UNIFORM ESTIMATES FOR POSITIVE SOLUTIONS OF A CLASS OF SEMILINEAR ELLIPTIC EQUATIONS

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Abstract. We consider the semilinear elliptic equation $\Delta u = W'(u)$ with Dirichlet boundary conditions in a Lipschitz, possibly unbounded, domain $\Omega \subset \mathbb{R}^n$. Under suitable assumptions on the potential W, including the double well potential that gives rise to the Allen-Cahn equation, we deduce a condition on the size of the domain that implies the existence of a positive solution satisfying a uniform pointwise estimate. Here, uniform means that the estimate is independent of Ω. The main advantage of our approach is that it allows us to remove a restrictive monotonicity assumption on W that was imposed in the recent paper by G. Fusco, F. Leonetti and C. Pignotti [89]. In addition, we can remove a non-degeneracy condition on the global minimum of W that was assumed in the latter reference. Furthermore, we can generalize an old result of P. Hess [102] and D. G. De Figueiredo [71], concerning semilinear elliptic nonlinear eigenvalue problems. Moreover, we study the boundary layer of global minimizers of the corresponding singular perturbation problem. For the above applications, our approach is based on a refinement of a useful result that dates back to P. Clément and G. Sweers [57], concerning the behavior of global minimizers of the associated energy over large balls, subject to Dirichlet conditions. Combining this refinement with global bifurcation theory and the celebrated sliding method, we can prove uniform estimates for solutions away from their nodal set, refining a lemma from a well known paper of H. Berestycki, L. A. Caffarelli and L. Nirenberg [28]. In particular, combining our approach with a-priori estimates that we obtain by blow-up, the doubling lemma of P. Polacik, P. Quittner, and P. Souplet [138] and known Liouville type theorems, we can give a new proof of a Liouville type theorem of Y. Du and L. Ma [76], without using boundary blow-up solutions. We can also provide an alternative proof of a Liouville theorem of H. Berestycki, F. Hamel, and H. Matano [32], involving the presence of an obstacle.

1. Introduction and statement of the main result

A problem that has received considerable attention in the literature is the study of the structure of solutions $(\lambda, u) \in \mathbb{R} \times C^{2,\alpha}(\bar{\mathcal{D}})$, $0 < \alpha < 1$, depending on the nonlinearity f, of the semilinear elliptic nonlinear eigenvalue problem

$$\Delta u + \lambda f(u) = 0,\ x \in \mathcal{D};\quad u(x) = 0,\ x \in \partial\mathcal{D}, \tag{1.1}$$

where $\mathcal{D}$ is typically a smooth bounded domain. To this end, the main approaches used include the method of upper and lower solutions, bifurcation techniques, as well as topological and variational methods (see [112], [122], [152], [155] and the references therein).

Recently, G. Fusco, F. Leonetti and C. Pignotti considered in [89] the semilinear elliptic problem

$$\begin{cases} \Delta u = W'(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is a domain with nonempty Lipschitz boundary (see for instance [80]), under the following assumptions on the C^2 function $W : \mathbb{R} \to \mathbb{R}$, which we will often refer to as a potential:

(a): There exists a constant $\mu > 0$ such that

$$0 = W(\mu) < W(t),\ t \in [0, \infty),\ t \neq \mu,$$
$$W(-t) \geq W(t),\ t \in [0, \infty);$$

(b): $W'(t) \leq 0,\ t \in (0, \mu)$;
(c): $W''(\mu) > 0$.

A model potential which satisfies the assumptions in [89] is the double well potential in (1.23) below, appearing frequently in the mathematical study of phase transitions, see [72]. Another, model example is given in (4.1). An example of an unbounded domain with nonempty Lipschitz boundary is (4.10) below, which was considered in [28]. We stress that, in the case where the domain is unbounded, the boundary conditions in (1.2) *do not* refer to $u(x) \to 0$ as $|x| \to \infty$ with $x \in \Omega$. Note that (1.1) can be related to (1.2) via a simple rescaling (see the relation between (9.9) and (9.10) below).

For $x \in \mathbb{R}^n$, $\rho > 0$, we let

$$B_\rho(x) = \{y \in \mathbb{R}^n \ : \ |y - x| < \rho\}, \quad B_\rho = B_\rho(0),$$
$$A + B = \{x + y \ : \ x \in A,\ y \in B\}, \quad A, B \subset \mathbb{R}^n,$$

and denote by $d(x, E)$ the Euclidean distance of the point $x \in \mathbb{R}^n$ from the set $E \subset \mathbb{R}^n$, and by $|E|$, unless specified otherwise, the n-dimensional Lebesgue measure of E (see [80]). By $\mathcal{O}(\cdot)$, $o(\cdot)$ we will denote the standard Landau's symbols.

The main result of [89] was the following:

Theorem 1.1. Assume Ω and W as above. There are positive constants R^*, $r^* \in (0, R^*)$, $a^* \in (0, \mu)$, k, K, depending only on W and n, such that if Ω contains a closed ball of radius R^*, then problem (1.2) has a solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ verifying

$$0 < u(x) < \mu, \quad x \in \Omega, \tag{1.3}$$

$$\mu - a^* < u(x), \quad x \in \Omega_{R^*} + B_{r^*}, \tag{1.4}$$

and

$$\mu - u(x) \leq K e^{-k d(x, \partial\Omega)}, \quad x \in \Omega, \tag{1.5}$$

where

$$\Omega_{R^*} = \{x \in \Omega \ : \ d(x, \partial\Omega) > R^*\}. \tag{1.6}$$

The approach of [89] to the proof of Theorem 1.1 is variational, involving the construction of various judicious radial comparison functions, see also [9]. In our opinion, their argument boils down to the construction of a lower solution to (1.2), see [26], whose building blocks, after a translation, are radial solutions of

$$\Delta u + a^2(\mu - u) = 0 \text{ in } B_R \text{ (with } a^2 < W''(\mu)), \quad -\Delta u = 0 \text{ in } \Omega \backslash B_R, \tag{1.7}$$

and the constant function 0, for large R (note that assumption **(b)** implies that solutions of (1.2) are supharmonic). We note that, once (1.4) is established, the proof of the exponential decay estimate (1.5), given in [89], can be simplified considerably by employing Lemma 4.2 in [84], making use of the non-degeneracy condition **(c)** (the constants in Theorem 1.1 can be chosen so that $W''(t) > 0,\ t \in [\mu - a^*, \mu]$). Moreover, an examination of the proof of Lemma 2.1 in [89] (see Lemma A.1 herein) shows that assumption **(a)** above can be relaxed to

(a'): There exists a constant $\mu > 0$ such that

$$0 = W(\mu) < W(t),\ t \in [0,\mu), \quad W(t) \geq 0,\ t \in \mathbb{R},$$
$$W(-t) \geq W(t),\ t \in [0,\mu].$$

For a typical example of such a potential, see Figure 1.1.

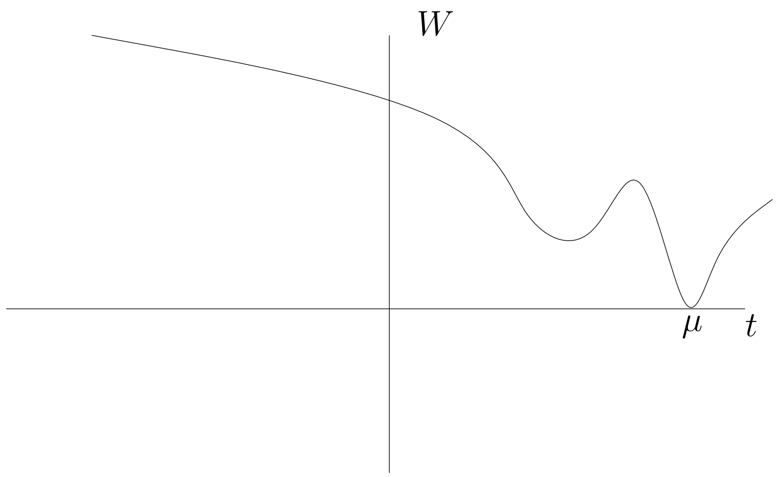

FIGURE 1.1. An example of a potential W satisfying hypothesis **(a')**.

If one further assumes that

$$W''(0) < 0 \ \text{ if } \ W'(0) = 0, \tag{1.8}$$

and

$$W'(t) < 0, \quad t \in (0,\mu), \tag{1.9}$$

then Theorem 1.1 can essentially be deduced from Lemmas 3.2–3.3 in the famous article [28] by Berestycki, Caffarelli and Nirenberg or Lemma 4.1 in the recent article [114] by Pacard, Kowalczyk and Liu, see also [90]. In fact, the latter lemmas hold for arbitrary positive solutions.

The main purpose of this article is to show that relation (1.4) can be established in a simple manner *without* assuming the monotonicity condition **(b)**, and in fact we will prove a stronger version of it. A well known nonlinearity which satisfies our assumptions but not **(b)** is

$$W'(u) = u(u-a)(u-\mu) \text{ with } 0 < a < \frac{\mu}{2}, \tag{1.10}$$

which arises in the mathematical study of population genetics (see [22]). Moreover, we remove completely the non-degeneracy condition **(c)** from the proof of (1.4). We will accomplish this, loosely speaking, by using translations of a positive solution of

$$\Delta u = W'(u),\ x \in B_R; \quad u(x) = 0,\ x \in \partial B_R,$$

which minimizes the associated energy, as a lower solution of (1.2) after we have extended it by zero outside of B_R. Actually, this approach will allow us to refine the results of [28], [114] that we mentioned earlier in relation to (1.8), (1.9). On the other side, assuming further that W' satisfies a scaling property and that the corresponding whole space problem (1.22) below does not have nontrivial entire solutions (a Liouville type theorem), we will use "blow-up" arguments from [92] together with a key "doubling lemma" from [138] to establish that Lemma 3.3 in [28] can be improved.

In passing, we remark that a similar monotonicity assumption to **(b)** also appears in a series of papers [9], [10], [13] in the context of variational elliptic systems of the form $\Delta u = \nabla_u W(u)$ with $W : \mathbb{R}^n \to \mathbb{R}$. In particular, these references employ comparison functions of the form (1.7). In this direction, see also Remarks 1.4, 2.9 below.

Our main result is

Theorem 1.2. Assume that Ω is as above, and that $W \in C^2$ satisfies **(a')**. Let $\epsilon \in (0,\mu)$ and $D > D'$, where D' is determined from the relation

$$\mathbf{U}(D') = \mu - \epsilon, \tag{1.11}$$

where in turn $\mathbf{U}$ is the only function in $C^2[0,\infty)$ that satisfies

$$\mathbf{U}'' = W'(\mathbf{U}),\ s>0;\quad \mathbf{U}(0)=0,\ \lim_{s\to\infty}\mathbf{U}(s)=\mu, \tag{1.12}$$

(see Remark 1.1 below). There exists an $R' > D$, depending only on ϵ, D, W, and n, such that if Ω contains some closed ball of radius R' then problem (1.2) has a solution $u \in C^2(\Omega)\cap C(\bar{\Omega})$ verifying (1.3), and

$$\mu - \epsilon \le u(x),\quad x \in \Omega_{R'} + B_{(R'-D)}, \tag{1.13}$$

where $\Omega_{R'}$ was previously defined in (1.6). Furthermore, it holds that

$$\min\{W(t)\ :\ t\in[0,u(x)]\} \le \frac{C}{\text{dist}(x,\partial\Omega)},\quad x\in\Omega_{R'}, \tag{1.14}$$

for some constant $C>0$ that depends only on W, n.
If $W''(\mu)>0$ then estimate (1.5) holds true.
If

$$W''(t)\ge 0\ \text{ for }\ \mu - t > 0\ \text{ small}, \tag{1.15}$$

then

$$-W'(u(x)) \le \frac{\tilde{C}}{(\text{dist}(x,\partial\Omega))^2},\quad x\in\Omega_{R'},\quad R\ge R', \tag{1.16}$$

for some constant $\tilde{C}>0$ that depends only on n, assuming that $W''\ge 0$ on $[\mu-\epsilon,\mu]$.
If there exist constants $c>0$ and $p>1$ such that

$$-W'(t)\ge c(\mu-t)^p\quad t\in[0,\mu], \tag{1.17}$$

and $\bar{\Omega}$ is disjoint from the closure of an infinite open connected cone, then

$$\mu - u \le \tilde{K}\text{dist}^{-\frac{2}{p-1}}(x,\partial\Omega),\quad x\in\Omega, \tag{1.18}$$

for some constant $\tilde{K}>0$ that depends only on c, p and n.

The method of our proof is quite flexible, and we came up with a variety of applications to related problems that can be found in the following sections and the included remarks (see the outline at the end of this section). As will be apparent from the proof, see in particular the comments leading to Proposition 10.1 below, a delicacy of our result is that the constant D' is independent of n.

Remark 1.1. The existence and uniqueness of such a solution $\mathbf{U}$ of the ordinary differential equation $u'' = W'(u)$ follows readily from **(a')** by phase plane analysis, using the fact that the latter equation has the conserved quantity $e(s) = \frac{1}{2}(u')^2 - W(u)$, see for instance Chapter 2 in [21] or page 135 in [164] (for a more analytic approach, we refer to [27]). We note that

$$\mathbf{U}'(s) > 0, \quad s \geq 0. \tag{1.19}$$

Remark 1.2. Similar assertions hold for the Robin boundary value problem:

$$\Delta u = W'(u), \ x \in \Omega; \ \ \frac{\partial u}{\partial \nu} + b(x)u = 0, \ x \in \partial\Omega,$$

where ν denotes the outward unit normal vector to the boundary of Ω, assuming here that the latter is at least C^1, with $b \in C^{1+\alpha}(\partial\Omega)$, $\alpha > 0$, being nonnegative (so that the constant μ is a positive upper solution, see [144]). Moreover, as in [89], we can possibly study some problems with mixed boundary conditions.

Remark 1.3. A sufficient, and easy to check, condition for the uniqueness of a positive solution of (1.2), in *any smooth bounded domain*, is

$$\frac{W'(t)}{t} \ \text{being strictly increasing in} \ \ (0, \infty), \tag{1.20}$$

see [41]. The above condition is clearly satisfied by the model double well potential in (1.23) below. Related conditions can be found in [160]. In certain cases, these type of conditions imply uniqueness of a positive solution in unbounded domains as well, see for example [69]. Another sufficient condition, which on the other hand depends partly on the smooth bounded domain Ω, is

$$W''(t) \geq -\lambda, \quad t \geq 0,$$

for some $\lambda < \lambda_1$, where $\lambda_1 > 0$ denotes the principal eigenvalue of $-\Delta$ in $W_0^{1,2}(\Omega)$ (see [15], [148]). This condition is clearly satisfied, with $\lambda = 0$, by the convex model potential in (4.1) below.

Let us mention that for a class of potentials, including (1.23), the dependence of the set of solutions of (1.2), in one space dimension, on the size of the interval was studied for the first time in [53] (see also the more up to date reference [55]).

In our opinion, Theorems 1.1 and 1.2 are important for the following reasons. If we additionally assume that W is even, namely

$$W(-t) = W(t), \quad t \in \mathbb{R}, \tag{1.21}$$

by means of these theorems, we can derive the existence of various sign-changing entire solutions for the problem

$$\Delta u = W'(u), \quad x \in \mathbb{R}^n. \tag{1.22}$$

This can be done by first establishing existence of a positive solution in a suitable large "fundamental" domain $\Omega_F \subset \mathbb{R}^n$, with Dirichlet boundary conditions on $\partial\Omega_F$, and then performing consecutive odd reflections to cover the entire space. In the case where

$$W(t) = \frac{1}{4}(t^2 - 1)^2, \quad t \in \mathbb{R}, \tag{1.23}$$

then (1.22) becomes the well known Allen-Cahn equation (see for instance [136]). Assuming that W is even, namely that (1.21) holds true, then (1.2) has always the trivial solution. In this regard, the purpose of estimate (1.13) is twofold: In the case where Ω_F is bounded, it ensures that the solution of (1.2) (on Ω_F), provided by Theorem 1.2, is nontrivial. The situation of unbounded domains Ω_F can be treated by exhausting them by an increasing (with respect to inclusions) sequence $\{\Omega_n\}$ of bounded ones, each containing the same ball $B_{R'}(x_0)$, and a standard compactness argument, making use of (1.3) together with elliptic estimates and a Cantor type diagonal argument. The fact that the region of validity of estimate (1.13) increases, as $n \to \infty$, rules out the possibility of subsequences of the (chosen) solutions u_n of $(1.2)_n$ on Ω_n converging, uniformly in compact subsets of Ω_F, to the trivial solution of (1.2) on Ω_F. Another approach for excluding this last scenario, which however does not seem to provide uniform estimates directly, can be found in the proof of Theorem 1.3 in [49], based on a similar relation to (2.64) below (see also [73] and [136]). In this fashion, and under more general assumptions on W than previous studies (conditions **(a')** and (1.21) suffice for most applications), one can construct a whole gallery of nontrivial sign-changing solutions of (1.22) that includes

- "saddle solutions" which vanish on the Simons cone $\{(x, y) \in \mathbb{R}^{2m} \ : \ |x| = |y|\} \subset \mathbb{R}^{2m} = \mathbb{R}^n$ if n is even (see [49], [69], [95], and [136]). (Some care is required near the vertex of the cone because there the boundary of the fundamental domain has merely the cone property, see [70], [94]). Estimate (1.5) implies that the corresponding saddle solution converges to $\pm\mu$ exponentially fast, as the signed distance from the Simons cone tends to plus/minus infinity respectively. Analogous solutions exist in odd dimensions, for example when $n = 3$ one can show that there exists a solution which vanishes on all coordinate planes (see the related discussion in [74]) In dimension $n = 2$, solutions whose zero level set has the symmetry of a regular $2k$–polygon and consists of k straight lines passing through the origin were found in [4]; such solutions can appropriately be named "pizza solutions", see also [150].
- "lattice solutions" which include solutions that are periodic in each variable x_i with period L_i, provided that L_i, $i = 1, \cdots, n$, are sufficiently large (see [13], [25], [85], and [108]). Another example, which is motivated from [125], are solutions in the plane whose nodal domains consist of sufficiently large (identical modulo translation and rotation) equilateral triangles tiling the plane (in relation to this, see also Remark 2.17 below). Under some additional hypotheses on W, planar lattice solutions can be constructed by local and global bifurcation techniques (see [85], [103], [108], and [125]).
- "tick saddle solutions" which have saddle (or pizza) structure in some coordinates while they are periodic in the remaining ones (see the introduction in [89]). For example, in $\mathbb{R}^2$, these solutions are odd with respect to both x and y, having as nodal curves the lines $x = 0$ and $y = kL$, $k \in \mathbb{Z}$, for L sufficiently large (so that the fundamental domain $\Omega_{F,L} \equiv \{x > 0, \ y \in (0, L)\}$ contains a sufficiently large closed ball). In fact, if $W''(0) < 0$ and (2.39) below hold, by modifying the approach of the current paper and using some ideas from Proposition 3.1 in [73] (which dealt with a problem of similar nature on an infinite half strip, see also Remark 2.10 below), it is plausible that there exists an explicit constant $L^* > 0$ such that (1.2) considered in $\Omega_{F,L}$ has a positive solution if and only if $L > L^*$ (see also Remark 2.6 below); a similar construction should also work in higher dimensions. We note that tick saddle solutions can be constructed as limits of appropriate lattice solutions by letting some of the periods tend to infinity (along a subsequence), see [13]. In the case where W is as in (1.23), and $n = 2$, the spectrum of the linearized operator about the

saddle solution of [69] has a unique negative eigenvalue (see [146]). Moreover, it has been shown recently that the saddle solution is non-degenerate, namely there are no decaying elements in the kernel of the linearized operator (see [113]). In view of these two properties it might also be possible to construct tick saddle solutions in $\mathbb{R}^3$, with W as in (1.23), by local bifurcation techniques (for example, by the ideas in [65]).

- "Screw–motion invariant solutions" whose nodal set is a helicoïd of $\mathbb{R}^3$, or analogous minimal surfaces in any odd dimension (see [73] and Remark 2.10 herein).

A completely different approach to the construction of sign-changing solutions of (1.22), mainly applied for potentials satisfying **(a)**, **(c)**, and (1.21) (the typical representative being (1.23)), is based on the implementation of an infinite dimensional Lyapunov–Schmidt reduction argument, see [72], [73], [74], [136], and the references therein. This approach produces solutions with less (or even without any) symmetry but is technically more involved.

Our Theorem 1.2 can also be used to construct multiple positive solutions of (1.2), using estimate (1.13) to make sure that they are distinct, see Section 9 below.

The outline of the paper is as follows: In Section 2, we will present the proof of our main result, with the exception of (1.18), by using two different approaches, both based on a special case of a radial lemma that we prove in Subsection 2.1. In Section 3, exploiting further this radial lemma, we prove uniform lower bounds for arbitrary positive solutions. In Section 4, we prove universal decay estimates for solutions, in the case where W is a model power nonlinearity potential, thereby generalizing the exponential decay estimate (1.5) by an algebraic one and relating the obtained result to a corresponding one in [28]. Moreover, this algebraic decay estimate allows us to show (1.18) and thus complete the proof of Theorem 1.2. In Section 5, under appropriate conditions on W, we will show that all entire solutions of (1.22) are uniformly bounded; combining this with the main result of Section 3, we can give a short self-contained proof of the main result in the paper of Du and Ma [76]. In Section 6, under a convexity assumption on W, we will obtain uniform estimates for possibly sign-changing solutions. In Section 7, we assume that W is monotone, as opposed to the previous convexity assumption, and show that the same estimates hold, provided that we restrict ourselves to one, two or three dimensions (we also assumed **(c)** for the latter case). In Section 8 we provide an alternative proof of a Liouville type result of Berestycki, Hamel and Matano, where an obstacle is present. In Section 9, we will show how our Theorem 1.2 can be used to produce multiple positive solutions of (1.2) and thus generalize an old result of P. Hess from 1981, where nonlinear eigenvalue problems were considered. In Section 10, we study the size of the boundary layer of global minimizers of the corresponding singular perturbation problem, in the context of nonlinear eigenvalue problems. In Appendix A, for completeness purposes, we will state some useful comparison lemmas that we will use in this article. In Appendix C, for the reader's convenience, we will state the useful doubling lemma of [138] that we mentioned earlier.

Remark 1.4. We recently found the paper [13], where it is stated that G. Fusco, in work in progress, has been able to remove the corresponding monotonicity assumption to **(b)** from the vector-valued Allen-Cahn type equation that was treated in [9]. After the first version of the current paper was completed, we were informed by G. Fusco that himself, F. Leonetti and C. Pignotti are working in a paper where, using the same technique developed for the vector case, they are in the process of extending the main result in [89] to more general potentials without assuming **(b)**. Their approach is certainly more elaborate but it is entirely self-contained, while we use in a simple and coordinate way various deep well known results.

2. Proof of the main result

2.1. **Minimizers of the energy functional on large balls.** In this subsection, we will mainly prove two lemmas concerning the asymptotic behavior of the minimizing (of the associated energy) solutions of (1.2) over large balls as their radius tends to infinity. The first one is essential for the proof of Theorem 1.2, and refines a result of P. Clément and G. Sweers [57]. The latter result is quite useful, and has been previously applied in singular perturbation problems (see [66], [115], and [118]). The second lemma, an extension of the first, is of independent interest and in particular allows for $W'(0)$ to be positive. Even though the first lemma is a special case of the second, we felt that it would be more instructive and more convenient for the reader to present them separately, since the more general second lemma is not needed for the proof of Theorem 1.2 and can be skipped at first reading.

The following is our first lemma, which is motivated from Lemma 2 in [115] and Lemma 2.2 in [118], whose origins can be traced back to [56, 57]. In these works, the weaker relation (2.12) below was established, which implies that assertion (2.3) holds *but* with constant D possibly *diverging* as $n \to \infty$ (see also Remark 2.2 below). Our improvement turns out to have interesting consequences in the study of the boundary layer of solutions of singular perturbation problems of the form (9.9) below, with $\lambda = \varepsilon^{-1} \to \infty$, see Remark 9.3 and Section 10 below. Moreover, estimate (2.3) will be used in a crucial way in Proposition 2.1 for studying the asymptotic stability of minimizing solutions that are provided by the following lemma or the more general Lemma 2.2 below.

Lemma 2.1. Assume that $W \in C^2$ satisfies condition **(a')**. Let $\epsilon \in (0, \mu)$ and $D > D'$, where D' is as in (1.11). There exists a positive constant $R' > D$, depending only on ϵ, D, W and n, such that there exists a global minimizer u_R of the energy functional

$$J(v; B_R) = \int_{B_R} \left\{ \frac{1}{2}|\nabla v|^2 + W(v) \right\} dx, \quad v \in W_0^{1,2}(B_R), \tag{2.1}$$

which satisfies

$$0 < u_R(x) < \mu, \ x \in B_R, \tag{2.2}$$

and

$$\mu - \epsilon \le u_R(x), \ x \in \bar{B}_{(R-D)}, \tag{2.3}$$

provided that $R \ge R'$. Moreover, there exists a constant C depending only on W, n such that

$$\min\{W(t) \ : \ t \in [0, u_R(r)]\} \le \frac{C}{R-r}, \quad r \in [0, R), \ \forall\ R \ge R'. \tag{2.4}$$

(If necessary, we assume that W is extended linearly outside of a large compact interval so that the above functional is well defined; clearly this modification does not affect the assertions of the lemma).

Proof. Under our assumptions on W, it is standard to show the existence of a global minimizer $u_R \in W_0^{1,2}(B_R)$ satisfying

$$0 \le u_R(x) \le \mu \ \text{ a.e. in } B_R, \tag{2.5}$$

see [89], [143], and Lemma A.1 herein (applied to the minimizing sequence converging, weakly in $W_0^{1,2}(B_R)$, to u_R). (The upper bound in (2.5) can also be derived from Lemma A.3 below, see also the second proof of Theorem 1.2). By standard elliptic regularity theory [93], this minimizer is a smooth solution, in $C^2(\bar{B}_R)$, of

$$\Delta u = W'(u) \ \text{ in } \ B_R; \ u = 0 \ \text{ on } \ \partial B_R. \tag{2.6}$$

By the strong maximum principle (see for example Lemma 3.4 in [93]), via (2.5) and (2.6), we deduce that $u_R(x) < \mu, \ x \in B_R$, and that either u_R is identically equal to zero or $u_R(x) > 0, \ x \in B_R$ (recall that assumption **(a')** implies that $W'(0) \leq 0$ and $W'(\mu) = 0$).

By adapting an argument from Section 4 in [136] (see also Theorem 1.13 in [143]), we will show that u_R is nontrivial, provided that R is sufficiently large (depending only on W and n). (This is certainly the case when $W'(0) < 0$). It is easy to cook up a test function, and use it as a competitor, to show that there exists a positive constant C_1, depending only on W and n, such that

$$J(u_R; B_R) \leq C_1 R^{n-1}, \ \text{ say for } \ R \geq 2. \tag{2.7}$$

(Plainly construct a function which interpolates smoothly from μ to 0 in a layer of size 1 around the boundary of B_R and which is identically equal to μ elsewhere, see also (2.63) below or Lemma 1 in [50]). In fact, as in Proposition 1 in [2] (see also [116]), it can be shown that

$$J(u_R; B_K) \leq \tilde{C}_1 K^{n-1} \ \ \forall\, K < R, \ \ R \geq 2, \tag{2.8}$$

where the constant $\tilde{C}_1 > 0$ depends only on W and n (see also Remark 2.11, and the arguments leading to relation (2.64) below). On the other hand, the energy of the trivial solution is

$$J(0; B_R) = \int_{B_R} W(0)dx = C_2 R^n,$$

where $C_2 > 0$ depends only on $W, \ n$. From (2.7), and the above relation, we infer that u_R is certainly not identically equal to zero for

$$R \geq C_1 C_2^{-1} + 2.$$

We thus conclude that (2.2) holds. (In the above calculation, we relied on the fact that **(a')** implies that $W(0) > 0$; in this regard, see Remark 2.8 below).

Since $u_R \in C^2(\bar{B}_R)$ is strictly positive in the ball B_R, by (2.6) and the method of moving planes [43, 91], we infer that u_R is radially symmetric and decreasing, namely

$$u_R'(r) < 0, \ \ r \in (0, R), \tag{2.9}$$

(with the obvious notation). In this regard, keep in mind that if $v \in W_0^{1,2}(B_R)$ is nonnegative, then its Schwarz symmetrization $v^* \in W_0^{1,2}(B_R)$, which is radially symmetric and decreasing, satisfies $J(v^*; B_R) \leq J(v; B_R)$ (see for example [45] and the references therein). We note that, since u_R is a global minimizer and thus stable (in the usual sense, as described in Remark 2.15 below), the radial symmetry of u_R, for $n \geq 2$, can also be deduced as in Lemma 1.1 in [6] (see also the related references in the proof of Lemma 2.2 below). In fact, the monotonicity property (2.9) can be alternatively derived by arguing as in Lemma 2 in [48], making use of the stability of the radial solution u_R (see also the proof of Lemma 2.2 below, and Lemma 1 in [5]). Now, relation (2.7) and the nonnegativity of W clearly imply that

$$\int_{B_R \setminus B_{\frac{R}{2}}} \left\{ \frac{1}{2}|\nabla u_R|^2 + W(u_R) \right\} dx \leq C_1 R^{n-1}, \ \ R \geq C_1 C_2^{-1} + 2. \tag{2.10}$$

Hence, by the mean value theorem and the radial symmetry of u_R, there exists a $\xi \in \left(\frac{R}{2}, R\right)$ such that

$$\left\{\frac{1}{2}[u_R'(\xi)]^2 + W\left(u_R(\xi)\right)\right\}\left|B_R \backslash B_{\frac{R}{2}}\right| \leq C_1 R^{n-1}, \quad R \geq C_1 C_2^{-1} + 2,$$

i.e.,

$$\frac{1}{2}[u_R'(\xi)]^2 + W\left(u_R(\xi)\right) \leq C_3 R^{-1}, \quad R \geq C_1 C_2^{-1} + 2, \tag{2.11}$$

where the positive constat C_3 depends only on W and n (for simplicity in notation, we have suppressed the obvious dependence of ξ on R). Hence, from assumption **(a')**, and relations (2.9), (2.11), we obtain that

$$u_R \to \mu, \text{ uniformly in } \bar{B}_{\frac{R}{2}}, \text{ as } R \to \infty. \tag{2.12}$$

In the sequel, we will prove that the stronger property (2.3) holds true.

For future reference, we note here that

$$[u_R'(R)]^2 \to 2W(0) \text{ as } R \to \infty. \tag{2.13}$$

Indeed, let

$$E_R(r) = \frac{1}{2}[u_R'(r)]^2 - W\left(u_R(r)\right), \quad r \in (0, R). \tag{2.14}$$

Thanks to (2.6), we find that

$$E_R'(r) = u_R'' u_R' - W'(u_R) u_R' = -\frac{n-1}{r}(u_R')^2, \quad r \in (0, R). \tag{2.15}$$

So,

$$E_R(R) = E_R(\xi) - \int_\xi^R \frac{n-1}{r}(u_R')^2 dr, \tag{2.16}$$

where $\xi \in \left(\frac{R}{2}, R\right)$ is as in (2.11). Now, observe that (2.10) and the nonnegativity of W imply that

$$\int_\xi^R r^{n-1}(u_R')^2 dr \leq C_4 R^{n-1}, \quad R \geq C_1 C_2^{-1} + 2,$$

with C_4 depending only on W and n. In turn, the above estimate clearly implies that

$$\int_\xi^R (u_R')^2 dr \leq 2^{n-1} C_4, \quad R \geq C_1 C_2^{-1} + 2,$$

and it follows that

$$\int_\xi^R \frac{n-1}{r}(u_R')^2 dr \leq 2^n C_4 (n-1) R^{-1}, \quad R \geq C_1 C_2^{-1} + 2. \tag{2.17}$$

The claimed relation (2.13) follows readily from (2.11), (2.14), (2.16), and (2.17). In fact, we have shown that $R|E_R(R)|$ remains uniformly bounded as $R \to \infty$. In relation to (2.13), see also Remark 10.5 below.

We also consider the following family of functions

$$U_R(s) = u_R(R - s), \quad s \in [0, R]. \tag{2.18}$$

We claim that

$$U_R \to \mathbf{U}, \text{ uniformly on compact intervals of } [0, \infty), \text{ as } R \to \infty, \tag{2.19}$$

where $\mathbf{U}$ is as in (1.12).

In view of (2.6), we get

$$U_R'' - \frac{n-1}{R-s}U_R' - W'(U_R) = 0, \quad s \in (0,R).$$

Making use of (2.2), the above equation, elliptic estimates [93], Arczela-Ascoli's theorem, and a standard diagonal argument, passing to a subsequence $R_i \to \infty$, we find that

$$U_{R_i} \to V \text{ and } U_{R_i}' \to V', \text{ uniformly on compact intervals of } [0,\infty), \text{ as } i \to \infty, \tag{2.20}$$

where $V \in C^2[0,\infty)$ is nonnegative and satisfies

$$V'' = W'(V), \ s > 0, \text{ and } V(0) = 0.$$

Moreover, by (2.13), (2.18), and (2.20), we see that

$$[V'(0)]^2 = 2W(0) > 0.$$

By the uniqueness of solutions of initial value problems of ordinary differential equations, see for example page 108 in [164], we deduce that

$$V \equiv \mathbf{U},$$

where $\mathbf{U}$ is as in (1.12). We also used that $\mathbf{U}$, V are nonnegative (which implies that $\mathbf{U}'(0)$, $V'(0)$ are also nonnegative), and the relation

$$[\mathbf{U}'(0)]^2 = 2W(0), \tag{2.21}$$

which follows from the identity

$$[\mathbf{U}'(s)]^2 - [\mathbf{U}'(0)]^2 = 2\int_0^s W'(\mathbf{U})\mathbf{U}'ds = 2W(\mathbf{U}(s)) - 2W(0), \quad s \geq 0,$$

and the fact that $\mathbf{U}(s) \to \mu$ as $s \to \infty$, recalling that $W(\mu) = 0$ (otherwise, $\mathbf{U}'(s)$ would tend to a nonzero number and in turn $|\mathbf{U}(s)|$ to infinity, as $s \to \infty$). Moreover, by the uniqueness of the limiting function, we infer that the limits in (2.20) hold for *all* $R \to \infty$. Consequently, the claimed relation (2.19) holds.

Having (2.13), (2.19) at our disposal, we can now proceed to the proof of (2.3). Let $\epsilon \in (0,\mu)$ and $D > D'$, where D' is as in (1.11). By virtue of (1.12), (1.19), and (2.19), there exists a sufficiently large R', depending only on ϵ, D, W, n, such that $U_R(D) \geq \mu - \epsilon$, and all the previous relations continue to hold, for $R > R'$. In other words, via (2.18), we have that

$$u_R(R-D) = U_R(D) \geq \mu - \epsilon, \quad R > R'. \tag{2.22}$$

The fact that u_R is radially decreasing, recall (2.9), and the above relation imply the validity of (2.3). As will be apparent from the Remarks 2.3 and 2.4 that follow, condition (2.9) is essential only when dealing with degenerate situations when there exists a sequence $t_j \to \mu^-$ such that $W'(t_{2j})W'(t_{2j+1}) < 0$ for large j; an example is a potential W that coincides with $(\mu - t)^2\left[\sin\left(\frac{1}{\mu-t}\right) + 2\right]$ near μ, in which case we can choose $t_j = \mu - \frac{1}{j\pi}$ (note that $W'(t) \sim \cos\left(\frac{1}{\mu-t}\right)$ as $t \to \mu^-$). It remains to prove (2.4). To this end, note that the nonnegativity of W and (2.7) imply that

$$\int_r^R s^{n-1}W(u_R(s))\,ds \leq \tilde{C}_1 R^{n-1}, \quad r \in (0,R),$$

where $\tilde{C}_1$ is independent of $R \geq R'$. It follows, via (2.9), that

$$\min \{W(t) \ : \ t \in [0, u_R(r)]\} (R^n - r^n) \leq n\tilde{C}_1 R^{n-1},$$

which clearly implies the validity of (2.4).

The proof of the lemma is complete. □

Remark 2.1. Our assumptions on the behavior of W near its global minimum at μ are quite weak, and in fact even allow for the potential W to have C^∞ contact with zero at the point μ, that is $W^{(i)}(\mu) = 0, \ i \geq 1$. This degeneracy translates into the absence of decay rates for the convergence of the "inner" approximate solution $\mathbf{U}(R - |x|)$ (in the sense of singular perturbation theory, see [84] and the related references that can be found in Remark 10.6 below), where $\mathbf{U}$ is as described in (1.12), to the "outer" one μ, away from the boundary of B_R, as $R \to \infty$ (see also the discussion leading to (9.10) below). This is the main reason why we have not attempted to apply a perturbation argument, see for instance [84] and the related references in Remark 10.6 below, in order to study the asymptotic behavior of u_R as $R \to \infty$. From the viewpoint of geometric singular perturbation theory, the case $W''(\mu) = 0$ corresponds to lack of normal hyperbolicity of the slow manifold corresponding to the equilibria with $(u, u') = (\mu, 0)$ (see [161]).

If $W''(\mu) > 0$, then the convergence of $\mathbf{U}$ to μ is of order $e^{-\sqrt{W''(\mu)}s}$ as $s \to \infty$ (by the stable manifold theorem, see [59]), and one can effectively interpolate between the outer and inner approximations in order to construct a smooth global approximation that is valid throughout B_R.

Remark 2.2. By the well known relations $|B_R| = c_n R^n$, $|\partial B_R| = nc_n R^{n-1}$, $R > 0, n \geq 2$, for some explicit constants c_n (independent of R), where $|\partial B_R|$ denotes the $(n-1)$–dimensional measure of ∂B_R, we find that

$$\frac{|\partial B_R|}{\left|B_R \backslash B_{\frac{R}{2}}\right|} = \frac{n2^n}{2^n - 1} R^{-1}, \quad R > 0.$$

We deduce that the constant R' in Lemma 2.1 diverges (at least linearly) as $n \to \infty$ (see in particular the relations leading to (2.11)).

Remark 2.3. If in addition to **(a')** we assume that there exists some $d \in (0, \mu)$ such that

$$W'(t) \leq 0, \quad t \in (\mu - d, \mu), \tag{2.23}$$

(note that this is very natural), then relation (2.3) can alternatively be shown, starting from (2.22), *without* assuming knowledge of (2.9), as follows: Assuming, without loss of generality, that $2\epsilon < d$, thanks to Lemma A.2 below, we can find a radial $\tilde{u} \in W^{1,2}(B_{R-D})$ such that

$$J(\tilde{u}; B_{R-D}) \leq J(u_R; B_{R-D}), \quad \tilde{u}(R-D) = u_R(R-D), \quad \text{and} \quad \tilde{u}(x) \in [\mu - \epsilon, \mu], \ x \in \bar{B}_{R-D}.$$

Thus, the function

$$\hat{u}(x) = \begin{cases} \tilde{u}(x), & x \in B_{R-D}, \\ \\ u_R(x), & x \in B_R \backslash B_{R-D}, \end{cases}$$

belongs in $W_0^{1,2}(B_R)$ and is a global minimizer of $J(\cdot; B_R)$ in $W_0^{1,2}(B_R)$ (since $J(\hat{u}; B_R) \leq J(u_R; B_R)$). In particular, it is smooth, radial (and by virtue of its construction), and solves (2.6). It follows from Lemma 3.1 in [106], which is in the spirit of Lemma A.3 below, that the function $u_R - \hat{u}$ is either strictly positive, strictly negative, or identically equal to zero in B_R, and obviously the latter case occurs. For completeness purposes, as well as for future reference, we will draw the same conclusion by an alternative and, to our opinion, more elementary approach: The function

$$v \equiv u_R - \hat{u}$$

solves the linear equation

$$\Delta v + Q(x)v = 0, \quad x \in B_R,$$

where

$$Q(x) = \begin{cases} \frac{W'(\hat{u}(x)) - W'(u_R(x))}{u_R(x) - \hat{u}(x)}, & \text{if } \hat{u}(x) \neq u_R(x), \\ \\ -W''(u_R(x)), & \text{if } \hat{u}(x) = u_R(x). \end{cases} \tag{2.24}$$

On the other hand, since

$$v(x) = 0, \quad x \in B_R \backslash B_{(R-D)},$$

and $Q \in L^\infty(B_R)$, the unique continuation principle (see for instance [104]) yields that

$$v(x) = 0, \; x \in B_R.$$

(In this simple case of radial symmetry, we can also make use of the uniqueness theorem of ordinary differential equations to show that $v \equiv 0$). Therefore, estimate (2.3) holds. We remark that, if W was *strictly* decreasing in $(\mu - d, \mu)$, then (2.3) follows at once from the general lemma in [11] (see also the second assertion of Lemma A.2 herein) and (2.22).

The approach that we just presented makes only partial use of the radial symmetry of the problem (in order to establish (2.22)), and may be applied to extend some results in [66] to the general case (without radial symmetry), see [156]. Moreover, it can be applied for the study of global minimizers of the analogous vector-valued energy functionals, as those appearing in [9], over B_R. In this case, it is known that global minimizers are radial, see [123], but monotonicity properties do not hold in general.

Remark 2.4. In the one dimensional case, i.e., when $n = 1$, the assertion of Remark 2.3 can be shown *without* assuming (2.23). As in the latter remark, we do not assume the monotonicity property (2.9) of u_R, just that it is even, and we will start from (2.22) which clearly implies that

$$u_R(R - D) \to \mu \;\; \text{as} \;\; R \to \infty. \tag{2.25}$$

Since the energy of u_R is not larger than that of the even function given by

$$\check{u}_R(x) = \begin{cases} u_R(x), & x \in [R - D, R], \\ \\ \frac{u_R(R-D)-\mu}{D}(x - R + D) + u_R(R - D), & x \in [R - 2D, R - D], \\ \\ \mu, & x \in [0, R - 2D], \end{cases} \tag{2.26}$$

it follows readily from **(a')** and (2.25) that

$$\int_{-R+D}^{R-D} \left\{ (u_R')^2 + W(u) \right\} dx \to 0 \;\; \text{as } R \to \infty. \tag{2.27}$$

Hence, by **(a')** and the *clearing–out* Lemma 1 in [36] (noting that it continues to apply in our possibly degenerate setting), we have that

$$u_R \to \mu, \; \text{uniformly in } [-R + D, R - D], \; \text{as } R \to \infty. \tag{2.28}$$

The intuition behind the latter lemma, as applied in the case at hand, is that if the energy is sufficiently small in some place, then there are no spikes located there. Note that from (2.2), (2.6), in arbitrary dimensions, via standard interior elliptic regularity estimates [93] (see also Lemma A.1 in [35]), applied on balls of radius $\frac{D}{4}$ covering $B_{(R-D)}$, we have that $|\nabla u_R|$ remains uniformly bounded in $B_{(R-D)}$ as $R \to \infty$ (or see the gradient bound in (2.51) below). Thus, relation (2.28) can also be derived from **(a')** and (2.27) similarly to Theorem III.3 in [37] (the point is that the "bad" intervals, where u_R is away from μ must have size of order one (by the uniform gradient estimate), as $R \to \infty$, which is not possible by **(a')** and (2.27)). In contrast to the one dimensional case, in $n \geq 2$ dimensions, by the analog of (2.27), i.e.,

$$R^{1-n} J(u_R; B_{(R-D)}) \to 0 \quad \text{as} \quad R \to \infty, \tag{2.29}$$

arguing again as in Theorem III.3 in [37], we can show the weaker property:

$$\text{Given } \alpha \in (0,1) \ \Rightarrow \ u_R \to \mu, \ \text{uniformly in } \bar{B}_{(R-D)} \backslash B_{\alpha R}, \ \text{as } R \to \infty. \tag{2.30}$$

We note that if $W''(\mu) > 0$, then (2.28) follows directly from (2.27), via (2.40) below and the Sobolev embedding

$$\|\mu - u_R\|_{L^\infty(-R+D,R-D)} \leq C \|\mu - u_R\|_{W^{1,2}(-R+D,R-D)},$$

with constant C independent of $R \geq 2D$ (see Corollary 5.16 in [1]). One might be curious whether this simple argument can be extended to $n \geq 2$ dimensions. In this direction, we would like to mention that by using the pointwise estimate

$$(\mu - u_R(r))^2 \leq C_n r^{1-n} \|\mu - u_R\|^2_{W^{1,2}\left(B_{(R-D)}\right)} + \left(\frac{R-D}{r}\right)^{n-1} (\mu - u_R(R-D))^2, \ \ r \in (0, R-D),$$

which can be proven similarly as the classical Strauss radial lemma (see [157]), relation (2.25), and (2.29), we arrive again at (2.30). On the other side, as in [8], fixing K and letting $R \to \infty$, we see from the monotonicity formula (2.52) below that $u_R \to \mu$, uniformly on $\bar{B}_K$, as $R \to \infty$ (see also Remark 2.12 below, and the compactness argument that follows). Suppose that for a sequence $R \to \infty$, there exist $r_R \in [0, R-D]$ such that $u_R(r_R) = \mu - 2\epsilon$. From (2.30) and our previous comment, we get that $R - r_R \to \infty$ and $r_R \to \infty$, as $R \to \infty$, respectively. As in the proof of Theorems 1.3 and 1.4 in [66], we let $v_R(s) = u_R(r_R + s)$, $s \in (-r_R, R - r_R)$, note that $v_R(0) = \mu - 2\epsilon$. Using (2.2), (2.6), together with standard elliptic regularity estimates and Sobolev embeddings (see [93]), passing to a subsequence, we find that $v_R \to V$ in $C^1_{loc}(\mathbb{R})$, where

$$V'' = W'(V), \quad 0 \leq V \leq \mu, \quad s \in \mathbb{R}, \quad V(0) = \mu - 2\epsilon. \tag{2.31}$$

Moreover, the solution V is a minimizer of the energy

$$I(v) = \int_{-\infty}^{\infty} \left[\frac{1}{2}(v')^2 + W(v)\right] ds,$$

in the sense that $I(V + \phi) \leq I(V)$ for every $\phi \in C_0^\infty(\mathbb{R})$, see page 104 in [66]. Arguing as in the proof of De Giorgi's conjecture in low dimensions (see [16], [29], [67], [83], [90], [136]), we can prove that either V is a constant with $W'(V) = 0$, $W''(V) \geq 0$ or V' is nontrivial and has fixed sign. Since we are assuming that $W''(\mu) > 0$, the first scenario is ruled out at once from the last condition in (2.31); in the second scenario, it follows from phase analysis (see [21], [164]) that V has to connect two equal wells of the potential W at respective infinities, one of them being μ, but this is impossible since $W(t) > 0, \ t \in [0, \mu)$. Consequently, if we assume that $W''(\mu) > 0$, assertion (2.3) can be deduced in this manner from (2.22) *without* making use of (2.9) for *all* $n \geq 1$.

A more direct approach, with the advantage of not making use of the radial symmetry of u_R, is the following: Observe that the function $v = (\mu - u_R)^2$ satisfies

$$|\nabla v| = 2(\mu - u_R)|\nabla u_R| \leq |\nabla u_R|^2 + (\mu - u_R)^2, \quad x \in B_{(R-D)}.$$

Then, via (2.29), and (2.40) below, we obtain that

$$R^{1-n} \int_{B_{(R-D)}} |\nabla v| dx \to 0 \quad \text{as} \quad R \to \infty.$$

By a useful imbedding theorem of Morrey (see Theorem 7.19 in [93]), and the above relation, we infer that

$$\text{osc}_{B_{(R-D)}} v = \max_{B_{(R-D)}} v - \min_{B_{(R-D)}} v \to 0 \quad \text{as} \quad R \to \infty,$$

which clearly implies the relation sought for.

Remark 2.5. If $W \in C^{2,\alpha}(\mathbb{R})$, $0 < \alpha < 1$, satisfies **(a')**,

$$W'(\rho_1) = 0, \quad W'(t) < 0, \ t \in (\rho_1, \mu), \ \text{for some} \ \rho_1 \in (0, \mu),$$

and (1.15), then Theorem 2 in [158] tells us that there exists a $\delta_1 \in (0, \mu)$ such that (2.6) has at most one solution such that

$$\max_{x \in \bar{B}_R} u(x) \in (\mu - \delta_1, \mu) \ \text{and} \ -\mu < u(x) < \mu, \ x \in B_R,$$

for all $R > 0$. Therefore, under these assumptions on W, in view of (2.2) and (2.3) which hold for all global minimizers (with the same R'), we conclude that there exists a unique global minimizer of (2.1), if R is sufficiently large.

On the other side, if in addition to **(a')**, the stronger assumption $W''(\mu) > 0$ holds (in other words **(c)**), then a simple proof of the uniqueness of the global minimizer, satisfying (2.2), for large R, can be given as follows: One first shows that if a solution of (2.6) satisfies (2.2), (2.3), and (2.19) (recall (2.18)), then it is *asymptotically* stable for large $R > 0$ (we will give a short self–contained proof of this in the sequel). Then, suppose that u_1 and u_2 are two distinct global minimizers of (2.1), satisfying (2.2). By the proof of Lemma 2.1, they satisfy (2.2), (2.3), and (2.19), uniformly (independent of the choice of minimizers) as $R \to \infty$. Thanks to Lemma 3.1 in [106] (see also Lemma A.3 herein), without loss of generality, we may assume that $u_1(x) < u_2(x)$, $x \in B_R$ (in the problem at hand, we can also assume this when dealing with stable solutions). On the other hand, by the mountain pass theorem or the theory of monotone dynamical systems (see [71], [126] respectively, and Section 9 herein), we infer that there exists an *unstable* solution $\hat{u}_1$ of (2.6) such that $u_1(x) < \hat{u}_1(x) < u_2(x)$, $x \in B_R$. In particular, the unstable solution enjoys the asymptotic behavior of global minimizers, as $R \to \infty$, and thus is asymptotically stable (by our previous discussion); a contradiction. A related uniqueness proof, based on a dynamical systems argument (but not of monotone nature), can be found in [5].

Here, for completeness, assuming that $W''(\mu) > 0$, we will show that solutions u_R of (2.6) which satisfy (2.2), (2.3), and (2.19) are *asymptotically* stable if R is sufficiently large. Our argument is inspired from [20] where, in particular, under the additional assumption **(b)** with strict inequality, it was applied to (9.9) below on a smooth bounded domain with large

λ. We will ague by contradiction. Suppose that, for a sequence $R \to \infty$, the principal eigenvalue μ_R of the linearized operator about u_R is non-positive, i.e,

$$\mu_R \leq 0. \tag{2.32}$$

It is well known that μ_R is simple and that the corresponding eigenfunction φ_R (modulo normalization) may be chosen to be positive in B_R, see for instance Theorem 8.38 in [93]. We have

$$-\Delta\varphi_R + W''(u_R)\varphi_R = \mu_R\varphi_R \text{ in } B_R;\ \varphi_R = 0 \text{ on } \partial B_R, \tag{2.33}$$

and we normalize φ_R by imposing that

$$\|\varphi_R\|_{L^\infty(B_R)} = 1. \tag{2.34}$$

We note that φ_R is radially symmetric (and so is every eigenfunction that is associated to a non-positive eigenvalue, see [101], [119]). For future reference, observe that testing (2.33) by φ_R yields the uniform (in R) lower bound:

$$\mu_R \geq -\max_{t\in[0,\mu]} |W''(t)|\,. \tag{2.35}$$

Now, by virtue of (2.3) and the positivity of $W''(\mu)$, there exists a constant $D > 0$ such that

$$W''(u_R) \geq \frac{W''(\mu)}{2} > 0 \ \text{ on } \ \bar{B}_{(R-D)},$$

for large $R > 0$. So, from (2.32), (2.33), and (2.34), we obtain that there exist $z_R \in [R-D, R]$ such that $\varphi_R(z_R) = 1$, $\varphi_R'(z_R) = 0$, and $\varphi_R''(z_R) \leq 0$, for large R (along the sequence). As in the proof of Lemma 2.1, making use of (2.19), (2.32), (2.33), (2.34), and (2.35), passing to a subsequence, we get that $\varphi_{R_i}(R_i - \cdot) \to \Phi(\cdot)$ in $C^1_{loc}[0,\infty)$, $\mu_{R_i} \to \mu_* \leq 0$, and $R_i - z_{R_i} \to \mathbf{z} \in [0, D]$, as $i \to \infty$, such that

$$-\Phi'' + W''(\mathbf{U}(r))\Phi = \mu_*\Phi,\ r \in (0,\infty);\ \Phi(0) = 0,\ \Phi(\mathbf{z}) = \|\Phi\|_{L^\infty(0,\infty)} = 1, \tag{2.36}$$

where $\mathbf{U}$ is as in (1.12). On the other hand, differentiating (1.12), multiplying the resulting identity by $\frac{\Phi^2}{\mathbf{U}'}$ (recall (1.19)) and integrating by parts over $(0,\infty)$, we arrive at $\mu_* \geq 0$ (see also Proposition 3.1 in [136]); to be more precise, one first multiplies by $\frac{\zeta_m^2}{\mathbf{U}'}$, with $\zeta_m \in C_0^\infty(0,\infty)$ such that $\zeta_m \to \Phi$ in $W_0^{1,2}(0,\infty)$, and then lets $m \to \infty$. A different way to see that $\mu_* \geq 0$ is to note that the linear operator defined by the lefthand side of (2.36) is an unbounded, self-adjoint operator in $L^2(0,\infty)$ with domain $W_0^{1,2}(0,\infty) \cap W^{2,2}(0,\infty)$, having as continuous spectrum the interval $[W''(\mu),\infty)$ and principal eigenvalue zero (by the positivity of $\mathbf{U}'$), see also Remark 2.8 in [7] or Proposition 1 in [100] or [146]. In other words, recalling (2.32), we have

$$-\Phi'' + W''(\mathbf{U}(r))\Phi = 0,\ \Phi > 0,\ r \in (0,\infty);\ \Phi(0) = 0,\ \Phi(\mathbf{z}) = \|\Phi\|_{L^\infty(0,\infty)} = 1.$$

The above linear second order equation has the following two independent solutions:

$$\mathbf{U}'(r) \ \text{ and } \ \mathbf{U}'(r)\int_0^r \frac{1}{[\mathbf{U}'(s)]^2}ds,$$

see for example Lemma 3.2 in [24]. It is easy to see that the second solution grows unbounded as $r \to \infty$ (plainly apply l'hospital's rule), and thus Φ has to be $\|\mathbf{U}'\|^{-1}_{L^\infty(0,\infty)}\mathbf{U}'$. Since $\Phi(0) = 0$, whereas $\mathbf{U}'(0) = \sqrt{2W(0)} > 0$, we have reached a contradiction.

For further information on "asymptotic" uniqueness of positive solutions, in arbitrary domains, we refer to Remark 9.2 below.

In the case where uniqueness of a stable solution, satisfying (2.2), holds for $R > R_0 \geq 0$ (recall Remark 1.3 and see Remark 9.2 below), it is easy to see that the family $\{u_R\}_{R>R_0}$ is nondecreasing with respect to R, namely

$$u_{R_2}(x) > u_{R_1}(x), \quad x \in \bar{B}_{R_1}, \quad \forall\ R_2 > R_1 > R_0, \tag{2.37}$$

see Lemma 1 in [69]. Moreover, as in Lemma 2 in [69], we have that

$$u_R(R-r) \leq \mathbf{U}(r), \quad r \in [0,R],\ \forall R > R_0.$$

Remark 2.6. If in addition to **(a')**, we assume that $W'(0) = 0$, $W''(0) < 0$, and $W \in C^3$, then (2.6) admits a nontrivial positive solution, which is a global minimizer of $J(\cdot; B_R)$ in $W_0^{1,2}(B_R)$, as long as $R > R_c$, where

$$R_c = \sqrt{-\frac{\lambda_1}{W''(0)}}, \tag{2.38}$$

and λ_1 denotes the principal eigenvalue of $-\Delta$ in $W_0^{1,2}(B_1)$ (an analogous result holds for (9.9) below). If we further assume that

$$W'(t) \geq W''(0)t, \quad t \geq 0, \tag{2.39}$$

then (2.6), for $R \in (0, R_c)$, has no positive solution. These assertions can be proven by adapting the proof of Lemma 2.1 in [73].

Under some different conditions, which are compatible with **(a')**, and are satisfied for example by the nonlinearity in (1.10), there exists an $R_c' > 0$ such that (2.6) has exactly one positive solution for $R = R_c'$ and exactly two for $R > R_c'$, the one is a global minimizer while the other is a mountain pass of the associated energy (see [134], [152], [165]).

Remark 2.7. By (2.10), via the coarea formula (see [80]), it follows that there exists a $\xi_R \in \left(\frac{R}{2}, R\right)$ such that

$$\int_{\partial B_{\xi_R}} \left\{\frac{1}{2}|\nabla u_R|^2 + W(u_R)\right\} dS \leq 2C_1R^{n-2}, \quad R \geq C_1C_2^{-1} + 2.$$

This observation makes no use of the radial symmetry of u_R, and is motivated from the proof of the corollary in [11]. In regard to the latter comment, it might be useful to recall our Remark 2.4 and compare with the arguments of [11].

Remark 2.8. In case a C^2 potential W satisfies $W(0) = 0$ and the domain Ω has C^1 boundary, is bounded, and star-shaped with respect to some point in its interior, the well known Pohozaev identity easily implies that there does not exist a nontrivial solution of (1.2) such that $W(u(x)) \geq 0,\ x \in \Omega$ (see for instance relation (11) in [14], a reference which is in accordance with our notation). Actually, relation (11) in the latter reference holds true for the elliptic system that corresponds to (1.2) (with the obvious notation), and an analogous nonexistence result holds in that situation as well.

Remark 2.9. Under the stronger assumptions **(a)** (or more generally **(a')**), **(b)**, and **(c)**, considered in [89] (recall the introduction herein), motivated from the proof of Lemma 3 in [120] (see also [147] and the remarks following Lemma 2.1 in [52]), we can give a streamlined proof of relation (2.12) as follows: Note first that, thanks to **(a')** and **(c)**, there exists a positive constant c_0 such that

$$W(t) \geq c_0(\mu - t)^2, \quad 0 \leq t \leq \mu. \tag{2.40}$$

Then, bounds (2.2), (2.7), and the above relation yield that

$$\int_{B_R} (\mu - u_R)^2 dx \leq c_1 R^{n-1}, \quad R \geq 2, \tag{2.41}$$

where the positive constant c_1 depends only on W and n. Next, note that assumption **(b)**, bound (2.2), and the equation in (2.6), imply that the function $\mu - u_R$ is subharmonic in B_R, and thus we have

$$\Delta(\mu - u_R)^2 \geq 0 \text{ in } B_R, \quad R \geq 2.$$

In other words, the function $(\mu - u_R)^2$ is also subharmonic in B_R. Consequently, by (2.41) and the mean value inequality of subharmonic functions (see Theorem 2.1 in [93]) together with a simple covering argument (see also the general Theorem 9.20 in [93] and Chapter 5 in [127]), we deduce that

$$\max_{\bar{B}_{\frac{R}{2}}} (\mu - u_R)^2 \leq c_2 R^{-n} \int_{B_R} (\mu - u_R)^2 dx \leq c_3 R^{-1}, \quad R \geq 2, \tag{2.42}$$

where the positive constants c_2, c_3 depend only on W and n. The latter inequality clearly implies the validity of (2.12). In passing, we note that the spherical mean of $(\mu - u_R)^2$ appearing in the above inequality is nondecreasing with respect to R, because of the subharmonic property, see [141].

The above argument makes no use of the fact that u_R is radially symmetric. Moreover, it works equally well if instead of (2.40) we had $W(t) \geq c(\mu - t)^p$, $t \in [0, \mu]$, for some constants $c > 0$ and $p > 2$. Hopefully, this approach may be adapted to simplify the arguments of Section 6 in [10], where the De Giorgi oscillation lemma for subharmonic functions was employed instead of the mean value inequality. The former lemma roughly says that if a positive subharmonic function is smaller than one in B_1 and is "far from one" in a set of non trivial measure, it cannot get too close to one in $B_{\frac{1}{2}}$ (see for example [51]). An intriguing application of the techniques in the current remark is given in the following Remark 2.10.

Remark 2.10. When seeking solutions of (1.22) in $\mathbb{R}^3$ which are invariant under screw motion and whose nodal set is a helicoïd, assuming that W is even, by introducing cylindrical coordinates, one is led to study positive solutions of

$$\partial_r^2 U + \frac{1}{r}\partial_r U + \left(1 + \frac{\lambda^2}{\pi^2 r^2}\right)\partial_s^2 U - W'(U) = 0, \tag{2.43}$$

in the infinite half strip $\{(r, s) \in (0, \infty) \times (0, \lambda)\}$, vanishing on the boundary of $[0, \infty) \times [0, \lambda]$, where λ corresponds to a dilation parameter of a fixed helicoïd. More specifically, such solutions U give rise to solutions u of (1.22) which vanish on the helicoïd that is parameterized by

$$\left\{ (r\cos\theta, r\sin\theta, z) \in \mathbb{R}^3 \ : \ z = \frac{\lambda}{\pi}\theta \right\},$$

see [73] for the details. In the latter reference, assuming that $W''(0) < 0$ and (2.39), it was shown that there exists an explicit constant $\lambda^* > 0$ such that the above problem has a positive solution U_λ if and only if $\lambda > \lambda^*$ (λ^* is actually equal to $2R_c$, where R_c is given from (2.38) with $n = 1$).

Here, motivated from our previous Remark 2.9, we will study this problem for large values of λ. Due to the presence of singularities in the equation (2.43) at $r = 0$, as in Lemma 3.4

in [79], we will first consider the approximate (regularized) problem

$$\Delta_{\mathbb{R}^2}U+\left(1+\frac{\lambda^2}{\pi^2|x|^2}\right)\partial_s^2U-W'(U)=0, \tag{2.44}$$

in $\{\xi<|x|,\ s\in(0,\lambda)\}$ with zero conditions on $\{|x|=\xi,\ s\in[0,\lambda]\}$ and $\{\lambda=0,\ |x|\geq\xi\}$, $\{\lambda=1,\ |x|\geq\xi\}$, with ξ small (this was skipped in [73]). Then, we consider equation (2.44) in the annular cylinder $\{\xi<|x|<R,\ s\in(0,\lambda)\}$, imposing that U also vanishes on $|x|=R$. Assuming **(a')**, as in Lemma 2.1, by minimizing the energy

$$E(V)=\frac{1}{2}\int\left\{|\nabla_xV|^2+\left(1+\frac{\lambda^2}{\pi^2|x|^2}\right)|\partial_sV|^2+2W(V)\right\}dxds \tag{2.45}$$

in $W_0^{1,2}\left((B_R\backslash B_\xi)\times(0,\lambda)\right)$ (with the obvious notation), but this time *in the radially symmetric class* with respect to $|x|$, we find a solution $U_{\xi,R,\lambda}$ of (2.44), satisfying the prescribed Dirichlet boundary conditions, such that $0\leq U_{\xi,R,\lambda}(|x|,s)\leq\mu$ on $\left(\bar{B}_R\backslash B_\xi\right)\times[0,\lambda]$ (see Lemma A.2 below). Moreover, as in the proof of Lemma 2.1, we have

$$E\left(U_{\xi,R,\lambda}\right)\leq CR\lambda,\quad R\geq2,\ \lambda\geq2,\ \xi\leq1, \tag{2.46}$$

with C independent of $\xi,\ R,\ \lambda$ (for this, it is convenient to use a separable test function of the form $\eta(r)\vartheta(s)$, see also [73]). Hence, again as in Lemma 2.1, we have that $0<U_{\xi,R,\lambda}(|x|,s)<\mu$, if $\xi\leq|x|\leq R,\ s\in[0,\lambda]$, for all $\xi\leq1,\ \lambda\geq2$, provided that R is sufficiently large (note that $E(0)=\lambda\pi(R^2-\xi^2)W(0)$). Using the standard compactness argument, letting $\xi\to0$ and $R\to\infty$ (along a sequence), we are left with a solution U_λ of (2.43) in the infinite half strip $(0,\infty)\times(0,\lambda)$, with zero conditions on its boundary, such that $0\leq U_\lambda\leq\mu$ on the half strip. The latter relation leaves open the possibility of U_λ being identically zero. However, U_λ is a minimizer of the energy in (2.45), in the sense of (2.65) below (since it is the limit of a family of minimizers, see also page 104 in [66]). So, with the help of a suitable energy competitor (see for example (2.26) or (2.63)), for any two–dimensional ball $B_{\frac{\lambda}{3}}(q)$ of radius $\frac{\lambda}{3}$ that is contained in $(\lambda,\infty)\times(1,\lambda-1)$, we have

$$\int_{B_{\frac{\lambda}{3}}(q)}W(U_\lambda)drds\leq C\lambda^2,$$

with constant $C>0$ independent of large λ. If we further assume that conditions **(b)** and **(c)** hold, noting that

$$\partial_r^2(\mu-U_\lambda)+\frac{1}{r}\partial_r(\mu-U_\lambda)+\left(1+\frac{\lambda^2}{\pi^2r^2}\right)\partial_s^2(\mu-U_\lambda)\geq0,$$

and that the coefficients of the elliptic operator above satisfy

$$\frac{1}{r}\leq\lambda^{-1},\quad 1\leq1+\frac{\lambda^2}{\pi^2r^2}\leq1+\pi^{-2}\ \text{ on } B_{\frac{\lambda}{3}}(q),$$

the arguments in Remark 2.9 can be applied to show that

$$U_\lambda\to\mu,\ \text{uniformly on } \bar{B}_{\frac{\lambda}{6}}(q),\ \text{ as } \lambda\to\infty.$$

Since q was any point with coordinates $r>\frac{4\lambda}{3}$ and $s\in\left(\frac{\lambda}{3}+1,\frac{2\lambda}{3}-1\right)$, we deduce that

$$U_\lambda\to\mu,\ \text{uniformly on } [2\lambda,\infty)\times\left[\frac{\lambda}{6}+1,\frac{5\lambda}{6}-1\right],\ \text{ as } \lambda\to\infty.$$

Studying the asymptotic behavior of U_λ, as $\lambda \to \infty$, assuming *only* **(a')**, is left as an interesting open problem.

An extension of Lemma 2.1 can be shown, allowing the possibility $W'(0) \geq 0$, provided that the potential W satisfies:

(a"): There exist constants $\mu_- \leq 0$ and $\mu > 0$ such that

$$0 = W(\mu) < W(t), \ t \in [\mu_-, \mu), \quad W(t) \geq 0, \ t \in \mathbb{R},$$
$$W(2\mu_- - t) \geq W(t), \ t \in [\mu_-, \mu].$$

Note that **(a")** reduces to **(a')** when $\mu_- = 0$. Below, we state such a result which seems to be new and of independent interest.

Lemma 2.2. Assume that $W \in C^2$ satisfies condition **(a")**. Let $\epsilon \in (0, \mu)$ and $D > D'$, where D' is as in (1.11). Then, there exists a positive constant $R' > D$, depending only on ϵ, D, W, and n, such that there exists a global minimizer u_R of the energy functional in (2.1) which satisfies (2.2), (2.3), and (2.4), provided that $R \geq R'$. (As before, we assume that W has been appropriately extended outside of a large compact interval). (We have chosen to keep some of the notation from Lemma 2.1).

Proof. The existence of a minimizer u_R, which solves (2.6), and satisfies

$$\mu_- < u_R(x) < \mu, \ x \in B_R,$$

follows as in the proof of Lemma 2.1. The main difference with the proof of Lemma 2.1 is that the above relation does not exclude the possibility of the minimizer u_R taking non-positive values. In particular, the method of moving planes (see [43], [91]) is not applicable in order to show that u_R is radially symmetric and decreasing. (Nevertheless, it is known that nonnegative solutions of (2.6), with $n \geq 2$, are actually positive in B_R and so the method of moving planes is still applicable in that situation, see [139] and the references therein). Not all is lost however. As we have already remarked in the proof of Lemma 2.1, if $n \geq 2$, the stability of u_R (as a global minimizer) implies that it is radially symmetric, see Lemma 1.1 in [6], Remark 3.3 in [47], Proposition 2.6 in [66], and [129]; for an elegant proof that exploits the fact that u_R is a global minimizer, see Corollary II.10 in [123] (see also [99]). In [60], see also Proposition 10.4.1 in [58], it has additionally been shown that stable solutions have constant sign, and hence are radially monotone by the method of moving planes. For the reader's convenience, we will show that $u_R(r)$ is a decreasing function of r, namely that (2.9) holds true, by a far more elementary argument. In view of (2.11), which still holds for the case at hand (by virtue of radial symmetry alone), it suffices to show that $u_R'(r) \neq 0, \ r \in (0, R]$. We will follow the part of the proof of Lemma 2 in [48] which dealt with problem (1.22) with $n \geq 3$, and in fact show that it continues to apply for $n \leq 2$. To this end, we have not been able to adapt the approach of Lemma 1 in [5], which basically consists in multiplying (2.49) below by $V^+ \equiv \max\{V, 0\} \in W^{1,2}(B_R)$ and integrating the resulting identity by parts over B_R, since in the problem at hand $V(R) = u_R'(R)$ may be positive. Let

$$V \equiv u_R',$$

and suppose, to the contrary, that $V(R_0) = 0$ for some $R_0 \in (0, R]$. We will show that the function

$$\tilde{V}(r) = \begin{cases} V(r), & r \in [0, R_0], \\ 0 & r \in [R_0, R], \end{cases} \tag{2.47}$$

belonging in $W_0^{1,2}(B_R)$, satisfies

$$\int_{B_R} \left\{ |\nabla \tilde{V}|^2 + W''(u_R)\tilde{V}^2 \right\} dx < 0, \tag{2.48}$$

which clearly contradicts the stability of u_R. Differentiating (2.6) with respect to r, we arrive at

$$-\Delta V + W''(u_R)V + \frac{n-1}{r^2}V = 0, \quad x \in B_R\backslash\{0\}. \tag{2.49}$$

Let ζ be a smooth function such that

$$\zeta(t) = \begin{cases} 0, & t \in [0,1], \\ 1, & t \in [2,\infty). \end{cases}$$

Multiplying (2.49) by $\zeta\left(\frac{r}{\varepsilon}\right)V(r)$, with $\varepsilon > 0$ small, and integrating the resulting identity by parts over B_{R_0} (recall that $V(R_0) = 0$), we find that

$$\int_{B_{R_0}} \left\{ \zeta\left(\frac{r}{\varepsilon}\right)|\nabla V|^2 + \frac{1}{\varepsilon}V\zeta'\left(\frac{r}{\varepsilon}\right)\left(\frac{x}{r}\cdot\nabla V\right) + \zeta\left(\frac{r}{\varepsilon}\right)W''(u_R)V^2 + \zeta\left(\frac{r}{\varepsilon}\right)\frac{n-1}{r^2}V^2 \right\} dx = 0. \tag{2.50}$$

Note that

$$\left| \int_{B_{R_0}} \frac{1}{\varepsilon}V\zeta'\left(\frac{r}{\varepsilon}\right)\left(\frac{x}{r}\cdot\nabla V\right) dx \right| \leq C\varepsilon^{-1}\int_{\varepsilon}^{2\varepsilon} r^{n-1}dr \to 0 \text{ as } \varepsilon \to 0,$$

since the constant $C > 0$ does not depend on ε. (Note that we have silently assumed that $N \geq 2$, since in the case $N = 1$ we can plainly multiply (2.49) by V and then integrate by parts over $(-R_0, R_0)$). So, letting $\varepsilon \to 0$ in (2.50), and employing Lebesgue's dominated convergence theorem (see for instance page 20 in [80]), it readily follows that

$$\int_{B_{R_0}} \left\{ |\nabla V|^2 + W''(u_R)V^2 + \frac{n-1}{r^2}V^2 \right\} dx = 0,$$

where in order to obtain the last term we used that $|V(r)| \leq C'r$, $r \in [0,R]$, with constant $C' > 0$ depending only on R (keep in mind that $u_R \in C^2[0,R]$ with $u_R''(0) = \frac{1}{n}W'(u_R(0))$, see for instance page 72 in [164]). From the above relation, via (2.47), we get (2.48). We have thus arrived at the desired contradiction. Consequently, the monotonicity relation (2.9) also holds for the more general case at hand. The rest of the argument follows word by word the proof of Lemma 2.1, and is therefore omitted.

The proof of the lemma is complete. □

Remark 2.11. Suppose that u_R is as in Lemma 2.1 or Lemma 2.2, and E_R as defined in (2.14). From (2.15), it follows that

$$E_R(r) < E_R(0) = -W(u_R(0)) < 0, \quad r \in [0,R],$$

i.e.,

$$\frac{1}{2}[u_R'(r)]^2 < W(u_R), \quad r \in (0,R], \tag{2.51}$$

recall **(a')** and that $u_R'(0)=0$, see also Remark 4 in [5] for a related discussion. In passing, we note that every bounded solution of (1.22) satisfies

$$\frac{1}{2}|\nabla u|^2 \le W(u), \quad x\in\mathbb{R}^n,$$

provided that W is nonnegative.The proof of this gradient bound, originally due to L. Modica, is much more complicated than that of its radially symmetric counterpart (2.51). We refer the interested reader to Lemma 4.1 in [54], and to the older references that can be found in [8]. In turn, making use of the gradient bound (2.51), we can establish the monotonicity formula

$$\frac{d}{dr}\left(\frac{1}{r^{n-1}}\int_{B_r}\left\{\frac{1}{2}|\nabla u_R|^2+W(u_R)\right\}dx\right)>0, \quad r\in(0,R), \tag{2.52}$$

see [8] for a modern approach as well as the older references therein. In passing, we note that a similar monotonicity formula holds true for solutions of (1.22), and a weaker one (with the exponent $n-1$ replaced by $n-2$) holds in the case of the corresponding systems, see again [8] and the references therein or [52]. Now, making use of (2.7) and the above relation, we find that

$$\frac{1}{K^{n-1}}\int_{B_K}\left\{\frac{1}{2}|\nabla u_R|^2+W(u_R)\right\}dx<C_1 \quad \forall\ K\in(0,R), \quad R\ge 2.$$

We have therefore provided a proof (of a sharper version) of (2.8). It also follows from (2.52) that $R^{1-n}J(u_R;B_R)$ remains bounded from below by some positive constant, as $R\to\infty$ (compare with (2.7)). If $W''(\mu)>0$, making use of (2.19), it is not hard to determine the constant to which $R^{1-n}J(u_R;B_R)$ converges as $R\to\infty$ (see [19]), recall also the last part of Remark 2.1. In this regard, we also refer to Theorem 7.10 in [38] where functionals of the form (2.1) are shown to converge (in an appropriate variational sense) to functionals involving the perimeter of the domain.

Remark 2.12. Here, for completeness, we sketch an argument related to the proof of Lemma 2.2. By (2.2), elliptic estimates (see [93]), and a standard compactness argument, it follows readily that u_R converges, up to a subsequence $R_i\to\infty$, uniformly on compact subsets of $\mathbb{R}^n$ to a radially symmetric solution U of (1.22) such that $0\le U(x)\le\mu,\ x\in\mathbb{R}^n$. Moreover, arguing as in page 104 of [66], this solution is a global minimizer of (1.22) in the sense of (2.65) below, with $\Omega=\mathbb{R}^n$, see also [49], [106].

On the other hand, it is known that (1.22), for *any* $W\in C^2$, does not have nonconstant bounded, radial global minimizers (see [163]). This property is also related to the nonexistence of nonconstant "bubble" solutions to (1.22) with $W\ge 0$ vanishing nondegenerately at a finite number of points, namely solutions that tend to one of these points as $|x|\to\infty$, see Chapter 4 in [154] and recall Remark 2.8 (keep in mind that stable solutions of (1.22) are radially monotone and tend to a local or global minimum of W, as $r\to\infty$, see [48]). In passing, we note that if $n\le 10$ then nonconstant radial solutions of (1.22), with $W\in C^2$ arbitrary, are unstable (see [48]). Under certain assumptions on W, satisfied by the Allen-Cahn potential (1.23) for example, it was shown in [96] that nonconstant radial solutions of (1.22) tend oscillatory to zero as $r\to\infty$ and thus are unstable. More generally, the nonexistence of nonconstant finite energy solutions to (1.22) with $W\ge 0$ holds, see [8] or [100] where this property is refereed to as a theorem of Derrick and Strauss. Related nonexistence results for nonnegative solutions can be found in Sections 5 and 8 herein.

Obviously $U \equiv \mu$ (recall **(a')**) and, by the uniqueness of the limit, the convergence holds for *all* $R \to \infty$. We conclude that, given any $K > 1$, we have $u_R \to \mu$, uniformly in B_K, as $R \to \infty$. The main advantage of this approach is that it continues to work when (2.6) is replaced by $\Delta u = F_R(|x|, u)$, with a suitable $F_R(|x|, u)$ which converges uniformly over compact sets of $[0, \infty) \times \mathbb{R}$ to a C^1 function $F(u)$ (the point being that $\frac{d}{dr} F_R(r, u)$ may be negative somewhere, and (2.9) may fail in B_R).

The following lemma is motivated from Lemma 3.3 in [28].

Lemma 2.3. Assume that W satisfies conditions **(a")** and (1.15). Let $\epsilon \in (0, \mu)$ be any number such that

$$W''(t) \geq 0 \quad \text{on} \quad [\mu - \epsilon, \mu]. \tag{2.53}$$

Then, the global minimizers u_R that are provided by Lemmas 2.1 and 2.2 satisfy

$$-W'(u_R(0)) \leq \tilde{C} R^{-2}, \tag{2.54}$$

where the constant $\tilde{C} > 0$ depends only on n, provided that $R \geq R'$, where R' is as in the latter lemmas.

Proof. Let D be as in the assertions of Lemmas 2.1 and 2.2. Thanks to (2.2), (2.3), (2.6), (2.9), and (2.53), we have

$$\Delta u_R = W'(u_R) \leq W'(u_R(0)) \quad \text{on} \quad \bar{B}_{(R-D)},$$

if $R \geq R'$.

For such R, let Z_R be the solution of

$$\Delta Z_R = W'(u_R(0)) \quad \text{in } B_{(R-D)}; \quad Z_R = 0 \quad \text{on } \partial B_{(R-D)}.$$

By scaling, one finds that

$$\max_{|x| \leq R-D} Z_R(x) = Z_R(0) = -z(0) W'(u_R(0)) (R-D)^2,$$

for $R \geq R'$, where z is the solution of

$$\Delta z = -1 \quad \text{in } B_1; \quad z = 0 \quad \text{on } \partial B_1.$$

By the maximum principle, we deduce that

$$Z_R(x) \leq u_R(x) < \mu, \quad x \in B_{(R-D)}, \quad R \geq R'.$$

In particular, by setting $x = 0$ in the above relation, we get (2.54).

The proof of the lemma is complete. □

Under conditions **(a")** and (1.15), the global minimizers that are provided by Lemmas 2.1 and 2.2 are *asymptotically* stable, if R is sufficiently large. This property is a direct consequence of the following proposition, which will play an essential role in the proof of Theorems 8.1 and 8.2 below.

Proposition 2.1. Assume that **(a")** and (1.15) hold, then any solution of (2.6) which satisfies (2.2), (2.3), and (2.20) with $V = \mathbf{U}$ for *all* $R \to \infty$ (keep in mind (2.18)) is linearly non–degenerate if R is sufficiently large.

Proof. We remark that in the case where $W''(\mu) > 0$, we have already seen in Remark 2.5 that any such solution is in fact asymptotically stable for large R.

To prove this proposition, we will argue once more by contradiction. Suppose that there exists a sequence $R \to \infty$ and solutions u_R of (2.6), as in the assertion of the proposition, such that there are nontrivial solutions φ_R of (2.33) with $\mu_R = 0$. Without loss of generality, we may assume that the normalization (2.34) holds. This time, the φ_R's may change sign but they are still radially symmetric (see again [101], [119]). By Lemma 2.1 in [110], the following identity holds

$$R^{-n}\int_0^R W'\left(u_R(r)\right)\varphi_R(r)r^{n-1}dr = -\frac{1}{2}u_R'(R)\varphi_R'(R). \tag{2.55}$$

In order to make the presentation as self-contained as possible, let us mention that a direct proof of (2.55) can be given by observing that the function

$$\zeta(r) = r^n\left[u'\varphi' - W'(u)\varphi\right] + (n-2)r^{n-1}u'\varphi, \quad r \in [0, R],$$

(having dropped the subscripts for the moment) introduced in [159], satisfies

$$\zeta'(r) = -2W'(u)\varphi r^{n-1}, \quad r \in (0, R).$$

Since $W'(\mu) = 0$, by (2.3), we deduce that

$$R^{-n}\int_0^R \left[W'\left(u_R(r)\right)\right]^2 r^{n-1}dr \to 0 \ \text{ as } R \to \infty,$$

see also Chapter 1 in [112] (a perhaps simpler proof was given in [111]). Hence, recalling (2.34), via the Cauchy–Schwarz inequality, we find that the lefthand side of (2.55) tends to zero as $R \to \infty$ (along the sequence). On the other side, from our assumption that (2.20) with $V = \mathbf{U}$ holds for all $R \to \infty$, we know that

$$u_R'(R) \to -\sqrt{2W(0)} < 0 \ \text{ as } \ R \to \infty.$$

So, from (2.55), we get that $\varphi_R'(R) \to 0$ as $R \to \infty$ (along the contradicting sequence). By the continuous dependence theory for systems of ordinary differential equations [21, 164] (applied to $\varphi_R(R-r)$), making use of (2.20) with $V = \mathbf{U}$ for all $R \to \infty$, we infer that for any $D > 0$ we have

$$|\varphi_R(r)| + |\varphi_R'(r)| \leq \frac{1}{2}, \quad r \in [R-D, R], \tag{2.56}$$

provided that R is sufficiently large (along this sequence). If D is chosen so that $W''(u_R) \geq 0$ on $\bar{B}_{(R-D)}$, which is possible by (2.2), (2.3) and (1.15), it follows from (2.33) with $\mu_R = 0$ that

$$\varphi_R\Delta\varphi_R = W''(u_R)\varphi_R^2 \geq 0 \ \text{ on } \bar{B}_{(R-D)},$$

for such large R. In particular, we find that φ_R cannot vanish in $B_{(R-D)}\backslash\{0\}$ (using the radial symmetry, and integrating by parts over B_z if $\varphi_R(z) = 0$). Furthermore, it cannot vanish at the origin by virtue of the uniqueness theorem for ordinary differential equations, which still holds despite of the singularity at $r = 0$ (see [69], [137], [164]). Without loss of generality, we may assume that $\varphi_R > 0$ in $B_{(R-D)}$. Therefore, the positive function φ_R is subharmonic in $B_{(R-D)}$, and not greater than $\frac{1}{2}$ on $\partial B_{(R-D)}$ (recall (2.56)), for R large along the contradicting sequence. The maximum principle (see for example Theorem 2.3 in [93]) yields that $0 < \varphi_R \leq \frac{1}{2}$ on $\bar{B}_{(R-D)}$. The latter relation together with (2.56) clearly contradict (2.34), and we are done.

The proof of the proposition is complete. □

Remark 2.13. We note that identity (2.55) has been generalized in Lemma 2.3 in [134] for the case of solutions of (1.2) on an arbitrary smooth bounded domain. This leaves open the possibility that Proposition 2.1 above can be generalized accordingly.

The following corollary is a simple consequence of the maximum principle.

Corrolarry 2.1. If $W''(\mu)>0$, then the solutions provided by Lemmas 2.1 and 2.2 satisfy

$$\mu-u_R(r)\leq C_5e^{-C_6(R-r)},\ \ r\in[0,R-2D]\ \text{ for }\ R\geq R',$$

and some positive constants C_5, C_6, depending on W and n.

Proof. Let $\varphi\equiv\mu-u_R$, where u_R is as in Lemma 2.1 or 2.2. By virtue of **(a')**, (2.2), and (2.3), we can choose ϵ sufficiently small such that that

$$W'(u_R(x))\leq\frac{W''(\mu)}{2}\left(u_R(x)-\mu\right),\ \ x\in B_{(R-D)},$$

provided that $R\geq R'$, where D, R' are as in the previously mentioned lemmas (having increased the value of R', if necessary, but still depending only on ϵ, D, W, and n). It follows from (2.6) that

$$-\Delta\varphi+\frac{W''(\mu)}{2}\varphi\leq0\ \text{ in } B_{(R-D)},\ R\geq R'.$$

Now, the desired assertion of the corollary follows from a standard comparison argument, see Lemma 2 in [35] or Lemma 4.2 in [84] (see also Lemma 2.5 in [79]).

The proof of the corollary is complete. □

Remark 2.14. A special case of Theorem 2.1 in [63] shows that the assertion of Corollary 2.1 above can be considerably refined to

$$\lim_{R\to\infty}R^{-1}\ln\left(\mu-u_R(Rs)\right)=-(1-s)\sqrt{W''(\mu)},\ \ \forall\ s\in[0,1],$$

see also [23].

2.2. **Proof of Theorem 1.2.** Once Lemma 2.1 is established, the proof of Theorem 1.2 proceeds in a rather standard way. We will present two different approaches, and leave it to the reader's personal taste. The first approach is based on the method of upper and lower solutions, while the second one is based on variational arguments.

`First proof of Theorem 1.2`: We will adapt an argument from the proof of Theorem 2.1 in [68], and prove existence of the desired solution to (1.2) by the method of upper and lower solutions (see for instance [126], [144]). Let $\epsilon\in(0,\mu)$ and $D>D'$, where D' is as in (1.11), and R' be the positive constant, depending only on ϵ, D, W, and n, that is described in Lemma 2.1. Suppose that Ω contains a closed ball of radius R'. We use $\bar{u}(x)\equiv\mu$, $x\in\Omega$, as an upper solution (recall that $W'(\mu)=0$), and as lower solution the function

$$\underline{u}_P(x)\equiv\begin{cases}u_{\text{dist}(P,\partial\Omega)}(x-P), & x\in B_{\text{dist}(P,\partial\Omega)}(P),\\ 0, & x\in\Omega\backslash B_{\text{dist}(P,\partial\Omega)}(P),\end{cases}\tag{2.57}$$

for some $P \in \Omega_{R'}$ (considered fixed for now), where u_R is as in Lemma 2.1 (here we used that $W'(0) \leq 0$ and Proposition 1 in [26] to make sure that $\underline{u}_P$ is a lower solution, see also Proposition 1 in [115]). In view of (2.2) and (2.3), keeping in mind that

$$\mathrm{dist}(P, \partial\Omega) > R', \tag{2.58}$$

it follows that

$$\underline{u}_P(x) < \bar{u}(x) \equiv \mu, \ \ x \in \Omega, \quad \text{and} \quad \mu - \epsilon < \underline{u}_P(x), \quad x \in B_{(\mathrm{dist}(P,\partial\Omega)-D)}(P). \tag{2.59}$$

In the case where Ω is bounded, it follows immediately from the method of monotone iterations, see Theorem 2.3.1 in [144] (see also the book [94] for the corresponding elliptic estimates that are required in case the domain has corners, and [70]), that there exists a solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ of (1.2) such that

$$\underline{u}_P(x) < u(x) < \bar{u}(x) \equiv \mu, \quad x \in \Omega, \tag{2.60}$$

(have in mind that the solution u depends on the choice of the center P). The same property can also be shown in the case where Ω is unbounded, by exhausting it with a sequence of bounded domains, see Theorem 2.10 in [128] (also recall our discussion following the statement of Theorem 1.2), see also [130, 132]. We have thus established the existence of a solution u of (1.2) that satisfies (1.3), and the lower bound (1.13) in the region $B_{(\mathrm{dist}(P,\partial\Omega)-D)}(P)$ (recall (2.2), (2.3), and (2.58)), or equivalently in $P + B_{(\mathrm{dist}(P,\partial\Omega)-D)} \supseteq P + B_{(R'-D)}$. It remains to show that the latter lower bound is valid in $\Omega_{R'} + B_{(R'-D)}$. Observe that, as we vary the point P in $\Omega_{R'}$, assuming for the moment that $\Omega_{R'}$ has a single arcwise connected component, the functions $\underline{u}_P$'s continue to be lower solutions of (1.2). Consequently, by Serrin's sweeping principle (see [57, 62, 105, 144], and the last part of the proof of Proposition 3.1 herein), we deduce that

$$\underline{u}_Q(x) < u(x), \quad x \in \Omega, \quad \forall \, Q \in \Omega_{R'}, \tag{2.61}$$

(see also the proof of Lemma 3.1 in [57], and note that $\underline{u}_Q$ varies continuously with respect to Q because of the connectedness of $\partial\Omega$). In fact, this is more in the spirit of the celebrated sliding method [30]: Let $\gamma(s)$, $s \in [0,1]$, be a smooth curve lying entirely in $\Omega_{R'}$ such that $\gamma(0) = P$ and $\gamma(1) = Q$ ($Q \in \Omega_{R'}$ arbitrary). It follows from (2.60) that $\underline{u}_{\gamma(0)} < u$ in Ω. We want to prove that $\underline{u}_{\gamma(s)} \leq u$ in Ω for all $s \in [0,1]$. If not, there would exist an $s_0 \in (0,1]$ such that $\underline{u}_{\gamma(s_0)} \leq u$ in Ω and $\underline{u}_{\gamma(s_0)}(x_0) = u(x_0)$ at some point $x_0 \in \Omega$. But we have

$$\Delta \left(u - \underline{u}_{\gamma(s_0)}\right) + q(x) \left(u - \underline{u}_{\gamma(s_0)}\right) = 0, \quad x \in \Omega, \ \text{with } q \in L^\infty(\Omega),$$

and the strong maximum principle implies that $u \equiv \underline{u}_{\gamma(s_0)}$. This is not possible, since $\underline{u}_{\gamma(s_0)}$ is compactly supported while u is not. We remark that a similar argument, in the case where the radius of the sliding ball is kept fixed, appears in the proof of Lemma 3.1 in [28]. The validity of the lower bound (1.13), over the whole specified domain, now follows from (2.3), (2.57), (2.58), and (2.61). In the case where the domain $\Omega_{R'}$ has numerable many arcwise connected components, we can use the function $\max\{\underline{u}_{P_i}, \ i = 1, \cdots\}$ as a lower solution, where the $\underline{u}_{P_i}$'s are as in (2.57) with each center P_i belonging to a different component of $\Omega_{R'}$. (We use again Proposition 1 in [26], keep in mind that the maximum is essentially chosen among finitely many functions). The case where there are denumerable many arcwise connected components of $\Omega_{R'}$ can be treated similarly. The validity of (1.14) follows at once from (2.4), (2.57), and (2.61).

If $W''(\mu) > 0$, the validity of (1.5) for $x \in \Omega_{R'}$ follows at once from Corollary 2.1 and relations (2.57), (2.61). If $\text{dist}(x, \partial\Omega) \leq R'$, then plainly observe that

$$\mu - u(x) \leq \mu = \mu e^{R'} e^{-R'} \leq \mu e^{R'} e^{-\text{dist}(x,\partial\Omega)}. \tag{2.62}$$

If relation (1.15) holds, then the validity of (1.16) follows from (2.54), (2.57), and (2.61), keeping in mind that $\mu - \epsilon \leq \underline{u}_P(P) \leq u(P)$, via (1.15), implies that $-W'(u(P)) \leq -W'(\underline{u}_P(P))$. We postpone the proof of relation (1.18) until Subsection 4.1.

The first proof of the theorem, with the exception of (1.18), is complete. □

Remark 2.15. Since it is constructed by the method of upper and lower solutions, we know that the obtained solution u is stable (with respect to the corresponding parabolic dynamics), see [126, 144], namely the principal eigenvalue of

$$-\Delta\varphi + W''(u)\varphi = \lambda\varphi, \ x \in \Omega; \quad \varphi = 0, \ x \in \partial\Omega,$$

in nonnegative. In the case of unbounded domains, some extra care is needed in the definition of stability, see [48, 67].

`Second proof of Theorem 1.2:` Assume first that Ω is bounded. As in the proof of Lemma 2.1, there exists a global minimizer u_{min} of the energy

$$J(v;\Omega) = \int_\Omega \left\{ \frac{1}{2}|\nabla v|^2 + W(v) \right\} dx, \quad v \in \ W_0^{1,2}(\Omega),$$

which furnishes a classical solution of (1.2) such that $0 \leq u_{min}(x) < \mu$, $x \in \Omega$. Again, by the strong maximum principle, either u_{min} is identically equal to zero or it is strictly positive in Ω. We intend to show that there exists an $R_* > 0$, depending only on W and n, such that u_{min} is nontrivial, provided that Ω contains some closed ball of radius R_*.

For the sake of our argument, suppose that u_{min} is the trivial solution. Then, motivated from Proposition 1 in [2] (see also [49], [116] and [131]), assuming without loss of generality that $\bar{B}_{R+2} \subset \Omega$ for some $R > 0$, we consider the function

$$Z(x) = \begin{cases} 0, & x \in \Omega \backslash B_{R+1}, \\ \mu(R+1-|x|), & x \in B_{R+1} \backslash B_R, \\ \mu, & x \in B_R. \end{cases} \tag{2.63}$$

Since $Z \in W_0^{1,2}(\Omega)$, from the relation $J(0;\Omega) \leq J(Z;\Omega)$, and recalling that $W(\mu) = 0$, we obtain that

$$J(0; B_{R+1}) \leq \int_{B_{R+1}\backslash B_R} \left\{ \frac{1}{2}|\nabla Z|^2 + W(Z) \right\} dx \leq C_0 R^{n-1}, \tag{2.64}$$

with C_0 depending only on W and n. In turn, the above relation implies that

$$|B_{R+1}| W(0) \leq C_0 R^{n-1},$$

which cannot hold if $R \geq R_*$ is sufficiently large, depending on W and n. Consequently, the minimizer u_{min} is nontrivial, provided that Ω contains some closed ball of radius R_*. From our previous discussion, we therefore conclude that u_{min} satisfies (1.3).

Let $\epsilon \in (0, \mu)$ and $D > D'$, where D' is as in (1.11). Suppose that Ω contains a closed ball of radius R', where R' is as in the assertion of Lemma 2.1; without loss of generality, we may assume that $R' > R_*$. Relation (1.13) now follows by applying Lemma A.3 below, over every closed ball of radius R' contained in Ω, and recalling Lemma 2.1. Note that, as in Remark 2.3, the unique continuation principle implies that

$$u_{min} \text{ minimizes } J(v;\mathcal{D}) \text{ in } v-u_{min}\in W_0^{1,2}(\mathcal{D}) \text{ for every smooth bounded domain } \mathcal{D}\subset\Omega. \tag{2.65}$$

The case where Ω is unbounded can be treated by exhausting it by an infinite sequence of bounded ones, where the above considerations apply (see also [131]). The minimizers over the bounded domains (extended by zero outside) converge locally uniformly to a solution u of (1.2) that satisfies (1.3) (the latter solution is nontrivial by virtue of the lower bound $u(x)\geq \mu-\epsilon,\ x\in B_{(R'-D)}(x_0)$ for some $x_0\in\Omega_{R'}$, which is valid since we may assume that each one of the bounded domains contains the same closed ball $\bar{B}_{R'}(x_0)$). This solution of (1.2), on the unbounded domain Ω, found in this way, may have infinite energy but is still a global minimizer in the sense of Definition 1.2 in [106], namely satisfies (2.65). As before, it satisfies (1.13).

The validity of (1.14) follows from (2.4) and Lemma A.3 below (applied on every ball $B_{\text{dist}(x,\partial\Omega)}(x),\ x\in\Omega_{R'}$). Similarly, if $W''(\mu)>0$, the validity of (1.5) follows from Corollary 2.1, Lemma A.3 below, and the observation in (2.62). The validity of relation (1.16) follows in the same manner, making use of (2.54). We postpone the proof of relation (1.18) until Subsection 4.1.

The second proof of the theorem, with the exception of (1.18), is complete. □

Remark 2.16. If W is as in Remark 2.5, and Ω is bounded with smooth boundary (at least C^3), in view of the latter remark and Theorem 2 in [158], the solutions of Theorem 1.2 that we found by the two different approaches are actually the same, if ϵ is chosen sufficiently small.

Remark 2.17. Assume that the domain Ω is symmetric with respect to the hyperplane $x_i=0$. Then, since the solution of (1.2), provided by the second proof of Theorem 1.2, is a global minimizer of the associated energy (in the sense described above, in case Ω is unbounded), it follows from Theorem II.5 in [123] (applied on symmetric bounded domains, with respect to the hyperplane $x_i=0$, exhausting Ω) that the latter solution is symmetric with respect to this hyperplane. Note that, if in addition the domain Ω is bounded and convex in the x_i direction, this assertion holds true for *any* positive solution of (1.2) by virtue Theorem 2 in [43] (proven by the method of moving planes). Clearly, if uniqueness holds for positive solutions of (1.2) (recall Remark 1.3), these assertions follow at once (see also Remark 1.3 in [89]).

3. Uniform estimates for positive solutions without specified boundary conditions

In this section, we will assume conditions **(a')**, (1.8), (1.9), and (1.15). These conditions, in particular, guarantee that problem (2.6) has at most one solution, satisfying (2.2), for sufficiently large R (see Remark 9.2 below, and recall Remarks 1.3, 2.6). Under these assumptions, we will establish uniform estimates for solutions of

$$\Delta u = W'(u), \tag{3.1}$$

provided that they are positive and less than μ over a sufficiently set. Our motivation comes from Lemmas 3.2–3.3 in [28] and Lemma 4.1 in [114] (see also Lemma 2.4 in [82]).

The next proposition and the corollary that follows refine the latter results, pretty much as (2.3) refined (2.12).

Proposition 3.1. Suppose that $W \in C^2$ satisfies **(a')**, (1.8), (1.9), and (1.15). Let $\epsilon \in (0, \mu)$ be any number such that $W'' \geq 0$ on $[\mu - \epsilon, \mu]$, and $D > D'$, where D' is given from (1.11). There exists a positive constant R', depending on ϵ, D, W, n, such that for *any* solution of (3.1) which satisfies

$$0 < u(x) < \mu, \quad x \in \bar{B}_R(P), \text{ for some } P \in \mathbb{R}^n, \quad \text{and } R \geq R', \tag{3.2}$$

we have

$$u(x) \geq \mu - \epsilon, \quad x \in \bar{B}_{(R-D)}(P). \tag{3.3}$$

Moreover, we have

$$\min \{W(t) \; : \; t \in [0, u(x)]\} \leq \frac{C}{R - |x - P|}, \quad x \in B_R(P), \tag{3.4}$$

for some positive constant C that depends only on W, n. Finally, we have that

$$-W'\left(u(P)\right) \leq \tilde{C}(R - |x - P|)^{-2}, \tag{3.5}$$

for some constant $\tilde{C} > 0$ that depends only on n.

Proof. Before we go into the proof, let us make some remarks. The point of this proposition is that we do not assume that the solutions under consideration are global minimizers, a case which can be handled similarly to the second proof of Theorem 1.2. The argument that was used for the related results in [28], [114] essentially consists in constructing a family of positive lower solutions of (3.1) of the form $s\varphi_R$, $s > 0$, where φ_R is the eigenfunction associated to the principal eigenvalue of the negative Dirichlet Laplacian over a fixed ball B_R, and sweeping á la Serrin with respect to s (see also Lemma 3.1 in [20], Lemma 2 in [62], and Theorem 2.1 in [87]). On the other hand, our argument consists in constructing a family of nonnegative lower solutions of (3.1) from the global minimizing solutions of (2.6) that are provided by Lemma 2.1, and sweeping with respect to the radius of the ball (a similar idea can be found in [69], see also the comments after Proposition 2.2 in [98]). The drawback of our approach is that, for purely technical reasons, we had to assume (1.15). We believe that our proof, except the part for (3.5), can be adapted to work without assuming the latter relation (see Remark 3.2 below), as is the case in the aforementioned references. Another slight drawback of our approach is discussed in Remark 3.3 below.

Observe that if u solves (2.6), the function $v(y) \equiv u(Ry)$, $y \in B_1$, satisfies

$$\Delta v = R^2 W'(v), \; y \in B_1; \quad v(y) = 0, \; y \in \partial B_1. \tag{3.6}$$

Since $W'(0) \leq 0$ (recall **(a')**), it follows from the results in [110] (see also Chapter 1 in [112]), which are based on the identity (2.55), that solutions of (3.6) lie on smooth curves in the (R, v) "plane", i.e. either solutions of (3.6) can be continued in R, or else there are simple turning points (see also [108] for the definitions and functional set up). We will distinguish two cases according to the sign of $W'(0)$:

- If $W'(0) < 0$, by a classical global result of Leray and Schauder (see [117] or page 65 in [18]), there exists an unbounded connected branch $\mathcal{C}_+$ of positive solutions of (3.6) in $(0, \infty) \times C(\bar{B}_1)$ that meets $(0, 0)$ (see also Lemma 5.1 in [17]). (As we have already discussed, thanks to [91], all positive solutions of (3.6) are radially symmetric and decreasing). In fact, the detailed behavior of that branch as $R \to 0$ is described in

Theorem 3.2 of [135]. By the strong maximum principle, we deduce that the solutions on $\mathcal{C}_+$ take values strictly less than μ (by a continuity argument, since they do so for small R, see also Lemma 1 in [102]). Thus, the projection of $\mathcal{C}_+$ onto $(0,\infty)$ is unbounded.

- If $W'(0)=0$ and $W''(0)<0$ (recall (1.8)), there is a global connected solution curve $\mathcal{C}_+$ in $(0,\infty)\times C(\bar{B}_1)$, emanating from $(R_c,0)$, where R_c was defined in (2.38), due to a bifurcation from a simple eigenvalue as R crosses R_c (see [108]). As before, the solutions on that branch are positive and strictly less than μ. It follows readily from Rabinowitz's global bifurcation theorem [142] (see also Chapter 4 in [18]) that the projection of $\mathcal{C}_+$ onto $(0,\infty)$ is unbounded (see the appendix in [151] for instance). (Keep in mind that $\mathcal{C}_+$ is bounded away from $R=0$, as can easily be seen by testing (3.6) by the principal eigenfunction φ_1 (see Lemma 6.17 in [135]); in fact, if (2.39) holds, the projection of $\mathcal{C}_+$ onto $(0,\infty)$ is (R_c,∞)).

From our assumptions on W, in view of Remark 9.2 below, we know that (3.6) has at most one solution with values in $(0,\mu)$, provided that R is sufficiently large. As a consequence, recalling the last part of Remark 2.5, these solutions are nondecreasing pointwise with respect to large R. Moreover, for large R, it follows that $(R,v)\in\mathcal{C}_+ \ \Rightarrow\ v(y)=u_R(Ry),\ y\in\bar{B}_1$, where u_R as in Lemma 2.1. As in [61, 110], we can parameterize smoothly $\mathcal{C}_+$ by $\{(R_\tau,v_\tau)\,,\ \tau\in(0,\mu)\}$, where τ is the maximum of v_τ, namely $v_\tau(0)=\tau$, also recall (2.3). The functions $\mathtt{u}_\tau(r)\equiv v_\tau\left(R_\tau^{-1}r\right)$, $\tau\in(0,\mu)$, define a smooth, with respect to τ, family of solutions of (2.6), satisfying (2.2), with $R=R_\tau$. Note that we have

$$R_\tau\to 0,\ \text{as}\ \tau\to 0,\ \text{if}\ W'(0)<0;\ R_\tau\to R_c,\ \text{as}\ \tau\to 0,\ \text{if (1.8) holds.} \tag{3.7}$$

On the other side, in both cases, we have $R_\tau\to\infty$ as $\tau\to\mu^-$. From the definition of τ, we see that

$$\mathtt{u}_\tau\to 0,\ \text{uniformly on}\ \bar{B}_{R_\tau},\ \text{as}\ \tau\to 0. \tag{3.8}$$

From our previous discussion, the solution $\mathtt{u}_\tau$ coincides with the minimizer u_{R_τ} for τ sufficiently close to μ^-. In particular, the behavior of $\mathtt{u}_\tau$ as $\tau\to\mu^-$ is known from Lemma 2.1.

Given $\epsilon\in(0,\mu)$ and $D>D'$, where D' as in (1.11), let R' be as in the assertion of Lemma 2.1. Suppose that a solution u of (3.1) satisfies (3.2) for some $\mathtt{R}>R'$ and $P\in\mathbb{R}^n$ (without loss of generality, we may assume that uniqueness for (2.6) in the class (2.2) holds for $R\geq R'$). The family of functions

$$\underline{\mathtt{u}}_\tau(x)=\begin{cases}\mathtt{u}_\tau(x-P), & x\in B_{R_\tau}(P),\\ \\ 0, & \text{elsewhere},\end{cases}$$

are lower solutions of (3.1) for all $\tau\in(0,\mu)$ (as the maximum of two lower solutions, recall that $W'(0)\leq 0$, see [26]). Moreover, we have

$$\underline{\mathtt{u}}_\tau(x)=0<u(x),\quad x\in\partial B_{R_\tau}(P),\quad \tau\in(0,\tau_*],$$

where τ_* is such that

$$R_{\tau_*}=\mathtt{R}.$$

Also, thanks to (3.7) and (3.8), we get

$$\underline{\mathtt{u}}_\tau(x)<u(x),\quad x\in\bar{B}_{\mathtt{R}}(P),\ \text{for}\ \tau\ \text{close to}\ 0.$$

Consequently, by Serrin's sweeping principle (see [57, 62, 144]), we deduce that $\underline{u}_{\tau_*}(x) \leq u(x)$, $x \in \bar{B}_{\mathtt{R}}(P)$, and thus

$$u_{\mathtt{R}}(x-P) = \underline{u}_{\tau_*}(x-P) \leq u(x), \quad x \in \bar{B}_{\mathtt{R}}(P). \tag{3.9}$$

To prove this, we let $\tilde{\tau} = \sup\left\{\tau \in (0, \tau_*] \ : \ u \geq \underline{u}_\tau \text{ on } \bar{B}_{\mathtt{R}}(P)\right\}$, note that $u \geq \mathtt{u}_{\tilde{\tau}}$ on $\bar{B}_{R_{\tilde{\tau}}}(P)$, and apply the strong maximum principle to $u - \mathtt{u}_{\tilde{\tau}}$ to deduce that this function has a positive lower bound on $\bar{B}_{R_{\tilde{\tau}}}(P)$; this implies that the same holds true for the function $u - \underline{u}_{\tilde{\tau}}$ on $\bar{B}_{\mathtt{R}}(P)$ which contradicts the maximality of $\tilde{\tau}$ if $\tilde{\tau} < \tau_*$. Relation (3.9), by virtue of (2.3) (recall that $\mathtt{R} > R'$), clearly implies the validity of (3.3). The validity of relation (3.4) follows at once from (2.4) and (3.9). Finally, relation (3.5) follows immediately from (2.54) and (3.9).

The proof of the proposition is complete. □

Remark 3.1. The assumption (1.8) is essential for our approach. Indeed, if $W'(0) = 0$ and $W''(0) = 0$ then there are no arbitrarily uniformly small positive solutions of (3.6) for any $R > 0$ (thanks to the implicit function theorem, see for example [108]). In fact, for the case $W'(t) = rt^p - t^q$, $t \geq 0$, with $p > q > 1$, $r > 0$, which satisfies **(a')**, (1.9), and (1.15) but not (1.8), the global bifurcation diagram of positive solutions of (3.6) has been shown in [134] to be qualitatively the same as the one corresponding to (1.10) that we described at the end of Remark 2.6, namely $\subset$-shaped. It might also be useful to see the condition on the behavior of W' near the origin for the so-called "hair–trigger effect" to take place in the parabolic equation $u_t = \Delta u - W'(u)$, see [22].

Remark 3.2. We believe that assumption (1.15) can be removed from the assertions of Proposition 3.1 (except possibly from the derivation of (3.5), see Remark 3.5 for this). A possible way to achieve this is to make heavier use of the results of [17], [110]. These together tell us that, in both cases $W'(0) < 0$ or $W'(0) = 0$, the branch $\mathcal{C}_+$ is unbounded. Loosely speaking, in the problem at hand, this implies intuitively that its projection has to cover $(0, \infty)$ or (R_c, ∞), depending on $W'(0)$.

Remark 3.3. In [28], the assumption (1.8) was replaced by the weaker one: W' being Lipschitz continuous and $-W'(t) \geq \delta_0 t$ on $[0, t_0]$ for some $\delta_0, t_0 > 0$. A possible "cure" for this could be the use of bifurcation theory for Hölder continuous nonlinearities (see Appendix B in [135] and the references therein).

The following corollary can be deduced from Proposition 3.1 by making use of the celebrated sliding method.

Corrolarry 3.1. Suppose that $W \in C^2$ satisfies **(a')**, (1.8), (1.9), and (1.15). Let $\epsilon \in (0, \mu)$ and $D > D'$, where D' is given from (1.11). There exists an $R' > D$, depending only on ϵ, D, W, and n, such that any solution u of (3.1) which satisfies (1.3) in a domain $\Omega \neq \mathbb{R}^n$ (open and connected set), containing some closed ball of radius R', satisfies (1.13), (1.14), and (1.16). In the case where $\Omega = \mathbb{R}^n$, the only solutions of (1.22) such that $0 \leq u(x) \leq \mu$, $x \in \mathbb{R}^n$, are the constant ones, namely $u \equiv 0$ and $u \equiv \mu$.

Proof. Suppose that ϵ, D, $\Omega \neq \mathbb{R}^n$, u are as in the first assertion of the corollary. From our assumptions, we know that Ω contains some closed ball $\bar{B}_{\mathtt{R}}(P)$ for some $\mathtt{R} \geq R'$ and $P \in \Omega_{\mathtt{R}}$. Since u satisfies (1.3) and (3.1) in Ω, it follows from the proof of Proposition 3.1 that relation (3.9) holds. As in the first proof of Theorem 1.2, we can use the sliding method to show that the latter relation actually holds for all $P \in \Omega_{\mathtt{R}}$. We point out that here we do not need that the boundary of Ω is continuous, since the radius of the ball is fixed, and we can apply directly Lemma 3.1 in [28]. The validity of the first assertion of (1.13) now follows at once form (2.3) (keep in mind that $\mathtt{R}$ could also be chosen as R'). Now, let $Q \in \Omega_{R'} + B_{(R'-D)}$. From the proof of Proposition 3.1, using Serrin's sweeping principle, we have that $u(x) \geq u_{\mathrm{dist}(Q,\partial\Omega)}(x-Q)$ in $B_{\mathrm{dist}(Q,\partial\Omega)}(Q)$. By means of (2.4) and (3.5), we infer that u also satisfies (1.14) and (1.16) respectively. Consequently, we have established the first assertion of the corollary.

The second assertion of the corollary follows easily. By the strong maximum principle, we deduce that either $u \equiv 0$, $u \equiv \mu$, or $0 < u(x) < \mu$, $x \in \mathbb{R}^n$. We will show that the latter alternative cannot happen. Suppose to the contrary that $0 < u(x) < \mu$, $x \in \mathbb{R}^n$. Then, we get that (3.9) holds for every $\mathtt{R} > 0$ and $P \in \mathbb{R}^n$. By fixing P and letting $\mathtt{R} \to \infty$, making use of (2.12), we obtain that $u \geq \mu$ in $\mathbb{R}^n$; a contradiction.

The proof of the corollary is complete. □

Remark 3.4. If we additionally assume that $W''(\mu) > 0$, then Proposition 3.1 and Corollary 3.1 can be derived from the exponential decay estimates of Lemma 4.2 in [114] (see also [90] and Lemma 2.4 in [97]).

Remark 3.5. Estimate (1.16) represents a slight improvement over estimate (3.3) in [28] (see also relation (4.11) below). We remark that the latter relation was shown in [28] without making use of (1.15).

4. Algebraic singularity decay estimates in the case of pure power nonlinearity, and completion of the proof of Theorem 1.2

The potential that comes first to mind when looking at **(a')** is

$$W(t) = |t-\mu|^{p+1}, \tag{4.1}$$

where $p \geq 1$.

If $p = 1$, the solutions provided by Theorem 1.2 satisfy the exponential decay estimate (1.5). In this section, we will show that a universal algebraic decay estimate holds for *all* solutions of (3.1) with potential as in (4.1), provided that $p > 1$. Although our arguments in this section rely on the specific form of the potential W, our results may be used together with a comparison argument to cover a broader class of potentials. In particular, as we will show in the following subsection, we can establish the remaining relation (1.18) from the proof of Theorem 1.2. Moreover, our main estimate in this section suggests that there is room for improvement over a result of the celebrated paper [28] by Berestycki, Caffarelli and Nirenberg, see Remark 4.3 below. This section is self-contained and can be studied independently of the rest of the paper.

The main result of this section is

Proposition 4.1. Let W be given from (4.1), with $p > 1$, and let $\Omega \neq \mathbb{R}^n$ be a domain of $\mathbb{R}^n$. There exists a positive constant C, depending only on p, n, such that *any* solution u of (3.1) in Ω satisfies

$$|\mu - u| + |\nabla u|^{\frac{2}{p+1}} \leq C\mathrm{dist}^{-\frac{2}{p-1}}(x,\partial\Omega), \quad x \in \Omega. \tag{4.2}$$

In particular, if Ω is an exterior domain, i.e., $\Omega \supset \{x \in \mathbb{R}^n \ : \ |x| > R\}$ for some $R > 0$, then

$$|\mu - u| + |\nabla u|^{\frac{2}{p+1}} \leq C|x|^{-\frac{2}{p-1}}, \quad |x| \geq 2R. \tag{4.3}$$

Proof. Our proof is modeled after that of Theorem 2.3 in [138] which dealt with focusing nonlinearities, making use of scaling (blow-up) arguments, inspired from [92], combined with a key "doubling" estimate. The main difference with [138] is in the Liouville type theorem that we will use to conclude, see Remark 4.2 below.

We will argue by contradiction. Suppose that estimate (4.2) fails. Then, there exist sequences of domains $\Omega_k \neq \mathbb{R}^n$, functions u_k, and points $y_k \in \Omega_k$, such that u_k solves (3.1) in Ω_k and the functions

$$M_k \equiv |\mu - u_k|^{\frac{p-1}{2}} + |\nabla u_k|^{\frac{p-1}{p+1}}, \quad k = 1, 2, \cdots ,$$

satisfy

$$M_k(y_k) > 2k\mathrm{dist}^{-1}(y_k, \partial\Omega_k), \quad k = 1, 2, \cdots .$$

From the *Doubling Lemma* of Polacik, Quittner and Souplet, see Lemma 5.1 and Remark 5.2 (b) in [138] or Lemma C.1 in the appendix below, it follows that there exist $x_k \in \Omega_k$ such that

$$M_k(x_k) \geq M_k(y_k), \quad M_k(x_k) > 2k\mathrm{dist}^{-1}(x_k, \partial\Omega_k), \tag{4.4}$$

and

$$M_k(z) \leq 2M_k(x_k), \quad |z - x_k| \leq kM_k^{-1}(x_k), \quad k = 1, 2, \cdots . \tag{4.5}$$

Note that $B_{kM_k^{-1}(x_k)}(x_k) \subset \Omega_k$. Now, we rescale u_k by setting

$$v_k(y) \equiv \lambda_k^{\frac{2}{p-1}} \left[\mu - u_k(x_k + \lambda_k y)\right], \quad |y| \leq k, \quad \text{with } \lambda_k = M_k^{-1}(x_k). \tag{4.6}$$

The function v_k solves

$$\Delta v_k(y) = (p+1)v_k(y)\,|v_k(y)|^{p-1}\,, \quad |y| \leq k.$$

Moreover,

$$\left[|v_k|^{\frac{p-1}{2}} + |\nabla v_k|^{\frac{p-1}{p+1}}\right](0) = \lambda_k M_k(x_k) = 1, \tag{4.7}$$

and

$$\left[|v_k|^{\frac{p-1}{2}} + |\nabla v_k|^{\frac{p-1}{p+1}}\right](y) \leq 2, \quad |y| \leq k.$$

By using elliptic L^q estimates and standard imbeddings (see [93]), we deduce that some subsequence of v_k converges in $C^1_{loc}(\mathbb{R}^n)$ to a (classical) solution $\mathbf{V}$ of

$$\Delta v = (p+1)v(y)\,|v(y)|^{p-1}\,, \quad y \in \mathbb{R}^n. \tag{4.8}$$

Moreover, thanks to (4.7), we have

$$\left[|\mathbf{V}|^{\frac{p-1}{2}} + |\nabla \mathbf{V}|^{\frac{p-1}{p+1}}\right](0) = 1,$$

so that $\mathbf{V}$ is nontrivial. On the other hand, by a result of Brezis [40], we know that there does not exist a nontrivial solution of (4.8) in $L^p_{loc}(\mathbb{R}^n)$ (in the sense of distributions), see also Theorems 4.6–4.7 in [83] or Theorem B.1 below. Consequently, we have arrived at the desired contradiction.

The proof of the proposition is complete. □

Remark 4.1. The powers $2/(p-1)$ and $(p+1)/(p-1)$ in (4.2) (for u and $|\nabla u|$ respectively) are sharp for $p \in \left(1, \frac{n}{n-2}\right)$ if $n \geq 3$ and $p > 1$ if $n = 2$, as can be seen from the explicit solution $u(x) = c(p,n)|x|^{-\frac{2}{p-1}}$ of

$$\Delta u = u^p, \tag{4.9}$$

in $\mathbb{R}^n \backslash \{0\}$, see for example [41].

In the latter reference, it was shown that every nonnegative solution $u \in C^2(B_R)$ of (4.9), with $p > 1$, satisfies

$$u(0) \le C(p,n)R^{-\frac{2}{p-1}},$$

where $C(p,n)$ is determined explicitly. This result, minus the explicit dependence of the constat on p, n, follows as a special case of our Proposition 4.1 if we choose $\mu = 0$. Moreover, it was shown in the same reference that every nonnegative solution $u \in C^2(B_R \backslash \{0\})$ of (4.9), with $p \in \left(1, \frac{n}{n-2}\right)$ if $n \ge 3$ and $p > 1$ if $n = 2$, satisfies

$$u(x) \le l(p,n)|x|^{-\frac{2}{p-1}} \left[1 + \frac{C(p,n)}{l(p,n)} \left(\frac{|x|}{R} \right)^{\beta} \right], \quad 0 < |x| \le \frac{R}{2},$$

where $\beta = \frac{4}{p-1} + 2 - n > \frac{2}{p-1}$, and $C(p,n), l(p,n)$ some explicitly determined constants. The validity of this estimate, minus the explicit dependence of the constat on p, n, follows for *all* the range $p > 1$ from our Proposition 4.1 with $\mu = 0$. It was shown in [39] that, if $n \ge 3$ and $p \ge \frac{n}{n-2}$, there exists a constant $A = A(p,n) > 0$ such that every nonnegative solution $u \in C^2(B_1 \backslash \{0\})$ of (4.9) satisfies

$$u(x) \le \frac{A}{|x|^{n-2}}, \quad 0 < |x| < \frac{1}{2}.$$

In turn, the latter estimate was used to show that the solution u has a removable singularity at the origin. Clearly, the above estimate follows from (4.2) with $\mu = 0$. The proofs in [39], [42], and [40] (where we referred to towards the end of the proof of Proposition 4.1), are based on the explicit knowledge of positive, radially symmetric upper solutions of the equation $-\Delta u + |u|^{p-1}u = 0$ on arbitrary open balls, with the further property that these functions blow up at the boundary of the considered balls. This fact is crucially related to the shape of the nonlinear function $|t|^{p-1}t$ and it does not easily extend to more general functions. We refer to [83] for a different approach for establishing the Liouville type theorem of [40], that we used towards the end of the proof of Proposition 4.1, with the advantage to apply to a larger class of problems (see Theorem B.1 below).

To the best of our knowledge, this is the first time that the doubling lemma of [138] has been used in relation with the previously mentioned papers.

Remark 4.2. In the problems studied in [92], [138] (see also [141]), the blowing-up argument leads to a positive solution of the whole space problem $\Delta v + v^p = 0$, which does not exist for the range of exponents $p \in \left(1, \frac{n+2}{n-2}\right)$ if $n \ge 3$, $p > 1$ if $n = 2$.

Remark 4.3. Assume that the potential $W \in C^2$ satisfies (1.9), $W'(t) \ge 0$ for $t \ge \mu$, $-W'(t) \ge \delta_0 t$ on $[0, t_0]$ for some $\delta_0, t_0 > 0$, and (1.15). Clearly, these conditions are satisfied by the model examples (1.23) and (4.1). Let Ω be the entire *epigraph*:

$$\Omega = \{ x \in \mathbb{R}^n \; : \; x_n > \varphi(x_1, \cdots, x_{n-1}) \}, \tag{4.10}$$

where $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a globally Lipschitz continuous function. It was shown in Lemma 3.2 in [28] that there are constants $\varepsilon_1, R_0 > 0$ with R_0, depending only on n, δ_0, such that any positive bounded solution of (1.2) satisfies $u < \mu$ in Ω, and

$$u(x) > \varepsilon_1 \;\; \text{if} \;\; x \in \Omega_{R_0}, \; \text{i.e.} \; \text{dist}(x, \partial\Omega) > R_0.$$

Moreover, setting

$$\delta(x) = \min \{ -W'(t) \; : \; t \in [\varepsilon_1, u(x)] \}, \quad x \in \Omega_{R_0},$$

there exists a constant C_1, depending only on n, such that

$$C_1\delta(x) \le \left[\mathrm{dist}(x,\partial\Omega) - R_0\right]^{-2}, \quad x \in \Omega_{R_0}, \tag{4.11}$$

recall also estimate (1.16) herein. In the case of the potential (4.1), the function $\delta(x)$ is plainly $\delta(x) = (p+1)\,(\mu - u(x))^p$, and estimate (4.11) says that

$$\mu - u(x) \le \left[C_1(p+1)\right]^{-\frac{1}{p}} \left[\mathrm{dist}(x,\partial\Omega) - R_0\right]^{-\frac{2}{p}}, \quad x \in \Omega_{R_0}.$$

Observe that our estimate (4.2) is an improvement of the above estimate, since $\frac{2}{p-1} > \frac{2}{p}$. Moreover, our estimate holds for *all* solutions, possibly sign changing or unbounded, without specified boundary conditions. Note also that these observations reveal that estimate (1.14) is far from optimal.

As in Theorem 2.1 in [138], we can generalize our Proposition 4.1 to

Proposition 4.2. Let $p > 1$, and assume that the smooth W satisfies

$$\lim_{|t|\to\infty} t^{-1}|t|^{1-p}W'(t+\mu) = \ell \in (0,\infty). \tag{4.12}$$

Let Ω be an arbitrary domain of $\mathbb{R}^n$. Then, there exists a constant $C = C(n,f) > 0$ (independent of Ω and u) such that, for any solution of (3.1), there holds

$$|\mu - u| + |\nabla u|^{\frac{2}{p+1}} \le C\left(1 + \mathrm{dist}^{-\frac{2}{p-1}}(x,\partial\Omega)\right), \quad x \in \Omega. \tag{4.13}$$

In particular, if $\Omega = B_R\backslash\{0\}$ then

$$|\mu - u| + |\nabla u|^{\frac{2}{p+1}} \le C\left(1 + |x|^{-\frac{2}{p-1}}\right), \quad 0 < |x| \le \frac{R}{2}.$$

Proof. Assume that estimate (4.13) fails. Keeping the same notation as in the proof of Proposition 4.1, we have sequences $\Omega_k,\ u_k,\ y_k \in \Omega_k$ such that u_k solves (3.1) in Ω_k and

$$M_k(y_k) > 2k\left(1 + \mathrm{dist}^{-1}(y_k,\partial\Omega_k)\right) > 2k\mathrm{dist}^{-1}(y_k,\partial\Omega_k).$$

Then, formulae (4.4)–(4.6) are unchanged but now the function v_k solves

$$\Delta_y v_k(y) = f_k\left(v_k(y)\right) \equiv -\lambda_k^{\frac{2p}{p-1}} W'\left(\mu - \lambda_k^{-\frac{2}{p-1}} v_k(y)\right), \quad |y| \le k.$$

Note that, since $M_k(x_k) \ge M_k(y_k) > 2k$, we also have

$$\lambda_k \to 0, \quad k \to \infty.$$

Since there exists a constant $C > 0$ such that $|W'(\mu - t)| \le C(1 + |t|^p)$, $t \in \mathbb{R}$, due to (4.12) (and W' being continuous), it follows that

$$|f_k(t)| \le C\lambda_k^{\frac{2p}{p-1}} + C|t|^p, \quad t \in \mathbb{R},\ k \ge 1.$$

By using elliptic L^q estimates, standard imbeddings, and (4.12), we deduce that some subsequence of v_k converges in $C^1_{loc}(\mathbb{R}^n)$ to a classical solution $\mathbf{V}$ of $\Delta v = \ell v|v|^{p-1}$ in $\mathbb{R}^n$. Moreover, we have that $|\mathbf{V}(0)|^{\frac{p-1}{2}} + |\nabla\mathbf{V}(0)|^{\frac{p-1}{p+1}} = 1$, so that $\mathbf{V}$ is nontrivial. As in Proposition 4.1, since $\ell > 0$, this contradicts the Liouville theorem in [43], [83], in particular Theorem B.1 below.

The proof of the proposition is complete. □

Remark 4.4. The same assertion of Proposition 4.2 holds, if we assume that the righthand side of (4.12) is as the function f in Theorem B.1 below.

4.1. **Proof of relation (1.18).** Based on Proposition 4.1, via a comparison argument, we can show relation (1.18) and thus complete the proof of Theorem 1.2.

`Proof of (1.18):` Let $0 < v < \mu$ be the solution of

$$\Delta v = -c(\mu - v)^p \ \text{ in } \Omega; \quad v = 0 \ \text{ on } \partial\Omega,$$

as provided by Theorem 1.2 (keep in mind the second part of Remark 1.3 which implies uniqueness), where $c > 0,\ p > 1$ are as in (1.17). From Proposition 4.1, we know that v satisfies

$$\mu - v \leq \tilde{K}\text{dist}^{-\frac{2}{p-1}}(x, \partial\Omega), \quad x \in \Omega,$$

for some constant $\tilde{K} > 0$ that depends only on c, p, and n.

On the other side, making use of (1.17), we find that the solutions of (1.2), provided by Theorem 1.2, satisfy

$$\Delta u \leq -c(\mu - u)^p \ \text{ in } \ \Omega.$$

We intend to show that

$$v \leq u \ \text{ in } \Omega, \tag{4.14}$$

from where relation (1.18) follows at once. Let $w = u - v$. We have

$$\Delta w \leq q(x)w \ \text{ in } \Omega,$$

where

$$q(x) = cp\int_0^1 (\mu - su - (1-s)v)^{p-1}\, ds.$$

This is a bit meshy but what matters is that q is continuous and nonnegative in Ω. Note that $w = 0$ on $\partial\Omega$ and w is bounded in Ω ($|w| \leq \mu$ to be more precise). Therefore, the assumption that $\bar{\Omega}$ is disjoint from the closure of an infinite open connected cone allows us to apply the maximum principle, even in the case where Ω is unbounded, to deduce that (4.14) holds (see Lemma 2.1 in [28]).

The proof of relation (1.18) is complete. □

Remark 4.5. We believe that an analog of estimate (1.18) should hold true in the case where relation (1.17) is assumed just for $\mu - t > 0$ small.

5. Bounds on entire solutions of $\Delta u = W'(u)$

In this subsection, we will assume that the C^2 potential W satisfies (4.12) for some $\mu \in \mathbb{R}$, and there exist $\mu_- < \mu_+$ such that

$$W'(\mu_-) = W'(\mu_+) = 0, \quad W'(t) < 0,\ t < \mu_-;\ W'(t) > 0,\ t > \mu_+. \tag{5.1}$$

We will utilize Propositions 3.1 and 4.2, together with the corresponding parabolic flow to (1.22), in order obtain the following result:

Proposition 5.1. Under the above assumptions, we have that *every* solution $u \in C^2(\mathbb{R}^n)$ of (1.22), which is not identically equal to μ_- or μ_+, satisfies

$$\mu_- < u(x) < \mu_+, \quad x \in \mathbb{R}^n. \tag{5.2}$$

Proof. From Proposition 4.2 with $\Omega = \mathbb{R}^n$, i.e. $\text{dist}^{-\frac{2}{p-1}}(x, \partial\Omega) = 0 \ \forall\, x \in \mathbb{R}^n$, we know that there exists a constant $C = C(W', n) > 0$ such that every solution of (1.22) satisfies

$$|u(x)| \leq C, \quad x \in \mathbb{R}^n.$$

We will show that

$$\mu_- \leq u(x) \leq \mu_+, \quad x \in \mathbb{R}^n. \tag{5.3}$$

Indeed, as in [20], [77], by the parabolic maximum principle, we infer that

$$u_-(t) \leq u(x) \leq u_+(t), \quad t \geq 0, \tag{5.4}$$

where $u_\pm$ are the solutions of the initial value problems

$$\dot{u}_\pm = W'(u_\pm), \quad t > 0, \quad u_\pm(0) = \pm 2C.$$

Note that $u_\pm(t)$ are solutions of $u_t = \Delta u - W'(u)$ on $\mathbb{R}^n \times (0, \infty)$, as is $u(x)$. From our assumptions on W, it follows that $u_-(t)$ is increasing and $u_+(t)$ is decreasing with respect to $t > 0$. Furthermore, it is easy to show that $u_\pm(t) \to \mu_\pm$ as $t \to \infty$, see also [21], [164]. Hence, letting $t \to \infty$ in (5.4), we find that relation (5.3) holds. By the strong maximum principle, it follows that (5.2) holds unless $u \equiv \mu_-$ or $u \equiv \mu_+$.

The proof of the proposition is complete. □

Remark 5.1. With trivial modifications, Proposition 5.1 can be applied in the case where there is an obstacle in $\mathbb{R}^n$, as in problem (8.1) below (see also Remark 8.2).

As a corollary to the above proposition, we can give a short proof of a Liouville type result in [76] (see Theorem 1.1 therein), where a squeezing argument involving boundary blow-up solutions (recall the discussion related to [40] at the end of Remark 4.1) was used instead (see also [77], [78]).

Corrolarry 5.1. Let $\lambda \in (-\infty, \infty)$, $p > 1$, and $u \in C^2(\mathbb{R}^n)$ be a nonnegative solution of

$$\Delta u = u^p - \lambda u \ \text{ in } \mathbb{R}^n.$$

Then, the solution u must be a constant.

Proof. If $\lambda \leq 0$, we have that $-\Delta u + u^p \leq 0$. Since $p > 1$, it follows from Keller-Osserman theory [107, 133] that $u \leq 0$ on $\mathbb{R}^n$ (see Theorem B.1 below). Hence, in this case, the solution u is identically zero.

If $\lambda > 0$, it follows readily from Proposition 5.1 that either $u \equiv 0$ or $u \equiv \lambda^{\frac{1}{p}}$ or $0 < u(x) < \lambda^{\frac{1}{p}}$, $x \in \mathbb{R}^n$. However, the latter alternative cannot occur, because of the second assertion of Corollary 3.1.

The proof of the proposition is complete. □

Remark 5.2. Our method of proof, as well as that of [76, 77, 78], work for a broader class of nonlinearities. In the special case of the Allen-Cahn equation

$$\Delta u = u^3 - u \ \text{ in } \mathbb{R}^n, \tag{5.5}$$

by making use of Kato's inequality and Keller-Osserman theory, it was shown in [44] (see also [81], [124]) that all solutions of this equation satisfy $|u(x)| \leq 1$, $x \in \mathbb{R}^n$ (for a different proof, see Lemma 4.1 in [54]). A parabolic version of this result can be found in [124].

The importance of the above results is that they imply that there is no need for the boundedness assumption is the well known statement of the famous De Giorgi's conjecture: *Let u be a bounded solution of equation (5.5) such that $u_{x_n} > 0$. Then the level sets $\{u = \lambda\}$*

are all hyperplanes, ***at least*** *for dimension $n \leq 8$.* There has been tremendous activity in the last years, and this conjecture has been completely resolved in dimensions $n \leq 3$ (see [90], [16]), and typically in dimensions $4 \leq n \leq 8$ (assuming that $u \to \pm 1$ pointwise as $x_n \to \pm\infty$, see [145]), while a counterexample which shows that the conjecture is false for $n \geq 9$ has been constructed in [72].

6. Uniform estimates in the case where W is convex

In this section, we will assume that the C^1 function

$$f(t) \equiv W'(t+\mu), \quad t \in \mathbb{R}, \tag{6.1}$$

satisfies

$$f(0) = 0, \quad tf(t) > 0 \text{ for } t \neq 0, \quad f \text{ is nondecreasing in } \mathbb{R}. \tag{6.2}$$

Observe that these conditions are clearly satisfied by the model potential (4.1). Moreover, our Theorem 1.2 and Proposition 3.1 are applicable when dealing with positive solutions of (1.2) and (3.1) respectively. Here, we will derive uniform estimates for, possibly sign-changing, uniformly bounded solutions of (3.1) in arbitrary domains. As in Section 4, we will use blow–up type arguments, relying on the fact that the corresponding "limit" whole space problem (1.22) has only the trivial bounded solution. We remark that this section is self-contained and can be studied independently of the rest of the paper.

Proposition 6.1. Assume that W satisfies (6.1) and (6.2). Given $\epsilon \in (0, \mu)$, $M > 0$, any solution u of (3.1) in any domain Ω such that $|u(x)| \leq M$, $x \in \bar{\Omega}$, satisfies

$$|\mu - u(x)| \leq \epsilon, \quad x \in \bar{\Omega}_{R_0},$$

where R_0 depends only on ϵ, M, W, n (keep in mind that the set Ω_{R_0}, defined in (1.6), might be empty).

Proof. We will argue by contradiction. Let $\epsilon > 0$ and $M > 0$. Assume that the assertion of the proposition fails for some $\epsilon \in (0, \mu)$. Then, there exist sequences of domains Ω_i, $u_i \in C^2(\Omega_i)$, $x_i \in \Omega_i$, $i \geq 1$, such that u_i solves (3.1) in Ω_i,

$$|u_i(x)| \leq M, \ x \in \bar{\Omega}_i, \ \ \text{dist}(x_i, \partial\Omega_i) \to \infty, \text{ as } i \to \infty, \ \ \text{while } |u_i(x_i) - \mu| \geq \epsilon. \tag{6.3}$$

Let

$$v_i(x) = u_i(x + x_i) - \mu, \quad x \in \Omega_i - x_i.$$

We have that

$$\Delta v_i = f(v_i) \ \text{ in } \Omega_i - x_i, \text{ and } |v_i(0)| \geq \epsilon, \ i \geq 1,$$

where f is as in (6.1). Since the solutions v_i are uniformly bounded on $\bar{\Omega}_i$ with respect to i, using elliptic estimates and the standard diagonal and compactness arguments, we can extract a subsequence which converges uniformly over compacts to a bounded function $\mathbf{v}$ which satisfies classically

$$\Delta \mathbf{v} = f(\mathbf{v}) \ \text{ in } \mathbb{R}^n. \tag{6.4}$$

(We have also used the second relation in (7.3) which implies that, for any $R > 0$, the domain $\Omega_i - x_i$ contains the ball B_R if $i \geq i(R)$ is sufficiently large). Moreover, we have that $|\mathbf{v}(0)| \geq \epsilon > 0$, which implies that $\mathbf{v}$ is nontrivial.

On the other hand, the following Liouville type theorem, due to J. Serrin [149], holds: Problem (6.4) does not have non-constant solutions $v \in C^2(\mathbb{R}^n)$ that satisfy $v = o(|x|^2)$, as $|x| \to \infty$, if $f : \mathbb{R} \to \mathbb{R}$ is continuous, nondecreasing, and nontrivial (see also Proposition 4.1

in [83]; a parabolic version can be found in [86]). Since f satisfies (6.2), this Liouville type theorem implies that the whole space problem (6.4) has only the trivial solution. We have thus arrived at a contradiction.

The proof of the proposition is complete. □

7. Uniform estimates in the case where W is monotone

In this section, we will assume that the C^1 function

$$f(t) \equiv W'(t+\mu), \quad t \in \mathbb{R}, \tag{7.1}$$

satisfies

$$f(0)=0, \quad f(t)<0 \text{ for } t \in [\mu_- - \mu, 0), \tag{7.2}$$

for some $\mu_- < \mu$.

We will derive uniform estimates for, possibly sign-changing, solutions $\mu_- \leq u \leq \mu$ of (3.1) in arbitrary domains of $\mathbb{R}^2$ or $\mathbb{R}^3$ (the same results naturally hold for $n=1$). As in Sections 4, 6, we will use blow–up type arguments, exploiting the fact that the corresponding "limit" whole space problem has only the trivial solution in an appropriate class. Again, this section is self-contained and can be studied independently of the rest of the paper.

Proposition 7.1. Assume that $W \in C^2$ satisfies (7.1) and (7.2). Given $\epsilon \in (\mu_-, \mu)$, any solution u of (3.1) in any domain $\Omega \subseteq \mathbb{R}^2$ such that $\mu_- \leq u(x) \leq \mu$, $x \in \bar{\Omega}$, satisfies

$$\mu - u(x) \leq \epsilon, \quad x \in \bar{\Omega}_{R_0},$$

where R_0 depends only on ϵ, W (keep in mind that the set Ω_{R_0}, defined in (1.6), might be empty).

Proof. Similarly to Proposition 6.1, we will argue by contradiction. Assume that the assertion of the proposition fails for some $\epsilon \in (\mu_-, \mu)$. Then, there exist sequences of domains $\Omega_i \subseteq \mathbb{R}^2$, $u_i \in C^2(\Omega_i)$, $x_i \in \Omega_i$, $i \geq 1$, such that u_i solves (3.1) in Ω_i,

$$\mu_- \leq u_i(x) \leq \mu, \ x \in \bar{\Omega}_i, \quad \text{dist}(x_i, \partial\Omega_i) \to \infty, \ \text{as } i \to \infty, \quad \text{while } \mu - u_i(x_i) \geq \epsilon. \tag{7.3}$$

Let $v_i(x) = u_i(x+x_i) - \mu, \quad x \in \Omega_i - x_i$. We have that

$$\Delta v_i = f(v_i) \ \text{ in } \Omega_i - x_i, \ \text{and } \ -v_i(0) \geq \epsilon, \ i \geq 1,$$

where f is as in (7.1). As in Proposition 6.1, letting $i \to \infty$ along a subsequence, we arrive at a nontrivial solution $\mu_- - \mu \leq \mathbf{v} \leq 0$ of (6.4) with $n=2$. Furthermore, assumption (7.2) implies that $\Delta \mathbf{v} \leq 0$ on $\mathbb{R}^2$. In other words, the function $\mathbf{v}$ is super-harmonic on $\mathbb{R}^2$. On the other hand, a well known Liouville theorem (see Theorem 3.1 in [83]), a consequence of Hadamard's three–circles theorem for sub-harmonic functions (see [140]), tells us that the only super-harmonic functions in $\mathbb{R}^2$ that are bounded from below are the constant ones. In our case, this implies that $\mathbf{v} \equiv 0$ (the inequality in (7.2) is strict), a contradiction.

The proof of the proposition is complete. □

Adapting an argument from [3], [67], we can see that the same result carries over to the three–dimensional case, provided that we impose a natural nondegeneracy condition for W.

Proposition 7.2. Under the additional assumption that $W''(\mu) > 0$ (in other words **(c)**), the same assertion of Proposition 7.1 holds true also if $\Omega \subseteq \mathbb{R}^3$.

Proof. Arguing by contradiction, exactly as in the proof of Proposition 7.1, we again arrive at a nontrivial solution $\mu_- - \mu \leq \mathbf{v} \leq 0$ of (6.4) with $n = 3$. As before, assumption (7.2) yields that $\Delta \mathbf{v} \leq 0$ on $\mathbb{R}^3$. We intend to show that there exists a constant $C > 0$ such that

$$\int_{B_R} \mathbf{v}^2 dx \leq CR^2 \quad \text{for every } R \geq 1. \tag{7.4}$$

Once this is shown, the desired assertion that $\mathbf{v}$ is a constant will follow at once from a well known Liouville theorem of [29] (see the formulation in Proposition 2.1 in [16], and [67], [83], [90], [136]), note also that

$$\mathbf{v}\Delta\mathbf{v} \geq 0 \quad \text{on } \mathbb{R}^3. \tag{7.5}$$

We remark that the latter Liouville theorem plays an essential role in the proof of De Giorgi's conjecture in low dimensions (recall our discussion in Remark 5.2).

With this in mind, let us proceed in validating estimate (7.4). By standard interior elliptic regularity estimates [93] (see also Lemma A.1 in [35]), applied on balls of radius one, we deduce that $|\nabla \mathbf{v}|$ is uniformly bounded on $\mathbb{R}^3$. Hence, by Green's theorem, we obtain that

$$-\int_{B_R} f(\mathbf{v}) dx = -\int_{\partial B_R} \frac{\partial \mathbf{v}}{\partial \nu} dS \leq C_2 R^2,$$

since the surface area of ∂B_R is $C_1 R^2$ (recall that $n = 3$); here we have denoted by ν the outer unit normal to ∂B_R, and the constants $C_i > 0$, here and in the sequel, are independent of R. Since $-f(\mathbf{v})$ is nonnegative and $-\mathbf{v}$ is bounded above by $\mu - \mu_-$, it follows that

$$\int_{B_R} \mathbf{v} f(\mathbf{v}) dx \leq (\mu - \mu_-) \int_{B_R} -f(\mathbf{v}) dx \leq C_3 R^2.$$

Our assumptions on W imply that $sf(s) \geq C_4 s^2$, $\mu_- - \mu \leq s \leq 0$. The validity of (7.4) follows at once.

The proof of the proposition is complete. ☐

8. A Liouville theorem arising in the study of traveling waves around an obstacle

In Theorem 6.1 of their article [32], H. Berestycki, F. Hamel, and H. Matano proved the following Liouville type theorem:

Theorem 8.1. Let Ω be a smooth, open, connected subset of $\mathbb{R}^n$, $n \geq 2$, with outward unit normal ν, and assume that $K = \mathbb{R}^n \backslash \Omega$ is compact. Let $0 \leq u \leq \mu$ be a classical solution of

$$\begin{cases} \Delta u = W'(u) & \text{in } \Omega, \\ \nu \nabla u = 0 & \text{on } \partial\Omega, \\ u(x) \to \mu & \text{as } |x| \to \infty, \end{cases} \tag{8.1}$$

where $W \in C^2$ satisfies conditions **(a')** and (1.15). If K is star-shaped, then

$$u \equiv \mu \quad \text{on } \bar{\Omega}. \tag{8.2}$$

In fact, the statement in [32] also requires that $W'(0) = 0$. In our statement, we assume that $W \in C^2$ but, as the reader can easily verify from the proofs throughout this paper, this is just for convenience and W' being Lipschitz is more than enough in most occasions. Below, we will provide an alternative proof of the above theorem. Loosely speaking, the approach of [32] consists in using a sweeping family of lower solutions of (8.1), built from the solution $\mathbf{U}$ of (1.12). Our proof is in the same spirit, but we build lower solutions out of one dimensional solutions of (2.6), capitalizing on the results of Subsection 2.1. In our opinion, our proof is simpler (having knowledge of Lemma 2.1 and Proposition 2.1) and more intuitive. In particular, our proof of Theorem 8.2 below is considerably simpler than the corresponding one of [32].

`Proof of Theorem 8.1:` Up to a shift of the origin, one can assume without loss of generality that K, if not empty, is star-shaped with respect to 0. By the strong maximum principle and Hopf's boundary point lemma, we deduce that $u > 0$ on $\bar{\Omega}$. From Proposition 2.1, there exists an $R_0 > 0$ such that the minimizing solutions of (2.6) with $n = 1$, provided by Lemma 2.1, are non-degenerate for $R \geq R_0$. Thus, via the implicit function theorem (see [108]), we can find a continuous family of such minimizing solutions u_R (for $R \geq R_0$, with respect to the uniform topology, as described in Corollary 2.2 in [105]). Increasing the value of R_0, if necessary, we may assume that

$$W'\left(u_R(0)\right) \leq 0, \quad R \geq R_0, \tag{8.3}$$

recall **(a')**, (1.15), (2.2), (2.3), and (2.9). By virtue of (2.2) and (8.1), there exists a large $T > R_0$ such that

$$u(x) > u_{R_0}(0) = \max_{\bar{B}_{R_0}} u_{R_0}, \quad x \in \mathbb{R}^n \backslash B_{(T-R_0)}. \tag{8.4}$$

Let

$$\underline{u}_R(r,\theta) = \begin{cases} 0, & r \in [0, \max\{T-R, 0\}]\,, \ \theta \in \mathbb{S}^{n-1}, \\ u_R(r-T), & r \in (\max\{T-R,0\}, T)\,, \ \theta \in \mathbb{S}^{n-1}, \\ u_R(0), & r \geq T, \ \theta \in \mathbb{S}^{n-1}, \end{cases}$$

with the obvious notation. Since $u_R'(0) = 0$, it follows that $\underline{u}_R \in C^1(\bar{\Omega})$. Using (2.6), we find that

$$\Delta \underline{u}_R - W'(\underline{u}_R) = \begin{cases} -W'(0), & r \in [0, \max\{T-R, 0\}]\,, \ \theta \in \mathbb{S}^{n-1}, \\ \frac{n-1}{r} u_R'(r-T) & r \in (\max\{T-R,0\}, T)\,, \ \theta \in \mathbb{S}^{n-1}, \\ -W'\left(u_R(0)\right) & r > T, \ \theta \in \mathbb{S}^{n-1}. \end{cases} \tag{8.5}$$

In particular, recalling that $W'(0) \leq 0$, (2.9) and (8.3), we find that

$$\underline{u}_R \ \text{ is a weak lower solution of (3.1) in } \Omega, \ \text{ if } R \geq R_0. \tag{8.6}$$

We claim that

$$\underline{u}_R \leq u \ \text{ on } \ \bar{\Omega}, \ \text{ for all } R \geq R_0. \tag{8.7}$$

Suppose that the claim is false, and let $R_* > R_0$ be such that $\underline{u}_R < u$ on $\bar{\Omega}$ for $R_0 < R < R_*$ (thanks to (8.4), this is certainly true if $R = R_0$), and there exists an $x_* \in \bar{\Omega}$ with $\underline{u}_{R_*}(x_*) = u(x_*)$. In view of (8.1) and (8.6), we find that

$$\Delta(u - \underline{u}_{R_*}) \leq Q(x)(u - \underline{u}_{R_*}) \ \text{ weakly in } \Omega, \tag{8.8}$$

where Q as in (2.24) with u in place of $\hat{u}$. Since $u \geq u_{R_*}$ on $\bar{\Omega}$, the weak Harnack inequality (see [93]) tells us that the point x_* must lie on the boundary of Ω (otherwise, $\underline{u}_{R_*} \equiv u$ which is not possible by (8.5)). If $x_* \in \partial\Omega = \partial K$, then by (8.8) and Hopf's boundary point lemma, we get that

$$0 > \nu\nabla(u - \underline{u}_{R_*}) = -\nu\nabla\underline{u}_{R_*} \quad \text{at } x_*. \tag{8.9}$$

On the other hand, since K is star-shaped with respect to the origin, we have that

$$x.\nu_x \leq 0, \quad x \in \partial K.$$

Also, relation (2.9) implies that

$$x\nabla\underline{u}_{R_*} > 0, \quad x \in B_T\backslash\{0\}.$$

The above two relations yield that $\nu\nabla\underline{u}_{R_*} \leq 0$ at x_*, which contradicts (8.9). Consequently, claim (8.7) holds.

Letting $R \to \infty$ in (8.7), thanks to (2.3), we arrive at (8.2).

The proof of the theorem is complete. □

Remark 8.1. In dimension $n = 1$, with K an open bounded interval, the same arguments can be adapted straightforwardly, and the conclusion of Theorem 8.1 holds.

Theorem 8.2. If in Theorem 8.1, we assume that $N \geq 1$ and that the obstacle K to be directionally convex instead of star–shaped, then conclusion (8.2) still holds.

Proof. Without loss of generality, we may assume that K is convex in the x_1 direction, which implies that

$$(x_1, \cdots, 0)\nu_x \leq 0 \quad \forall\, x = (x_1, \cdots, x_n) \in \partial K, \tag{8.10}$$

where ν denotes again the unit outer normal to $\partial\Omega$ (i.e. inner to ∂K). The proof proceeds along the same lines as that of Theorem 8.1. Let u_R denote the solution of (2.6) with $n = 1$, provided by Lemma 2.1. For $R,\ T > 0$, let

$$\underline{u}_R(x) = \begin{cases} u_R(x_1 - T), & x_1 \in (\max\{T - R, 0\}, T + R)\,, \\ u_R(x_1 + T), & x_1 \in (-T - R, \min\{-T + R, 0\})\,, \\ 0, & \text{otherwise}. \end{cases}$$

From (2.6), we have that $\underline{u}_R$ is a weak lower solution of (3.1) in Ω (see [26]). As before, there exist large $R_0,\ T > R_0$ such that $\underline{u}_{R_0} < u$ on $\bar{\Omega}$ and the minimizers u_R vary smoothly with respect to $R \geq R_0$.

We claim that

$$\underline{u}_R \leq u \quad \text{on } \bar{\Omega} \quad \text{for all} \quad R \geq R_0. \tag{8.11}$$

Arguing by contradiction, as in the proof of Theorem 8.1, we get the existence of analogous $R_* > R_0$ and $x_* \in \bar{\Omega}$. To reach a contradiction, it boils down to exclude the case $x_* \in \partial K$. In that case, Hopf's boundary point lemma tells us that (8.9) holds. On the other hand, recalling (2.9), at the point x_* we have that

$$(x_1, \cdots, 0)\nabla\underline{u}_{R_*} = x_1\partial_{x_1}\underline{u}_{R_*} = \begin{cases} x_1 u'_{R_*}(x_1 - T) & \text{if } \; 0 < x_1 < T - R_*, \\ x_1 u'_{R_*}(x_1 + T) & \text{if } \; -T + R_* < x_1 \leq 0. \end{cases}$$

Hence, relation (2.9) yields that $(x_1, \cdots, 0)\nabla \underline{u}_{R_*} \geq 0$ at x_*. However, from (8.9) and the latter relation, we get that $\nu\nabla u_{R_*} \leq 0$ at x_*; a contradiction. So, we have shown that claim (2.19) holds.

Letting $R \to \infty$ in (8.11), as before, we arrive at (8.2).

The proof of the theorem is complete. □

Remark 8.2. If in addition W satisfies a relation of the form (4.12), and (5.1) with $\mu_- = 0$, $\mu_+ = \mu$, then there is no need to assume that $0 \leq u \leq \mu$ in the assertions of Theorems 8.1, 8.2 (recall the proof of Proposition 5.1).

9. Extensions

Suppose that $W : \mathbb{R} \to \mathbb{R}$ is C^2 and there are positive numbers

$$\mu_1 < \cdots < \mu_m, \quad m \geq 2,$$

such that

$$W(\mu_1) > \cdots > W(\mu_m), \quad W'(0) \leq 0, \quad W'(\mu_i) = 0, \quad i = 1 \cdots, m,$$

and

$$W(t) > W(\mu_i), \quad t \in [0, \mu_i), \; i = 1, \cdots, m.$$

Note that at the points μ_i, the potential W has either minima or saddles. Obviously, we can extend W outside of $[0, \mu_i]$, to a C^2 potential $\tilde{W}_i$, in such a way that condition **(a')** is satisfied with $\tilde{W}_i(t) - W(\mu_i)$ in place of W and μ_i in place of μ, $i = 1, \cdots, m$. Next, consider any

$$\epsilon \in \left(0, \min_{i=1,\cdots,m} (\mu_i - \mu_{i-1})\right), \tag{9.1}$$

with the convention that $\mu_0 = 0$, and any

$$D_i > D_i' \;\text{ where }\; D_i' \;\text{ solve }\; \mathbf{U}_i\left(D_i'\right) = \mu_i - \epsilon, \quad i = 1, \cdots, m, \tag{9.2}$$

where

$$\mathbf{U}_i''(s) = W'\left(\mathbf{U}_i(s)\right), \; s > 0; \quad \mathbf{U}_i(0) = 0, \; \lim_{s\to\infty} \mathbf{U}_i(s) = \mu_i. \tag{9.3}$$

By means of Theorem 1.2, there exist positive numbers $R_i' > D_i$, depending only on ϵ, D_i, $\tilde{W}_i$, $i = 1, \cdots, m$, and n, such that if Ω has nonempty Lipschitz boundary and contains a closed ball of radius R_i' then there exists a solution u_i of

$$\Delta u = \tilde{W}_i'(u), \; x \in \Omega, \quad u(x) = 0, \; x \in \partial\Omega, \tag{9.4}$$

satisfying

$$0 < u_i(x) < \mu_i, \quad x \in \Omega, \tag{9.5}$$

and

$$\mu_{i-1} < \mu_i - \epsilon < u_i(x), \quad x \in \Omega_{R_i'} + B_{(R_i' - D_i)}, \quad i = 1, \cdots, m. \tag{9.6}$$

In view of (9.5), we conclude that u_i solves the original problem (1.2). Thus, given ϵ and D_i as in (9.1) and (9.2) respectively, if Ω contains a closed ball of radius R'', where $R'' = \max_{i=1,\cdots,m} R_i'$, we find that (1.2) has at least m positive solutions which satisfy (9.5)–(9.6). Moreover, keeping in mind Remark 2.15, we know that these solutions are stable.

These solutions may be chosen to be ordered, in the usual sense. In other words, given ϵ and D_i as in (9.1) and (9.2) respectively, there are at least m positive, stable solutions of (1.2) such that

$$u_1(x) < \cdots < u_m(x), \quad x \in \Omega, \quad 1 \le i \le m, \tag{9.7}$$

and (9.5)–(9.6) hold (we have chosen to keep the same notation). Indeed, the solution u_i can be captured by using the constant function μ_i as an upper solution; and the function $\max\{u_{i-1}(x),\ \underline{u}_P^i\}$ as lower solution, where $\underline{u}_P^i$ is the lower solution in (2.57) but with $\tilde{W}_i(t) - W(\mu_i)$ in place of $W(t)$, $i = 1, \cdots, m$, and $u_0 \equiv 0$. (We use again Proposition 1 in [26], see also Proposition 1 in [115], to make sure that it is a lower solution). As in the first proof of Theorem 1.2, we can sweep with the family of lower solutions $\underline{u}_Q^i$, $Q \in \Omega_{R_i'}$ to extend the lower bound on u_i (due to (2.3)) from $B_{(R_i'-D_i)}(P)$ to $\Omega_{R_i'} + B_{(R_i'-D_i)}$. Moreover, the strong inequalities in (9.7) follow from the strong maximum principle. Naturally, the obtained solutions are stable (recall Remark 2.15).

We have just proven the following:

Theorem 9.1. Suppose that Ω and W are as described in this section. Let ϵ and D_i be as in (9.1) and (9.2) respectively. There exist positive constants $R_i' > D_i$, $i = 1, \cdots, m$, depending only on ϵ, D_i, W and n, such that if Ω contains a closed ball of radius $R'' = \max_{i=1,\cdots,m} R_i'$, then problem (1.2) has at least m stable solutions u_i, ordered as in (9.7), such that (9.5)–(9.6) hold true.

On the other hand, assuming that Ω is bounded and smooth (a C^3 boundary suffices), the theory of monotone dynamical systems (see Theorem 4.4 in [126]) guarantees the existence of at least $m-1$ unstable solutions $\hat{u}_i$, $i = 1, \cdots, m-1$, of (1.2) such that

$$u_i(x) < \hat{u}_i(x) < u_{i+1}(x), \quad x \in \Omega, \quad i = 1, \cdots, m-1. \tag{9.8}$$

This can also be shown by the well known mountain pass theorem, see [71].

Remark 9.1. The extra assumptions on Ω were only required in order to apply Theorem 4.4 in [126].

In summary, we have

Theorem 9.2. Suppose that, in addition to the hypotheses of Theorem 9.1, the domain Ω is assumed to be smooth and bounded. Then, besides of the m stable solutions u_i that are provided by Theorem 9.1, there exist at least $m-1$ unstable solutions $\hat{u}_i$ of (1.2), ordered as in (9.8) (keep in mind (9.7)).

The above theorem extends an old result of P. Hess [102], in the context of nonlinear eigenvalue problems (which are included in our setting, see below), where the additional assumption that $W'(0) < 0$ was imposed (see also [46] for an earlier result in the case $n = 1$). In the same context, the case $W'(0) = 0$ was allowed in [71], at the expense of assuming that $W'(\mu_i) \neq 0$, $i = 1, \cdots, m$, and some geometric restrictions on the domain. All these references considered nonlinear eigenvalue problems of the form

$$\Delta u = \lambda^2 W'(u),\ x \in \mathcal{D}, \quad u(x) = 0,\ x \in \partial\mathcal{D}, \tag{9.9}$$

where $\mathcal{D}$ is a smooth bounded domain of $\mathbb{R}^n$. By stretching variables $x \mapsto \lambda^{-1}x$, assuming that $0 \in \mathcal{D}$ (this we can do without loss of generality), keeping the same notation, we are led to the equivalent problem:

$$\Delta u = W'(u),\ x \in \Omega, \quad u(x) = 0,\ x \in \partial\Omega, \tag{9.10}$$

where $\Omega \equiv \lambda\mathcal{D}$, for $\lambda > 0$, which is plainly problem (1.2). If λ is sufficiently large, then certainly the domain Ω contains the ball $B_{R''}$, appearing in the assertion of Theorem 9.1, but not the other way around. In contrast to our approach of using upper and lower solutions, De Figueiredo in [71] obtained the corresponding stable solutions as minimizers of the associated energy functionals (with W suitably modified outside of $[0, \mu_i]$, $i = 1, \cdots, m$), and a geometric condition had to be imposed on the domain in order to ensure that they are distinct for large λ. In our case, the fact that they are distinct follows at once from (9.5) and (9.6). As we have already pointed out, in [71], the unstable solutions were constructed as mountain passes (saddle points of the energy).

Remark 9.2. It has been proven in [62] that if $W'(t) < 0$, $t \in (0, \mu)$, $W'(0) < 0$, or $W'(0) = 0$ but $W''(0) < 0$, $W'(\mu) = 0$, and $W'' \geq 0$ near μ, then (9.9) has a unique solution with values $(0, \mu)$ when λ is large, see also [20].

Remark 9.3. If $\mathcal{D}$ is a bounded domain with boundary satisfying the interior ball condition (see [93], a sufficient condition for this is that the boundary is C^2), it follows from the proof of Theorem 1.2 that the corresponding stable solutions of (9.9), provided by Theorem 9.1, develop a boundary layer of size $\mathcal{O}(\lambda^{-1})$, as $\lambda \to \infty$, along the boundary of $\mathcal{D}$ (see Proposition 10.1 below for more details, and compare with the proof of Theorem 1.1 in [118], as well as with Theorem 4 in [71] and Lemma 2 in [115]). Loosely speaking, this means that the stable solutions u_i converge uniformly to μ_i on the domain $\mathcal{D}$ excluding the strip that is described by $\mathrm{dist}(x, \partial\mathcal{D}) \leq |\ln\lambda|^\alpha|\lambda^{-1}$, $\alpha > 0$, as $\lambda \to \infty$. It follows from (9.8) that the corresponding unstable solutions of (9.9), provided by Theorem 9.2, also develop a (local) boundary layer behavior. In fact, if $W''(\mu_i) > 0$, the fine structure of the boundary layer of the stable solution u_i is determined by the unique solution of the problem (9.3), see [20] and Remark 10.4 below. On the other side, under some restrictions on $\mathcal{D}$ and W, unstable solutions possessing an upward sharp spike layer on top of u_i, located near the most centered part of the domain, have been constructed in [63], [64], and [105] (see also [67]). The fine structure of this interior spike layer is determined by the problem

$$\Delta V = W'(V + \mu_i) \ \text{ in } \mathbb{R}^n; \quad V(x) \to 0, \ |x| \to \infty.$$

10. On the boundary layer of global minimizers of singularly perturbed elliptic equations

In this section, assuming only **(a')**, we will prove a general result on the size of the boundary layer of solutions of (9.9), which minimize the associated energy functional, as $\lambda \to \infty$ (recall also Remark 9.3). Setting $\varepsilon = \lambda^{-1} \to 0$, gives rise to a singular perturbation problem of the form

$$\varepsilon^2\Delta u = W'(u), \quad x \in \mathcal{D}; \quad u(x) = 0, \ x \in \partial\mathcal{D}, \tag{10.1}$$

and in this regard it might be helpful to recall Remark 2.1.

We emphasize that, in contrast to previous results in this direction, as Theorem 1.1 in [118], here the size of the boundary layer is shown to be *independent of the dimension n*. This is due to our previous improvement over Lemma 2.2 in [118] that was made in Lemma 2.1 herein (recall the discussion preceding it, and also see Remark 10.3 below). The point is that we have not assumed any nondegeneracy on W at μ; in the case where $W''(\mu) > 0$ or $n = 2$, the structure of the boundary layer is well understood (recall Remark 9.3 and see Remark 10.2 below). For a different possible approach to this, see Remark 10.4 below.

The main result in this section is

Proposition 10.1. Suppose that $\mathcal{D}$ is a bounded domain in $\mathbb{R}^n$, $n \geq 1$, whose boundary satisfies the interior ball condition (recall Remark 9.3), and let W satisfy assumption **(a')**. Consider any $\epsilon \in (0, \mu)$ and $D > D'$, where D' as in (1.11). There exists a positive constant λ_*, depending only on ϵ, D, $\mathcal{D}$, and W, such that there exists a solution u_λ of (9.9), which minimizes the associated energy functional, satisfies

$$0 < u_\lambda(x) < \mu, \quad x \in \mathcal{D}, \tag{10.2}$$

and

$$u_\lambda(x) \geq \mu - \epsilon, \ x \in \bar{\mathcal{D}}_{(D\lambda^{-1})}, \tag{10.3}$$

provided that $\lambda \geq \lambda_*$ (recall the definition (1.6), and note that $\mathcal{D}_{(D\lambda^{-1})}$ is a connected domain for large λ). (See also the comments at the end of the assertion of Lemma 2.1).

Proof. As in the second proof of Theorem 1.2, recalling the discussion leading to (9.10), there exists a smooth solution of (9.9), which minimizes the associated energy and satisfies (10.2), provided that λ is sufficiently large, say $\lambda \geq \lambda_0$, depending not just on W but this time also on the domain $\mathcal{D}$.

By the properties of the domain, there exists a radius $r_0 > 0$ and a family of balls $B_{r_0}(q) \subseteq \mathcal{D}$, $q \in \partial\mathcal{D}_{r_0}$ (i.e. $q \in \mathcal{D}$ with $\text{dist}(q, \partial\mathcal{D}) = r_0$) such that, for each such q, the closed ball $\bar{B}_{r_0}(q)$ touches $\partial\mathcal{D}$ at exactly one point.

Let $\epsilon \in (0, \mu)$ and $D > D'$, where D' as in (1.11). It follows from Lemma 2.1 (after a simple rescaling) that there exists a $\lambda_* > 0$, depending only on ϵ, D, W, and $\mathcal{D}$ (in terms of r_0), and a global minimizer $u_{r_0,q}$ of the associated energy to the equation of (9.9) in $W_0^{1,2}\left(B_{r_0}(q)\right)$ such that

$$0 < u_{r_0,q}(x) < \mu, \ x \in B_{r_0}(q), \quad \text{and} \quad u_{r_0,q}(x) \geq \mu - \epsilon, \quad x \in \bar{B}_{(r_0 - D\lambda^{-1})}(q),$$

provided that $\lambda \geq \lambda_*$. (Without loss of generality, we may assume that $\lambda_* > \lambda_0$). Thanks to Lemma A.3 below, we obtain that $u_\lambda(x) \geq u_{r_0,q}(x)$, $x \in B_{r_0}(q)$. Since the center q was any point on $\partial\mathcal{D}_{r_0}$, it follows that assertion (10.3) holds true for $x \in \mathcal{D}$ such that

$$D\lambda^{-1} \leq \text{dist}(x, \partial\mathcal{D}) \leq 2r_0 - D\lambda^{-1}. \tag{10.4}$$

If $W'(t) < 0$, $t \in [\mu - 2\epsilon, \mu)$, then the validity of (10.3), over the entire specified domain, follows at once via the second assertion of Lemma A.2 (this is also the case when relation (2.23) holds, recall Remark 2.3). Otherwise, we proceed as follows, see also Lemma 2 in [115]: Firstly, we cover $\bar{\mathcal{D}}_{r_0}$ by a finite number of balls of radius $\frac{r_0}{2}$ with centers on $\bar{\mathcal{D}}_{r_0}$. Secondly, if necessary, we increase the value of λ_* such that $D\lambda_*^{-1} < \frac{r_0}{2}$. Lastly, we apply Lemma A.3 to show that

$$u_\lambda(x) \geq u_{r_0,p}(x) \geq \mu - \epsilon, \quad x \in \bar{B}_{(r_0 - D\lambda^{-1})}(p) \supseteq \bar{B}_{\frac{r_0}{2}}(p),$$

for every center p of the finite covering of $\bar{\mathcal{D}}_{r_0}$, if $\lambda \geq \lambda_*$. We point out that this last part could have also been obtained from the weaker relation (2.12) (with the obvious modifications). The desired estimate (10.3) now follows from the comments leading to (10.4) and the above relation.

The proof of the proposition is complete. □

Remark 10.1. A similar result also holds if the domain $\mathcal{D}$ is unbounded.

Remark 10.2. The asymptotic behavior, as $\lambda \to \infty$, of uniformly bounded from above and below (with respect to λ), stable solutions of (9.9), where $\mathcal{D} \subseteq \mathbb{R}^n$ is bounded and smooth, has been studied in [67] in dimensions $n = 2, 3$ by techniques related to the proof of De Giorgi's conjecture in low dimensions. In fact, since global minimizers are stable, and since assumption **(a')** implies that $W'(0) \leq 0$, the assertions of Proposition 10.1 when $n = 2$ follow readily from Theorem 6 in [67]; this is also the case when $n = 3$, provided that the monotonicity assumption **(b)** from our introduction is imposed.

Remark 10.3. Let $\epsilon,\ D,\ R' > 0$ be related as in the assertion of Lemma 2.1. By means of a simple rescaling argument (see also the proof of Theorem 1.1 in [118]), Lemmas 2.1 and A.3 yield that the solution of (9.9), described in Proposition 10.1, satisfies $\mu - u_\lambda(x) \geq \epsilon$, if $\text{dist}(x, \partial\mathcal{D}) > D\lambda^{-1}$, provided that λ is sufficiently large (depending on ϵ, W, and $\mathcal{D}$). Note that relation (2.12) yields the same estimate but over the smaller region that is described by $\text{dist}(x, \partial\mathcal{D}) > \frac{R'}{2}\lambda^{-1}$, which *depends on* n, see [118].

Remark 10.4. Blowing up a global minimizer u_λ of Proposition 10.1 at a point $x_0 \in \partial\mathcal{D}$, up to a subsequence, we find that

$$u_\lambda\left(x_0 + \lambda^{-1}y\right) \to U(y),$$

uniformly on compacts, as $\lambda \to \infty$, where U is a nonnegative, global minimizer (in the sense of (2.65), this can be seen as in page 104 of [66]) of the following half-space problem

$$\Delta u = W'(u),\ y \in \mathbb{R}^n_+;\quad u(y) = 0,\ y \in \partial\mathbb{R}^n_+,$$

see [20], [67] for more details, where $\mathbb{R}^n_+ = \{(y_1, \cdots, y_n) \ :\ y_n > 0\}$. Furthermore, this solution is nontrivial by virtue of Remark 10.3. Hence, by the strong maximum principle, recall **(a')**, we deduce that U is positive in $\mathbb{R}^n_+$. As before, combining Lemmas 2.1 and A.3, we obtain that

$$u(y) \to \mu \ \text{ as } \ y_n \to \infty,\ \text{uniformly in } (y_1, \cdots, y_{n-1}) \in \mathbb{R}^{n-1},$$

(the weaker assertion (2.12) is sufficient for this). It follows from Theorem 1.4 in [29] that U depends only on the y_n variable and therefore coincides with $\mathbf{U}(y_n)$ that was described in (1.12). (If $W''(\mu) > 0$ then this has been shown earlier in [20], see also [31], [57] for the weaker case (1.15)).

Remark 10.5. In [153], the author established an asymptotic expansion of $\nu\nabla u_\varepsilon(P)$, $P \in \partial\mathcal{D}$, as $\varepsilon \to 0$, where u_ε solves (10.1) for a class of nonlinearities which in particular satisfy **(c)** and (1.9) (see also [87]). As usual, the vector ν denotes the unit outer normal to $\partial\mathcal{D}$ (having assumed that it is smooth and bounded). This expansion reveals that if P_1 is the only point which attains the minimum of the mean curvature of $\partial\mathcal{D}$, then P_1 is the steepest point of the boundary layer.

Remark 10.6. By adapting the proof of Lemma 2.3 in [118], and that of our Proposition 10.1, we can study the boundary layer of globally minimizing solutions of inhomogeneous singular perturbation problems of the form

$$\varepsilon^2\Delta u = W_u(u, x),\ x \in \mathcal{D};\quad u(x) = 0,\ x \in \partial\mathcal{D},$$

as $\varepsilon \to 0$, for appropriate righthand side that is more general than those that were considered in [33, 34, 75, 118], see also Lemma 7.13 in [78] and Section 13.3 in [121] (roughly, we want **(a')** to hold with $a(x)$ instead of μ, for every fixed $x \in \bar{\mathcal{D}}$, for a smooth positive function a).

Appendix A. Some useful "comparison" lemmas of the calculus of variations

The following is essentially Lemma 2.1 in [89].

Lemma A.1. Let $\mathcal{O} \subset \mathbb{R}^n$ be an open set and let $v \in W^{1,2}(\mathcal{O})$. Define $\tilde{v} : \mathcal{O} \to \mathbb{R}$ as

$$\tilde{v}(x) = \begin{cases} v(x) & \text{if } v(x) \in [0, \mu], \\ \mu & \text{if } v(x) \in (-\infty, -\mu) \cup (\mu, \infty), \\ -v(x) & \text{if } v(x) \in (-\mu, 0). \end{cases}$$

Then $\tilde{v} \in W^{1,2}(\mathcal{O})$ and, if W is C^2 and satisfies **(a')**, we have

$$\int_{\mathcal{O}} \left\{ \frac{1}{2}|\nabla \tilde{v}|^2 + W(\tilde{v}) \right\} dx \leq \int_{\mathcal{O}} \left\{ \frac{1}{2}|\nabla v|^2 + W(v) \right\} dx.$$

Proof. (Sketch) Firstly, note that $\tilde{v} = G(v)$, $x \in \mathcal{O}$, for some Lipschitz (piecewise linear) function $G : \mathbb{R} \to \mathbb{R}$. Thus, $\tilde{v} \in W^{1,2}(\mathcal{O})$, see for instance [80]. Then, to finish, note that

$$|\nabla \tilde{v}| \leq |\nabla v| \ \text{ and, thanks to } \mathbf{(a')}, \ W(\tilde{v}) \leq W(v) \ \text{ a.e. in } \mathcal{O}, \tag{A.1}$$

(the former inequality may be proven as in page 93 in [109]). □

The following is an extension of Lemma A.1, and is motivated from [11] (see also [12] for an extension). Our proof follows [156].

Lemma A.2. Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded domain with Lipschitz boundary, and $W : \mathbb{R} \to \mathbb{R}$ be a C^2 potential such that conditions **(a')** and (2.23) hold. Further, let $\mathcal{A} \subset \Omega$ be a bounded domain with Lipschitz boundary such that $\bar{\mathcal{A}} \subset \Omega$. Moreover, assume that

- $u \in W^{1,2}(\Omega)$, $0 \leq u \leq \mu$ a.e. in Ω
- $\mu - u \leq \eta$ a.e. on $\partial\mathcal{A}$, in the sense of Sobolev traces (see [80]), for some $\eta \in \left(0, \frac{d}{2}\right)$.

Then, there exists $\tilde{u} \in W^{1,2}(\Omega)$ such that

$$\begin{cases} \tilde{u}(x) = u(x), \quad x \in \Omega \backslash \mathcal{A}, \\ \mu - \eta \leq \tilde{u}(x) \leq \mu, \quad x \in \mathcal{A}, \\ \int_\Omega \left\{ \frac{1}{2}|\nabla \tilde{u}|^2 + W(\tilde{u}) \right\} dx \leq \int_\Omega \left\{ \frac{1}{2}|\nabla u|^2 + W(u) \right\} dx. \end{cases} \tag{A.2}$$

If condition (2.23) holds with strict inequality, and there exists a set $\mathcal{B} \subset \mathcal{A}$ of nonzero measure such that

$$u < \mu - \eta \ \text{ a.e. on } \ \mathcal{B},$$

then the last relation in (A.2) holds with a strict inequality.

Proof. (Sketch) The first assertion of the lemma can be deduced similarly to Lemma A.1. Indeed, the desired function is

$$\tilde{u}(x) = \begin{cases} \min\{\mu, \max\{u(x),\ 2\mu - 2\eta - u(x)\}\}, & x \in \mathcal{A}, \\ u(x), & x \in \Omega \backslash \mathcal{A}. \end{cases} \tag{A.3}$$

We point out that $\tilde{u} \in W^{1,2}(\mathcal{A})$ similarly to Lemma A.1, and $\tilde{u} \in W_0^{1,2}(\Omega)$ because $\mathcal{A}$ has Lipschitz boundary and $\tilde{u} = u$ on $\partial\mathcal{A}$ in the sense of Sobolev traces (see again [80]). Note that if $\mu - 2\eta \leq u(x) \leq \mu$ then $\mu - d < u(x) \leq \tilde{u}(x) \leq \mu$, so relation (2.23) implies that $W(\tilde{u}(x)) \leq W(u(x))$. Furthermore, if $0 \leq u(x) \leq \mu - 2\eta$ then $\tilde{u}(x) = \mu$ and $W(\tilde{u}(x)) = 0 \leq W(u(x))$. Also keep in mind the first relation in (A.1).

The second assertion can be shown with a little more care. Replacing u by the minimizer of the corresponding energy functional $J(\cdot;\mathcal{A})$ (recall (2.1)) among functions $v \in W^{1,2}(\mathcal{A})$ such that $v - u \in W_0^{1,2}(\mathcal{A})$, we may assume that u is a smooth solution of (3.1) in $\mathcal{A}$. Firstly, we consider the case where

$$\mu - 2\eta \leq u(x) \leq \mu \quad \text{on } \bar{\mathcal{A}}.$$

In that case, we have that $\mu - d < u < \tilde{u} \leq \mu$ on $\mathcal{B}$. In turn, from the assumption that the inequality in (2.23) is strict, we obtain that $W(\tilde{u}) < W(u)$ on $\mathcal{B}$. Since the set $\mathcal{B}$ has positive measure, taking into account our previous discussion for the first assertion, we arrive at

$$\int_\Omega W(\tilde{u})dx < \int_\Omega W(u)dx. \tag{A.4}$$

Hence, the second assertion holds in this case. On the other side, if

$$0 \leq u(x_0) < \mu - 2\eta \quad \text{for some } x_0 \in \mathcal{A},$$

then $0 \leq u \leq \mu - 2\eta \leq \tilde{u} = \mu$ in an open neighborhood of x_0. In this neighborhood, via **(a')**, we have that $W(u) \geq \min_{t\in[0,\mu-2\eta]} W(t) > 0$ while $W(\tilde{u}) = 0$. It follows that relation (A.4) holds in this case as well. Keeping in mind the first relation in (A.1), we conclude that the second assertion of the lemma holds.

The sketch of proof of the lemma is complete. □

The following is Lemma 2.3 in [66], which is reproduced in Lemma 1 in [115] and Lemma 2.1 in [118], see also Theorem 1.4 in [88] and Lemma 3.1 in [106].

Lemma A.3. Let $\mathcal{D}$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Let $g_1(x,t), g_2(x,t)$ be locally Lipschitz functions with respect to t, measurable functions with respect to x, and for any bounded interval I there exists a constant C such that $\sup_{x\in\mathcal{D},t\in I} |g_i(x,t)| \leq C$, $i = 1, 2$, holds. Let

$$G_i(x,t) = \int_0^t g_i(x,s)ds, \ i = 1, 2.$$

For $\eta_i \in W^{1,2}(\mathcal{D}),\ i = 1, 2$, consider the minimization problem:

$$\inf \left\{ J_i(u;\mathcal{D}) \mid u - \eta_i \in W_0^{1,2}(\mathcal{D}) \right\}, \quad \text{where } \ J_i(u;\mathcal{D}) = \int_\mathcal{D} \left\{ \frac{1}{2}|\nabla u|^2 - G_i(x,u) \right\} dx.$$

Let $u_i \in W^{1,2}(\mathcal{D}),\ i = 1, 2$, be minimizers to the minimization problems above. Assume that there exist constants $m < M$ such that

- $m \leq u_i(x) \leq M$ a.e. for $i = 1, 2,\ x \in \mathcal{D}$,
- $g_1(x,t) \geq g_2(x,t)$ a.e. for $x \in \mathcal{D},\ t \in [m, M]$,
- $M \geq \eta_1(x) \geq \eta_2(x) \geq m$ a.e. for $x \in \mathcal{D}$.

Suppose further that $\eta_i \in W^{2,p}(\mathcal{D})$ for any $p > 1$, and that they are *not identically equal* on $\partial\mathcal{D}$. Then, we have

$$u_1(x) \geq u_2(x), \quad x \in \mathcal{D}.$$

Appendix B. A Liouville-type theorem

Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous, and

$$\begin{cases} f(0) = 0, \\ \\ f(t) > 0, \ t > 0, \\ \\ f \text{ is non–decreasing and convex on } [0, \infty), \\ \\ \int_{t_0}^{\infty} \left[\int_{t_0}^{t} f(s) ds \right]^{-\frac{1}{2}} dt < \infty \quad \forall \ t_0 > 0. \end{cases}$$

In the mathematical literature, the above integral condition is known as Keller-Osserman condition, see [83], [107] and [133]. These conditions are clearly satisfied for

$$f(t) = t|t|^{p-1} \text{ with } p > 1. \tag{B.1}$$

The following is Theorem 4.7 in the review article [83]. As we have already discussed at the end of Remark 4.1, it was originally proven in [40] for the special case of the power nonlinearity (B.1).

Theorem B.1. Let f satisfy the above properties.

(i): Suppose $u \in L^1_{loc}(\mathbb{R}^n)$ is such that $f(u) \in L^1_{loc}(\mathbb{R}^n)$ and

$$-\Delta u + f(u) \leq 0 \text{ in } \mathcal{D}'(\mathbb{R}^n) \text{ (distributionally)}.$$

Then $u \leq 0$ a.e. on $\mathbb{R}^n$.

(ii): Assume also that f is an odd function. Suppose $u \in L^1_{loc}(\mathbb{R}^n)$ is such that $f(u) \in L^1_{loc}(\mathbb{R}^n)$ and

$$-\Delta u + f(u) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^n).$$

Then $u = 0$ a.e. on $\mathbb{R}^n$.

Appendix C. A doubling lemma

The following is a very useful result from [138].

Lemma C.1. Let (X, d) be a complete metric space, $\Gamma \subset X$, $\Gamma \neq X$, and $\gamma : X \backslash \Gamma \to (0, \infty)$. Assume that γ is bounded on all compact subsets of $X \backslash \Gamma$. Given $k > 0$, let $y \in X \backslash \Gamma$ be such that

$$\gamma(y) \text{dist}(y, \Gamma) > 2k.$$

Then, there exists $x \in X \backslash \Gamma$ such that

- $\gamma(x) \text{dist}(x, \Gamma) > 2k$,
- $\gamma(x) \geq \gamma(y)$,
- $2\gamma(x) \geq \gamma(z) \quad \forall \ z \in B_{\frac{k}{\gamma(x)}}$.

We remark that this doubling lemma is proven similarly as Baire's category theorem.

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