ON THE NON-DEGENERACY OF LEAST ENERGY SOLUTIONS OF A CLASS OF NONLINEAR SCHRODINGER EQUATIONS

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Abstract. We derive a new necessary condition on the potential $V$, so that every least energy ground state of $\Delta u - V(|x|)u + u^p = 0$ in $\mathbb{R}^N$, $p$ subcritical, is non-degenerate.

We consider the problem

$$\Delta u - V(|x|)u + u^p = 0, \quad u > 0 \text{ in } \mathbb{R}^N, \quad \lim_{|x| \to \infty} u(x) = 0.$$ (0.1)

We assume that $N \geq 2$, and

$$p \in \left(1, \frac{N + 2}{N - 2}\right) \text{ if } N \geq 3 \text{ and } p \in (1, \infty) \text{ if } N = 2,$$ (0.2)

$$V \in C([0, \infty)) \cap C^2((0, \infty)), \quad \liminf_{|x| \to \infty} V(x) > 0, \text{ and } \lambda_1 (-\Delta + V(|x|)) > 0,$$ (0.3)

where $\lambda_1 (-\Delta + V(|x|))$ is the minimum value of the spectrum of $-\Delta + V(|x|)$,

$$V(r) \text{ is non-decreasing and non-constant on } (0, \infty),$$ (0.4)

$$rV'(r) + 2V(r) \text{ is non-decreasing on } (0, \infty).$$ (0.5)

It is well known (see [2]) that, under (0.2) and (0.3), there exists a least energy solution $w$ of (0.1). Furthermore, by (0.4), we have that $w$ is radial and

$$w'(r) < 0, \quad r > 0 \text{ (see [2]).}$$ (0.6)

In this note we will show

Proposition 0.1. Under assumptions (0.2), (0.3), (0.4), and (0.5) we have that $w$ is a linearly non-degenerate solution of (0.1), i.e., the spectrum of the linearized operator, in $L^2(\mathbb{R}^N)$,

$$L(\varphi) = -\Delta \varphi + \left(V(|x|) - pw^{p-1}\right) \varphi$$

does not contain 0.

Proof. We follow Lemma 3.1 of [1] where the authors gave a short and new proof of non-degeneracy of the least energy solution of (0.1) with $V = 1$. As in [1], this method of proof is restricted to the power nonlinearity.

Since $w$ is a least energy solution, it has Morse index one. Hence, the principal eigenvalue of $L$ is negative and the rest of the spectrum is non-negative (see also [1]).

To prove the proposition, we will argue by contradiction. Suppose that there exists a nontrivial $\phi \in W^{1,2}(\mathbb{R}^N)$ such that

$$\Delta \phi - V(|x|)\phi + pw^{p-1}\phi = 0 \text{ in } \mathbb{R}^N.$$ (0.7)

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By the previous remark, this implies that zero is the second eigenvalue of $L$ and $\phi$ is a corresponding eigenfunction. By Lemma A.5 of [2], we infer that $\phi$ is radially symmetric. Hence, since $\phi$ is the second radial eigenfunction, we deduce that $\phi$ changes sign once. So, we may assume that

$$\phi < 0 \text{ for } 0 \leq r < r_0 \text{ and } \phi > 0 \text{ for } r > r_0. \tag{0.8}$$

Now, as in [3], we consider the function

$$\eta(r) = rw'(r) - \beta w(r), \quad r = |x|,$$

with $\beta \in \mathbb{R}$ to be determined. Then, a short calculation shows that $\eta$ satisfies

$$\Delta \eta - V(|x|)\eta + pw^{p-1}\eta = 2V(r)w + rV'(r)w + (\beta(1-p) - 2)w^p. \tag{0.9}$$

We choose $\beta$ such that

$$\beta(1-p) - 2 = -\frac{2V(r_0) + r_0V'(r_0)}{w^{p-1}(r_0)}.$$

Hence, in view of (0.5) and (0.6), we have

$$2V(r)w + rV'(r)w + (\beta(1-p) - 2)w^p < 0 \quad \text{for } 0 \leq r < r_0,$$

$$2V(r)w + rV'(r)w + (\beta(1-p) - 2)w^p > 0 \quad \text{for } r > r_0. \tag{0.10}$$

Multiplying (0.7) by $\eta$, (0.9) by $\phi$, subtracting and integrating the resulting identity over $\mathbb{R}^N$, we arrive at

$$\int_{\mathbb{R}^N} [2V(r)w + rV'(r)w + (\beta(1-p) - 2)w^p] \phi = 0$$

which is impossible by (0.8), (0.10).

Thus $\phi \equiv 0$ and this completes the proof. \qed

**Remark 0.2.** If $N = 2$, $p$ as in 0.2 and $V$ satisfies only (0.3), (0.4), then (0.1) has a unique solution, which is linearly non-degenerate (see Proposition 4.1 of [2]).

If $N \geq 3$, $p$ as in 0.2 and $V$ satisfies (0.3), (0.4) and

$$rV'(r) + \gamma V(r) - \frac{L(2-\gamma)}{r^2} \text{ is strictly increasing},$$

where

$$\gamma = \frac{2(N-1)(p-1)}{p+3} \in (0,2),$$

$$L = \frac{2(N-1)((N-2)p + N-4)}{(p+3)^2} > 0,$$

then it follows from the proof of Proposition 4.2 of [2] that (0.1) has a unique solution, which is linearly non-degenerate. Note that, since $\gamma \in (0,2)$, our assumption (0.5) is not covered in [2].
REFERENCES


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