TRAVELING KINKS IN A CHAIN OF COUPLED OSCILLATORS: THE ANTI-INTEGRABLE LIMIT

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Abstract. We prove the existence of heteroclinic traveling waves (kinks) in a chain of coupled oscillators. We apply a perturbation argument from the anti-integrable limit.

Introduction and main result

0.1. **The model.** Consider an array of particles attached at equal distances h to a rigid background by bistable springs with energy density $F(u_n)$. By u_n we denote the vertical displacement of the particle with index n. If we assume that the neighboring particles interact through standard harmonic forces, characterized by an elastic modulus ε; then the potential and kinetic energies of the chain can be written in the form

$$\mathbf{V} = \sum_n \frac{1}{2}\varepsilon h \left(\frac{u_{n+1}-u_n}{h}\right)^2 + hF(u_n), \quad \mathbf{K} = \sum_n \frac{1}{2}\rho h \dot{u}_n^2$$

where ρ is the mass density per unit spring length and $\dot{} = \frac{d}{dt}$. The Euler-Lagrange equations are then

$$\rho h \ddot{u}_n - \frac{\varepsilon}{h}(u_{n+1} - 2u_n + u_{n-1}) + hf(u_n) = 0, \quad n \in \mathbb{Z}, \tag{1}$$

where $f = F_u$. We refer the reader to [KT] for more details on the physical background of the problem.

We shall construct solutions of the above equation in the form of traveling waves. So let $x = hn$, write $u_n(t) = u(x,t)$, and seek a solution of the form $u(x,t) = u(\xi) = u(x - ct)$ satisfying the equation

$$\rho c^2 \frac{d^2u}{d\xi^2} - \frac{\varepsilon}{h^2}[u(\xi+h) - 2u(\xi) + u(\xi-h)] + f(u) = 0.$$

If we rescale ξ such that $z = \xi/h$, the equation for $u(z) = u(h\xi)$ becomes

$$\rho c^2 \frac{d^2u}{dz^2} - \varepsilon[u(z+1) - 2u(z) + u(z-1)] + h^2 f(u) = 0. \tag{2}$$

System (1) has been studied, for various types of functions f, by many authors. In [MA] the implicit function theorem was used to show the existence of breathers in the anti-integrable limit $\varepsilon \to 0$. Homoclinic traveling wave solutions were studied in [IK] by a center manifold reduction. The phenomenon of blue sky catastrophe has been investigated in [Fe]. We refer the reader to [BZ] for traveling pulses in the integrable limit ($h \to 0$) where also many further references can be found.

0.2. **The main result.** In the current paper, for given $c \neq 0$, we study heteroclinic waves for h fixed and sufficiently small ε and nonlinearity f satisfying (H1) below. We also allow infinite range and not just nearest neighbor or finite length interaction, although those are included as special cases.

Hence, the equation we will be dealing with (after a relabeling) is

$$u'' - \epsilon \sum_{k=-\infty}^{\infty} a_k u(z-k) + f(u) = 0, \quad z \in \mathbb{R} \tag{3}$$

together with the conditions

$$\lim_{z \to -\infty} u(z) = 0, \quad \lim_{z \to \infty} u(z) = 1. \tag{4}$$

Here $\epsilon \geq 0$ is small, and

(H1) $f \in C^2(\mathbb{R}),\ f(0) = f(1) = 0,\ f_u(0), f_u(1) < 0;$

$$F(1) = 0,\ F(u) \neq 0\ \ \forall u \in (0,1)$$

where $F(u) = \int_0^u f(s)ds$.

(H2) $\sum_{k=-\infty}^{\infty} a_k = 0,\ a_0 < 0,\ a_k = a_{-k},$ and $\sum_{k \geq 1} |a_k| k^2 < \infty$.

Remark 1. *We will assume these hypotheses throughout the rest of the paper.*

When $\epsilon = 0$, (3) becomes

$$u'' + f(u) = 0. \tag{5}$$

Under the hypotheses (H1), the above equation has a heteroclinic solution u_0 satisfying (4) (see [Ar]).

Our result is

Theorem 1. *If $\epsilon > 0$ is sufficiently small, then there exists a solution u_ϵ of (3) such that*

$$||u_\epsilon - u_0||_{H^2(\mathbb{R})} \leq C\epsilon$$

($C > 0$ is a constant independent of ϵ).

To prove this we adapt a technique from an earlier paper of ours (cf. [AFFS]). We use two important properties:

(i) Nondegeneracy of u_0 The operator obtained by linearizing the left-hand side of (5) at u_0 has 0 as a simple isolated eigenvalue, the remaining spectrum being in the open left half-plane ([He], Section 5.4).

(ii) Hamiltonian form of the problem If u satisfies (4) and $u' \in H^1(\mathbb{R})$, then

$$\int_{-\infty}^{\infty} \left(u'' - \epsilon \sum_{k=-\infty}^{\infty} a_k u(z-k) + f(u) \right) u' dz = 0.$$

In Section 1 we present the proof of Theorem 1, in Section 2 we make some comments on the application of the standard Lyapunov-Schmidt reduction, and in the Appendix's we prove some technical Lemmas.

Remark 2. *In what follows, $|| \cdot ||_{L^2}$, $|| \cdot ||_{L^\infty}$, and $|| \cdot ||_{H^i}$, $(i = 1, 2)$ denote the norms of the spaces $L^2(\mathbb{R})$, $L^\infty(\mathbb{R})$, and $H^i(\mathbb{R})$, respectively. Also,*

$$(\phi, \psi) \equiv \int_{\mathbb{R}} \phi\psi dz; \quad \phi \perp \psi \ \Leftrightarrow\ (\phi, \psi) = 0.$$

Unless specified otherwise C/c denotes a large/small positive constant independent of $\epsilon > 0$ whose value will change from line to line. In many cases we will not explicitly write the obvious dependence of functions on ϵ.

1. Proof of Theorem 1

1.1. **Properties of the linear operator Δ.** Consider the linear operator Δ defined via

$$\Delta u := \sum_{k=-\infty}^{\infty} a_k u(z-k).$$

Then one can verify that if $u, v \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap C(\mathbb{R})$, then

$$||\Delta u||_{L^2} \leq C||u||_{L^2} \quad (C \text{ independent of } u), \tag{6}$$

and

$$(\Delta u, v) = (u, \Delta v).$$

We also have the following Lemma whose proof will be given in Appendix A.

Lemma 1. *If $u \in C^1(\mathbb{R})$, $u' \in L^2(\mathbb{R})$;*

$$\lim_{z\to-\infty} u(z) = u(-\infty) \in \mathbb{R} \quad \text{and} \quad \lim_{z\to\infty} u(z) = u(\infty) \in \mathbb{R}, \tag{7}$$

then

$$\Delta u \in L^2(\mathbb{R}) \quad \text{and} \quad (\Delta u, u') = 0.$$

1.2. **Properties of the heteroclinic u_0.** It is well known (see [Ar], [HK]) that u_0 is the unique (up to translation) solution of (5), (4). Furthermore $u_0' > 0$, u_0 approaches its limits exponentially and

$$u_0'(z),\ |u_0''(z)|,\ |u_0'''(z)| \leq Ce^{-c|z|}, \quad z \in \mathbb{R}.$$

The linear operator L^0 with $D(L^0) = H^2(\mathbb{R})$ and

$$L^0\phi = \phi'' + f_u(u_0(z))\phi$$

is self-adjoint in $L^2(\mathbb{R})$ and $\sigma(-L^0) \subseteq \{0\} \cup [c, \infty)$ with 0 a simple eigenvalue corresponding to u_0'. (We refer to [SF] for an elementary proof of this or note that since $f_u(0),\ f_u(1) < 0$ then $\sigma_{ess}(-L^0) \subseteq [c, \infty)$; thus the only thing left to prove is that 0 is the principal eigenvalue which follows from $u_0' > 0$.)

These properties imply

Proposition 1. *Let $g \in L^2(\mathbb{R})$ with $g \perp u_0'$, then there exists a unique $\phi \in H^2(\mathbb{R})$ with $\phi \perp u_0'$ such that*

$$L^0\phi = g.$$

Furthermore we have

$$||\phi||_{H^2} \leq C||g||_{L^2}$$

with C independent of g.

1.3. **The perturbation argument.** We search for a solution of (3) in the form $u_\epsilon = u_0 + \phi_\epsilon$ with $\phi_\epsilon \in H^2(\mathbb{R})$ and $\phi_\epsilon \perp u_0'$. Then ϕ_ϵ must satisfy

$$L^0\phi = \epsilon\Delta\phi + N(\phi) + \epsilon\Delta u_0$$

with

$$N(\phi) = -f(u_0+\phi) + f(u_0) + f_u(u_0)\phi.$$

We define a mapping $\mathbf{T} : H^2(\mathbb{R}) \cap \{u_0'\}^\perp \to H^2(\mathbb{R}) \cap \{u_0'\}^\perp$ via $\mathbf{T}\phi = \bar{\phi}$ where

$$L^0\bar{\phi} = -b(\phi)u_0' + \epsilon\Delta\phi + N(\phi) + \epsilon\Delta u_0 \tag{8}$$

and

$$b(\phi) = \frac{1}{||u_0'||^2_{L^2}}\left(\epsilon\Delta\phi + N(\phi) + \epsilon\Delta u_0, u_0'\right). \tag{9}$$

Note that $\mathbf{T}$ is well defined via Proposition 1 since the right hand side of (8) is orthogonal to u_0'. (Note also that by Lemma 1 we have $\Delta u_0 \in L^2(\mathbb{R})$.)

Let

$$B_\epsilon = \{\phi \in H^2(\mathbb{R}) \cap \{u_0'\}^\perp : \ ||\phi||_{H^2} \leq M\epsilon\}$$

with M a positive constant independent of $\epsilon > 0$ to be determined later.

We will show that there exists a large $M > 0$ such that, provided $\epsilon > 0$ is sufficiently small, $\mathbf{T}$ maps B_ϵ into itself and is a contraction. Thus $\mathbf{T}$ possesses a fixed point, which, after adding u_0, gives a solution to

$$u'' - \epsilon\Delta u + f(u) = -b_\epsilon u_0' \tag{10}$$

for some $b_\epsilon \in \mathbb{R}$. Then multiplying the above equation by u', integrating and using Lemma 1 and (H1) we find that $b_\epsilon = 0$, which gives a solution to (3), (4).

We now present the details. Let $\phi \in B_\epsilon$, then via (8), (9) and Proposition 1,

$$\begin{aligned} ||\bar{\phi}||_{H^2} &\leq C\left(|b(\phi)| + \epsilon||\Delta\phi||_{L^2} + ||N(\phi)||_{L^2} + \epsilon||\Delta u_0||_{L^2}\right) \\ &\overset{(9)}{\leq} C\left(\epsilon||\Delta\phi||_{L^2} + ||N(\phi)||_{L^2} + \epsilon||\Delta u_0||_{L^2}\right). \end{aligned} \tag{11}$$

The first and third term will be estimated from (6) and Lemma 1 respectively. To estimate the nonlinear term $N(\phi)$, we first recall the embedding $||\phi||_{L^\infty} \leq C||\phi||_{H^1}$ for every $\phi \in H^1(\mathbb{R})$. Hence, setting $C = \sup_{|s|\leq 2}|f_{uu}(s)|$, we have

$$|N(\phi)| \leq CM\epsilon|\phi| \ \text{ and } \ |N(\phi_1) - N(\phi_2)| \leq CM\epsilon|\phi_1 - \phi_2| \tag{12}$$

pointwise for all $\phi, \phi_1, \phi_2 \in B_\epsilon$. Thus, (11) yields

$$\begin{aligned} ||\bar{\phi}||_{H^2} &\leq C\left(\epsilon||\phi||_{L^2} + M\epsilon||\phi||_{L^2} + \epsilon\right) \\ &\leq C\left(M\epsilon + M^2\epsilon + 1\right)\epsilon. \end{aligned}$$

Choosing a large M (say $M = 2C$), then $||\bar{\phi}||_{H^2} \leq M\epsilon$ provided $\epsilon > 0$ is sufficiently small, i.e. $\mathbf{T} : B_\epsilon \to B_\epsilon$. Similarly we can show that $\mathbf{T}$ is a contraction in the H^2 norm. Hence, since B_ϵ is closed with respect to this norm, the Banach fixed point theorem gives us a fixed point $\phi_* \in B_\epsilon$ of $\mathbf{T}$. Then

$$u_\epsilon = u_0 + \phi_*^\epsilon \tag{13}$$

satisfies (10) for some $b_\epsilon \in \mathbb{R}$ ($b_\epsilon = b(\phi_*^\epsilon)$). Multiplying (10) by u'_ϵ and integrating over $\mathbb{R}$ yields

$$\int_{\mathbb{R}} u'u''dz - \epsilon(\Delta u, u') + \int_{\mathbb{R}} f(u)u'dz = -b_\epsilon \int_{\mathbb{R}} u'_0 u' dz.$$

Since $\phi_* \in H^2(\mathbb{R})$, we have that $u(-\infty) = 0$, $u(\infty) = 1$ and $u'(\pm\infty) = 0$. The left hand side of the above equation is 0 (see Lemma 1 and recall that $F(0) = F(1)$) and we get that

$$b_\epsilon \int_{\mathbb{R}} u'_0 u' dz = 0.$$

This implies that $b_\epsilon = 0$ since

$$\begin{aligned} \int_{\mathbb{R}} u'_0 u' dz &= \int_{\mathbb{R}} u'_0(u'_0 + \phi'_*)dz \geq \int_{\mathbb{R}} u_0'^2 dz - ||u'_0||_{L^2}||\phi'_*||_{L^2} \geq \\ &\geq \int_{\mathbb{R}} u_0'^2 dz - C\epsilon > 0 \end{aligned}$$

provided $\epsilon > 0$ is sufficiently small.

Therefore u given by (13) is a solution of (3) satisfying the estimate of Theorem 1, thereby completing the proof.

Remark 3. *If the $\epsilon = 0$ equation (3) has a unique even homoclinic solution u_0 (see* [BL] *for necessary and sufficient conditions on f), then the proof of persistence for ϵ small is considerably simplified by seeking $u_\epsilon = u_0 + \phi$ with $\phi \in H^2(\mathbb{R})$ even. Note that given an even $g \in L^2(\mathbb{R})$, there exists a unique even $\phi \in H^2(\mathbb{R})$ such that $\phi'' + f_u(u_0)\phi = g$. Moreover $||\phi||_{H^2} \leq C||g||_{L^2}$ for some $C > 0$ independent of g.*

2. Some remarks on the standard Lyapunov-Schmidt approach

In this section we make some remarks on a difficulty that arises when trying to prove Theorem 1 using the standard Lyapunov-Schmidt reduction.

We have seen that u_0 satisfies (3) up to an order of ϵ. We begin by refining this approximation so that $u_{ap} = u_0 + \epsilon u_1$ satisfies (3) up to an order of ϵ^2. We choose $u_1 \in H^2(\mathbb{R})$, $u_1 \perp u'_0$ such that

$$u''_1 + f_u(u_0)u_1 = \Delta u_0 \tag{14}$$

(this is possible via Lemma 1 and Proposition 1). Then, a simple calculation gives

$$-G(\epsilon) := u''_{ap} - \epsilon\Delta u_{ap} + f(u_{ap}) = -\epsilon^2 \Delta u_1 - N(\epsilon u_1)$$

and thus from (12):

$$||G(\epsilon)||_{L^2} \leq C\epsilon^2. \tag{15}$$

We seek a solution of (3) in the form $u_\epsilon = u_{ap} + \psi_\epsilon$ with $\psi_\epsilon \in H^2(\mathbb{R})$. Then ψ_ϵ must satisfy

$$L^\epsilon \psi = N_{ap}(\psi) + G(\epsilon) \tag{16}$$

where $L^\epsilon \psi = \psi'' + f_u(u_{ap})\psi - \epsilon\Delta\psi$ and $N_{ap}(\psi) = -f(u_{ap} + \psi) + f(u_{ap}) + f_u(u_{ap})\psi$.

Since $\Delta : H^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a bounded linear operator and $||u_{ap} - u_0||_{L^\infty} \leq C\epsilon$, L^ϵ is a regular $O(\epsilon)$ perturbation of L^0. From the special form of the perturbation, however, the simple eigenvalue 0 of L^0 is perturbed to an $O(\epsilon^2)$ simple eigenvalue of L^ϵ (this is the the source of the difficulty). More precisely we have the following Proposition whose proof we postpone to Appendix B.

Proposition 2. *If $\epsilon \geq 0$ is sufficiently small then $\sigma(-L^\epsilon) \subset \{\lambda_1(\epsilon)\} \cup [c, \infty)$ with $\lambda_1(\epsilon)$ simple corresponding to $\phi_1(\epsilon) \in H^2(\mathbb{R})$ with $||\phi_1(\epsilon)||_{H^2} = 1$. Moreover $\lambda_1(\epsilon)$, $\phi_1(\epsilon)$ depend smoothly on ϵ up to $\epsilon = 0$ and*

$$\begin{aligned} &\lambda_1(\epsilon) = O(\epsilon^2) \\ &\phi_1(\epsilon) = \tfrac{u_0'}{||u_0'||_{H^2}} + O(\epsilon) \qquad (\textit{here } ||O(\epsilon)||_{H^2} \leq C\epsilon). \end{aligned} \tag{17}$$

Define the orthogonal projection P onto the span of ϕ_1 by

$$P\psi = (\psi, \phi_1(\epsilon)) \frac{\phi_1(\epsilon)}{||\phi_1(\epsilon)||^2_{L^2}}.$$

According to this projection we have

$$H^2(\mathbb{R}) = \text{span}\{\phi_1\} \oplus X_1, \quad L^2(\mathbb{R}) = \text{span}\{\phi_1\} \oplus Y_1,$$

where $X_1,\ Y_1$ are respectively the kernel of P in $H^2(\mathbb{R})$ and $L^2(\mathbb{R})$. By decomposing ψ as $\psi = a\phi_1(\epsilon) + v$ $(a \in \mathbb{R}, v \in X_1)$, one finds that (16) is equivalent to

$$\begin{aligned} L^\epsilon v = &\quad (I - P)\{N_{ap}(a\phi_1(\epsilon) + v) + G(\epsilon)\} \\ -a\lambda_1(\epsilon)\phi_1(\epsilon) = &\quad P\{N_{ap}(a\phi_1(\epsilon) + v) + G(\epsilon)\}. \end{aligned} \tag{18}$$

Applying Proposition 2, using (15), and the Banach fixed point theorem, we can uniquely solve $(18)_{(i)}$ for $v = v^*(a, \epsilon)$ in a neighborhood of $(a, v) = (0, 0)$. This solution depends smoothly on a, ϵ and satisfies $||v^*(a, \epsilon)||_{H^2} = O(a^2 + \epsilon^2)$ if $|a|, \epsilon \geq 0$ small. Using this in $(18)_{(ii)}$ and taking the L^2 inner product with $\phi_1(\epsilon)$ yields

$$B(a, \epsilon) := -a\lambda_1(\epsilon)||\phi_1(\epsilon)||^2_{L^2} - (N_{ap}(a\phi_1(\epsilon) + v^*) + G(\epsilon), \phi_1(\epsilon)) = 0 \tag{19}$$

i.e.

$$B(a, \epsilon) = -a\lambda_1(\epsilon)||\phi_1(\epsilon)||^2_{L^2} + \frac{1}{2}\left(f_{uu}(u_{ap})\phi_1^2, \phi_1\right) a^2 - (G(\epsilon), \phi_1) + O(a^\mu \epsilon^\nu) = 0$$

as $a, \epsilon \to 0$ with $\mu + \nu \geq 3$. If $\lambda_1(\epsilon) = d\epsilon + O(\epsilon^2)$ with $d \neq 0$ (indep. of ϵ), then we could apply the implicit function theorem to $\epsilon^{-2}B(\epsilon\tilde{a}, \epsilon) = 0$ and find an $a_* = O(\epsilon)$ satisfying (19). However, since by $(17)_{(i)}$ we have d=0, this analysis breaks down.

Appendix A. Proof of Lemma 1

$u \in L^\infty(\mathbb{R})$ and (H2) imply that $\Delta u \in L^\infty(\mathbb{R})$ and

$$\Delta u(z) = \sum_{k=-\infty}^{\infty} a_k[u(z - k) - u(z)] = \sum_{k=-\infty}^{\infty} a_k \int_0^{-k} u'(z + t)dt, \quad z \in \mathbb{R}.$$

$$(\Delta u(z))^2 \leq \sum_{k=-\infty}^{\infty} |a_k| \sum_{k=-\infty}^{\infty} |a_k| \left(\int_0^{-k} u'(z + t)dt\right)^2 \overset{(H2)}{\leq} C \sum_{k=-\infty}^{\infty} |a_k|(-k) \int_0^{-k} u'^2(z+t)dt.$$

$$\int_{-\infty}^{\infty} (\Delta u(z))^2\, dz \leq C \int_{-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} |a_k|(-k) \int_0^{-k} u'^2(z + t)dt\right) dz =$$

$$= C \sum_{k=-\infty}^{\infty} |a_k|(-k) \int_{-\infty}^{\infty} \int_0^{-k} u'^2(z+t)dtdz = C \sum_{k=-\infty}^{\infty} |a_k|(-k) \int_0^{-k} \int_{-\infty}^{\infty} u'^2(z+t)dzdt =$$

$$= C\sum_{k=-\infty}^{\infty}|a_k|k^2||u'||^2_{L^2} \overset{(H2)}{\leq} C||u'||^2_{L^2} < \infty.$$

Thus, $\Delta u \in L^2(\mathbb{R})$.

We have

$$(\Delta u, u') = \int_{-\infty}^{\infty} u'(z)\sum_{k=-\infty}^{\infty} a_k[u(z-k)-u(z)]dz =$$

$$= \sum_{k=-\infty}^{\infty} a_k \int_{-\infty}^{\infty} u'(z)[u(z-k)-u(z)]dz = \sum_{k=-\infty}^{\infty} a_k \int_{-\infty}^{\infty} u'(z+k)[u(z)-u(z+k)]dz =$$

$$\overset{a_k=a_{-k}}{=} \sum_{k=-\infty}^{\infty} a_k \int_{-\infty}^{\infty} u'(z-k)[u(z)-u(z-k)]dz.$$

Thus,

$$2(\Delta u, u') = \sum_{k=-\infty}^{\infty} a_k \int_{-\infty}^{\infty} \frac{d}{dz}\left\{u(z)u(z-k) - \frac{u^2(z)}{2} - \frac{u^2(z-k)}{2}\right\} dz \overset{(7)}{=} 0.$$

Appendix B. Proof of Proposition 2

Since zero is a simple eigenvalue of $-L^0$, it follows from regular perturbation theory (cf. Sec. 14.3 of [CH]) that it perturbs smoothly to a simple eigenvalue $\lambda(\epsilon)$ of $-L^\epsilon$. The corresponding eigenfunction $\phi(\epsilon)$ with $||\phi(\epsilon)||_{H^2} = 1$ also depends smoothly (in the H^2 norm) on $\epsilon \geq 0$ small and $\phi(0) = \frac{u_0'}{||u_0'||_{H^2}}$. It is easy to show that $\lambda(\epsilon)$ is the principal eigenvalue of $-L^\epsilon$; we denote it by $\lambda_1(\epsilon)$ and the corresponding H^2 normalized eigenfunction by $\phi_1(\epsilon)$. (Recall that $(-L^0\phi, \phi) \geq c||\phi||^2_{L^2},\ \forall \phi \in H^2(\mathbb{R}),\ \phi \perp u_0'$, and $\left|\left|\phi_1(\epsilon) - \frac{u_0'}{||u_0'||_{H^2}}\right|\right|_{H^2} \leq C\epsilon$, to obtain $(-L^\epsilon\phi, \phi) \geq c||\phi||^2_{L^2},\ \forall \phi \in H^2(\mathbb{R}),\ \phi \perp \phi_1(\epsilon)$.) We have

$$\phi_1'' + f_u(u_{ap})\phi_1 - \epsilon\Delta\phi_1 = -\lambda_1\phi_1$$

and

$$-\lambda_1(\phi_1, u_0') = (\phi_1, u_0''' + f_u(u_{ap})u_0' - \epsilon\Delta u_0') = (\phi_1, [f_u(u_0+\epsilon u_1) - f_u(u_0)]u_0' - \epsilon\Delta u_0')\,.$$

Since $\phi_1(\epsilon) \overset{H^2}{\to} \frac{u_0'}{||u_0'||_{H^2}}$ as $\epsilon \to 0$, we get

$$-\lim_{\epsilon\to 0}\frac{\lambda_1(\epsilon)}{\epsilon} = \frac{1}{||u_0'||^2_{L^2}}\left(u_0', f_{uu}(u_0)u_1u_0' - \Delta u_0'\right).$$

Differentiating (14) yields

$$u_1''' + f_{uu}(u_0)u_0'u_1 + f_u(u_0)u_1' = \Delta u_0'$$

i.e.

$$(u_0', f_{uu}(u_0)u_1u_0' - \Delta u_0') = -(u_0', u_1''' + f_u(u_0)u_1') = -(u_0''' + f_u(u_0)u_0', u_1') = 0.$$

This and the smoothness of $\lambda_1(\epsilon)$ gives us $(17)_{(i)}$. From the smoothness of $\phi_1(\epsilon)$ (in the H^2 norm) we have $(17)_{(ii)}$.

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