A SPECTRAL BOUND FOR A CLASS OF SCHRÖDINGER OPERATORS

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We present a different proof and a generalization of a result of Fusco and Pignotti [4] concerning the lower bound of the spectrum of a class of Schrödinger operators of the form $L^\varepsilon := -\varepsilon^2 \frac{d^2}{ds^2} + V$, where $V : \mathbb{R} \to \mathbb{R}$ is a nonnegative function singular (vanishes) at finite points $s_1 < \ldots < s_N$. We also examine the case of multidimensional Schrödinger operators possessing radial symmetry. Sharp bounds are obtained in the case of one singularity.

Keywords: Schrödinger operators, singular perturbations, Schwarz Rearrangement

1. A spectral bound in the case of multiple singularities

Consider, in $L^2(\mathbb{R})$, the Schrödinger operator

$$L^\varepsilon w := -\varepsilon^2 w'' + q^2 w$$

under the hypotheses on $q \in C(\mathbb{R})$:

(H1) $q(s) > 0$, $s \neq s_k$, $q(s_k) = 0$, $k = 1, \ldots, N$ (N independent of ε),

(H2) $q(s) \geq c|s - s_k|^p$ if $|s - s_k| < \delta$, $k = 1, \ldots, N$, for some $c, p, \delta > 0$ indep. of ε,

(H3) $q(s) \geq \tilde{c}$ if $s \in \mathbb{R} - \cup_{k=1}^N B(s_k, \delta)$ for some $\tilde{c} > 0$ independent of ε, where $B(s_k, \delta) = (s_k - \delta, s_k + \delta)$.

The case $N = 1$ was studied in [4], where (among other things) a lower bound for the principal eigenvalue of L^ε was obtained. This followed from sharp upper bounds for the fundamental solution of L^ε; with $q \in C(\mathbb{R})$ satisfying (H1) with $N = 1$ ($s_1 = 0$), $q(s) = c|s|^p$, $|s| < \delta$, and (H3).

The purpose of this section is to give a simple proof of this spectral bound which applies directly to the case $N \geq 1$. If we assume that $q(s) = c|s|^p$ for all $s \in \mathbb{R}$, then, by setting $s = \varepsilon^{\frac{1}{p+1}} r$ and $E_n = \varepsilon^{\frac{2p}{p+1}} c_n$, the eigenvalue equation $L^\varepsilon w = Ew$ can be rescaled to $-\frac{d^2w}{dr^2}+c^2|r|^{2p}w=c_nw$ which is independent of ε. The idea is to apply this re-scaling to the general case (see also [9]). Our result is

Theorem 1 If $\varepsilon>0$ is sufficiently small, the spectrum $\sigma(L^\varepsilon)$ of L^ε satisfies

$$\sigma(L^\varepsilon)\subseteq\left[c_0\varepsilon^{\frac{2p}{p+1}},\infty\right)$$

for some $c_0>0$ independent of ε.

Proof. By the Friedrichs extension, L^ε defines a self-adjoint operator in $L^2(\mathbb{R})$ with domain $D(L^\varepsilon)\subseteq H^1(\mathbb{R})$ (cf. [10] Ch. 5). Also (cf. [5] pg. 55):

$$\sigma(L^\varepsilon)\subseteq\left[\inf_{0\neq w\in D(L^\varepsilon)}\frac{(L^\varepsilon w,w)}{(w,w)},\infty\right).$$

Moreover, given $w\in D(L^\varepsilon)$, there exist $\varphi_n\in C_0^\infty(\mathbb{R})$ such that $\varphi_n\overset{L^2(\mathbb{R})}{\to}w$ and $(L^\varepsilon\varphi_n,\varphi_n)\to(L^\varepsilon w,w)$ as $n\to\infty$.

Note that (H1), (H2), (H3) imply that, for $\varepsilon>0$ sufficiently small,

$$q(s)\geq q_1(s)=\begin{cases} c|s-s_k|^p, & |s-s_k|\leq\varepsilon^{\frac{1}{p+1}},\ k=1,\dots,N,\\ c\varepsilon^{\frac{p}{p+1}}, & \text{otherwise.}\end{cases}$$

Thus, from the above, it suffices to show that

$$\int_{-\infty}^{\infty}\varepsilon^2\varphi'^2+q_1^2(s)\varphi^2ds\geq c_0\varepsilon^{\frac{2p}{p+1}}\int_{-\infty}^{\infty}\varphi^2ds,\quad\forall\varphi\in C_0^\infty(\mathbb{R}).\tag{1}$$

Let $\varphi\in C_0^\infty(\mathbb{R})$, then

$$\int_{-\infty}^{\infty}\varepsilon^2\varphi'^2+q_1^2(s)\varphi^2ds=$$

$$=\sum_{k=1}^N\int_{B(s_k,\varepsilon^{\frac{1}{p+1}})}\varepsilon^2\varphi'^2+c^2|s-s_k|^{2p}\varphi^2ds+\int_{\mathbb{R}-\cup_{k=1}^NB(s_k,\varepsilon^{\frac{1}{p+1}})}\varepsilon^2\varphi'^2+c^2\varepsilon^{\frac{2p}{p+1}}\varphi^2ds\geq$$

$$\left(\text{Set } r=\varepsilon^{-\frac{1}{p+1}}(s-s_k) \text{ and write } \dot{}=\frac{d}{dr}\right)$$

$$\geq\sum_{k=1}^N\varepsilon^{\frac{1}{p+1}}\int_{B(0,1)}\varepsilon^{2-\frac{2}{p+1}}\dot\varphi^2+c^2\varepsilon^{\frac{2p}{p+1}}|r|^{2p}\varphi^2dr+c^2\varepsilon^{\frac{2p}{p+1}}\int_{\mathbb{R}-\cup_{k=1}^NB(s_k,\varepsilon^{\frac{1}{p+1}})}\varphi^2ds=$$

$$=\sum_{k=1}^N\varepsilon^{\frac{2p}{p+1}}\varepsilon^{\frac{1}{p+1}}\int_{B(0,1)}\dot\varphi^2+c^2|r|^{2p}\varphi^2dr+c^2\varepsilon^{\frac{2p}{p+1}}\int_{\mathbb{R}-\cup_{k=1}^NB(s_k,\varepsilon^{\frac{1}{p+1}})}\varphi^2ds\geq$$

$\big($Let $\mu_1 > 0$ be the principal eigenvalue of $-\ddot{\varphi} + c^2|r|^{2p}\varphi = \mu\varphi,\ r \in (-1,1),$ with Neumann B.C's$\big)$

$$\geq \sum_{k=1}^{N} \varepsilon^{\frac{2p}{p+1}} \varepsilon^{\frac{1}{p+1}} \mu_1 \int_{B(0,1)} \varphi^2 dr + c^2 \varepsilon^{\frac{2p}{p+1}} \int_{\mathbb{R} - \cup_{k=1}^N B(s_k, \varepsilon^{\frac{1}{p+1}})} \varphi^2 ds =$$

$$= \sum_{k=1}^{N} \varepsilon^{\frac{2p}{p+1}} \mu_1 \int_{B(s_k, \varepsilon^{\frac{1}{p+1}})} \varphi^2 ds + c^2 \varepsilon^{\frac{2p}{p+1}} \int_{\mathbb{R} - \cup_{k=1}^N B(s_k, \varepsilon^{\frac{1}{p+1}})} \varphi^2 ds \geq$$

$$\geq c_0 \varepsilon^{\frac{2p}{p+1}} \int_{-\infty}^{\infty} \varphi^2 ds.$$

Hence, (1) is true. □

Remark 1 In the case that L^ε has essential spectrum, (H3) yields $\sigma_{ess}(L^\varepsilon) \subseteq [\tilde{c}^2, \infty)$ (cf. [2]).

One can easily obtain (see also [4])

Corollary 1 Let $g \in C(\mathbb{R})$ be such that

$$g(s) \geq -C_1 \varepsilon^{\frac{2p}{p+1}}, \quad s \in \mathbb{R}, \tag{2}$$

for some $C_1 > 0$ independent of ε. Then, if $C_1 > 0$ and $\varepsilon > 0$ are sufficiently small, we have

$$\sigma(L^\varepsilon + g) \subseteq \left[\tilde{c_0} \varepsilon^{\frac{2p}{p+1}}, \infty\right) \tag{3}$$

for some $\tilde{c_0}$ independent of ε.

Remark 2 As it can be seen from the proofs, the continuity assumption on q and g can be relaxed to $L^1_{loc}(\mathbb{R})$.

Remark 3 In [9], for the analysis of the linearization of a singular perturbation problem, we studied a linear operator of the form $L^\varepsilon + g$; with g satisfying (2) ($p = \frac{1}{2}$) but with $C_1 > 0$ a fixed constant (*not* necessarily small) (see next section).

Remark 4 The proof of Theorem 1 seems to extend to the case of multidimensional Scrödinger operators of the form:

$$\mathbf{L}^\varepsilon u = -\varepsilon^2 \Delta u + q^2(|\mathbf{x}|)u, \quad \mathbf{x} \in \mathbb{R}^m,$$

with $q \in C[0,\infty)$ satisfying $q(0) = 0,\ q(s) \geq cs^p,\ 0 \leq s \leq \delta$ and $q(s) \geq \tilde{c},\ s \geq \delta$ ($c, p, \delta, \tilde{c}$ independent of ε). If $\varepsilon > 0$ is sufficiently small, then $q(s) \geq q_1(s), s \geq 0$, where $q_1(s) := cs^p,\ 0 \leq s \leq \varepsilon^{\frac{1}{p+1}}$, $q_1(s) := c\varepsilon^{\frac{p}{p+1}},\ s \geq \varepsilon^{\frac{1}{p+1}}$. The main observation is that to obtain the inequality

$$\int_{\mathbb{R}^m} \varepsilon^2 |\nabla u|^2 + q_1^2(|\mathbf{x}|)u^2 d\mathbf{x} \geq c_0 \varepsilon^{\frac{2p}{p+1}} \int_{\mathbb{R}^m} u^2 d\mathbf{x}, \quad \forall u \in C_0^\infty(\mathbb{R}^m),$$

it is enough to show it for the Schwarz Rearrangement u^* of u. Recall that u^* is radial, $u^* \in H^1(\mathbb{R}^m)$, $\int_{\mathbb{R}^m} |\nabla u^*|^2 d\mathbf{x} \leq \int_{\mathbb{R}^m} |\nabla u|^2 d\mathbf{x}$, $\int_{\mathbb{R}^m} q_1^2(|\mathbf{x}|) {u^*}^2 d\mathbf{x} \leq \int_{\mathbb{R}^m} q_1^2(|\mathbf{x}|) u^2 d\mathbf{x}$ (because q_1^2 is continuous and nondecreasing), and $\int_{\mathbb{R}^m} {u^*}^2 d\mathbf{x} = \int_{\mathbb{R}^m} u^2 d\mathbf{x}$ (cf. [7] and Appendix D of [6]).

2. On the critical eigenvalues in the case of one singularity

Consider the Schrödinger operator

$$L^\varepsilon = -\varepsilon^2 \frac{d^2}{ds^2} + q_\varepsilon(s)I.$$

Here I denotes the identity operator, the parameter $\varepsilon > 0$ is small, and the function q_ε satisfies

(i) $q_\varepsilon \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ (this is for convenience purposes only) and

$$|q_\varepsilon(s)| \leq C(|s|^\alpha + \varepsilon^{\frac{2\alpha}{2+\alpha}}), \quad s \in \mathbb{R}, \ \ (C > 0 \text{ independent of } \varepsilon), \tag{4}$$

(ii) $\varepsilon^{-\frac{2\alpha}{2+\alpha}} q_\varepsilon(\varepsilon^{\frac{2}{2+\alpha}} s) \to f(s)$ in $C_{loc}(\mathbb{R})$ as $\varepsilon \to 0$, for some constant $\alpha > 0$ (independent of ε) and $f(s) \to \infty$ as $|s| \to \infty$,

(iii) There exist constants $c, C, D_0 > 0$ (indep. of ε) such that, for small ε,

$$q_\varepsilon(s) \geq c|s|^\alpha \ \text{ if } \ D_0 \varepsilon^{\frac{2}{2+\alpha}} \leq |s| \leq \frac{1}{o_\varepsilon(1)} \varepsilon^{\frac{2}{2+\alpha}}, \tag{5}$$

$$q_\varepsilon(s) \geq c o_\varepsilon(1)^{-\alpha} \varepsilon^{\frac{2\alpha}{2+\alpha}} \ \text{ if } \ |s| \geq \frac{1}{o_\varepsilon(1)} \varepsilon^{\frac{2}{2+\alpha}}, \tag{6}$$

where $o_\varepsilon(1) \to 0^+$ as $\varepsilon \to 0$, and

$$\varepsilon^{-\frac{2\alpha}{2+\alpha}} q_\varepsilon(\varepsilon^{\frac{2}{2+\alpha}} s) \geq -C, \quad s \in \mathbb{R}. \tag{7}$$

The critical eigenvalues (the e.v.'s that approach 0 as $\varepsilon \to 0^+$) have been studied in the recent paper [4] using the classical Prüfer transformation. It is proved there that for each $n \in \mathbb{N}$ L^ε admits at least n eigenvalues if $\varepsilon > 0$ is sufficiently small and the first n eigenvalues satisfy

$$\lambda_i^\varepsilon \geq c(i\varepsilon)^{\frac{2\alpha}{2+\alpha}}, \ \ i = 1, \cdots, n, \ \ (c > 0 \ \text{ indep. of } \varepsilon, n),$$

(they assume $f(s) = (\text{const.})|s|^\alpha$). Our contribution in the present note is that we provide sharp information for λ_i^ε as $\varepsilon \to 0$ and for the corresponding eigenfunctions. We also allow more general potential q_ε. This is necessary for applications to nonlinear singularly perturbed boundary value problems involving corner layers (cf. [1], [9]). To accomplish this we use re-scaling arguments and perturbation theory for self-adjoint operators. We remark that our proof can be trivially modified to treat the case of bounded intervals with Dirichlet or Neumann boundary conditions.

Throughout this section we will denote by C/c a large/small positive generic constant independent of ε whose value will change from line to line.

It is well known that L^ε defines a self-adjoint operator in $L^2(\mathbb{R})$ with domain $D(L^\varepsilon) = H^2(\mathbb{R})$ (from (i)) and essential spectrum $\sigma_{ess}(L^\varepsilon) \subseteq [c o_\varepsilon(1)^{-\alpha} \varepsilon^{\frac{2\alpha}{2+\alpha}}, \infty)$ (from (6), cf. [2]).

The fact that $f(s) \to \infty$ as $|s| \to \infty$ implies that the spectrum, in $L^2(\mathbb{R})$, of the operator

$$\mathbb{M} = -\frac{d^2}{ds^2} + f(s)I$$

consists only of simple eigenvalues $\mu_1 < \mu_2 < \cdots$ with $\mu_i \to \infty$ as $i \to \infty$. The corresponding eigenfunctions ψ_i satisfy

$$|\psi_i(s)| \leq C_i e^{-c_i|s|}, \quad s \in \mathbb{R}, \;\; i = 1, 2, \cdots \; (C_i, c_i > 0 \text{ constants}), \tag{8}$$

and each ψ_i has exactly $i-1$ zeroes in $\mathbb{R}$ (obviously simple). We assume that $\|\psi_i\|_{L^2(\mathbb{R})} = 1$ and $\psi_i(s) > 0$ for $s > 0$ sufficiently large (sign normalization).

Our main result is

Theorem 2 For each $n \in \mathbb{N}$ there exists $\varepsilon_n > 0$ such that L^ε admits at least n eigenvalues if $0 < \varepsilon < \varepsilon_n$. The first n eigenvalues $\lambda_1^\varepsilon < \cdots < \lambda_n^\varepsilon$ satisfy

$$\lim_{\varepsilon \to 0} \varepsilon^{-\frac{2\alpha}{2+\alpha}} \lambda_i^\varepsilon = \mu_i, \;\; i = 1, \cdots, n,$$

and the corresponding L^2-normalized eigenfunctions φ_i^ε with $\varphi_i^\varepsilon(s) > 0$ for $s > 0$ sufficiently large satisfy

$$\left|\left| \varepsilon^{\frac{1}{2+\alpha}} \varphi_i^\varepsilon(\varepsilon^{\frac{2}{2+\alpha}} s) - \psi_i(s) \right|\right|_{L^2(\mathbb{R})} \to 0 \;\text{ as } \varepsilon \to 0, \;\; i = 1, \cdots, n.$$

Proof. Note that $\varphi \in H^2(\mathbb{R}),\ \lambda \in \mathbb{R}$ satisfy $L^\varepsilon \varphi = \lambda \varphi$, $\|\varphi\|_{L^2(\mathbb{R})} = 1$, if and only if

$$\tilde{\varphi}(s) = \varepsilon^{\frac{1}{2+\alpha}} \varphi(\varepsilon^{\frac{2}{2+\alpha}} s), \;\; \tilde{\lambda} = \varepsilon^{-\frac{2\alpha}{2+\alpha}} \lambda$$

satisfy $M^\varepsilon \tilde{\varphi} = \tilde{\lambda} \tilde{\varphi}$, $\|\tilde{\varphi}\|_{L^2(\mathbb{R})} = 1$; where

$$M^\varepsilon = -\frac{d^2}{ds^2} + \varepsilon^{-\frac{2\alpha}{2+\alpha}} q_\varepsilon(\varepsilon^{\frac{2}{2+\alpha}} s) I.$$

Fix $n \in \mathbb{N}$. Let

$$\delta_i^\varepsilon = \|M^\varepsilon \psi_i - \mu_i \psi_i\|_{L^2(\mathbb{R})}, \quad i = 1, \cdots, n.$$

We have

$$\delta_i^\varepsilon = \left|\left| \left(\varepsilon^{-\frac{2\alpha}{2+\alpha}} q_\varepsilon(\varepsilon^{\frac{2}{2+\alpha}} s) - f(s) \right) \psi_i \right|\right|_{L^2(\mathbb{R})} \to 0 \;\text{ as }\; \varepsilon \to 0 \;\; (\text{from } \; (4), (\text{ii}), (8)).$$

It follows from regular perturbation theory for self-adjoint operators (cf. [5]) that $\sigma(M^\varepsilon) \cap (\mu_i - 2\delta_i^\varepsilon, \mu_i + 2\delta_i^\varepsilon) \neq \emptyset$, $i = 1, \cdots, n$. Indeed, suppose that $dist(\mu_i, \sigma(M^\varepsilon)) \geq 2\delta_i^\varepsilon$ for some i. Then, thanks to the inequality (cf. [8])

$$\|(M^\varepsilon - \mu_i I)^{-1}\|_{L^2 \to L^2} \leq dist^{-1}(\mu_i, \sigma(M^\varepsilon)) \leq \frac{1}{2\delta_i^\varepsilon},$$

we get

$$1 = \|\psi_i\|_{L^2(\mathbb{R})} \leq \frac{1}{2\delta_i^\varepsilon}\|(M^\varepsilon - \mu_i I)\psi_i\|_{L^2(\mathbb{R})} = \frac{1}{2},$$

a contradiction. Hence, if ε is small, M^ε has at least n eigenvalues in $(\mu_1 - 1, \mu_n + 1)$ (note that $\sigma_{ess}(M^\varepsilon) \subseteq [co_\varepsilon(1)^{-\alpha}, \infty)$). Let $\tilde{\lambda}_1^\varepsilon < \cdots < \tilde{\lambda}_n^\varepsilon$ denote the first n eigenvalues of M^ε (recall that they are simple) and $\tilde{\varphi}_i^\varepsilon \in H^2(\mathbb{R})$, $i = 1, \cdots, n$ their corresponding eigenfunctions. Each $\tilde{\varphi}_i^\varepsilon$ has exactly $i-1$ zeroes and we assume that $\|\tilde{\varphi}_i^\varepsilon\|_{L^2(\mathbb{R})} = 1$ and $\tilde{\varphi}_i^\varepsilon(s) > 0$ for large $s > 0$.

To prove the theorem, we will show by induction on i that

$$\tilde{\lambda}_i^\varepsilon \to \mu_i \text{ and } \|\tilde{\varphi}_i^\varepsilon - \psi_i\|_{L^2(\mathbb{R})} \to 0 \text{ as } \varepsilon \to 0, \tag{9}$$

$i = 1, \cdots, n$.

Step 1 We have

$$-\frac{d^2}{ds^2}\tilde{\varphi}_1^\varepsilon + \varepsilon^{-\frac{2\alpha}{2+\alpha}} q_\varepsilon(\varepsilon^{\frac{2}{2+\alpha}} s)\tilde{\varphi}_1^\varepsilon = \tilde{\lambda}_1^\varepsilon \tilde{\varphi}_1^\varepsilon, \quad s \in \mathbb{R}. \tag{10}$$

Multiplying both sides with $\tilde{\varphi}_1^\varepsilon$, integrating by parts and using (7), $\|\tilde{\varphi}_1^\varepsilon\|_{L^2(\mathbb{R})} = 1$, $\tilde{\lambda}_1^\varepsilon < \mu_n + 1$ yields $\|\tilde{\varphi}_1^\varepsilon\|_{H^1(\mathbb{R})} \leq C$ and $\tilde{\lambda}_1^\varepsilon \geq -C$. As a result, we may assume that there are $\tilde{\lambda}_1^0 \in [-C, \mu_n + 1]$, $\tilde{\varphi}_1^0 \geq 0$ in $H^1(\mathbb{R})$ such that

$$\tilde{\lambda}_1^{\varepsilon_j} \to \tilde{\lambda}_1^0, \tag{11}$$

$$\tilde{\varphi}_1^{\varepsilon_j} \rightharpoonup \tilde{\varphi}_1^0 \text{ weakly in } H^1(\mathbb{R}),$$

$$\tilde{\varphi}_1^{\varepsilon_j} \to \tilde{\varphi}_1^0 \text{ in } C_{loc}(\mathbb{R}) \tag{12}$$

as $j \to \infty$ for some sequence $\varepsilon_j \to 0$. We see from (ii), (10), (11), (12) that

$$\tilde{\varphi}_1^{\varepsilon_j} \to \tilde{\varphi}_1^0 \text{ in } C^2_{loc}(\mathbb{R}) \text{ as } j \to \infty.$$

Hence, by letting $\varepsilon_j \to 0$ in (10) we infer that

$$-\frac{d^2}{ds^2}\tilde{\varphi}_1^0 + f(s)\tilde{\varphi}_1^0 = \tilde{\lambda}_1^0\tilde{\varphi}_1^0, \quad s \in \mathbb{R}. \tag{13}$$

On the other hand, (5), (6) imply that $\forall K > 0$ (indep. of ε), $\exists\, D_K, \varepsilon_K > 0$ (indep. of ε) such that

$$\varepsilon^{-\frac{2\alpha}{2+\alpha}} q_\varepsilon(\varepsilon^{\frac{2}{2+\alpha}} s) \geq K, \ \ |s| \geq D_K, \ 0 < \varepsilon < \varepsilon_K \tag{14}$$

(if $|s| \geq D > D_0$ then either $\varepsilon^{-\frac{2\alpha}{2+\alpha}} q_\varepsilon(\varepsilon^{\frac{2}{2+\alpha}} s) \geq cD^\alpha$ or $\varepsilon^{-\frac{2\alpha}{2+\alpha}} q_\varepsilon(\varepsilon^{\frac{2}{2+\alpha}} s) \geq co_\varepsilon(1)^{-\alpha}$). Making use of the inequality $\|\tilde{\varphi}_1^\varepsilon\|_{L^\infty(\mathbb{R})} \leq C\|\tilde{\varphi}_1^\varepsilon\|_{H^1(\mathbb{R})} \leq C$, (10), (14), and a standard comparison argument we obtain

$$|\tilde{\varphi}_1^\varepsilon(s)| \leq Ce^{-c|s|} \text{ for } |s| \geq D, \ \varepsilon \text{ small}, \tag{15}$$

with constants $c, C, D > 0$ independent of ε. For any fixed $L > 0$, via (12), (15),

$$\begin{aligned} \|\tilde{\varphi}_1^{\varepsilon_j} - \tilde{\varphi}_1^0\|_{L^2(\mathbb{R})} &\leq \|\tilde{\varphi}_1^{\varepsilon_j} - \tilde{\varphi}_1^0\|_{L^2(|s|\leq L)} + \|\tilde{\varphi}_1^{\varepsilon_j}\|_{L^2(|s|\geq L)} + \|\tilde{\varphi}_1^0\|_{L^2(|s|\geq L)} \\ &= o_{\varepsilon_j}(1) + o_L(1) \end{aligned}$$

where $o_{\varepsilon_j}(1) \to 0$ as $j \to \infty$ and $o_L(1) \to 0$ as $L \to \infty$, i.e., $\|\tilde{\varphi}_1^{\varepsilon_j} - \tilde{\varphi}_1^0\|_{L^2(\mathbb{R})} \to 0$ as $j \to \infty$. It follows that $\|\tilde{\varphi}_1^0\|_{L^2(\mathbb{R})} = 1$ and, from $\tilde{\varphi}_1^0 \geq 0$, (13), we obtain $\tilde{\lambda}_1^0 = \mu_1$ and $\tilde{\varphi}_1^0 = \psi_1$. By the uniqueness of the limit we deduce that (9) holds for $i = 1$.
Step 2 Suppose now that (9) is true for $i = 1, \cdots, k-1$ with $1 < k < n$. We have $M^\varepsilon \tilde{\varphi}_k^\varepsilon = \tilde{\lambda}_k^\varepsilon \tilde{\varphi}_k^\varepsilon$ and as in Step 1,

$$\tilde{\lambda}_k^{\varepsilon_j} \to \tilde{\lambda}_k^0, \tag{16}$$

$$\tilde{\varphi}_k^{\varepsilon_j} \rightharpoonup \tilde{\varphi}_k^0 \text{ weakly in } H^1(\mathbb{R}),$$

$$\tilde{\varphi}_k^{\varepsilon_j} \to \tilde{\varphi}_k^0 \text{ in } L^2(\mathbb{R}) \text{ and } C^2_{loc}(\mathbb{R}) \tag{17}$$

as $j \to \infty$ for a sequence $\varepsilon_j \to 0$ and $\tilde{\lambda}_k^0 \in [-C, \mu_n + 1]$, $\tilde{\varphi}_k^0 \in H^1(\mathbb{R})$ satisfying $\mathbb{M}\tilde{\varphi}_k^0 = \tilde{\lambda}_k^0 \tilde{\varphi}_k^0$. Since $\tilde{\varphi}_k^0$ has at most $k-1$ zeroes (from (17)), we conclude that $\tilde{\lambda}_k^0 = \mu_{i_0}$ and $\tilde{\varphi}_k^0 = \psi_{i_0}$ or $\tilde{\varphi}_k^0 = -\psi_{i_0}$ for some $i_0 \in \{1, \cdots, k\}$. If $i_0 < k$ then letting $j \to \infty$ in the relation

$$\int_{\mathbb{R}} \tilde{\varphi}_{i_0}^{\varepsilon_j} \tilde{\varphi}_k^{\varepsilon_j} ds = 0, \quad j \geq 1,$$

and using $(9)_{i_0}$ yields $\int_{\mathbb{R}} \psi_{i_0}^2 ds = 0$. Thus, $i_0 = k$ and $\tilde{\varphi}_k^0 = \psi_{i_0}$ (from the sign normalization). Again the uniqueness of the limit implies that the convergence in (16)-(17) is valid for every sequence $\varepsilon_j \to 0$. We conclude that (9) is true for $i = k$.

The proof of the theorem is complete. □

Remark 5 We have

$$|\varepsilon^{-\frac{2\alpha}{2+\alpha}} \lambda_i^\varepsilon - \mu_i| + \|\varepsilon^{\frac{1}{2+\alpha}} \varphi_i^\varepsilon(\varepsilon^{\frac{2}{2+\alpha}} s) - \psi_i(s)\|_{L^2(\mathbb{R})} \leq C\|(\varepsilon^{-\frac{2\alpha}{2+\alpha}} q_\varepsilon(\varepsilon^{\frac{2}{2+\alpha}} s) - f(s)) e^{-c|s|}\|_{L^2(\mathbb{R})}$$

$i = 1, \cdots, n$, $\varepsilon > 0$ small.

Indeed, if $i = 1, \cdots, n$,

$$|\varepsilon^{-\frac{2\alpha}{2+\alpha}} \lambda_i^\varepsilon - \mu_i| \leq 2\delta_i^\varepsilon \overset{(8)}{\leq} C\|(\varepsilon^{-\frac{2\alpha}{2+\alpha}} q_\varepsilon(\varepsilon^{\frac{2}{2+\alpha}} s) - f(s)) e^{-c|s|}\|_{L^2(\mathbb{R})}. \tag{18}$$

Write

$$\tilde{\varphi}_i^\varepsilon - \psi_i = a_i^\varepsilon \psi_i + p_i^\varepsilon,$$

$a_i^\varepsilon \in \mathbb{R}$, $\int_{\mathbb{R}} \psi_i p_i^\varepsilon$ds=0. We have

$$\mathbb{M}p_i^\varepsilon - \mu_i p_i^\varepsilon = \left(f(s) - \varepsilon^{-\frac{2\alpha}{2+\alpha}} q_\varepsilon(\varepsilon^{\frac{2}{2+\alpha}} s)\right) \tilde{\varphi}_i^\varepsilon + (\tilde{\lambda}_i^\varepsilon - \mu_i)\tilde{\varphi}_i^\varepsilon.$$

Using the inequality

$$\langle \mathbb{M}p_i^\varepsilon - \mu_i p_i^\varepsilon, p_i^\varepsilon \rangle_{L^2(\mathbb{R})} \geq c\|p_i^\varepsilon\|^2_{L^2(\mathbb{R})},$$

the analog of (15) for $\tilde{\varphi}_i^\varepsilon$, and (18) we get

$$\|p_i^\varepsilon\|_{L^2(\mathbb{R})} \leq C\|(\varepsilon^{-\frac{2\alpha}{2+\alpha}} q_\varepsilon(\varepsilon^{\frac{2}{2+\alpha}} s) - f(s))e^{-c|s|}\|_{L^2(\mathbb{R})}. \tag{19}$$

From the relation

$$\|\tilde{\varphi}_i^\varepsilon - \psi_i\|^2_{L^2(\mathbb{R})} = a_i^{\varepsilon 2} + \|p_i^\varepsilon\|^2_{L^2(\mathbb{R})},$$

we obtain $a_i^\varepsilon \to 0$ as $\varepsilon \to 0$ (see Theorem 1). Since $\tilde{\varphi}_i^\varepsilon = (a_i^\varepsilon + 1)\psi_i + p_i^\varepsilon$,

$$1 = (a_i^\varepsilon + 1)^2 + \|p_i^\varepsilon\|^2_{L^2(\mathbb{R})},$$

i.e., $-a_i^\varepsilon(2 + a_i^\varepsilon) = \|p_i^\varepsilon\|^2_{L^2(\mathbb{R})}$. Thus, $|a_i^\varepsilon| \leq C\|(\varepsilon^{-\frac{2\alpha}{2+\alpha}} q_\varepsilon(\varepsilon^{\frac{2}{2+\alpha}} s) - f(s))e^{-c|s|}\|^2_{L^2(\mathbb{R})}$, which combined with (19) proves of the bound.

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