EXISTENCE AND UNIQUENESS OF ODD INCREASING SOLUTIONS FOR A CLASS OF INHOMOGENEOUS ALLEN-CAHN EQUATIONS IN $\mathbb{R}$

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ABSTRACT. We prove existence and uniqueness of odd increasing solutions to a class of inhomogeneous Allen-Cahn type equations. A typical example is $u'' + |x|^\alpha(u-u^3) = 0,\ \alpha > 0$. We use the fact that this is well known when $\alpha = 0$ and a continuation argument. Our motivation is the study of layered solutions for $\varepsilon^2 u'' + h(x)(u-u^3) = 0$, $h(x) = |x-x_1|^\alpha + o(|x-x_1|^\alpha),\ x \to x_1$, $h > 0,\ x \neq x_1$ in a bounded interval with Neumann boundary conditions and $\varepsilon > 0$ small (see [3]).

1. Introduction and main result

We consider the problem of finding increasing solutions to the problem

$$(1.1)\qquad \begin{cases} u'' + |x|^\alpha f(u) = 0 & x \in \mathbb{R} \\ u(x) \to -1 \text{ as } x \to -\infty & u(x) \to 1 \text{ as } x \to \infty, \\ u \text{ odd}, \end{cases}$$

where $\alpha > 0$ and $f \in C^2(\mathbb{R})$ is odd, $f(\pm 1) = 0,\ f_u(1) < 0,\ f(u) > 0,\ u \in (0,1)$,

$$(1.2)\qquad f(\delta u) \geq \delta f(u)\quad \forall u \in (0,1),\ \delta \in (0, \delta_0],$$

for some small $\delta_0 > 0$. A typical example is $f(u) = u - u|u|^p,\ p > 0$.

Remark 1.1. *In* [3] *we did not require (1.2) but $f_u(0) > 0$. Existence and asymptotic stability of a solution were established by topological arguments.*

Let u_0 be the unique odd solution of

$$(1.3)\qquad \begin{cases} u'' + f(u) = 0 & x \in \mathbb{R} \\ u(x) \to -1 \text{ as } x \to -\infty & u(x) \to 1 \text{ as } x \to \infty. \end{cases}$$

We have $u_0' > 0$.

By odd reflection, (1.1) is equivalent to

$$(1.4)\qquad \begin{cases} u'' + x^\alpha f(u) = 0 & x > 0 \\ u(0) = 0 & u(x) \to 1 \text{ as } x \to \infty. \end{cases}$$

Proposition 1.1. *There exists a solution u of (1.4) such that $u'(x) > 0,\ x \geq 0$.*

Partly supported by grant FONDECYT 3085026.

Proof. Note that

$$u_\delta(x)=\begin{cases}\delta u_0(x-1) & x\geq 1\\ \\ 0 & 0\leq x\leq 1,\end{cases}\tag{1.5}$$

$\delta\geq 0$ is a nontrivial subsolution of (1.4) if $\delta\in(0,\delta_0]$.
Indeed, in [0,1) we have $u_\delta=0$. So,

$$u_\delta''+x^\alpha f(u_\delta)=x^\alpha f(0)=0,\quad x\in[0,1).$$

In $(1,\infty)$ we have, for small $\delta\in(0,\delta_0]$,

$$\begin{aligned} u_\delta''+x^\alpha f(u_\delta) &= \delta u_0''(x-1)+x^\alpha f(\delta u_0(x-1))\\ \text{(by (1.2))} &\geq \delta u_0''(x-1)+\delta x^\alpha f(u_0(x-1))\\ &\geq \delta u_0''(x-1)+\delta f(u_0(x-1))=0.\end{aligned}$$

Moreover, $u_\delta(0)=0,\ u_\delta(\infty)=\delta<1$. Finally, the function u_δ is the maximum of two regular sub-solutions, and therefore u_δ itself must be a subsolution.

Let ϕ_ε solve

$$\begin{cases} -\phi_\varepsilon''-f_u(1)|x|^\alpha\phi_\varepsilon=-\phi_\varepsilon & x>\varepsilon^{-\frac{2}{\alpha}}\\ \\ \phi_\varepsilon(\varepsilon^{-\frac{2}{\alpha}})=1-u_0(\varepsilon^{-\frac{2+\alpha}{\alpha}})>0 & \phi_\varepsilon'(\varepsilon^{-\frac{2}{\alpha}})=-\varepsilon^{-1}u_0'(\varepsilon^{-\frac{2+\alpha}{\alpha}})<0.\end{cases}$$

We have $\phi_\varepsilon>0,\ \phi_\varepsilon'<0,\ \phi_\varepsilon''>0, x>\varepsilon^{-\frac{2}{\alpha}}$, and a standard barrier argument yields

$$0<\phi_\varepsilon(x)\leq Ce^{-\frac{1}{C\varepsilon^{\frac{2+\alpha}{\alpha}}}}e^{-\left(x-\varepsilon^{-\frac{2}{\alpha}}\right)},\quad x>\varepsilon^{-\frac{2}{\alpha}},$$

provided $\varepsilon>0$ is small (recall that $f_u(1)<0$ and $1-u_0(x)\leq Ce^{-|x|/C}$ for a generic constant C).

We claim that, for small ε,

$$v_\varepsilon=\begin{cases}1-\phi_\varepsilon(x) & x>\varepsilon^{-\frac{2}{\alpha}}\\ \\ u_0\left(\frac{x}{\varepsilon}\right) & 0<x\leq\varepsilon^{-\frac{2}{\alpha}}\end{cases}$$

is a supersolution of (1.4). Note that $v_\varepsilon>u_\delta$ if $\delta\leq\delta_0$ and $\varepsilon>0$ is small. We have $v_\varepsilon\in C^1(\mathbb{R})$ and $v_\varepsilon(0)=0,\ v_\varepsilon(\infty)=1$. In $(0,\varepsilon^{-\frac{2}{\alpha}})$,

$$v_\varepsilon''+|x|^\alpha f(v_\varepsilon)=\left(-\frac{1}{\varepsilon^2}+|x|^\alpha\right)f(v_\varepsilon)\leq 0.$$

In $(\varepsilon^{-\frac{2}{\alpha}},\infty)$,

$$\begin{aligned} v_\varepsilon'' + |x|^\alpha f(v_\varepsilon) &= -\phi'' + |x|^\alpha f(1-\phi) \\ &= -\phi'' - f_u(1)|x|^\alpha \phi + |x|^\alpha O(\phi^2) \\ &= \left(-1 + |x|^\alpha O(|\phi|)\right)\phi \\ &\le \left(-1 + C\left(|x-\varepsilon^{-\frac{2}{\alpha}}|^\alpha + \varepsilon^{-2}\right) e^{-\frac{1}{C\varepsilon^{\frac{2+\alpha}{\alpha}}}} e^{-\left(x-\varepsilon^{-\frac{2}{\alpha}}\right)}\right)\phi < 0 \end{aligned}$$

if ε is small.

Hence, there exists a solution u of the differential equation of (1.4) such that

$$u_\delta \le u \le v_\varepsilon, \quad x \ge 0.$$

It is easy to see that $u'(x) > 0,\ x \ge 0$ and, thus, u solves (1.4). The proof of the proposition is complete. □

Remark 1.2. *Alternatively, one can get existence using the continuation argument we will use for uniqueness (starting from $\alpha = 0$).*

Lemma 1.1. *Every increasing solution u of (1.4) satisfies*

$$u(x) \ge \delta_0 u_0(x-1), \quad x \ge 1.$$

Proof. We prove the result by using the subsolutions u_δ defined in (1.5) and Serrin's sweeping technique (cf. [2]).

By the proof of Proposition 1.1, u_δ is a subsolution of (1.4) if $\delta \in [0, \delta_0]$. Suppose that u is an increasing solution of (1.4). Then, if $\delta = 0$, $u \ge u_\delta$ in $(0,\infty)$. By the Serrin sweeping principle, and since $u(0) = 0,\ u(\infty) = 1$ while $u_\delta(0) = 0,\ u_\delta(\infty) = \delta < 1,\ u_\delta'(0) = 0$, we see that

$$u(x) \ge u_{\delta_0}(x) = \delta_0 u_0(x-1)_+ \ \text{ in } (0,\infty).$$

To prove this, we let $\tilde{\delta} = \sup\{\delta \in [0,\delta_0] :\ u \ge u_\delta \ \text{ in } (0,\infty)\}$, note that $u \ge u_{\tilde{\delta}}$ in $(0,\infty)$, and apply the maximum principle to $u - u_{\tilde{\delta}}$ to deduce that this function has a positive lower bound in $(0,\infty)$. This contradicts the maximality of $\tilde{\delta}$ if $\tilde{\delta} < \delta_0$. The proof of the lemma is complete. □

Lemma 1.2. *There exists $\alpha_0 > 0$ such that (1.4) has a unique increasing solution for each $0 < \alpha \le \alpha_0$.*

Proof. The idea is to obtain good asymptotics for the solution and then use this to prove uniqueness.

Step 1 Suppose that u_i are increasing solutions of (1.4) for α_i where $\alpha_i > 0$ and $\alpha_i \to 0$ as $i \to \infty$.

We have $0 < u_i < 1$ in $(0,\infty)$ and by equation

$$u_i'' + x^{\alpha_i} f(u_i) = 0, \quad x \in (0,\infty), \tag{1.6}$$

we see that

$$\|u_i\|_{C^2[0,L]} \le C(L), \quad L > 0,\ i \ge 1. \tag{1.7}$$

Using the Arczela-Ascoli theorem and the standard diagonal argument we obtain that, for a subsequence,

$$u_i \to u_* \ \text{ in } C^2_{loc}[0,\infty),$$

for some nondecreasing $0 \leq u_* \leq 1$ bounded solution of

$$u'' + f(u) = 0, \quad x > 0, \quad u(0) = 0,$$

(we also used that $x^\alpha \to 1$ in $C_{loc}[0,\infty)$ as $\alpha \to 0$).

By Lemma 1.1, we also have $u_* \geq \delta_0 u_0(x-1),\ x \geq 1$, i.e, u_* is nontrivial. Hence, $u_* \equiv u_0$, By the uniqueness of the limit, we conclude that the every subsequence of u_i satisfies

$$u_i \to u_0 \ \text{ in } C^2_{loc}[0,\infty) \text{ as } i \to \infty. \tag{1.8}$$

Step 2 Suppose that the lemma is false and that $u_i,\ v_i$ are two distinct increasing solutions of (1.4) with $\alpha = \alpha_i$ and $\alpha_i \to 0$ as $i \to \infty$. Let

$$\phi_i = \frac{u_i - v_i}{\|u_i - v_i\|_{L^\infty(0,\infty)}}.$$

Then ϕ_i satisfies

$$\begin{cases} \phi_i'' + V_i(x)\phi_i = 0 & x \in (0,\infty) \\ \\ \phi_i(0) = 0 & \|\phi_i\|_{L^\infty(0,\infty)} = 1,\ \phi_i(\infty) = 0, \end{cases} \tag{1.9}$$

where

$$V_i(x) = \begin{cases} x^{\alpha_i} \frac{f(u_i)-f(v_i)}{u_i - v_i} & \text{if } u_i(x) \neq v_i(x) \\ \\ x^{\alpha_i} f_u(u_i) & \text{if } u_i(x) = v_i(x). \end{cases}$$

which converges in $C_{loc}[0,\infty)$ to $f_u(u_0)$ by Step 1. Using once more the Arczela-Ascoli theorem and the standard diagonal argument we see that, for a subsequence,

$$\phi_i \to \phi_* \ \text{ in } C^2_{loc}[0,\infty) \text{ as } i \to \infty, \tag{1.10}$$

where ϕ_* is bounded in $(0,\infty)$ and solves

$$\phi'' + f_u(u_0)\phi = 0, \quad x > 0, \quad u(0) = 0,$$

Hence,

$$\phi_* \equiv 0. \tag{1.11}$$

(To see this note that the boundedness of ϕ and $f_u(1) < 0$ imply that $\phi_* \in H^2(0,\infty)$. So, its odd reflection is in the kernel of the operator $B\phi = \phi'' + f_u(u_0)\phi,\ \phi \in H^2(\mathbb{R})$. Thus, $\phi_* = \lambda u_0'$, and by setting $x = 0$ we get $\lambda = 0$).

There exist $c,\ d > 0$ such that $f_u(u) < -c$ if $|u - 1| \leq d$. Moreover, there is an $L_0 > 0$ such that $0 < 1 - u_0(x) < d$ if $x \geq L_0$. Since $u_i,\ v_i$ are increasing and converge to u_0 uniformly in compact intervals, we get that $1 - d < u_i(x),\ v_i(x) < 1,\ x \geq L_0$ if i is sufficiently large. By the mean value theorem,

$$V_i(x) = x^{\alpha_i} f_u\left(\theta u_i + (1-\theta)v_i\right), \quad \theta \in [0,1].$$

Hence,

$$V_i(x) \leq -cx^{\alpha_i}, \quad x \geq L_0. \tag{1.12}$$

There exist $x_i > 0$ such that, without loss of generality, $\phi_i(x_i) = 1,\ \phi_i'(x_i) = 0,\ \phi_i''(x_i) \leq 0,\ i \geq 1$. We claim that the sequence $\{x_i\}$ is bounded. Indeed, if not, by (1.9), (1.12) we find that $\phi_i''(x_i) \to \infty$ as $i \to \infty$, a contradiction.

By passing to a subsequence we may assume that $x_i \to x_* \geq 0$. In view of (1.10), we get

$$\phi_*(x_*) = 1,$$

which contradicts (1.11). The proof of the lemma is complete. □

Lemma 1.3. *If u is an increasing solution of (1.1) then u is nondegenerate. In particular it is asymptotically stable, i.e, if*

$$-\psi_1'' - |x|^\alpha f_u(u)\psi_1 = \mu_1\psi_1 \quad \text{in } \mathbb{R}, \tag{1.13}$$

$\psi_1 \in L^2(\mathbb{R})$, $\psi_1 > 0$ in $\mathbb{R}$, then $\mu_1 > 0$.

Proof. We will show that the spectrum, in $L^2(\mathbb{R})$, of the operator defined by the left-hand side of (1.13) is strictly positive. Since $-|x|^\alpha f_u(u) \to \infty$ as $|x| \to \infty$, its spectrum consists of discrete simple eigenvalues $\mu_1 < \mu_2 < \cdots$ with $\mu_i \to \infty$ as $i \to \infty$ and corresponding eigenfunctions $\psi_1, \psi_2, \cdots$. Each ψ_i, $i \geq 1$ has exactly $i-1$ nodes. Without loss of generality we can assume that $\psi_1 > 0$ is an even function of x. It is standard to show that $1-u, \psi_1$ tend to 0 super-exponentially as $|x| \to \infty$ (see [3]). Note that $w = u' > 0$ solves

$$-w'' - |x|^\alpha f_u(u)w = \alpha|x|^{\alpha-2}xf(u), \quad x \in \mathbb{R}. \tag{1.14}$$

By multiplying (1.13) with w, (1.14) with ψ_1 and subtracting, we find that

$$\mu_1\int_{-\infty}^{\infty}\psi_1 w dx = \alpha\int_{-\infty}^{\infty}|x|^{\alpha-2}xf(u)\psi_1 dx = 2\alpha\int_0^\infty x^{\alpha-1}f(u)\psi_1 dx > 0.$$

The proof is complete. □

Lemma 1.4. *If U is an increasing solution of $(1.1)_A$ (this means $\alpha = A$) then there exists a $\delta > 0$ and an increasing solution u_α of $(1.1)_\alpha$ for $|\alpha - A| \leq \delta$. Moreover,*

$$\|u_\alpha - u_A\|_{L^\infty(0,\infty)} \to 0 \quad \text{as } \alpha \to A. \tag{1.15}$$

Proof. We seek a solution u of $(1.1)_\alpha$ in the form $u = u_A + \varphi$, $\varphi \in H^2(\mathbb{R})$ odd. In terms of φ, $(1.1)_\alpha$ becomes

$$L(\varphi) = |x|^\alpha N(\varphi) + E,$$

where

$$L(\varphi) = -\varphi'' - |x|^\alpha f_u(U)\varphi$$

$$N(\varphi) = f(U+\varphi) - f(U) - f_u(U)\varphi$$

$$E = U'' + |x|^\alpha f(U).$$

We introduce the norm $\|\varphi\|_w = \|e^{|x|}\varphi\|_{L^\infty(\mathbb{R})}$ and the Banach space

$$X = \{\varphi : \ \varphi \text{ is odd}, \ \|\varphi\|_w < \infty\}.$$

Arguing as in Lemma 1.2 we see that L is nonsingular if α is close to A. By rather standard arguments we see that if $f \in X$ then there exists a unique $\varphi \in H^2(\mathbb{R}) \cap X$ such that $L(\varphi) = f$. Moreover $\|\varphi\|_w \leq C\|f\|_w$ (C is a generic constant). Note that (if $\|\varphi\|_{L^\infty} \leq 1$) we have

$$|x|^\alpha|N(\varphi)| \leq C|x|^\alpha\varphi^2 \leq C|x|^\alpha e^{-2|x|}\|\varphi\|_w^2$$

$$|E| = \left|\left(|x|^\alpha - |x|^A\right)f(U)\right| \leq C\left||x|^\alpha - |x|^A\right|e^{-|x|^{1+\alpha/2}},$$

($1-U$ decays super-exponentially). Now the existence of a solution u of $(1.1)_\alpha$ with $|\alpha - A|$ small satisfying (1.15) follows from Banach's fixed point theorem. We can show that $|u_\alpha'(x) - U'(x)| \to 0$ as $\alpha \to A$. It follows that $u \geq 0$ in $(0,\infty)$ and, thus, $u' > 0$ in $\mathbb{R}$ provided α is close to A. □

Lemma 1.5. *If u_α are increasing solutions of $(1.1)_\alpha$, $\alpha > \alpha_*$ then there exists $\delta > 0$ and increasing solutions u_α of $(1.1)_\alpha$ for $\alpha_* - \delta \leq \alpha \leq \alpha_*$. Moreover u_α are continuous functions of $[\alpha_* - \delta, \alpha]$ to $L^\infty(\mathbb{R})$ (assumming that they are for $\alpha > \alpha_*$).*

Proof. We know that there exists a sequence of increasing solutions u_i of (1.1) with $\alpha_i \to \alpha_*$ as $i \to \infty$ and $\alpha_i > \alpha_*$. Since $|u_i| \leq 1$, we can employ Arczela-Ascoli's theorem and the standard diagonal argument again to show that there exists a nondecreasing solution u_* of the equation of (1.1) with $\alpha = \alpha_*$. By Lemma 1.1 we get that u_* is nontrivial and, thus, is an increasing solution of (1.1). By the C_{loc} convergence and the fact that u_a are increasing we have that the convergence is uniform in $\mathbb{R}$. This establishes continuity from $[\alpha_*, \alpha]$ to $L^\infty(\mathbb{R})$. By Lemma 1.4 we obtain increasing solutions for $\alpha^* - \delta \leq \alpha \leq \alpha^*$. The continuity assertion follows by substracting two different equations and working as in the proof of Lemma 1.4. □

Theorem 1.1. *If $\alpha > 0$ then there exists a unique increasing solution of (1.1).*

Proof. Existence was established in Proposition 1.1. For the uniqueness we will argue by contradiction. Suppose that u_1, u_2 are two distinct increasing solutions of $(1.1)_{\alpha^0}$ for some $\alpha^0 > 0$. By Lemmas 1.4, 1.5 we can construct increasing solutions $u_i(\alpha)$, $i = 1, 2$ of $(1.1)_\alpha$ for $0 < \alpha < \alpha^0$ such that $u_i : (0, \alpha^0] \to L^\infty(\mathbb{R})$ is continuous. Lemma 1.2 implies that there exists an α_1 such that $u_1(\alpha_1) = u_2(\alpha_1)$. The continuity of the branches yields that the linearization of (1.1) at $u_1(\alpha_1)$ is singular (as in 1.2). On the other hand, since $u_1(\alpha_1)$ is increasing, we get a contradiction by Lemma 1.3. Therefore, the proof of Theorem 1.1 is complete. □

Remark 1.3. *Our approach for uniqueness is motivated from* [1].

Remark 1.4. *If instead of (1.2) we had the stronger*

$$f(\delta u) \geq \delta f(u), \quad u \in [0, 1], \quad \delta \in [0, 1],$$

as in the case $f(u) = u - u|u|^p$, $p > 0$, the uniqueness proof is very simple. Suppose that u, v are two increasing solutions of (1.4). Then δu, $\delta \in [0, 1]$ is a family of subsolutions and $\delta u \leq v$ in $(0, \infty)$ if $\delta = 0$. The Serrin sweeping principle yields $u \leq v$ in $(0, \infty)$. Similarly, $v \leq u$ in $(0, \infty)$.

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