A PROOF OF HALPERN - LÄUCHLI PARTITION THEOREM.

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ABSTRACT. A proof of Halpern - Läuchli partition theorem and its version for strong subtrees is given. We prove a general statement which has as immediate conequence the above mentioned results. The proof of this is direct and avoids metamathematical arguments. Some consequences for partitions of finite products of metric spaces are also presented.

Introduction

Halpern-Läuchli partition theorem ([HL]) is a fundamental Ramsey type principle concerning partitions of finite products of trees. This partition theorem plays an important role in the development of the infinite Ramsey theory for trees ([M], [LSV]). In the present paper we also present some more applications of this theorem in the area of metric spaces. Its original proof makes use of certain metamathematical tools difficult to be followed by unfamiliar readers. The main purpose of the present paper is to give a proof of this theorem using exclusively standard mathematical arguments.

Before presenting the statement of the main theorem we state the following result which actually is the equivalent formulation of it in the context of metric spaces. We recall that a subset F of a metric space (X, ρ) is said to be $\varepsilon-$dense for some $\varepsilon > 0$, provided that $\rho(x, F) < \varepsilon$ for all $x \in X$. Also (X, ρ) is totally bounded (or precompact) if for each $\varepsilon > 0$ there exists a finite $\varepsilon-$dense subset F_ε of X.

Theorem 0.1. *Let (X_i, ρ_i), $i = 1, ..., d$ be a finite family of totally bounded metric spaces. For each $n \in \mathbb{N}$ and $i = 1, ..., d$ choose an $F_{n,i}$ $\frac{1}{n}-$dense subset of X_i and set $\mathcal{F}_n = \prod_{i=1}^d F_{n,i}$. Then for every finite partition*

$$\cup_{n=1}^{\infty} \mathcal{F}_n = C_1 \cup ... \cup C_p$$

we have the following:

There exists an increasing sequence $(n_k)_{k \in \mathbb{N}}$, $B_{n_k,i} \subset F_{n_k,i}$ and $j \in \{1, ..., p\}$ such that

$$\cup_{k=1}^{\infty} \prod_{i=1}^d B_{n_k,i} \quad \text{is } j-\text{ homogeneous.}$$

Moreover,

(a) If $j = 1$ then for all $k \in \mathbb{N}$, $i = 1, ..., d$, $B_{n_k,i}$ is $\frac{1}{k}-$dense in X_i.

(b) If $j > 1$ then for all $i = 1, ..., d$ there exists a non empty open neighborhood V_i of X_i such that $B_{n_k,i}$ is $\frac{1}{k}-$ dense in V_i for all $k \in \mathbb{N}$.

Some comments are in order concerning the above stated theorem. First observe that the conclusion is not symmetric with respect the members $(C_j)_{j=1}^p$ of the partition of $\bigcup_{n=1}^{\infty} F_n$. Also we should point out the double character of the conclusion. Namely, beyond the $j-$ homogenuity of $\prod_{i=1}^d B_{n_k,i}$ we also obtain the $\frac{1}{k}-$ density. Thus a more accurate description of this theorem is as a Baire-Ramsey result.

Several difficulties occur if someone attempted to proceed to a direct proof of the above theorem. In particular if he wants to proceed by induction on the number

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$X_1,...,X_d$ of the metric spaces, then for $d \geq 2$ will face the difficulty of how the inductive assumption could be used. This is one of the points where the statement of the theorem in terms of trees appears to be very useful.

The statement of the main result requires some terminology, stated briefly below and more carefully in the first section. Throughout the paper by the term a tree $\mathcal{T}$ we mean a finitely branching tree of height ω with finitely many roots and without maximal elements.

Let $\mathcal{W}$ be a subset of $\mathcal{T}$. The k^{th} *level* $\mathcal{W}(k)$ of $\mathcal{W}$ is the set $\{t \in \mathcal{W}\colon |t| = k\}$ and the *level set* the set $L(\mathcal{W}) = \{k \in \mathbb{N} : \mathcal{W}(k) \neq \emptyset\}$.

A subset $\mathcal{W}$ of $\mathcal{T}$ is said to be *dense* in $\mathcal{T}$ if $L(\mathcal{W}) = \{l_k\}_{k=1}^{\infty}$ and for all $k \in \mathbb{N}$, the set $\mathcal{W}(l_k)$ *dominates* the set $\mathcal{T}(k)$. (i.e. for every $t \in \mathcal{T}(k)$ there exists $s \in \mathcal{W}(l_k)$ with $t \prec s$). Further for $t \in \mathcal{T}$ the set $\mathcal{W}$ is $t-$dense in $\mathcal{T}$ provided that $\mathcal{W}(l_k)$ dominates $\mathcal{T}_t(k) = \{s \in \mathcal{T} : |s| = k, t \prec s\}$.

Let $\vec{\mathcal{T}} = (\mathcal{T}_1, ..., \mathcal{T}_d)$ be a $d-$tuple of trees. The **level product** of $\vec{\mathcal{T}}$ denoted by $\otimes\vec{\mathcal{T}}$ is the set $\bigcup_{k=1}^{\infty} \prod_{i=1}^{d} \mathcal{T}_i(k)$. For $\mathcal{W}_i \subset \mathcal{T}_i$, $i = 1, ..., d$, the $d-$tuple $\vec{\mathcal{W}} = (\mathcal{W}_1, ..., \mathcal{W}_d)$ is said to be compatible if $L(\mathcal{W}_i) = L(\mathcal{W}_j)$ for all $1 \leq i, j \leq d$. For a $\vec{\mathcal{W}} = (\mathcal{W}_1, ..., \mathcal{W}_d)$ the level product $\otimes\vec{\mathcal{W}}$ is also defined in a similar manner as $\otimes\vec{\mathcal{T}}$.

For a compatible $\vec{\mathcal{W}} = (\mathcal{W}_1, ..., \mathcal{W}_d)$ and $\vec{t} = (t_1, ..., t_d)$ we say that $\vec{\mathcal{W}}$ is dense ($\vec{t}-$dense) in $\vec{\mathcal{T}}$ provided that $\mathcal{W}_i$ is dense (t_i-dense) in $\mathcal{T}_i$ for all $i = 1, ..., d$. Finally we adopt the following notation. For $\vec{\mathcal{W}} = (\mathcal{W}_1, ..., \mathcal{W}_d)$ and $\vec{\mathcal{W}}' = (\mathcal{W}'_1, ..., \mathcal{W}'_d)$ we denote by $\vec{\mathcal{W}}' \prec \vec{\mathcal{W}}$ the relation $\mathcal{W}'_i \subset \mathcal{W}_i$ for all $i = 1, ..., d$.

The main result of the paper is the following:

Theorem 0.2. *Let $\vec{\mathcal{T}} = (\mathcal{T}_1, ..., \mathcal{T}_d)$ be a $d-$tuple of trees and $\vec{\mathcal{W}} = (\mathcal{W}_1, ..., \mathcal{W}_d)$ a compatible dense in $\vec{\mathcal{T}}$. Then for every finite partition $\otimes\vec{\mathcal{W}} = C_1 \cup ... \cup C_p$ one of the following holds:*
(a) There exists compatible $\vec{\mathcal{W}}' \prec \vec{\mathcal{W}}$ with $\vec{\mathcal{W}}'$ dense in $\vec{\mathcal{T}}$ and $\otimes\vec{\mathcal{W}}'$ is 1-homogeneous.
(b) There exists $j > 1$, $\vec{t} = (t_1, ..., t_d)$ and compatible $\vec{\mathcal{W}}' \prec \vec{\mathcal{W}}$ which is $\vec{t}-$dense in $\vec{\mathcal{T}}$ and $\otimes\vec{\mathcal{W}}'$ is j-homogeneous.

The basic steps in the proof of the above theorem are the following:

First by an easy inductive argument we reduce the problem to the case of partitions with two elements. Hence assume that $\otimes\vec{\mathcal{W}} = C_1 \cup C_2$. Then we proceed by induction. The proof for the case $d = 1$ is given in the Proposition 2.2. For the general case we consider a $d+1-$tuple $(\mathcal{S}, \vec{\mathcal{T}})$ with $\vec{\mathcal{T}} = (\mathcal{T}_1, ..., \mathcal{T}_d)$ and $\mathcal{S}$ a tree. We denote by $\mathcal{C}_\infty(\mathcal{S})$ the set of all infinite chains of $\mathcal{S}$. For $c \in \mathcal{C}_\infty(\mathcal{S})$ and $\vec{\mathcal{W}}$ dense in $\vec{\mathcal{T}}$ such that $(c, \vec{\mathcal{W}})$ is compatible, the inductive assumption yields $c' \subset c$, $\vec{\mathcal{W}}' \prec \vec{\mathcal{W}}$ with either $\vec{\mathcal{W}}'$ dense in $\vec{\mathcal{T}}$ and $c' \otimes \vec{\mathcal{W}}'$ is 1-homogeneous or $\vec{\mathcal{W}}'$ $\vec{t}-$dense in $\vec{\mathcal{T}}$ and $c' \otimes \vec{\mathcal{W}}'$ is 2-homogeneous (Proposition 2.3).

The next step is to consider a statement denoted by $\mathbf{Q}\Big(\varepsilon, (s, \vec{t}), (\mathcal{U}, \vec{\mathcal{W}})\Big)$ where $\varepsilon \in \{1, 2\}$, $s \in \mathcal{S}$, $\vec{t} \in \vec{\mathcal{T}}$, $(\mathcal{U}, \vec{\mathcal{W}})$ dense in $(\mathcal{S}, \vec{\mathcal{T}})$. Also the corresponding $\mathbf{Q}(\varepsilon, \vec{0}, (\mathcal{U}, \vec{\mathcal{W}}))$ is similarly defined.

Related to this we show the next two

(A) If $\mathbf{Q}\Big(\varepsilon, (s, \vec{t}), (\mathcal{U}, \vec{\mathcal{W}})\Big)$ holds for some $\varepsilon, (s, \vec{t}), (\mathcal{U}, \vec{\mathcal{W}})$ then there exists $(\mathcal{U}', \vec{\mathcal{W}}') \prec (\mathcal{U}, \vec{\mathcal{W}})$ $(s, \vec{t})$ dense in $(\mathcal{S}, \vec{\mathcal{T}})$ such that $\mathcal{U}' \otimes \vec{\mathcal{W}}'$ is $\varepsilon-$homogeneous. (If $(s, \vec{t}) = \vec{0}$ then $(\mathcal{U}', \vec{\mathcal{W}}')$ is obtained to be dense in $(\mathcal{S}, \vec{\mathcal{T}})$.)

(B) For $(\mathcal{U}, \vec{\mathcal{W}})$ dense in $(\mathcal{S}, \vec{\mathcal{T}})$ either $\mathbf{Q}(1, \vec{0}, (\mathcal{U}, \vec{\mathcal{W}}))$ holds or else there exists $(\mathcal{U}', \vec{\mathcal{W}}') \prec (\mathcal{U}, \vec{\mathcal{W}})$ and $(s, \vec{t})$ such that $\mathbf{Q}\Big(2, (s, \vec{t}), (\mathcal{U}', \vec{\mathcal{W}}')\Big)$ holds.

(A) is proved in Proposition 2.4 and **(B)** in Proposition 2.5. Clearly **(A)** and **(B)** yield the complete proof of Theorem 0.2.

It is worth pointing out that **(A)** constitutes the "Ramsey" part of the proof of the main theorem while **(B)**, the proof of which involves a Baire category argument, can be considered as the "Baire part" of the proof. We also should say that the requirement for the density of the homogeneous subset, which at the beginning appears as an additional difficulty, turns out to be very useful to overcome certain combinatorial difficulties.

In the last section we present a proof of the Laver-Pincus theorem [PH] for strong subtrees and a proof of Theorem 0.1 stated above.

An earlier version of the present paper appeared in 1998 under the names of the first two coauthors. Recently the referee pointed out to us an error in that proof. Actually the arguments presented in that paper could derive a proof of Theorem 0.2 for $d \leq 2$. But as it is also mentioned in Halpern-Läuchli's paper the full complexity of the proof appears for $d > 2$. The present version is the result of the collaboration with the third named coauthor.

1. Definitions and Notations

By the term tree $\mathcal{T}$ we will mean a finitely branching tree $\mathcal{T}$ with finitely many roots and without maximal elements. Namely, the tree $\mathcal{T}$ has finitely many minimal elements, for each $t \in \mathcal{T}$ the set $\{s \in \mathcal{T} : s \prec t\}$ is finite and also the set of immediate successors of t is finite and non empty.

Definition 1.1. *Let $\mathcal{T}$ be a tree. Then*
(i) for each $t \in \mathcal{T}$ we define the ***order*** *$|t|$ of t to be the cardinality of the set $\{s \in \mathcal{T} : s \prec t\}$.*
(ii) For each $n \in \mathbb{N}$, we define the n^{th} ***level*** *$\mathcal{T}(n)$ of $\mathcal{T}$ to be the set $\mathcal{T}(n) = \{t \in \mathcal{T} : |t| = n\}$.*
(iii) For each $\mathcal{W} \subset \mathcal{T}$, $t \in \mathcal{T}$ and $n \in \mathbb{N}$ we set $\mathcal{W}(n) = \{t \in \mathcal{W} : |t| = n\} = \mathcal{W} \cap \mathcal{T}(n)$ $\mathcal{T}_t = \{s \in \mathcal{T} : t \prec s, |t| < |s|\}$, $\mathcal{W}_t = \{s \in \mathcal{W} : t \prec s, |t| < |s|\} = \mathcal{W} \cap \mathcal{T}_t$.
By $L(\mathcal{W})$ we denote the set $L(\mathcal{W}) = \{n \in \mathbb{N} : \mathcal{W}(n) \neq \emptyset\}$ and we call it the ***level set*** *of $\mathcal{W}$.*
(iv) For each $M \subset \mathbb{N}$ and $\mathcal{W} \subset \mathcal{T}$, we set $\mathcal{W}|_M = \bigcup_{m \in M} \mathcal{W}(m)$.
(v) For every $\mathcal{W}_1, \mathcal{W}_2$ subsets of $\mathcal{T}$ we say that $\mathcal{W}_2$ ***dominates*** *$\mathcal{W}_1$ if for each $t \in \mathcal{W}_1$ there exists a $t_2 \in \mathcal{W}_2$ such that $t_1 \prec t_2$.*

Definition 1.2. *Let $\mathcal{T}$ be a tree, $\mathcal{W}$ a subset of $\mathcal{T}$ with level set $L(\mathcal{W}) = \{l_n\}_n$ and $t \in \mathcal{T}$.*
The set $\mathcal{W}$ is called ***dense ($t-$dense)*** *in $\mathcal{T}$ if $\mathcal{W}(l_n)$ dominates $\mathcal{T}(n)$ ($\mathcal{W}(l_n)$ dominates $\mathcal{T}_t(n)$) for all $n \in \mathbb{N}$.*

Dense and $t-$dense subsets of $\mathcal{T}$ have a central role in the statements and the results appeared in this paper. So we next state some permanence properties of them.

Proposition 1.1. *Let $\mathcal{W}$ be a dense (or $t-$dense) in $\mathcal{T}$. Then for every infinite subset M of the level set $L(\mathcal{W})$ the set $\mathcal{W}' = \mathcal{W}|_M$ remains also dense (or $t-$dense) in $\mathcal{T}$.*

The proof of the above proposition follows immediately from the Definition 1.2.

Definition 1.3. *Let $\mathcal{T}_1, \ldots, \mathcal{T}_d$ be a finite sequence of trees. The* **vector tree** $\vec{\mathcal{T}}$ *is the ordered $d-$tuple $\vec{\mathcal{T}} = (\mathcal{T}_1, \ldots, \mathcal{T}_d)$. A* **vector subset** $\vec{\mathcal{W}}$ *of $\vec{\mathcal{T}}$ is an ordered $d-$tuple $\vec{\mathcal{W}} = (\mathcal{W}_1, \ldots, \mathcal{W}_d)$ such that $\mathcal{W}_i \subset \mathcal{T}_i$ for all $i = 1, \ldots, d$. A* **vector element** $\vec{t}$ *of $\vec{\mathcal{T}}$ is also an ordered $d-$tuple $\vec{t} = (t_1, \ldots, t_d)$ with $t_i \in \mathcal{T}_i$ for all $i = 1, \ldots, d$.*

Notation. Let $\vec{\mathcal{W}}^1 = (\mathcal{W}_1^1, \ldots, \mathcal{W}_d^1)$, $\vec{\mathcal{W}}^2 = (\mathcal{W}_1^2, \ldots, \mathcal{W}_d^2)$ vector subsets of $\vec{\mathcal{T}}$. By $\vec{\mathcal{W}}^1 \prec \vec{\mathcal{W}}^2$ we shall denote the relation $\mathcal{W}_i^1 \subset \mathcal{W}_i^2$ for all $i = 1, \ldots, d$.

Definition 1.4. *Let $\vec{\mathcal{T}} = (\mathcal{T}_1, \ldots, \mathcal{T}_d)$ be a vector tree, $\vec{\mathcal{W}} = (\mathcal{W}_1, \ldots., \mathcal{W}_d)$ a vector subset of $\vec{\mathcal{T}}$ and $\vec{t} = (t_1, \ldots, t_d)$ a vector element of $\vec{\mathcal{T}}$. The vector subset $\vec{\mathcal{W}}$ is called* **dense** *(**$\vec{t}-$dense**) in $\vec{\mathcal{T}}$ if $\mathcal{W}_i$ is dense in $\mathcal{T}_i$ ($\mathcal{W}_i$ is t_i-dense in $\mathcal{T}_i$) for all $i = 1, \ldots, d$.*

Definition 1.5. *Let $\vec{\mathcal{T}}$ be a vector tree. A vector subset $\vec{\mathcal{W}} = (\mathcal{W}_1, \ldots., \mathcal{W}_d)$ of $\vec{\mathcal{T}}$ is called* **compatible** *if for all $i, j \in \{1, \ldots, d\}$ $L(\mathcal{W}_i) = L(\mathcal{W}_j)$. If $\vec{\mathcal{W}}$ is compatible then the* **level set** *of $\vec{\mathcal{W}}$, denoted by $L(\vec{\mathcal{W}})$, is the common level set of each component of $\vec{\mathcal{W}}$.*

Definition 1.6. *Let $\vec{\mathcal{T}} = (\mathcal{T}_1, \ldots, \mathcal{T}_d)$ be a vector tree and $\vec{\mathcal{W}} = (\mathcal{W}_1, \ldots., \mathcal{W}_d)$ a vector subset of $\vec{\mathcal{T}}$. Then for each $n \in \mathbb{N}$, we set $\vec{\mathcal{T}}(n) = (\mathcal{T}_1(n), \ldots, \mathcal{T}_d(n))$ and we call $\vec{\mathcal{T}}(n)$ the n^{th}* **level** *of $\vec{\mathcal{T}}$.*
Similarly we set $\vec{\mathcal{W}}(n) = (\mathcal{W}_1(n), \ldots., \mathcal{W}_d(n))$.
If M is an infinite subset of $\mathbb{N}$ we set $\vec{\mathcal{W}}|_M = \bigcup_{m \in M} \vec{\mathcal{W}}(m)$.

The following proposition is a generalization of Proposition 1.1.

Proposition 1.2. *Let $\vec{\mathcal{W}}$ be a compatible vector subset of a vector tree $\vec{\mathcal{T}}$ dense (or $\vec{t}-$dense) in $\vec{\mathcal{T}}$. Then for every infinite subset M of the level set $L(\vec{\mathcal{W}})$ of $\vec{\mathcal{W}}$, the vector subset $\vec{\mathcal{W}}|_M$ is dense (or $\vec{t}-$dense) in $\vec{\mathcal{T}}$ as well.*

Definition 1.7. *Let $\vec{\mathcal{T}}$ be a vector tree and $\vec{\mathcal{W}}^1 = (\mathcal{W}_1^1, \ldots, \mathcal{W}_d^1)$, $\vec{\mathcal{W}}^2 = (\mathcal{W}_1^2, \ldots, \mathcal{W}_d^2)$ vector subsets of $\vec{\mathcal{T}}$. Then $\vec{\mathcal{W}}^2$ is said to* **dominate** *$\vec{\mathcal{W}}^1$ if $\mathcal{W}_i^2$ dominates $\mathcal{W}_i^1$ for all $i = 1, \ldots, d$.*

Definition 1.8. *Let $\vec{\mathcal{T}}$ be a vector tree.*
(a) For a vector element $\vec{t} = (t_1, \ldots, t_d\ )$ of $\vec{\mathcal{T}}$ we define the **order** *$|\vec{t}|$ of $\vec{t}$ to be the number $|\vec{t}| = \max\{|t_i| : i = 1, \ldots, d\}$.*
(b) For a vector element $\vec{t} = (t_1, \ldots, t_d\ )$ of $\vec{\mathcal{T}}$ and a vector subset $\vec{\mathcal{W}} = (\mathcal{W}_1, \ldots, \mathcal{W}_d)$ of $\vec{\mathcal{T}}$ we set $\vec{\mathcal{W}}_t = (\mathcal{W}_1', \ldots, \mathcal{W}_d')$ where for all $i = 1, \ldots, d$, $\mathcal{W}_i' = \{s \in \mathcal{W}_i : t_i \prec s \text{ and } |\vec{t}| < |s|\}$.

By Proposition 1.2, if $\vec{\mathcal{W}}$ is a compatible dense vector subset of $\vec{\mathcal{T}}$ then $\vec{\mathcal{W}}_{\vec{t}}$ is a compatible $\vec{t}-$dense vector subset of $\vec{\mathcal{T}}$.

Proposition 1.3. *Let $\vec{\mathcal{T}}$ be a vector tree and $\vec{\mathcal{W}}$ a compatible vector subset of $\vec{\mathcal{T}}$ with level set $L(\vec{\mathcal{W}}) = \{l_n\}_n$. Then*

(a) The vector subset $\vec{\mathcal{W}}$ is dense in $\vec{\mathcal{T}}$ if and only if $\vec{\mathcal{W}}(l_n)$ dominates $\vec{\mathcal{T}}(n)$ for all $n \in \mathbb{N}$.

(b) If for all $n \in \mathbb{N}$, $\vec{\mathcal{W}}(l_n)$ dominates $\vec{\mathcal{T}}_{\vec{t}}(|\vec{t}| + n)$ then $\vec{\mathcal{W}}$ is $\vec{t}-$dense in $\vec{\mathcal{T}}$.

Proof. The statement (a) is obvious. We shall prove (b).

Let $d \geq 1$, $\vec{\mathcal{T}} = (\mathcal{T}^1, \ldots, \mathcal{T}^d)$, $\vec{\mathcal{W}}=(\mathcal{W}^1, \ldots, \mathcal{W}^d)$ and $\vec{t} = (t_1, \ldots., t_d)$. We observe that for all $n \in \mathbb{N}$, $\vec{\mathcal{T}}_{\vec{t}}(|\vec{t}| + n) = \left(\mathcal{T}_{t_1}^1(|\vec{t}| + n), \ldots, \mathcal{T}_{t_d}^d(|\vec{t}| + n)\right)$. If $\vec{\mathcal{W}}(l_n)$ dominates $\vec{\mathcal{T}}_{\vec{t}}(|\vec{t}| + n)$ then $\mathcal{W}^i(l_n)$ dominates $\mathcal{T}_{t_i}^i(|\vec{t}| + n)$ and hence $\mathcal{T}_{t_i}^i(n)$ for all $i = 1, \ldots, d$ and $n \in \mathbb{N}$. Therefore $\mathcal{W}^i$ is t_i-dense in $\mathcal{T}^{\ i}$ that is $\vec{\mathcal{W}}$ is $\vec{t}-$dense in $\vec{\mathcal{T}}$. $\square$

Halpern-Läuchli partition theorem and our main theorem concerns finite partitions of the level product of any $d-$vector tree $\vec{\mathcal{T}}$. The definition of the level product has as follows:

Definition 1.9. *Let $\vec{\mathcal{T}}=(\mathcal{T}_1,...,\mathcal{T}_d)$ be a $d-$vector tree. We denote by $\otimes\vec{\mathcal{T}}$ or $\otimes_{i=1}^d\vec{\mathcal{T}}$ the* ***level product*** *of $\vec{\mathcal{T}}$ which is equal to $\bigcup_{k=0}^\infty\prod_{i=1}^d\mathcal{T}_i(k)$.*
Similarly, for a compatible vector subset $\vec{\mathcal{W}}=(\mathcal{W}_1,....,\mathcal{W}_d)$ of $\vec{\mathcal{T}}$ the level product $\otimes\vec{\mathcal{W}}$ or $\otimes_{i=1}^d\mathcal{W}_i$ is also defined.

For $\vec{t}\in\otimes\vec{\mathcal{T}}$ we observe that $\vec{t}$ is a vector element of $\vec{\mathcal{T}}$ and $|\vec{t}|$ is the unique $k\in\mathbb{N}$ such that $\vec{t}\in\prod_{i=1}^d\mathcal{T}_i(k)$.

In the sequel we shall deal exclusively with compatible vector subsets of $\vec{\mathcal{T}}$. Hence whenever we say that "$\vec{\mathcal{W}}$ is a vector subset of $\vec{\mathcal{T}}$" we shall always mean that $\vec{\mathcal{W}}$ is additionally compatible.

2. The proof of Theorem 0.2

Let us observe that the statement of the theorem allows us to reduce, by induction on p, the proof to the case $p=2$. We recall that for $\vec{\mathcal{W}}$ vector subset of $\vec{T}$, the level product $\otimes\vec{\mathcal{W}}$ is called 1-*homogeneous* or 2-*homogeneous* provided that $\otimes\vec{\mathcal{W}}\subset C_1$ or $\otimes\vec{\mathcal{W}}\subset C_2$ respectively. By the above it is clear that it suffices to prove the following:

Theorem 2.1. *Let $\vec{\mathcal{T}}$ be a $d-$vector tree and $\vec{\mathcal{W}}$ a dense vector subset of $\vec{\mathcal{T}}$. Then for each partition $\otimes\vec{\mathcal{W}}=C_1\cup C_2$ one of the following holds:*
(a) There exists $\vec{\mathcal{W}}'\prec\vec{\mathcal{W}}$ dense in $\vec{\mathcal{T}}$ such that $\otimes\vec{\mathcal{W}}'$ is 1-homogeneous.
(b) There exists $\vec{t}\in\vec{\mathcal{T}}$ and $\vec{\mathcal{W}}'\prec\vec{\mathcal{W}}$ $\vec{t}-$dense in $\vec{\mathcal{T}}$ such that $\otimes\vec{\mathcal{W}}'$ is 2-homogeneous.

The proof of the above theorem will be given by induction on d. The case of $d=1$ is well known. For the sake of completeness we present a proof of it.

Proposition 2.2. *Let $\mathcal{T}$ be a tree and $\mathcal{W}$ a dense subset of $\mathcal{T}$. Then for each partition $\mathcal{W}=C_1\cup C_2$ either there exists an 1-homogeneous $\mathcal{W}'\subset\mathcal{W}$ dense in $\mathcal{T}$, or there exists $t\in\mathcal{T}$ and a 2-homogeneous $\mathcal{W}'\subset\mathcal{W}$ $t-$dense in $\mathcal{T}$.*

Proof. Let $L(\mathcal{W})=\{l_k\}_k$ be the level set of $\mathcal{W}$. We consider the following two alternative cases:

Case 1. For each $t\in\mathcal{T}$ there exists $k_t\in\mathbb{N}$ such that for all $k\geq k_t$, $\mathcal{W}(l_k)\cap\mathcal{T}_t\cap C_1\neq\emptyset$.

Case 2. There exist $t\in T$ and a strictly increasing sequence $(k_n)_n$ such that for all $n\in\mathbb{N}$, $\mathcal{W}(l_{k_n})\cap\mathcal{T}_t\subset C_2$.

In case 1, we can easily construct by induction a subset $\mathcal{W}'$ of $\mathcal{W}$ dense in $\mathcal{T}$ and $1-$homogeneous.

In case 2, we set $\mathcal{W}'=\cup_{n\in\mathbb{N}}\mathcal{W}(l_{k_n})\cap\mathcal{T}_t$. Then $\mathcal{W}'$ is $t-$dense in $\mathcal{T}$ and $\mathcal{W}'$ is 2-homogeneous. □

Notation. (i) Let $\mathcal{T}$ be a tree and $\mathcal{W}\subset\mathcal{T}$. We denote by $\mathcal{C}_\infty(\mathcal{W})$ the set of all infinite linearly ordered subsets of $\mathcal{W}$. Every element of $\mathcal{C}_\infty(\mathcal{W})$ will be called a *chain* of $\mathcal{W}$. For each $c\in\mathcal{C}_\infty(\mathcal{W})$ the level set $L(c)$ of c is the set $L(c)=\{|t|:t\in c\}$. The notation $c'\prec c$ denotes that c' is an infinite subset of c and hence c' is also a chain. Observe that if $\mathcal{W}$ is dense or $t-$dense in $\mathcal{T}$ for some $t\in\mathcal{T}$ then $\mathcal{C}_\infty(\mathcal{W})$ is not empty.

(ii) In the sequel it will be notational convenient to denote any $d+1-$vector tree by $(\mathcal{S},\vec{\mathcal{T}})$ where $\mathcal{S}$ is a tree and $\vec{\mathcal{T}}=(\mathcal{T}_1,....,\mathcal{T}_d)$ is a $d-$vector tree. A vector subset of $(\mathcal{S},\vec{\mathcal{T}})$ will be denoted by $(\mathcal{V},\vec{\mathcal{W}})$ and by this we shall mean that $\mathcal{V}$ is a subset of $\mathcal{S}$, $\vec{\mathcal{W}}$ is a vector subset of $\vec{\mathcal{T}}$ and $(\mathcal{V},\vec{\mathcal{W}})$ is compatible. Also, whenever we say that $(\mathcal{V},\vec{\mathcal{W}})$ is $(s,\vec{t})$ dense in $(\mathcal{S},\vec{\mathcal{T}})$ we shall mean that $\mathcal{V}$ is $s-$dense in $\mathcal{S}$ and $\vec{\mathcal{W}}$ is $\vec{t}-$dense in $\vec{\mathcal{T}}$.

Hereafter we assume that Theorem 2.1 has been established for some $d\geq 1$. We fix a $d+1-$vector tree $(\mathcal{S},\vec{\mathcal{T}})$, a vector subset $(\mathcal{V}_0,\vec{\mathcal{W}}_0)$ of $(\mathcal{S},\vec{\mathcal{T}})$ dense in $(\mathcal{S},\vec{\mathcal{T}})$, and a partition $\mathcal{V}_0\otimes\vec{\mathcal{W}}_0=C_1\cup C_2$. We shall prove that Theorem 2.1 holds for this case as well.

Our inductive assumption yields the following proposition which plays key role in the rest of the proof.

Proposition 2.3. *Let $c\in\mathcal{C}_\infty(\mathcal{V}_0)$ and $\vec{\mathcal{W}}\prec\vec{\mathcal{W}}_0$ dense in $\vec{\mathcal{T}}$ such that $(c,\vec{\mathcal{W}})$ is compatible. Then either*
(a) There exist $c'\prec c$ and $\vec{\mathcal{W}}'\prec\vec{\mathcal{W}}$ dense in $\vec{\mathcal{T}}$ such that $(c',\vec{\mathcal{W}}')$ is compatible and $c'\otimes\vec{\mathcal{W}}'$ is 1-homogeneous, or
(b) There exist $c'\prec c$, a vector element $\vec{t}$ of $\vec{\mathcal{T}}$ and $\vec{\mathcal{W}}'\prec\vec{\mathcal{W}}$ $\vec{t}-$dense in $\vec{\mathcal{T}}$ such that $(c',\vec{\mathcal{W}}')$ is compatible and $c'\otimes\vec{\mathcal{W}}'$ is 2-homogeneous.

Proof. Let $L(\vec{\mathcal{W}})=\{l_k\}_k$ be the level set of $\vec{\mathcal{W}}$. Then $c=\{s_k\}$ where $s_k\in\mathcal{V}_0$ and $|s_k|=l_k$ for all $k\in\mathbb{N}$. We consider the following partition of $\otimes\vec{\mathcal{W}}$:
$\otimes\vec{\mathcal{W}}=\tilde{C}_1\cup\tilde{C}_2$ where
$\tilde{C}_1=\Big\{\vec{t}:\vec{t}\in\otimes\vec{\mathcal{W}},\ |\vec{t}|=k \text{ for some } k\in\mathbb{N} \text{ and } (s_k,\vec{t})\in C_1\Big\}$
$\tilde{C}_2=\Big\{\vec{t}:\vec{t}\in\otimes\vec{\mathcal{W}},\ |\vec{t}|=k \text{ for some } k\in\mathbb{N} \text{ and } (s_k,\vec{t})\in C_2\Big\}$.

By our inductive assumption (recall that $\vec{\mathcal{W}}$ is dense in the $d-$vector tree $\vec{\mathcal{T}}$) either there exists $\vec{\mathcal{W}}'\prec\vec{\mathcal{W}}$ dense in $\vec{\mathcal{T}}$ such that $\otimes\vec{\mathcal{W}}'$ is 1-homogeneous or there exist $\vec{t}\in\vec{\mathcal{T}}$ and $\vec{\mathcal{W}}'\prec\vec{\mathcal{W}}$ $\vec{t}-$dense in $\vec{\mathcal{T}}$ such that $\otimes\vec{\mathcal{W}}'$ is 2-homogeneous.

Hence, if $c'=c|_{L(\vec{\mathcal{W}}')}$ then either $c'\otimes\vec{\mathcal{W}}'$ is 1-homogeneous or $c'\otimes\vec{\mathcal{W}}'$ is 2-homogeneous and the proof of Proposition 2.3 is complete. □

For each $(\mathcal{V},\vec{\mathcal{W}})\prec(\mathcal{V}_0,\vec{\mathcal{W}}_0)$ dense vector subset of $(\mathcal{S},\vec{\mathcal{T}})$, for each vector element $(s,\vec{t})$ of $(\mathcal{S},\vec{\mathcal{T}})$ and for $\varepsilon\in\{1,2\}$ we consider the following two statements:

$\mathbf{Q}\left(\varepsilon,(s,\vec{t}),(\mathcal{V},\vec{\mathcal{W}})\right)$: For each $s'\in\mathcal{V}_s$ and each $\vec{\mathcal{W}}'\prec\vec{\mathcal{W}}$ with $\vec{\mathcal{W}}'$ dense in $\vec{\mathcal{T}}$ there exist $c\in\mathcal{C}_\infty(\mathcal{V}_{s'})$ and $\vec{\mathcal{W}}''\prec\vec{\mathcal{W}}'$ dense in $\vec{\mathcal{T}}$ such that $\left(c,\vec{\mathcal{W}}''_{\vec{t}}\right)$ is compatible and $c\otimes\vec{\mathcal{W}}''_{\vec{t}}$ is $\varepsilon-$homogeneous.

$\mathbf{Q}(\varepsilon,\vec{0},(\mathcal{V},\vec{\mathcal{W}}))$: For each $s'\in\mathcal{V}$ and each $\vec{\mathcal{W}}'\prec\vec{\mathcal{W}}$ with $\vec{\mathcal{W}}'$ dense in $\vec{\mathcal{T}}$ there exist $c\in\mathcal{C}_\infty(\mathcal{V}_{s'})$ and $\vec{\mathcal{W}}''\prec\vec{\mathcal{W}}'$ dense in $\vec{\mathcal{T}}$ such that $\left(c,\vec{\mathcal{W}}''\right)$ is compatible and $c\otimes\vec{\mathcal{W}}''$ is $\varepsilon-$homogeneous.

The first of the two main steps of the proof for the case $d+1$ is the following:

Proposition 2.4. *Let $(s,\vec{t})$ be a vector element of $(\mathcal{S},\vec{\mathcal{T}})$, $(\mathcal{V},\vec{\mathcal{W}})\prec(\mathcal{V}_0,\vec{\mathcal{W}}_0)$ dense vector subset of $(\mathcal{S},\vec{\mathcal{T}})$ and $\varepsilon\in\{1,2\}$ such that $\mathbf{Q}\Big(\varepsilon,(s,\vec{t}),(\mathcal{V},\vec{\mathcal{W}})\Big)$ holds. Then there* *exist $(\mathcal{V}',\vec{\mathcal{W}}') \prec (\mathcal{V},\vec{\mathcal{W}})$, $(s,\vec{t})-$dense in $(\mathcal{S},\vec{\mathcal{T}})$ such that $\mathcal{V}'\otimes\vec{\mathcal{W}}'$ is $\varepsilon-$homogeneous. In the case where $\boldsymbol{Q}(\varepsilon,\vec{0},(\mathcal{V},\vec{\mathcal{W}}))$ holds the resulting $(\mathcal{V}',\vec{\mathcal{W}}')$ is dense in $(\mathcal{S},\vec{\mathcal{T}})$.*

Proof. Suppose that $Q(\varepsilon,(s,,\vec{t}),(\mathcal{V},\vec{\mathcal{W}}))$ holds, for ε, $(s,\vec{t})$, $(\mathcal{V},\vec{\mathcal{W}})$ as in the statement of the proposition. By induction we construct a strictly increasing sequence $(m_n)_n$ of positive integers, a sequence $(\mathcal{V}_n)_n$ of subsets of $\mathcal{V}$ and a sequence $(\vec{\mathcal{W}}_n)_n$ of vector subsets of $\vec{\mathcal{W}}$ such that for all $n\in\mathbb{N}$:

(i) The set $\mathcal{V}_n$ dominates $\mathcal{S}_s(|s|+n)$ and $\mathcal{V}_n\subseteq\mathcal{V}_s(m_n)$

(ii) The vector set $\vec{\mathcal{W}}_n$ dominates $\vec{\mathcal{T}}_{\vec{t}}(|\vec{t}|+n)$ and $\vec{\mathcal{W}}_n\prec\vec{\mathcal{W}}_{\vec{t}}(m_n)$ (We recall that $|\vec{t}|=\max\{|t_i|:i=1,\dots,d\}$ where $\vec{t}=(t_1,\dots,t_d)$)

(iii) The level product $\mathcal{V}_n\otimes\vec{\mathcal{W}}_n$ is $\varepsilon-$homogeneous.

If the above construction has been done, then we set

$$\mathcal{V}'=\bigcup_{n\in\mathbb{N}}\mathcal{V}_n,\ \vec{\mathcal{W}}'=\bigcup_{n\in\mathbb{N}}\vec{\mathcal{W}}_n.$$

From (i), (ii) and Proposition 1.3 (b) it is clear that $\left(\mathcal{V}',\vec{\mathcal{W}}'\right)\prec\left(\mathcal{V},\vec{\mathcal{W}}\right)$ and $\left(\mathcal{V}',\vec{\mathcal{W}}'\right)$ is $\left(s,\vec{t}\right)$ dense in $\left(\mathcal{S},\vec{\mathcal{T}}\right)$. Also, $\mathcal{V}'\otimes\vec{\mathcal{W}}'=\bigcup_{n\in\mathbb{N}}\mathcal{V}_n\otimes\vec{\mathcal{W}}_n$ and hence from (iii) it is $\varepsilon-$homogeneous.

The general inductive step for our construction goes as follows:

Assume that $(m_i)_{i=1}^n$, $(\mathcal{V}_i)_{i=1}^n$, $(\vec{\mathcal{W}}_i)_{i=1}^n$ have been constructed so that (i)-(iii) are fulfilled. Since $\mathcal{V}$ is dense in $\mathcal{S}$ there exist a $m\in L(\mathcal{V})$ such that $\mathcal{V}(m)$ dominates $\mathcal{S}(|s|+n+1)$. Then actually $\mathcal{V}_s(m)$ is non empty and dominates $\mathcal{S}_s(|s|+n+1)$. Let $\mathcal{V}_s(m)=\{s_k\}_{k=1}^r$. Since $Q(\varepsilon,(s,\vec{t}),(\mathcal{V},\vec{\mathcal{W}}))$ holds, by induction we construct

(i') A finite sequence $(c_k)_{k=1}^r$ of chains such that $c_k\in\mathcal{C}_\infty(\mathcal{V}_{s_k})$ for all $1\le k\le r$,

(ii') A decreasing finite sequence $(\vec{\mathcal{W}}^k)_{k=1}^r$ of dense vector subsets of $\vec{\mathcal{T}}$ such that $\vec{\mathcal{W}}\succ\vec{\mathcal{W}}^1\succ\ldots\succ\vec{\mathcal{W}}^r$ with the property that:

For all $1\le k\le r$, $(c_k,\vec{\mathcal{W}}^k_{\vec{t}})$ is compatible and $c_k\otimes\vec{\mathcal{W}}^k_{\vec{t}}$ is $\varepsilon-$ homogeneous.

We observe that if $c'_k=c_k|_{L(\vec{\mathcal{W}}^r_{\vec{t}})}$ then $c'_k\otimes\vec{\mathcal{W}}^r_{\vec{t}}$ is $\varepsilon-$homogeneous for all $1\le k\le r$.

Choose $m_{n+1}\in L(\vec{\mathcal{W}}^r)$ such that $m_{n+1}>m_n$ and $\vec{\mathcal{W}}^r(m_{n+1})$ dominates $\vec{\mathcal{T}}(|\vec{t}|+n+1)$. Then also $\vec{\mathcal{W}}^r_{\vec{t}}(m_{n+1})$ dominates $\vec{\mathcal{T}}_{\vec{t}}(|\vec{t}|+n+1)$. We set

$$\mathcal{V}_{n+1}=\bigcup_{k=1}^r c'_k(m_{n+1})\text{ and }\vec{\mathcal{W}}_{n+1}=\vec{\mathcal{W}}^r_{\vec{t}}(m_{n+1})$$

Observe that $\mathcal{V}_{n+1}$ dominates $\mathcal{V}_s(m)$ and hence it dominates $\mathcal{S}_s(|s|+n+1)$. Also, $\mathcal{V}_{n+1}\otimes\vec{\mathcal{W}}_{n+1}\subset\bigcup_{k=1}^r c'_k\otimes\vec{\mathcal{W}}^r_{\vec{t}}$ and so by the preceding construction, $\mathcal{V}_{n+1}\otimes\vec{\mathcal{W}}_{n+1}$ is $\varepsilon-$homogeneous.

The case where $Q(\varepsilon,\vec{0},(\mathcal{V},\vec{\mathcal{W}}))$ holds, is similarly treated. Namely we construct sequences $(m_n)_n$, $(\mathcal{V}_n)_n$ and $(\vec{\mathcal{W}})_n$ such that $(m_n)_n$ is strictly increasing and for all $n\in\mathbb{N}$, $\mathcal{V}_n\subset\mathcal{V}(m_n)$, $\vec{\mathcal{W}}_n\subset\vec{\mathcal{W}}(m_n)$, $\mathcal{V}_n$ dominates $\mathcal{S}(n)$, $\vec{\mathcal{W}}_n$ dominates $\vec{\mathcal{T}}(n)$ and $\mathcal{V}_n\otimes\vec{\mathcal{W}}_n$ is $\varepsilon-$homogeneous. After this we set $\mathcal{V}'=\bigcup_n\mathcal{V}_n$, $\vec{\mathcal{W}}'_n=\bigcup_n\vec{\mathcal{W}}_n$ and it is clear that $(\mathcal{V}',\vec{\mathcal{W}}')$ is dense in $(\mathcal{S},\vec{\mathcal{T}})$ and $\mathcal{V}'\otimes\vec{\mathcal{W}}'$ is $\varepsilon-$homogeneous. □

The second step of the proof is the next.

Proposition 2.5. *For each $(\mathcal{V},\vec{\mathcal{W}})\prec(\mathcal{V}_0,\vec{\mathcal{W}}_0)$ dense vector subset of $(\mathcal{S},\vec{\mathcal{T}})$ either*

(a) $\mathbf{Q}(1,\vec{0},(\mathcal{V},\vec{\mathcal{W}}))$ holds, or

(b) There exist $(s,\vec{t})$ vector element of $(\mathcal{S},\vec{\mathcal{T}})$ and $(\mathcal{V}',\vec{\mathcal{W}}')\prec(\mathcal{V},\vec{\mathcal{W}})$ dense in $\vec{\mathcal{T}}$ such that $\mathbf{Q}(2,(s,\vec{t}),(\mathcal{V}',\vec{\mathcal{W}}'))$ holds.

Proof. Assume on the contrary that none of the cases (a) and (b) holds. Then choose $s_0 \in \mathcal{V}$ and $\vec{\mathcal{W}}^0 \prec \vec{\mathcal{W}}$ dense in $\vec{\mathcal{T}}$ witnessing the failure of $\mathbf{Q}(1, \vec{0}, (\mathcal{V}, \vec{\mathcal{W}}))$. Next fix an enumeration $(\vec{t}_n)_n$ of all the vector elements of $\vec{\mathcal{T}}$. Since (b) fails, we construct by induction two sequences $(s'_n)_{n=1}^{\infty}$, $(s_n)_{n=1}^{\infty}$ in $\mathcal{S}$, as well as, a sequence $\left(\vec{\mathcal{W}}^n\right)_{n=1}^{\infty}$ of vector subsets of $\vec{\mathcal{T}}$ and a strictly increasing sequence $(m_n)_{n=1}^{\infty}$ of natural numbers such that the following are fulfilled for all $n = 1, 2, ...$

(i) The elements s'_n, s_n belong to $\mathcal{V}_{s_0}$, $s_n \in \mathcal{V}(m_n)$ and $s_0 \prec s'_1 \prec s_1 \prec ... \prec s'_n \prec s_n$.

(ii) The vector subset $\vec{\mathcal{W}}^n$ of $\vec{\mathcal{T}}$ is dense in $\vec{\mathcal{T}}$, $\vec{\mathcal{W}}^n(m_n)$ dominates $\vec{\mathcal{T}}(n)$ and $\vec{\mathcal{W}}^0 \succ \vec{\mathcal{W}}^1 \succ ... \succ \vec{\mathcal{W}}^n$.

(iii) If $\mathcal{V}^n = \mathcal{V}|_{L(\vec{\mathcal{W}}^n)}$ then for all $c \in \mathcal{C}_\infty\left(\mathcal{V}^n_{s'_n}\right)$ and all $\vec{\mathcal{W}}' \prec \vec{\mathcal{W}}^n$ dense in $\vec{\mathcal{T}}$ such that $\left(c, \vec{\mathcal{W}}'_{\vec{t}_n}\right)$ is compatible, then $c \otimes \vec{\mathcal{W}}'_{\vec{t}_n}$ is not 2-homogeneous.

(iv) The integer m_n is greater than the order $|\vec{t}_n|$ of $\vec{t}_n$.

The general inductive step goes as follows.

Suppose that $(s'_k)_{k=1}^n$, $(s_k)_{k=1}^n$, $\left(\vec{\mathcal{W}}^k\right)_{k=1}^n$, $(m_k)_{k=1}^n$ have been defined so that (i)-(iv) hold. Since (b) fails we can choose an $s'_{n+1} \in \mathcal{V}^n_{s_n}$ and a $\vec{\mathcal{W}}^{n+1} \prec \vec{\mathcal{W}}^n$ dense in $\vec{\mathcal{T}}$ witnessing the failure of $\mathbf{Q}\Big(2, (s_n, \vec{t}_{n+1}), (\mathcal{V}^n, \vec{\mathcal{W}}^n)\Big)$.

Let $\mathcal{V}^{n+1} = \mathcal{V}|_{L(\vec{\mathcal{W}}^{n+1})}$ and observe that since $\mathcal{V}^{n+1} \subset \mathcal{V}^n$, condition (iii) remains valid with $n+1$ in place of n. It is clear that we can choose a sufficiently large integer m_{n+1} such that $m_{n+1} \in L\left(\vec{\mathcal{W}}^{n+1}\right)$, $m_{n+1} > \max\left\{|s'_{n+1}|, |\vec{t}_{n+1}|\right\}$, and also $\mathcal{V}(m_{n+1})$ dominates $\{s'_{n+1}\}$ and also $\vec{\mathcal{W}}^{n+1}(m_{n+1})$ dominates $\vec{\mathcal{T}}(n+1)$. Finally, we pick an $s_{n+1} \in \mathcal{V}_{s'_{n+1}}(m_{n+1})$. This completes the inductive construction.

We set $\vec{\mathcal{W}}_\infty = \bigcup_{n=1}^{\infty} \vec{\mathcal{W}}^n(m_n)$ and $c_\infty = \{s_n\}_{n=1}^{\infty}$.

From the above (i) and (ii) we obtain that $\vec{\mathcal{W}}_\infty \prec \vec{\mathcal{W}}^0$, $\vec{\mathcal{W}}_\infty$ is dense in $\vec{\mathcal{T}}$, $c_\infty \in \mathcal{C}_\infty(\mathcal{V}_{s_0})$ and $\left(c_\infty, \vec{\mathcal{W}}_\infty\right)$ is compatible. By our assumption about s_0, $\vec{\mathcal{W}}^0$ we have that for all $c \prec c_\infty$, $\vec{\mathcal{W}} \prec \vec{\mathcal{W}}_\infty$ dense in $\vec{\mathcal{T}}$ with $(c, \vec{\mathcal{W}})$ compatible, the level product $c \otimes \vec{\mathcal{W}}$ is not 1-homogeneous. Hence, by Proposition 2.3, there exist $c \prec c_\infty$, a vector element $\vec{t}$ of $\vec{\mathcal{T}}$ and a $\vec{\mathcal{W}} \prec \vec{\mathcal{W}}_\infty$ $\vec{t}-$dense in $\vec{\mathcal{T}}$, such that $(c, \vec{\mathcal{W}})$ is compatible and $c \otimes \vec{\mathcal{W}}$ is 2-homogeneous. There exists $n \in \mathbb{N}$ such that $\vec{t} = \vec{t}_n$.

Let $L = \{m_n, m_{n+1}, ...\}$, $c' = c|_L$, $\vec{\mathcal{W}}' = \vec{\mathcal{W}}|_L$.

Then by (i) and (ii), we have that $c' \in \mathcal{C}_\infty\left(\mathcal{V}^n_{s'_n}\right)$ and $\vec{\mathcal{W}}' \prec \vec{\mathcal{W}}^n$ is $\vec{t}_n-$ dense in $\vec{\mathcal{T}}$.

It is easy to see that $\vec{\mathcal{W}}'$ is extended to a dense in $\vec{\mathcal{T}}$ vector subset $\vec{\mathcal{W}}''$ of $\vec{\mathcal{W}}_\infty$ such that $L(\vec{\mathcal{W}}'') = L(\vec{\mathcal{W}}')$ and further $\mathcal{W}''_{\vec{t}_n} = \mathcal{W}'_{\vec{t}_n}$. Observe that $(c', \mathcal{W}'')$ is compatible and also $c' \otimes \vec{\mathcal{W}}''_{\vec{t}_n} = c' \otimes \vec{\mathcal{W}}'_{\vec{t}_n}$ is 2-homogeneous. This contradicts the assumption (iii) of the inductive construction and the proof is complete. $\square$

Proof of Theorem 0.2: Proposition 2.4, 2.5 immediately yield a proof of Theorem 2.1 for partitions with two elements. As we have already pointed out, the proof for an arbitrary finite partition follows by an easy inductive argument.

3. Consequences

This section contains some consequences of Theorem 0.2

We start with the proof of Theorem 0.1 stated in the introduction and which as we have mentioned is the equivalent statement of Theorem 0.2 in the context of metric spaces. Its proof requires the following lemmas. The first is well known and the proof follows by a direct inductive argument.

Lemma 3.1. *Let (X,ρ) be a totally bounded metric space. Then there exist a finitely branching tree $\mathcal{T}$ and a family of open non empty subsets of X, denoted by $\{G_t\}_{t\in\mathcal{T}}$, such that the following properties are fulfilled:*

(1) The tree $\mathcal{T}$ has a unique root denoted by $\rho(\mathcal{T})$ and $G_{\rho(\mathcal{T})}=X$
(2) If $t\in\mathcal{T}$, $|t|>1$, then $diam(G_t)<\frac{1}{|t|}$
(3) If $t,s\in\mathcal{T}$, $s\prec t$ then $G_t\subset G_s$
(4) $G_t=\bigcup_{s\in IS(t)}G_s$, for all $t\in\mathcal{T}$.

By $IS(t)$ we denote the set of all immediate successors of t in $\mathcal{T}$.

Lemma 3.2. *Let $\{(X_i,\rho_i)\}_{i=1}^d$ be a finite family of totally bounded metric spaces and $\{G_t\}_{t\in\mathcal{T}_i}$ be the corresponding families of open sets resulting from the previous lemma. For $n\in\mathbb{N}$, $i=1,\dots,d$, let $F_{n,i}$ be an $\frac{1}{n}-$dense subset of X_i. Then there exist families $\{\{x_t\}_{t\in\mathcal{T}_i}\}_{i=1}^d$ and a strictly increasing sequence $(m_k)_k$ such that for all $i=1,\dots,d$ the following are fulfilled :*

(1) For all $t\in\mathcal{T}_i$, $x_t\in G_t$.
(2) For all $k\in\mathbb{N}$ $\{x_t:\ t\in\mathcal{T}_i,\ |t|=k\}\subset F_{m_k,i}$.

Proof. For each $i=1,\dots,d$ and $t\in\mathcal{T}_i$ we choose $y_t\in G_t$ and $\delta_t>0$ such that $B(y_t,\delta_t)\subset G_t$. Next for each $k\in\mathbb{N}$ we set $\delta_k=\min\{\delta_t:\ t\in\mathcal{T}_i,\ |t|=k, i=1,\dots .d\}$. Finally we select a strictly increasing sequence $(m_k)_{k\in\mathbb{N}}$ such that $\frac{1}{m_k}<\delta_k$ for all $k\in\mathbb{N}$. Observe that for $t\in\mathcal{T}_i$ with $|t|=k$, $F_{m_k,i}\cap B(y_t,\delta_t)$ is non empty and hence $F_{m_k,i}\cap G_t$ is non empty as well. We set x_t be any element of $F_{m_k,i}\cap G_t$. It can be readily checked that $\{\{x_t\}_{t\in\mathcal{T}_i}\}_{i=1}^d$ and $(m_k)_{k\in\mathbb{N}}$ have the desired properties. □

Proof of Theorem 0.1. By the preceding two lemmas, for each (X_i,ρ_i) we have a tree $\mathcal{T}_i$, a family $\{G_t\}_{t\in\mathcal{T}_i}$ and a family $\{x_t\}_{t\in\mathcal{T}_i}$ with the listed properties. Let $\vec{\mathcal{T}}=(\mathcal{T}_1,\dots,\mathcal{T}_d)$. Observe that the partition $\cup_{n=1}^{\infty}F_n=C_1\cup\dots C_p$ induces a corresponding partition of the level product $\otimes\vec{\mathcal{T}}$ which is as follows: $\otimes\vec{\mathcal{T}}=C_1'\cup\dots\cup C_p'$ where for each $j\in\{1,\dots,p\}$, $C_j'=\{\vec{t}:\vec{t}\in\otimes\vec{\mathcal{T}},\ \vec{t}=(t_1,\dots,t_d)\ and\ (x_{t_1},\dots,x_{t_d})\in C_j\}$.

By Theorem 0.2 either there exists a dense vector subset $\vec{\mathcal{W}}$ of $\vec{\mathcal{T}}$ such that $\otimes\vec{\mathcal{W}}$ is 1-homogeneous or there exist a $\vec{t}=(t_1,\dots,t_d)$ and a $\vec{t}$-dense vector subset $\vec{\mathcal{W}}$ of $\vec{\mathcal{T}}$ such that $\otimes\vec{\mathcal{W}}$ is $j-homogeneous$ for some $j\in\{2,\dots,p\}$. Let $\vec{\mathcal{W}}=(\mathcal{W}_1,\dots,\mathcal{W}_d)$ and $L(\vec{\mathcal{W}})=\{\ell_k\}_{k\in\mathbb{N}}$ be the level set of $\vec{\mathcal{W}}$. Let also $(m_k)_k$ be the resulting sequence from the above lemma. If 1-homogenuity occurs then for each $i=1,\dots,d$ and $k\in\mathbb{N}$ we set $n_k=m_{\ell_k}$ and $B_{n_k,i}=\{x_t:t\in\mathcal{W}_i(\ell_k)\}$. We observe that the properties of $\{G_t\}_{t\in\mathcal{T}_i}$, $\{x_t\}_{t\in\mathcal{T}_i}$ yield that $B_{n_k,i}$ is $\frac{1}{k}$-dense in X_i, $B_{n_k,i}\subset F_{n_k,i}$ for all $1\le i\le d$ and since $\otimes\vec{\mathcal{W}}\subset\mathcal{C}_1'$, $\cup_{k=1}^{\infty}\prod_{i=1}^{d}B_{n_k,i}$ is 1-homogeneous. This proves Theorem 0.1 if the first alternative occurs.

If $\vec{\mathcal{W}}$ is $\vec{t}-$dense in $\vec{\mathcal{T}}$ and $\otimes\vec{\mathcal{W}}$ is $j-$homogeneous,for some $j>1$, then setting $\vec{t}=(t_1,\dots,t_d)$and also,

$$n_k=m_{\ell_k},\quad B_{n_k,i}=\{x_t:t\in\mathcal{W}_i(\ell_k),\ t\succ t_i\}\ \text{ and }\mathcal{V}_i=G_{t_i},$$

the properties of $\{G_t\}$, $\{x_t\}$ yield that $B_{n_k,i}\subset F_{n_k,i}$, $B_{n_k,i}$ is $\frac{1}{k}$-dense in $\mathcal{V}_i$ and in addition $\cup_{k=1}^{\infty}\prod_{i=1}^{d}B_{n_k,i}$ is $j-$homogeneous. This completes the proof of Theorem 0.1 □

We conclude this section with the Laver - Pincus theorem concerning strong subtrees of a vector tree $\vec{\mathcal{T}}$. We begin by recalling the definition of a strong subtree.

Definition 3.1. *A subtree $\mathcal{W}$ of a tree $\mathcal{T}$ is called a **strong subtree** of $\mathcal{T}$ if the following conditions are fulfilled.*
(i) For every $k \in \mathbb{N}$ there exists a $n_k \in \mathbb{N}$ such that $\mathcal{W}(k) \subset \mathcal{T}(n_k)$.
(ii) For every $t \in \mathcal{W}$ and $s \in IS(t,\mathcal{T})$ there exists a unique $s' \in IS(t,\mathcal{W})$ with $s \prec s'$.

An easy inductive argument yields that every dense in $\vec{\mathcal{T}}$ or $\vec{t}-$dense in $\vec{\mathcal{T}}$ subset of a finitely branching tree $\vec{\mathcal{T}}$ contains a strong subtree. This remark with Theorem 0.2 yield a proof of Laver- Pincus theorem which states the following:

Theorem 3.3 (Laver-Pincus)**.** *Let $(\mathcal{T}_i)_{i=1}^d$ be a finite sequence of trees and $\bigcup_{n=0}^{\infty}\prod_{i=1}^d \mathcal{T}_i(n) = \bigcup_{j=1}^p C_j$ for some $p \in \mathbb{N}$. Then there exist $j \in \{1,...,p\}$ and a sequence $(\mathcal{W}_i)_{i=1}^d$ with $\mathcal{W}_i$ strong subtree of $\mathcal{T}_i$ such that $\bigcup_{n=0}^{\infty}\prod_{i=1}^d \mathcal{W}_i(n) \subset C_j$.*

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