ON THE NON-DEGENERACY OF LEAST ENERGY SOLUTIONS OF A CLASS OF NONLINEAR SCHRÖDINGER EQUATIONS

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ABSTRACT. We derive a new necessary condition on the potential V, so that every least energy ground state of $\Delta u - V(|x|)u + u^p = 0$ in $\mathbb{R}^N$, p subcritical, is non-degenerate.

We consider the problem

$$\Delta u - V(|x|)u + u^p = 0, \quad u > 0 \;\text{ in }\; \mathbb{R}^N, \quad \lim_{|x|\to\infty} u(x) = 0. \tag{0.1}$$

We assume that $N \geq 2$, and

$$p \in \left(1, \frac{N+2}{N-2}\right) \text{ if } N \geq 3 \;\text{ and }\; p \in (1,\infty) \text{ if } N = 2, \tag{0.2}$$

$$V \in C\left([0,\infty)\right) \cap C^2\left((0,\infty)\right), \quad \liminf_{|x|\to\infty} V(x) > 0, \text{and} \quad \lambda_1\left(-\Delta + V(|x|)\right) > 0, \tag{0.3}$$

where $\lambda_1\left(-\Delta + V(|x|)\right)$ is the minimum value of the spectrum of $-\Delta + V(|x|)$,

$$V(r) \;\text{ is non-decreasing and non-constant on } (0,\infty), \tag{0.4}$$

$$rV'(r) + 2V(r) \;\text{ is non-decreasing on } (0,\infty). \tag{0.5}$$

It is well known (see [2]) that, under (0.2) and (0.3), there exists a least energy solution w of (0.1). Furthermore, by (0.4), we have that w is radial and

$$w'(r) < 0, \quad r > 0 \quad \text{(see [2])}. \tag{0.6}$$

In this note we will show

Proposition 0.1. *Under assumptions (0.2), (0.3), (0.4), and (0.5) we have that w is a linearly non-degenerate solution of (0.1), i.e., the spectrum of the linearized operator, in $L^2(\mathbb{R}^N)$,*

$$L(\varphi) = -\Delta\varphi + \left(V(|x|) - pw^{p-1}\right)\varphi$$

does not contain 0.

Proof. We follow Lemma 3.1 of [1] where the authors gave a short and new proof of non-degeneracy of the least energy solution of (0.1) with $V = 1$. As in [1], this method of proof is restricted to the power nonlinearity.

Since w is a least energy solution, it has Morse index one. Hence, the principal eigenvalue of L is negative and the rest of the spectrum is non-negative (see also [1]).

To prove the proposition, we will argue by contradiction. Suppose that there exists a nontrivial $\phi \in W^{1,2}(\mathbb{R}^N)$ such that

$$\Delta\phi - V(|x|)\phi + pw^{p-1}\phi = 0 \;\text{ in }\; \mathbb{R}^N. \tag{0.7}$$

Key words and phrases. Nonlinear Schrödinger equation, least energy solution, non-degeneracy, radial solutions.

By the previous remark, this implies that zero is the second eigenvalue of L and ϕ is a corresponding eigenfunction. By Lemma A.5 of [2], we infer that ϕ is radially symmetric. Hence, since ϕ is the second radial eigenfunction, we deduce that ϕ changes sign once. So, we may assume that

$$\phi < 0 \text{ for } 0 \leq r < r_0 \quad \text{and} \quad \phi > 0 \text{ for } r > r_0. \tag{0.8}$$

Now, as in [3], we consider the function

$$\eta(r) = rw'(r) - \beta w(r), \quad r = |x|,$$

with $\beta \in \mathbb{R}$ to be determined. Then, a short calculation shows that η satisfies

$$\Delta\eta - V(|x|)\eta + pw^{p-1}\eta = 2V(r)w + rV'(r)w + (\beta(1-p) - 2)\, w^p. \tag{0.9}$$

We choose β such that

$$\beta(1-p) - 2 = -\frac{2V(r_0) + r_0 V'(r_0)}{w^{p-1}(r_0)}.$$

Hence, in view of (0.5) and (0.6), we have

$$\begin{aligned} &2V(r)w + rV'(r)w + (\beta(1-p) - 2)\, w^p < 0 \quad \text{for } 0 \leq r < r_0, \\ &2V(r)w + rV'(r)w + (\beta(1-p) - 2)\, w^p > 0 \quad \text{for } r > r_0. \end{aligned} \tag{0.10}$$

Multiplying (0.7) by η, (0.9) by ϕ, subtracting and integrating the resulting identity over $\mathbb{R}^N$, we arrive at

$$\int_{\mathbb{R}^N} \left[2V(r)w + rV'(r)w + (\beta(1-p) - 2)\, w^p\right] \phi = 0$$

which is impossible by (0.8), (0.10).

Thus $\phi \equiv 0$ and this completes the proof. □

Remark 0.2. If $N = 2$, p as in 0.2 and V satisfies only (0.3, (0.4), then (0.1) has a unique solution, which is linearly non-degenerate (see Proposition 4.1 of [2]).

If $N \geq 3$, p as in 0.2 and V satisfies (0.3), (0.4) and

$$rV'(r) + \gamma V(r) - \frac{L(2-\gamma)}{r^2} \quad \text{is strictly increasing,}$$

where

$$\gamma = \frac{2(N-1)(p-1)}{p+3} \in (0,2),$$

$$L = \frac{2(N-1)\left\{(N-2)p + N - 4\right\}}{(p+3)^2} > 0,$$

then it follows from the proof of Proposition 4.2 of [2] that (0.1) has a unique solution, which is linearly non-degenerate. Note that, since $\gamma \in (0,2)$, our assumption (0.5) is not covered in [2].

References

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