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On the existence of dark solitons of the defocusing cubic nonlinear Schrödinger equation with periodic inhomogeneous nonlinearity

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ABSTRACT

We provide a simple proof of the existence of dark solitons of the defocusing cubic nonlinear Schrödinger equation with periodic inhomogeneous nonlinearity. Moreover, our proof allows for a broader class of inhomogeneities and gives some new properties of the solutions. We also apply our approach to the defocusing cubic–quintic nonlinear Schrödinger equation with a periodic potential.

We consider the following defocusing cubic nonlinear Schrödinger equation with inhomogeneous nonlinearity on $\mathbb{R}$:

$$i\psi_t = -\frac{1}{2}\psi_{xx} + g(x)|\psi|^2\psi, \tag{1}$$

with g reasonably smooth,

$$g \text{ being } T\text{-periodic} \tag{2}$$

and satisfying

$$0 < g_{min} \le g(x) \le g_{max}. \tag{3}$$

The above problem arises in a variety of physical situations such as Bose–Einstein condensates and nonlinear optics (we refer the interested reader to [1–3] and the references therein for more details).

The solitary wave solutions of (1) are given by

$$\psi(x,t) = e^{i\lambda t}\phi(x), \tag{4}$$

where ϕ is real valued and solves

$$-\frac{1}{2}\phi_{xx} + \lambda\phi + g(x)\phi^3 = 0. \tag{5}$$

A solitary wave solution ψ of (1) is called a *dark soliton* if the associated ϕ satisfies

$$\frac{\phi(x)}{\phi_p(x)} \to \pm 1 \quad \text{as } x \to \pm\infty, \tag{6}$$

where the function ϕ_p is a positive, T-periodic solution of (5) (keep in mind that $-\phi_p$ is a negative, T-periodic solution).

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It is worth mentioning that, in the case where g is a constant, the dark soliton can be represented explicitly and its stability has been a topic of extensive investigations in recent years (see [4] and the references therein).

If $\lambda \geq 0$, it was shown in [2] that the only bounded solution of (5) is the trivial one. Therefore, we will assume that

$$\lambda < 0. \tag{7}$$

Since the constant functions

$$\underline{\phi} = \sqrt{-\frac{\lambda}{g_{max}}} \quad \text{and} \quad \bar{\phi} = \sqrt{-\frac{\lambda}{g_{min}}}$$

are T-periodic lower and upper solutions respectively of (5), a well known result [5,6] guarantees that (5) has a T-periodic solution ϕ_p such that

$$\sqrt{-\frac{\lambda}{g_{max}}} < \phi_p(x) < \sqrt{-\frac{\lambda}{g_{min}}}. \tag{8}$$

By combining several techniques from the classical theory of ODE's, such as topological degree and free homeomorphisms (applied to the Poincaré map), it was shown in [2] (see also [3,7]) that (5) has a solution that verifies (6), under the additional assumptions that g is even and

$$g_{min} > \frac{g_{max}}{3} \tag{9}$$

(in particular, the latter relation implies the uniqueness of a T-periodic solution of (5) satisfying (8)).

In this note, we will give a simple and elementary variational proof of this result which, in fact, *works without assuming the last two restrictions*. At the same time, it gives some new monotonicity properties of the obtained solutions.

Our main observation is to adapt to this setting a remarkable identity, due to [8], which reduces (5) to a weighted Allen–Cahn equation. This technique has become a standard tool in the study of vortices in inhomogeneous equations of Ginzburg–Landau type (see for instance [9] and the references therein). In the context of (1), in the semiclassical regime, a related idea appears in the recent paper [10] which studies the case of positive g that diverges at respective infinities. The main drawback to this approach is that, in principle, it works only in the case of power nonlinearities. On the other hand, this was not an issue for the approach in [2,7]. We believe that this approach could also be useful for the study of the stability of the dark soliton in the spirit of [4] and the references therein.

More precisely, our main result is the following. After its proof, we will remark on how to apply this approach to the defocusing cubic–quintic nonlinear Schrödinger equation with a periodic potential that was considered in [7].

Theorem 1. *Under the assumptions $g \in C(\mathbb{R})$, (2), (3) and (7), Eq. (5) admits a solution ϕ that satisfies (6). Moreover, it holds that*

$$\left(\frac{\phi}{\phi_p}\right)_x > 0, \quad x \in \mathbb{R}, \tag{10}$$

and there exists a constant $C_0 > 0$ such that

$$\left|\phi(x) \mp \phi_p(x)\right| + \left|\left(\phi(x) \mp \phi_p(x)\right)_x\right| \leq e^{-C_0|x|}, \quad \pm x > 0. \tag{11}$$

Proof. Motivated from [8], we set

$$w = \frac{\phi}{\phi_p}.$$

Then, it follows readily that Eq. (5) is equivalent to

$$\left(\phi_p^2 w'\right)' = 2g(x)\phi_p^4 w(w^2 - 1), \tag{12}$$

while the asymptotic boundary conditions (6) are equivalent to

$$w(x) \to \pm 1 \quad \text{as } x \to \pm\infty. \tag{13}$$

It is important to observe that, thanks to (8), the differential operator in (12) is nonsingular.

Eq. (12) is the Euler–Lagrange equation of the energy

$$E(w) = \int_{-\infty}^{\infty} \left\{\phi_p^2 w_x^2 + g(x)\phi_p^4(1 - w^2)^2\right\} dx.$$

Standard variational techniques can be then employed to find a solution of (12)–(13), a heteroclinic orbit that is, as a minimizer of E in the set

$$w \in W^{1,2}_{loc}(\mathbb{R}) \quad \text{and} \quad w(x) \to \pm 1 \quad \text{as } x \to \pm\infty, \tag{14}$$

see for example [11–13] and the references therein. Loosely speaking, conditions (3) and (8) prevent the minimizing sequences from becoming too flat, while the T-periodicity of g, ϕ_p can be used to "pull them back", without affecting their energy, should their "center of mass" escapes to infinity.

It holds that

$$|w| < 1, \tag{15}$$

as can be verified by observing that $\min\{w, 1\}$ and $\max\{w, -1\}$ have less or equal energy than w and then applying the strong maximum principle. We claim that

$$w_x \geq 0. \tag{16}$$

Indeed, let $w(x_1) = w(x_2)$ for some $x_1 < x_2$. Without loss of generality, we may assume that $w(x_1) > 0$. Since the function in $W^{1,2}_{loc}(\mathbb{R})$ which coincides with w in $\mathbb{R} \setminus (x_1, x_2)$ and is equal to $|w|$ in (x_1, x_2) has less or equal energy than w, we may also assume that $w \geq 0$ in (x_1, x_2). Then, the function in $W^{1,2}_{loc}(\mathbb{R})$ which coincides with w in $\mathbb{R} \setminus (x_1, x_2)$ and is equal to $w(x_1)$ in (x_1, x_2) has less energy than w which is impossible. In fact, we can show that $w_x > 0$, which implies at once (10), by using (12), (15) and (16). Finally, the convergence in (13) can be shown to be exponentially fast, which easily implies (11), by a standard barrier argument.

The proof of the theorem is complete. □

Remark 1. In [7], the authors studied dark soliton solutions of the cubic–quintic NLS

$$i\psi_t + \psi_{xx} + V(x)\psi - g_1|\psi|^2\psi - |\psi|^4\psi = 0,$$

where the potential V is smooth, real and T-periodic, while g_1 is a real constant. Plugging the ansatz (4) into the above equation yields that ϕ must satisfy

$$\phi_{xx} + (V(x) - \lambda)\,\phi - g_1\phi^3 - \phi^5 = 0. \tag{17}$$

If

$$\lambda < \min V, \tag{18}$$

it was shown as before, by the method of upper and lower solutions, that (17) has a T-periodic solution ϕ_p such that

$$\rho_1 < \phi_p < \rho_2,$$

where

$$\rho_j = \frac{1}{\sqrt{2}}\sqrt{\sqrt{g_1^2 + 4\lambda_j^2} - g_1} \quad \text{with } \lambda_1^2 = \min V - \lambda \quad \text{and} \quad \lambda_2^2 = \max V - \lambda.$$

If V is assumed to be even, an analogous condition to (9), but considerably more complicated, was found in [7] that guarantees the existence of a solution to (17) that verifies (6).

Our approach carries over to this problem without too much effort. Indeed, proceeding as in the proof of our main result, we find that the corresponding w should satisfy

$$(\phi_p^2 w')' = g_1\phi_p^4 w(w^2 - 1) + \phi_p^6 w(w^4 - 1)$$

together with the asymptotic behavior (13). Now, the associated energy is

$$E(w) = \int_{-\infty}^{\infty} \left\{ \frac{1}{2}\phi_p^2 w_x^2 + \phi_p^4 \left[\frac{g_1}{4} + \frac{\phi_p^2}{6}(2 + w^2) \right] (1 - w^2)^2 \right\} dx.$$

If $g_1 \geq 0$, the aforementioned variational arguments apply directly to show that there exists a minimizer of the energy in the set described in (14), without the need of imposing any further condition other than (18). The case $g_1 < 0$ is more subtle and requires a further investigation. Clearly, a sufficient condition for the minimization argument to go through and produce the desired solution is

$$\frac{g_1}{4} + \frac{\rho_1^2}{3} \geq 0,$$

which can always be made possible by choosing $-\lambda > 0$ sufficiently large.

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