EXISTENCE AND UNIQUENESS OF ODD INCREASING SOLUTIONS FOR A CLASS OF INHOMOGENEOUS ALLEN-CAHN EQUATIONS IN $\mathbb{R}$

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Abstract. We prove existence and uniqueness of odd increasing solutions to a class of inhomogeneous Allen-Cahn type equations. A typical example is $u'' + |x|^\alpha (u - u^3) = 0$, $\alpha > 0$. We use the fact that this is well known when $\alpha = 0$ and a continuation argument. Our motivation is the study of layered solutions for $\varepsilon^2 u'' + h(x)(u - u^3) = 0$, $h(x) = |x - x_1|^{\alpha} + o(|x - x_1|^\alpha)$, $x \to x_1$, $\varepsilon > 0$ small (see [3]).

1. Introduction and main result

We consider the problem of finding increasing solutions to the problem

$$
\begin{aligned}
&u'' + |x|^\alpha f(u) = 0 \quad x \in \mathbb{R} \\
u(x) \to -1 \text{ as } x \to -\infty \quad u(x) \to 1 \text{ as } x \to \infty,
\end{aligned}
$$

(1.1)

where $\alpha > 0$ and $f \in C^1(\mathbb{R})$ is odd, $f(\pm 1) = 0$, $f_u'(1) < 0$, $f(u) > 0$, $u \in (0, 1)$,

$$
\begin{aligned}
f(\delta u) \geq \delta f(u) \quad \forall u \in (0, 1), \quad \delta \in (0, \delta_0],
\end{aligned}
$$

(1.2)

for some small $\delta_0 > 0$. A typical example is $f(u) = u - u|u|^p$, $p > 0$.

Remark 1.1. In [3] we did not require (1.2) but $f_u'(0) > 0$. Existence and asymptotic stability of a solution were established by topological arguments.

Let $u_0$ be the unique odd solution of

$$
\begin{aligned}
&u'' + f(u) = 0 \quad x \in \mathbb{R} \\
u(x) \to -1 \text{ as } x \to -\infty \quad u(x) \to 1 \text{ as } x \to \infty,
\end{aligned}
$$

(1.3)

We have $u_0' > 0$.

By odd reflection, (1.1) is equivalent to

$$
\begin{aligned}
&u'' + x^\alpha f(u) = 0 \quad x > 0 \\
u(0) = 0 \quad u(x) \to 1 \text{ as } x \to \infty.
\end{aligned}
$$

(1.4)

Proposition 1.1. There exists a solution $u$ of (1.4) such that $u'(x) > 0$, $x \geq 0$. Partly supported by grant FONDECYT 3085026.
Proof. Note that
\[
(1.5) \quad u_\delta(x) = \begin{cases} 
\delta u_0(x - 1) & x \geq 1 \\
0 & 0 \leq x \leq 1,
\end{cases}
\]
\(\delta \geq 0\) is a nontrivial subsolution of (1.4) if \(\delta \in (0, \delta_0]\).
Indeed, in \([0,1)\) we have \(u_\delta = 0\). So,
\[
u_\delta'' + x^\alpha f(u_\delta) = x^\alpha f(0) = 0, \quad x \in [0,1).
\]
In \((1, \infty)\) we have, for small \(\delta \in (0, \delta_0]\),
\[
u_\delta'' + x^\alpha f(u_\delta) \geq \delta \nu_0''(x - 1) + x^\alpha f(u_0(x - 1)) \geq \delta \nu_0''(x - 1) + \delta f(u_0(x - 1)) = 0.
\]
Moreover, \(u_\delta(0) = 0\), \(u_\delta(\infty) = \delta < 1\). Finally, the function \(u_\delta\) is the maximum of two regular sub-solutions, and therefore \(u_\delta\) itself must be a subsolution.

Let \(\phi_\varepsilon\) solve
\[
\begin{cases}
-\phi_\varepsilon'' - f_u(1)|x|^{\alpha}\phi_\varepsilon = -\phi_\varepsilon & x > \varepsilon^{-\frac{2}{\alpha}} \\
\phi_\varepsilon(\varepsilon^{-\frac{2}{\alpha}}) = 1 - u_0(\varepsilon^{-\frac{2+\alpha}{\alpha}}) > 0 & \phi_\varepsilon'(\varepsilon^{-\frac{2}{\alpha}}) = -\varepsilon^{-1} u_0'(\varepsilon^{-\frac{2+\alpha}{\alpha}}) < 0.
\end{cases}
\]
We have \(\phi_\varepsilon > 0\), \(\phi_\varepsilon' < 0\), \(\phi_\varepsilon'' > 0\), \(x > \varepsilon^{-\frac{2}{\alpha}}\), and a standard barrier argument yields
\[
0 < \phi_\varepsilon(x) \leq C e^{-\frac{1}{C^2 \alpha} e^{-\frac{2}{\alpha}}(x - \varepsilon^{-\frac{2}{\alpha}})}, \quad x > \varepsilon^{-\frac{2}{\alpha}}.
\]
provided \(\varepsilon > 0\) is small (recall that \(f_u(1) < 0\) and \(1 - u_0(x) \leq C e^{-|x|/C}\) for a generic constant \(C\)).

We claim that, for small \(\varepsilon\),
\[
v_\varepsilon(x) = \begin{cases} 
1 - \phi_\varepsilon(x) & x > \varepsilon^{-\frac{2}{\alpha}} \\
u_\varepsilon(\varepsilon^{-\frac{2}{\alpha}}) & 0 < x \leq \varepsilon^{-\frac{2}{\alpha}}
\end{cases}
\]
is a supersolution of (1.4). Note that \(v_\varepsilon > u_\delta\) if \(\delta \leq \delta_0\) and \(\varepsilon > 0\) is small. We have \(v_\varepsilon \in C^1(\mathbb{R})\) and \(v_\varepsilon(0) = 0\), \(v_\varepsilon(\infty) = 1\). In \((0, \varepsilon^{-\frac{2}{\alpha}})\),
\[
v_\varepsilon'' + |x|^{\alpha} f(v_\varepsilon) = \left(-\frac{1}{\varepsilon^2} + |x|^\alpha\right) f(v_\varepsilon) \leq 0.
\]
In \((\varepsilon^{-\frac{2}{\alpha}}, \infty)\),

\[ v''_\varepsilon + |x|^{\alpha} f(v_\varepsilon) = -\phi'' + |x|^{\alpha} f(1 - \phi) \]

\[ = -\phi'' - f_u(1)|x|^\alpha \phi + |x|^\alpha O(\phi^2) \]

\[ = (-1 + |x|^\alpha O(|\phi|)) \phi \]

\[ \leq \left(-1 + C \left(|x - \varepsilon^{-\frac{2}{\alpha}}|^{\alpha} + \varepsilon^{-2}\right) e^{-\frac{1}{c_\varepsilon} \frac{1}{\varepsilon^2}} e^{-\frac{1}{c_\varepsilon} \frac{1}{\varepsilon^{\frac{\alpha}{2}}}} \right) \phi < 0 \]

if \( \varepsilon \) is small.

Hence, there exists a solution \( u \) of the differential equation of (1.4) such that

\[ u_\delta \leq u \leq v_\varepsilon, \quad x \geq 0. \]

It is easy to see that \( u'(x) > 0, \quad x \geq 0 \) and, thus, \( u \) solves (1.4). The proof of the proposition is complete. \[ \square \]

**Remark 1.2.** Alternatively, one can get existence using the continuation argument we will use for uniqueness (starting from \( \alpha = 0 \)).

**Lemma 1.1.** Every increasing solution \( u \) of (1.4) satisfies

\[ u(x) \geq \delta_0 u_0(x - 1), \quad x \geq 1. \]

**Proof.** We prove the result by using the subsolutions \( u_\delta \) defined in (1.5) and Serrin’s sweeping technique (cf. [2]).

By the proof of Proposition 1.1, \( u_\delta \) is a subsolution of (1.4) if \( \delta \in [0, \delta_0] \). Suppose that \( u \) is an increasing solution of (1.4). Then, if \( \delta = 0 \), \( u \geq u_\delta \) in \((0, \infty)\). By the Serrin sweeping principle, and since \( u(0) = 0, \quad u(\infty) = 1 \) while \( u_\delta(0) = 0, \quad u_\delta(\infty) = \delta < 1, \quad u'_\delta(0) = 0 \), we see that

\[ u(x) \geq u_{\delta_0}(x) = \delta_0 u_0(x - 1)_{+} \quad \text{in} \quad (0, \infty). \]

To prove this, we let \( \delta = \sup \{ \delta \in [0, \delta_0] : u \geq u_{\delta} \quad \text{in} \quad (0, \infty) \} \), note that \( u \geq u_{\delta} \) in \((0, \infty)\), and apply the maximum principle to \( u - u_{\delta} \) to deduce that this function has a positive lower bound in \((0, \infty)\). This contradicts the maximality of \( \delta \) if \( \delta < \delta_0 \).

The proof of the lemma is complete. \[ \square \]

**Lemma 1.2.** There exists \( \alpha_0 > 0 \) such that (1.4) has a unique increasing solution for each \( 0 < \alpha \leq \alpha_0 \).

**Proof.** The idea is to obtain good asymptotics for the solution and then use this to prove uniqueness.

**Step 1** Suppose that \( u_i \) are increasing solutions of (1.4) for \( \alpha_i > 0 \) and \( \alpha_i \to 0 \) as \( i \to \infty \).

We have \( 0 < u_i < 1 \) in \((0, \infty)\) and by equation

(1.6) \[ u''_i + x^{\alpha_i} f(u_i) = 0, \quad x \in (0, \infty), \]

we see that

(1.7) \[ \|u_i\|_{C^2[0,L]} \leq C(L), \quad L > 0, \quad i \geq 1. \]

Using the Arzela-Ascoli theorem and the standard diagonal argument we obtain that, for a subsequence,

\[ u_i \to u_* \quad \text{in} \quad C^2_{loc}[0, \infty), \]
for some nondecreasing $0 \leq u_* \leq 1$ bounded solution of
\[ u'' + f(u) = 0, \quad x > 0, \quad u(0) = 0, \]

(we also used that $x^\alpha \to 1$ in $C_{loc}(0, \infty)$ as $\alpha \to 0$).

By Lemma 1.1, we also have $u_* \geq \delta_0 u_0(x-1), \quad x \geq 1$, i.e., $u_*$ is nontrivial. Hence, $u_* \equiv u_0$. By the uniqueness of the limit, we conclude that the every subsequence of $u_i$ satisfies
\[ u_i \to u_0 \text{ in } C^2_{loc}([0, \infty)) \text{ as } i \to \infty. \]

**Step 2** Suppose that the lemma is false and that $u_i, \ v_i$ are two distinct increasing solutions of (1.4) with $\alpha = \alpha_1$ and $\alpha_1 \to 0$ as $i \to \infty$. Let
\[ \phi_i = \frac{u_i - v_i}{\|u_i - v_i\|_{L^\infty(0, \infty)}}. \]

Then $\phi_i$ satisfies
\[ \begin{aligned}
\phi_i'' + V_i(x)\phi_i &= 0 \quad x \in (0, \infty) \\
\phi_i(0) &= 0 \quad \|\phi_i\|_{L^\infty(0, \infty)} = 1, \quad \phi_i(\infty) = 0,
\end{aligned} \tag{1.9} \]
where
\[ V_i(x) = \begin{cases}
x^\alpha \frac{f(u_i) - f(v_i)}{u_i - v_i} & \text{if } u_i(x) \neq v_i(x) \\
x^\alpha f(u_i) & \text{if } u_i(x) = v_i(x).
\end{cases} \]

which converges in $C_{loc}[0, \infty)$ to $f_u(u_0)$ by Step 1. Using once more the Arzela-Ascoli theorem and the standard diagonal argument we see that, for a subsequence,
\[ \phi_i \to \phi_* \text{ in } C^2_{loc}(0, \infty) \text{ as } i \to \infty, \]
where $\phi_*$ is bounded in $(0, \infty)$ and solves
\[ \phi'' + f_u(u_0)\phi = 0, \quad x > 0, \quad u(0) = 0, \]

Hence,
\[ \phi_* \equiv 0. \tag{1.11} \]

(To see this note that the boundedness of $\phi$ and $f_u(1) < 0$ imply that $\phi_* \in H^2(0, \infty)$. So, its odd reflection is in the kernel of the operator $B\phi = \phi'' + f_u(u_0)\phi, \phi \in H^2(\mathbb{R})$. Thus, $\phi_* = \lambda u'_0$, and by setting $x = 0$ we get $\lambda = 0$).

There exist $c, d > 0$ such that $f_u(u) < -c$ if $|u - 1| \leq d$. Moreover, there is an $L_0 > 0$ such that $0 < 1 - u_0(x) < d$ if $x \geq L_0$. Since $u_i, \ v_i$ are increasing and converge to $u_0$ uniformly in compact intervals, we get that $1 - d < u_i(x), \ v_i(x) < 1, \ x \geq L_0$ if $i$ is sufficiently large. By the mean value theorem,
\[ V_i(x) = x^\alpha f_u(\theta u_i + (1 - \theta)v_i), \quad \theta \in [0, 1]. \]

Hence,
\[ V_i(x) \leq -cx^\alpha, \quad x \geq L_0. \tag{1.12} \]

There exist $x_i > 0$ such that, without loss of generality, $\phi_i(x_i) = 1, \ \phi_i'(x_i) = 0, \ \phi_i''(x_i) \leq 0, \ i \geq 1$. We claim that the sequence $\{x_i\}$ is bounded. Indeed if not, by (1.9), (1.12) we find that $\phi_i''(x_i) \to \infty$ as $i \to \infty$, a contradiction.

By passing to a subsequence we may assume that $x_i \to x_* \geq 0$. In view of (1.10), we get
\[ \phi_*(x_*) = 1, \]
which contradicts (1.11). The proof of the lemma is complete. \qed
Lemma 1.3. If \( u \) is an increasing solution of \((1.1)\) then \( u \) is nondegenerate. In particular it is asymptotically stable, i.e., if
\[
-\psi'' - |x|^\alpha f_u(u)\psi = \mu_1 \psi_1 \quad \text{in } \mathbb{R},
\]
\( \psi_1 \in L^2(\mathbb{R}) \), \( \psi_1 > 0 \) in \( \mathbb{R} \), then \( \mu_1 > 0 \).

Proof. We will show that the spectrum, in \( L^2(\mathbb{R}) \), of the operator defined by the left-hand side of \((1.13)\) is strictly positive. Since \(-|x|^\alpha f_u(u) \to \infty \) as \(|x| \to \infty \), its spectrum consists of discrete simple eigenvalues \( \mu_1 < \mu_2 < \cdots \) with \( \mu_i \to \infty \) as \( i \to \infty \) and corresponding eigenfunctions \( \psi_1, \psi_2, \cdots \). Each \( \psi_i, i \geq 1 \) has exactly \( i - 1 \) nodes. Without loss of generality we can assume that \( \psi_1 > 0 \) is an even function of \( x \). It is standard to show that \( 1 - u, \psi_1 \) tend to 0 super-exponentially as \(|x| \to \infty \) (see [3]). Note that \( w = u' > 0 \) solves
\[
-\psi'' - |x|^\alpha f_u(u)\psi = \alpha |x|^\alpha - 2 x f(u), \quad x \in \mathbb{R}.
\]

By multiplying \((1.13)\) with \( w \), \((1.14)\) with \( \psi_1 \) and subtracting, we find that
\[
\mu_1 \int_{-\infty}^{\infty} \psi_1 w dx = \alpha \int_{-\infty}^{\infty} |x|^\alpha - 2 x f(u) \psi_1 dx = 2 \alpha \int_{0}^{\infty} x^{\alpha - 1} f(u) \psi_1 dx > 0.
\]
The proof is complete. \( \square \)

Lemma 1.4. If \( U \) is an increasing solution of \((1.1)_A \) (this means \( \alpha = A \)) then there exists a \( \delta > 0 \) and an increasing solution \( u_\alpha \) of \((1.1)_\alpha \) for \(|\alpha - A| \leq \delta \). Moreover,
\[
(1.15) \quad |u_\alpha - u_A|_{L^\infty(0, \infty)} \to 0 \quad \text{as } \alpha \to A.
\]

Proof. We seek a solution \( u \) of \((1.1)\) in the form \( u = u_A + \varphi, \varphi \in H^2(\mathbb{R}) \) odd. In terms of \( \varphi, (1.1)_\alpha \) becomes
\[
L(\varphi) = |x|^\alpha N(\varphi) + E,
\]
where
\[
L(\varphi) = -\varphi'' - |x|^\alpha f_u(U)\varphi
\]
\[
N(\varphi) = f(U + \varphi) - f(U) - f_u(U)\varphi
\]
\[
E = U'' + |x|^\alpha f(U).
\]
We introduce the norm \( \|\varphi\|_w = \|e^{|x|^2}\varphi\|_{L^\infty(\mathbb{R})} \) and the Banach space
\[
X = \{ \varphi : \varphi \text{ is odd, } \|\varphi\|_w < \infty \}.
\]

Arguing as in Lemma 1.2 we see that \( L \) is nonsingular if \( \alpha \) is close to \( A \). By rather standard arguments we see that if \( f \in X \) then there exists a unique \( \varphi \in H^2(\mathbb{R}) \cap X \) such that \( L(\varphi) = f \). Moreover \( \|\varphi\|_w \leq C\|f\|_w \) (\( C \) is a generic constant). Note that \( (1 - U \text{ decays super-exponentially}) \). Now the existence of a solution \( u \) of \((1.1)_\alpha \) with \(|\alpha - A| \) small satisfying \((1.15)\) follows from Banach’s fixed point theorem. We can show that \(|u_\alpha(x) - U'(x)| \to 0 \) as \( \alpha \to A \). It follows that \( u \geq 0 \) in \((0, \infty)\) and, thus, \( u' > 0 \) in \( \mathbb{R} \) provided \( \alpha \) is close to \( A \). \( \square \)
Lemma 1.5. If \( u_\alpha \) are increasing solutions of \((1.1)_\alpha\), \( \alpha > \alpha_* \) then there exists \( \delta > 0 \) and increasing solutions \( u_\alpha \) of \((1.1)_\alpha\) for \( \alpha_* - \delta \leq \alpha \leq \alpha_* \). Moreover \( u_\alpha \) are continuous functions of \([\alpha_* - \delta, \alpha]\) to \( L^\infty(\mathbb{R}) \) (assuming that they are for \( \alpha > \alpha_* \)).

Proof. We know that there exists a sequence of increasing solutions \( u_i \) of \((1.1)\) with \( \alpha_i \to \alpha_* \) as \( i \to \infty \) and \( \alpha_i > \alpha_* \). Since \( |u_i| \leq 1 \), we can employ Arzela-Ascoli’s theorem and the standard diagonal argument again to show that there exists a nondecreasing solution \( u_* \) of the equation of \((1.1)\) with \( \alpha = \alpha_* \). By Lemma 1.1 we get that \( u_* \) is nontrivial and, thus, is an increasing solution of \((1.1)\). By the \( C_{loc} \) convergence and the fact that \( u_\alpha \) are increasing we have that the convergence is uniform in \( \mathbb{R} \). This establishes continuity from \([\alpha_* - \delta, \alpha]\) to \( L^\infty(\mathbb{R}) \). By Lemma 1.4 we obtain increasing solutions for \( \alpha_* - \delta \leq \alpha \leq \alpha_* \). The continuity assertion follows by substracting two different equations and working as in the proof of Lemma 1.4. □

Theorem 1.1. If \( \alpha > 0 \) then there exists a unique increasing solution of \((1.1)\).

Proof. Existence was established in Proposition 1.1. For the uniqueness we will argue by contradiction. Suppose that \( u_1, u_2 \) are two distinct increasing solutions of \((1.1)_{\alpha_0}\) for some \( \alpha_0 > 0 \). By Lemmas 1.4, 1.5 we can construct increasing solutions \( u_i(\alpha), i = 1, 2 \) of \((1.1)_\alpha\) for \( 0 < \alpha < \alpha_0 \) such that \( u_i : (0, \alpha_0] \to L^\infty(\mathbb{R}) \) is continuous. Lemma 1.2 implies that there exists an \( \alpha_1 \) such that \( u_1(\alpha_1) = u_2(\alpha_1) \). The continuity of the branches yields that the linearization of \((1.1)\) at \( u_1(\alpha_1) \) is singular (as in 1.2). On the other hand, since \( u_1(\alpha_1) \) is increasing, we get a contradiction by Lemma 1.3. Therefore, the proof of Theorem 1.1 is complete. □

Remark 1.3. Our approach for uniqueness is motivated from [1].

Remark 1.4. If instead of \((1.2)\) we had the stronger
\[
  f(\delta u) \geq \delta f(u), \quad u \in [0, 1], \quad \delta \in [0, 1],
\]
as in the case \( f(u) = u - u|u|^p, \ p > 0 \), the uniqueness proof is very simple. Suppose that \( u, v \) are two increasing solutions of \((1.4)\). Then \( \delta u, \ \delta \in [0, 1] \) is a family of subsolutions and \( \delta u \leq v \) in \((0, \infty)\) if \( \delta = 0 \). The Serrin sweeping principle yields \( u \leq v \) in \((0, \infty)\). Similarly, \( v \leq u \) in \((0, \infty)\).

References


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