Άσκηση 1

$$\frac{dy}{dx} = \frac{x y^3}{\sqrt{1+x^2}} \Rightarrow \frac{dy}{y^3} = \frac{x\,dx}{\sqrt{1+x^2}} \Rightarrow \int \frac{dy}{y^3} = \int \frac{x\,dx}{(1+x^2)^{1/2}}$$

$$\Rightarrow -\frac{1}{2}\frac{1}{y^2} = (1+x^2)^{1/2} + C$$

$$y(0) = 1 \Rightarrow -\frac{1}{2} = 1 + C \Rightarrow C = -3/2$$

$$\Rightarrow \frac{1}{y^2} = 3 - 2\sqrt{1+x^2} \Rightarrow y(x) = \frac{1}{\sqrt{3-2(1+x^2)^{1/2}}}$$

Για να ορίζεται η λύση πρέπει $3-2(1+x^2)^{1/2} > 0 \Leftrightarrow$

$$\Leftrightarrow (1+x^2)^{1/2} < 3/2 \Leftrightarrow 1+x^2 < 9/4 \Leftrightarrow x^2 < 5/4$$

$$\Leftrightarrow |x| < \sqrt{5}/2$$

Άσκηση 2

~~Μία (απλή) δοκιμή~~ $y_1 = 1$. Η εξίσωση είναι ~~Riccati~~ Riccati

Αναζητώ τη γενική λύση στη μορφή $y = 1 + \frac{1}{u}$

$y' = -\frac{u'}{u^2}$ και η Δ.Ε. γράφεται:

$$-\frac{u'}{u^2} + (1-2x)\left(1+\frac{1}{u}\right) + (x-1)\left(1+\frac{2}{u}+\frac{1}{u^2}\right) + x = 0$$

$$\Leftrightarrow -u' + (1-2x)u + (x-1)\cdot 2\cdot u + (x-1) = 0$$

$$\Leftrightarrow -u' - u + (x-1) = 0 \Leftrightarrow \boxed{u' + u = x - 1}$$

$$\Rightarrow \quad (e^x u)' = e^x (x-1) \Rightarrow e^x u = \int e^x (x-1) + C$$

$$\int e^x (x-1)\,dx \underset{\substack{\uparrow \\ \text{ολοκλήρωση} \\ \text{κατά μέρη}}}{=} -\int e^x + e^x (x-1) = e^x (x-2)$$

Άρα $\quad e^x u = e^x (x-2) + C \Rightarrow u(x) = (x-2) + C e^{-x}$

και τελικά $\quad y(x) = 1 + (x-2+Ce^{-x})^{-1}$

Άσκηση 3

$$x^2 y'' - 2x y' + 2y = x \ln x, \quad x>0$$

Μη ομογενής Euler. $\quad x=e^t,\ t=\ln x$

$y(x) = z(t)$

$$y' = z'(t)\frac{1}{x}, \quad y'' = z''\frac{1}{x^2} - \frac{1}{x^2} z'$$

$$\Rightarrow (z''-z') - 2z' + 2z = te^t \Leftrightarrow \boxed{z''-3z'+2z = te^t.}$$

χ.π. $\quad k^2-3k+2 = (k-1)(k-2) = 0 \Rightarrow k = 1, 2$

Άρα $\quad z_{ομογ}(t) = c_1 e^t + c_2 e^{2t} \quad (\Rightarrow y_{ομ}(x) = c_1 x + c_2 x^2)$

Αναζητούμε μερική λύση $z_\mu(t)$, στη μορφή

$$z_\mu = t e^t (B_0 t + B_1) \quad (\text{το } \lambda=1 \text{ είναι λύση της χ.π.})$$

$$= B_0 t^2 e^t + B_1 t e^t$$

$$z' = 2B_0 t e^t + B_0 t^2 e^t + B_1 t e^t + B_1 e^t$$
$$= B_0 t^2 e^t + (2B_0 + B_1) t e^t + B_1 e^t$$

$$z'' = B_0 t^2 e^t + 2B_0 t e^t + (2B_0 + B_1) e^t + (2B_0 + B_1) t e^t$$
$$+ B_1 e^t = B_0 t^2 e^t + (4B_0 + B_1) t e^t + (2B_0 + 2B_1) e^t$$

$\Rightarrow$

$$z'' - 3z' + 2z = t e^t \quad \Leftrightarrow$$

$$B_0 t^2 e^t + (4B_0 + B_1) t e^t + (2B_0 + 2B_1) e^t - 3B_0 t^2 e^t -$$
$$- 3(2B_0 + B_1) t e^t - 3B_1 e^t + 2B_0 t^2 e^t + 2B_1 t e^t =$$
$$= t e^t$$

$$\Rightarrow (-2B_0) t e^t + (2B_0 - B_1) e^t = t e^t$$

$$\Rightarrow \begin{array}{l} -2B_0 = 1 \\ 2B_0 - B_1 = 0 \end{array} \Rightarrow \begin{array}{l} B_0 = -1/2 \\ B_1 = -1 \end{array}$$

$$\Rightarrow z_\mu = -\frac{1}{2} t^2 e^t - t e^t$$

Γυρίζω στις αρχικές μεταβλητές του προβλήματος

$$y_\gamma = y_{o\mu} + y_\mu = c_1 x + c_2 x^2 - \frac{1}{2} x (\ln x)^2 - x \ln x, \quad x > 0$$

Άσκηση 4

Αφού $\mu = \mu(z)$, $z = xy$ ολοκλ. παράγοντας $\Rightarrow$

$$(\mu y(1-xy))_y = (\mu x)_x \Rightarrow$$

$$\Rightarrow \mu' xy(1-xy) + \mu(1-xy) - \mu xy = \mu' xy + \mu$$

$$\Rightarrow -\mu'(xy)^2 - 2\mu xy = 0 \Rightarrow \mu' z + 2\mu = 0$$

$$\Rightarrow \frac{\mu'}{\mu} = -\frac{2}{z} \Rightarrow \ln\mu = \ln z^{-2} \Rightarrow \boxed{\mu(z) = \frac{1}{z^2} = \frac{1}{x^2y^2}}$$

Οπότε η εξίσωση γίνεται:

$$\frac{1-xy}{x^2y}dx + \frac{1}{xy^2}dy = 0$$

$$u_y = \frac{1}{xy^2} \Rightarrow u(x,y) = -\frac{1}{xy} + c(x)$$

$$u_x = \frac{1}{x^2y} + c'(x) = \frac{1}{x^2y} - \frac{1}{x} \Rightarrow c'(x) = -\frac{1}{x} = (-\ln x)'$$

$$\Rightarrow c(x) = -\ln|x| + C$$

$$\Rightarrow u(x,y) = -\frac{1}{xy} - \ln|x| = C$$

για $x = 1$, $y = 1$ $\Rightarrow -1 = C \Rightarrow c = -1$

$$\Rightarrow \frac{1}{xy} = 1 - \ln|x| \Rightarrow \boxed{y(x) = \frac{1}{x(1-\ln|x|)}}$$