therein, in the case where that minimum is non-degenerate). In this regard, we highlight that our main result implies that, at least when $n = 2$, this Liouville property holds when W is merely monotone near its global minimum, in the sense of (13), and with less regularity in fact.

2. The main result

In this note, we will establish in a simple and elementary way the corresponding optimal growth lower bound for *just the potential energy term* when $n = 2$, that is (5) in the case of bounded and minimal solutions. In particular, our result strengthens the above lower bound of [6] when $n = 2$, even under weaker assumptions on W, and at the same time refines it (see Remark 4).

Our approach combines ideas from two disciplines: We adapt to this setting clearing-out arguments from the study of the Ginzburg-Landau system, see [9]. We elaborate around ideas from some variational maximum principles for globally minimizing solutions that have been recently devised and used for the study of the vectorial Allen-Cahn equation in [5].

Our proof seems unlikely to extend for $n \geq 3$, as will be apparent. However, keep in mind that the dimensions of physical interest are only $n = 1, 2, 3$. Nevertheless, we stress that the assumption $n = 2$ is used only at the first part of the proof; more precisely for reaching the conclusion (18). The arguments in the second part hold in all dimensions and can simplify some technical aspects in the proofs of the aforementioned variational maximum principles (at least when dealing with classical solutions and $W \in C^1$, see Remark 1).

Our main result is the following.

Theorem 1. *Assume that $W \in C^1(\mathbb{R}^m;\mathbb{R})$, $m \geq 1$, and that there exist finitely many $N \geq 1$ points $a_i \in \mathbb{R}^m$ such that*

$$W(u) > 0 \quad in \quad \mathbb{R}^m \setminus \{a_1, \cdots, a_N\}, \tag{12}$$

and there exists small $r_0 > 0$ such that the functions

$$r \mapsto W(a_i + r\nu), \quad where \quad \nu \in \mathbb{S}^{m-1}, \quad are\ nondecreasing\ for\ r \in (0, r_0), \quad i = 1, \cdots, N. \tag{13}$$

If $u \in C^2(\mathbb{R}^2;\mathbb{R}^m)$ is a bounded, nonconstant, and globally minimizing solution to the elliptic system

$$\Delta u = W_u(u) \quad in \quad \mathbb{R}^2, \tag{14}$$

for any $x_0 \in \mathbb{R}^2$, there exist constants $c_0, R_0 > 0$ such that

$$\int_{B(x_0,R)} W(u(x))\,dx \geq c_0 R \quad for \quad R \geq R_0. \tag{15}$$

Proof. Since the problem is translation invariant, without loss of generality, we may carry out the proof for $x_0 = 0$. Suppose, to the contrary, that there exists a bounded, nonconstant, and globally minimizing solution u of (14) such that

$$\int_{B(0,R_j)} W(u(x))\,dx = o(R_j), \quad \text{for some radii} \quad R_j \to \infty, \quad \text{as} \quad j \to \infty. \tag{16}$$

By making use of polar coordinates (see for instance [17, Appx. C]), the nonnegativity of W, and the integral mean value theorem, we infer that there exist

$$s_j \in \left(\frac{R_j}{2}, R_j\right)$$

such that

$$\int_{\partial B(0,s_j)} W(u(x))\,dS(x) = o(1) \quad \text{as} \quad j \to \infty. \tag{17}$$

We claim that this implies that

$$\max_{|x|=s_j} W(u(x)) = o(1) \quad \text{as} \quad j \to \infty. \tag{18}$$

Indeed, if not, passing to a subsequence if necessary, there would exist a constant $c_1 > 0$ such that

$$W\left(u(x_j)\right) \geq c_1 \quad \text{for some} \quad x_j \in \partial B(0, s_j), \quad j \geq 1.$$

On the other hand, since u is bounded in $\mathbb{R}^2$, by standard interior elliptic regularity estimates (see [17, 24]), we deduce that

$$|\nabla u| \in L^\infty(\mathbb{R}^2). \tag{19}$$

Hence, as in the clearing-out lemma of [9] (see Thm. III.3 therein), there exists $r_* > 0$ such that

$$W\left(u(x)\right) \geq \frac{c_1}{2}, \quad x \in B(x_j, r_*), \quad \text{for} \quad j \geq 1,$$

which, thanks to the nonnegativity of W, yields that

$$\int_{\partial B(0,s_j)} W\left(u(x)\right) dS(x) \geq \frac{c_1}{2}\mathcal{H}^1\left\{B(x_j, r_*) \cap \partial B(0, s_j)\right\} \geq c_2 \quad \text{for} \quad j \geq 1,$$

for some fixed constant $c_2 > 0$, where $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure. Clearly, the above relation contradicts (17), and thus the thesis of the claim follows. We stress that the assumption $n = 2$ is used *only* at this point in the proof.

Since u is bounded, relation (18) implies that there exist $i_j \in \{1, \cdots, N\}$ such that

$$\max_{|x|=s_j} \left|u(x) - a_{i_j}\right| = o(1) \quad \text{as} \quad j \to \infty. \tag{20}$$

We point out that we have not used so far the assumption that u is globally minimizing or the monotonicity property (13) of W near its wells. We will show that these force

$$\max_{|x|\leq s_j} \left|u(x) - a_{i_j}\right| = o(1) \quad \text{as} \quad j \to \infty. \tag{21}$$

As the proof of the above estimate is rather technical, in order not to interrupt the line of thought, let us first show how it can be used to conclude the proof of the theorem. To this end, observe that the above two relations imply the existence of an $i_0 \in \{1, \cdots, N\}$ such that

$$\max_{|x|\leq s_j} |u(x) - a_{i_0}| = o(1) \quad \text{as} \quad j \to \infty. \tag{22}$$

Now, letting $j \to \infty$ in the above relation yields that $u \equiv a_{i_0}$ which contradicts our assumption that u is nonconstant.

It remains to establish (21). The idea is that, if this is violated, one can construct a suitable competitor which agrees with u on $\partial B(0, s_j)$ but has less energy, thus contradicting the minimality of u. For this purpose, let us suppose to the contrary that

$$\max_{|x|\leq s_j} \left|u(x) - a_{i_j}\right| \geq r_1, \quad j \gg 1, \tag{23}$$

for some fixed constant $r_1 > 0$ (having passed to a further subsequence if needed). Then, let us consider a sequence $r_{2,j} \in \left(\frac{1}{100}\min\{r_0, r_1\}, \frac{1}{50}\min\{r_0, r_1\}\right)$ with the property that the following sets have smooth boundary:

$$\mathcal{A}_j = \left\{x \in B(0, s_j) \ : \ \left|u(x) - a_{i_j}\right| > 2r_{2,j}\right\},$$

$$\mathcal{B}_j = \left\{x \in B(0, s_j) \ : \ r_{2,j} < \left|u(x) - a_{i_j}\right| \leq 2r_{2,j}\right\}, \quad \mathcal{C}_j = \left\{x \in B(0, s_j) \ : \ \left|u(x) - a_{i_j}\right| \leq r_{2,j}\right\};$$

this is indeed possible by Sard's lemma [45] (note also that (20), (23) imply that the above sets are nonempty, and that $\bar{\mathcal{A}}_j, \bar{\mathcal{B}}_j$ are contained in $B(0, s_j)$, while $\partial B(0, s_j) \subset \bar{\mathcal{C}}_j$, for $j \gg 1$). To arrive at a contradiction, motivated from [5], we define the following auxiliary map:

$$v_j(x) = \begin{cases} u(x), & x \in \mathcal{C}_j, \\ a_{i_j} + \left(2r_{2,j} - \left|u(x) - a_{i_j}\right|\right)\frac{u(x)-a_{i_j}}{\left|u(x)-a_{i_j}\right|}, & x \in \mathcal{B}_j, \\ a_{i_j}, & x \in \mathcal{A}_j. \end{cases} \tag{24}$$

It follows at once that $v_j \in W^{1,2}\left(B(0, s_j); \mathbb{R}^m\right) \cap L^\infty\left(B(0, s_j); \mathbb{R}^m\right)$ and $v_j \equiv u$ on $\partial B(0, s_j)$. Moreover, similarly as in [5], we find that

$$E\left(v_j; B(0, s_j)\right) - E\left(u; B(0, s_j)\right) \overset{(13)}{\leq} E\left(v_j; \mathcal{A}_j\right) - E\left(u; \mathcal{A}_j\right) = -E\left(u; \mathcal{A}_j\right) < 0.$$

This contradicts the minimality of u, yielding the validity of (21).

The proof of the theorem is complete.

Remark 1. *Actually, with regards to (20), a careful examination of our arguments renders the following 'maximum principle':*

$$\max_{|x|\le s_j}\left|u(x)-a_{i_j}\right|\le \max_{|x|=s_j}\left|u(x)-a_{i_j}\right|, \tag{25}$$

as soon as j is large enough so that the righthand side, let's call it ρ_j, is smaller than $r_0/2$. Indeed, if the above relation was violated at a point $z_j\in B(0,s_j)$ such that $\left|u(z_j)-a_{i_j}\right|>2\rho_j$, then our arguments apply directly. It therefore remains to handle the case where $\left|u-a_{i_j}\right|\le 2\rho_j$ on $\bar{B}(0,s_j)$. But this is easy, since (13) yields that $|u-a_{i_j}|^2$ is sub-harmonic since $2\rho_j<r_0$. This variational maximum principle was originally proven recently in [5], within the more general context of $W^{1,2}$ maps (see also [7]). On the other hand, they assumed that strict monotonicity holds in (13) (see [40] for a small comment on this). The main difference with [5] is that they used the 'polar' representation of u (in analogy to the second branch in (24)) in the whole domain, even when $u=a_{i_j}$. This point, however, demanded a rather involved justification which our argument bypasses (a related observation was made independently in [7] very recently).

Remark 2. *It is worth mentioning that the passage from (20) to (21) if $n=1$ or $m=1$ can be shown by a similar 'local replacement' argument* which does not require condition (13), *see [37] and [44] respectively.*

Remark 3. *A careful examination of the proof of Theorem 1 reveals that the assumption of the boundedness of u, as well as that of the finiteness of the number of wells, can be dropped at the expense of imposing some natural uniformity conditions on W and compromising with the softer estimate (11) with $n=2$ as the conclusion. More precisely, we further assume that (13) holds with the same r_0 for all $i=1,\cdots$, and that the set $W^{-1}([0,\rho])$ shrinks uniformly to $\{a_1,a_2,\cdots\}$ as $\rho\to 0$. The main observation is the following. Writing u in polar coordinates as $u(x)=u(r,\theta)$, $r\ge 0$, $\theta\in\mathbb{S}^1$, and making use of the identity*

$$|\nabla u|^2=|u_r|^2+\frac{1}{r^2}|u_\theta|^2\quad (see\ [43, Ch.8]),$$

the corresponding relation to (17) becomes

$$\int_0^{2\pi}\left\{\frac{1}{2s_j}|u_\theta|^2+s_jW(u)\right\}d\theta=o(1),\quad at\ \ r=s_j,\quad as\ \ j\to\infty. \tag{26}$$

Hence, as in the clearing-out lemma in [11] (applied to $u\left(r_j,s_j^{-1}z\right)$), we deduce that the analog of (20) holds, and we can conclude as before.

Remark 4. *Under the assumptions of Theorem 1, the following refinement of (11) for $n=2$ holds:*

$$\liminf_{R\to\infty}\int_{\partial B(x_0,R)}\left\{\frac{1}{2}|\nabla u|^2+W(u)\right\}dx>0,\quad \forall\ x_0\in\mathbb{R}^2.$$

Indeed, this is a direct consequence of (15) and the Pohozaev identity for the system (1), that is

$$\int_{B(x_0,R)}\left\{\frac{n-2}{2}|\nabla u|^2+nW(u)\right\}dx=\int_{\partial B(x_0,R)}\left\{R\frac{|\nabla_T u|^2}{2}-\frac{R}{2}\left|\frac{\partial u}{\partial\nu}\right|^2+RW(u)\right\}dS(x)$$

(see [33, 34]).

Remark 5. *Actually, it is not hard to see that, under the assumptions of Theorem 1, the lower bound (15) continues to hold even if W in (15) is replaced by an arbitrary nonnegative, continuous $V:\mathbb{R}^m\to\mathbb{R}$ which vanishes only at a subset of $\{a_1,\cdots,a_N\}$. In fact, this property holds also for bounded, minimal solutions to the Ginzburg-Landau system (10) with $n=2$ (in that case V has to vanish at a finite number of points of $\mathbb{S}^1$). In the latter problem, the only essential modification in the proof of Theorem 1 is that one has to use Lemma 8 of [8] instead of arguing in the spirit of the maximum principle of [5].*

Remark 6. *There is an extension of the variational maximum principle (25), where the balls centered at a_{i_j} are replaced by smooth, convex bounded domains (with the obvious analog of (13), at least in the case of strict monotonicity). We refer the interested reader to [2] for the case $n=1$ and to the recent paper [7] for $n\ge 2$. Based on this, we expect that the assertion of Theorem 1 remains true if $W\ge 0$ vanishes only on a finite number of such domains and satisfies the analogous monotonicity property to (13) across their boundaries. It is worthwhile mentioning that the equation (1) with related potentials has been recently considered in [27].*

Open problems

In view of the aforementioned result of [6] and our theorem, it is natural to conjecture that the assertion of the latter holds in all dimensions, at least when the global minima of W are non-degenerate. In this regard, we note that the presence of the gradient in the energy was crucial in the corresponding proofs (there were two) of the latter reference.

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