Applications of Dynamics to Compact Manifolds of Negative Curvature

FRANÇOIS LEDRAPPIER

CNRS, Centre de Mathématiques
École Polytechnique
F-91128 Palaiseau, France

and

Laboratoire de Probabilités
Université Paris VI
F-75230 Paris, France

ABSTRACT. Consider closed Riemannian manifolds with negative sectional curvature. There are three natural dynamics associated with the Riemannian structure: the geodesic flow on the unit tangent bundle, the dynamics of the invariant foliations of the geodesic flow, and the Brownian motion on the universal cover of the manifold. These dynamics define global asymptotic objects such as growth rates or measures at infinity. For locally symmetric negatively curved spaces, these objects are easy to compute and to describe. In this paper, we survey some of their properties and relations in the general case.

1 Measures at infinity

Let (M, g) be a closed Riemannian manifold with negative sectional curvature and let $\pi : (\widetilde{M}, \tilde{g}) \to (M, g)$ be the universal cover of M, endowed with the canonically lifted metric $\tilde{g}$. The space $(\widetilde{M}, \tilde{g})$ is a simply connected Riemannian manifold with negative curvature; in particular, the space $(\widetilde{M}, \tilde{g})$ is a Hadamard manifold and the geometric boundary $\partial\widetilde{M}$ is defined as the space of ends of geodesics (see e.g. [BGS]). The geometric boundary $\partial\widetilde{M}$ is homeomorphic to a sphere. For any x in $\widetilde{M}$ write τ_x for the homeomorphism between the unit sphere $S_x\widetilde{M}$ in the tangent space at x and $\partial\widetilde{M}$ defined by associating to a unit vector v in $S_x\widetilde{M}$ the end $\tau_x(v)$ of the geodesic σ_v starting at v. In this section are defined natural families of finite positive measures on the boundary indexed by $x, x \in \widetilde{M}$.

(a) Lebesgue visibility measures. Let λ_x denote the image measure under τ_x of the Lebesgue measure on the unit sphere $S_x\widetilde{M}$. It follows from [A], [ASi] that for x and y in $\widetilde{M}$, the measures λ_x and λ_y have the same negligible sets and that the density $\frac{d\lambda_y}{d\lambda_x}$ admits a (Hölder) continuous version on $\partial\widetilde{M}$ (the metric on $\partial\widetilde{M}$ will be recalled below). Write λ for the common measure class of the $\lambda_x, x \in \widetilde{M}$.

(b) Harmonic measures. Let Δ be the Laplace-Beltrami operator on C^2-functions on $\widetilde{M}$, $\Delta =$ div grad. A function u on $\widetilde{M}$ is called harmonic if $\Delta u = 0$. The Dirichlet problem is solvable on $\widetilde{M} \cup \partial\widetilde{M}$ ([An], [S]): let f be a continuous function on $\partial\widetilde{M}$; there is a unique harmonic function u_f on $\widetilde{M}$ such that for all ξ in $\partial\widetilde{M}$,

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$\lim_{z \to \xi} u_f(z) = f(\xi)$. For any x in $\widetilde{M}$, the mapping $f \to u_f(x)$ defines a probability measure ω_x on $\partial\widetilde{M}$. The measure ω_x is called the harmonic measure of the point x. For x and y in $\widetilde{M}$, the measures ω_x and ω_y have the same negligible sets and the density $\frac{d\omega_y}{d\omega_x}$ admits a Hölder continuous version on $\partial\widetilde{M}$ called the Poisson kernel and denoted $k(x, y, \cdot)$ ([ASn], [Aa]). Write ω for the common measure class of the $\omega_x, x \in \widetilde{M}$.

(c) Margulis-Patterson measures. For two points (ξ, η) in $\partial\widetilde{M}$, and x in $\widetilde{M}$, define the Gromov product $(\xi, \eta)_x$ by:

$$(\xi, \eta)_x = \lim_{\substack{y \to \xi \\ z \to \eta}} \frac{1}{2} \left(d(x, y) + d(x, z) - d(y, z) \right)$$

(see e.g. [GH]). Set $d_x(\xi, \eta) = \exp -(\xi, \eta)_x$ and define balls, spherical Hausdorff measures, and spherical Hausdorff dimension as if d_x was a distance on $\partial\widetilde{M}$ (in fact, there is $\alpha > 0$ so that d_x^α is a distance on $\partial\widetilde{M}$). Let H be the spherical Hausdorff dimension of $\partial\widetilde{M}$. The spherical H-Hausdorff measure ν_x is positive and finite and the measure ν_x is called the Margulis-Patterson measure of the point x. For x, y in $\widetilde{M}$, the measures ν_x and ν_y have the same negligible sets.

Recall that for x in $\widetilde{M}$, ξ in $\partial\widetilde{M}$, the Busemann function $b_{x,\xi}$ is a function on $\widetilde{M}$ defined by

$$b_{x,\xi}(y) = \lim_{t \to \infty} d(y, \sigma_v(t)) - t \ ,$$

where σ_v is the geodesic in $\widetilde{M}$ starting at $v = \tau_x^{-1}\xi$. Then the density $\frac{d\nu_y}{d\nu_x}$ is given by

$$\frac{d\nu_y}{d\nu_x}(\xi) = \exp -Hb_{x,\xi}(y) \ .$$

Write ν for the common measure class of the $\nu_x, x \in \widetilde{M}$. The construction of this measure is essentially given in [M2]. The presentation and the properties given here are derived from [H1], [Ka3], and [L3].

(d) General properties. Let γ be an isometry of $\widetilde{M}$. Then the action of γ extends to $\partial\widetilde{M}$ and to measures on $\partial\widetilde{M}$. By naturality for $\mu = \lambda, \omega$, or ν:

$$\mu_{\gamma x} = \gamma \mu_x \ .$$

For $\mu = \lambda, \omega$, or ν, define a positive Radon measure $\tilde{\mu}$ on $S\widetilde{M}$ by setting

$$\int df\tilde{\mu} = \int \left(\int_{\partial\widetilde{M}} f(\tau_x^{-1}\xi)\, d\mu_x(\xi) \right) d\mathrm{vol}\ (x)$$

for any continuous function f on $S\widetilde{M}$ with compact support. The measure $\tilde{\mu}$ is invariant under the action of γ, and therefore defines a finite positive measure $\bar{\mu}$ on the quotient space SM.

In the case when $\widetilde{M}$ is a symmetric space of negative curvature, there is a compact group K_x of isometries of $\widetilde{M}$ that fixes x and acts transitively on $\partial\widetilde{M}$. Let m_x be the unique K_x-invariant probability measure on $\partial\widetilde{M}$. It follows from the above invariance relation that if (M, g) is locally symmetric there are constants a, b such that for all x in $\widetilde{M}$

$$a\lambda_x = \omega_x = b\nu_x = m_x \ .$$

Conversely, assume that there is a constant a, b, or c such that one of the following equalities of measures holds for all x in $\widetilde{M}$:

$$a\lambda_x = \omega_x, b\nu_x = \omega_x \text{ or } c\lambda_x = \nu_x \ .$$

Then the space (M, g) is locally symmetric. In order to prove this result, set for x in $\widetilde{M}$ and ξ in $\partial\widetilde{M}$:

$$B(x,\xi) = \Delta_y \ b_{x,\xi} \ (y)_{|y=x} \ ,$$

and observe that either hypothesis implies that B is constant ([L3], [Y]). A key result is that the function B is constant if and only if the space (M, g) is locally symmetric. This is immediate in dimension 2 and can be checked directly in dimension 3 (see e.g. [Kn]). In higher dimensions, the proof combines results from [FL], [BFL], and [BCG]. This result is used in the other characterizations of locally symmetric spaces that are given below.

2 Geodesic flow

The geodesic flow $(\theta_t)_{t\in\mathbf{R}}$ is a one-parameter group of diffeomorphisms of the unit tangent bundle SM, defined as follows: for v in SM, write $\{\sigma_v(t), t \in \mathbf{R}\}$ for the unit-speed geodesic starting at v. Then for any real t, $\theta_t v$ is the speed vector of the geodesic σ at $\sigma_v(t)$. A flot $(\theta_t)_{t\in\mathbf{R}}$ is called Anosov if there exist a metric $|| \ ||$ on TSM, numbers $C > 0$ and $\chi < 1$, and a Whitney decomposition of TSM as $E^{ss} \oplus E^{uu} \oplus \mathbf{R}X$, where X is the vector field generating the flow and for v in $E^{ss}, t > 0, ||D\theta_t v|| \leq C\chi^t ||v||$, for v in $E^{uu}, t > 0, ||D\theta_{-t} v|| \leq C\chi^t ||v||$.

Because of negative curvature, the geodesic flow is Anosov ([A]).

(a) Topological entropy. The number H is the topological entropy of the geodesic flow ([B]). There is a function c on M such that, uniformly on $\widetilde{M}$,

$$\lim_{R\to\infty} \exp(-HR) \text{ vol } B(x, R) = c(\pi x),$$

where $B(x, R)$ is the ball of radius R about x in $(\widetilde{M}, \tilde{g})$ and vol $B(x, R)$ its Riemannian volume ([M1]). Because $c(\pi x)$ is proportional to $\nu_x(\partial\widetilde{M})$, the function c is C^∞. The function c is in general not constant ([Kn]).

(b) Metric entropy. The measure $\bar\lambda$ is the Liouville measure; it is finite and invariant under the geodesic flow. Write $h_{\bar\lambda}$ for the Kolmogorov-Sinai entropy of the system

$$\left(SM, \frac{\bar\lambda}{\bar\lambda(SM)}, \theta_1\right) \;;\; h_{\bar\lambda} = \frac{\int B\, d\bar\lambda}{\bar\lambda(SM)} \qquad \text{(see[ASi])} .$$

From [LY] it follows that $h_{\bar\lambda}$ is the Hausdorff dimension of the λ measure class on $\partial\widetilde{M}$, i.e.

$$h_{\bar\lambda} = \inf \left\{\text{Hausdorff dimension } (A) : A \subset \partial\widetilde{M}, \lambda(A) > 0\right\} .$$

From the variational principle ([BR]) it follows that $h_{\bar\lambda} \leq H$ with equality if and only if the measure classes λ and ν coincide. In dimension 2, $h_{\bar\lambda} = H$ if and only if the curvature is constant ([K1]). In higher dimensions, the entropy rigidity problem is whether $h_\lambda = H$ if and only if the space (M, g) is locally symmetric.

(c) Regularity of the stable direction. In general, the distribution $E^s = E^{ss} \ominus \mathbf{R}X$ is only Hölder continuous. If the distribution is C^2, then $h_{\bar\lambda} = H$ ([H5]). If the distribution is C^∞, then the space (M, g) is locally symmetric (this follows again from [BFL] and [BCG]). The properties are more precise in the case of surfaces: the distribution E^s in C^1 ([Ho]), even $C^{1+\Lambda_*}$ ([HK]). If the distribution is $C^{1+o(s\cdot|\log s|)}$, then it is C^∞ ([HK]) and the curvature is constant ([Gh]). This discussion is a particular case of the analogous discussion for general Anosov flows (see [Gh], [BFL], [H4], and [F]).

3 Brownian motion on $\widetilde{M}$

Recall that Δ is the Laplace-Beltrami operator on $\widetilde{M}$ and write $p(t, x, y)$ for the fundamental solution of the equation $\frac{\partial u}{\partial t} = \Delta u$. The properties below reflect asymptotic properties of the Brownian motion on $\widetilde{M}$.

(a) Growth rates. There is a positive number ℓ such that, for all x in $\widetilde{M}$,

$$\ell = \lim_{t\to\infty} \frac{1}{t} \int_{\widetilde{M}} d(x, y) p(t, x, y)\, d\text{vol } (y)$$

(see [Gu]) and ℓ is given by $\ell = \frac{\int B d\bar\omega}{\bar\omega(SM)}$ ([Ka1]). There is another positive number h such that for all x in $\widetilde{M}$,

$$h = \lim_{t\to\infty} -\frac{1}{t} \int_{\widetilde{M}} p(t, x, y) \log p(t, x, y)\, d\text{vol } (y)$$

([Ka1]). Finally denote δ the spectral gap of Δ:

$$\delta = \inf_{f \in C_K^2(\widetilde{M})} \frac{-\int f\Delta f\, d\text{vol}}{\int f^2\, d\text{vol}} .$$

(b) Relations. The following inequalities hold:

(1) $h \leq \ell H$ with equality if and only if the measure classes ω and ν coincide ([L2]),

(2) $\ell^2 \leq h$ with equality if and only if the space (M, g) is locally symmetric ([Ka1]), and

(3) $4\delta \leq h$ with equality if and only if the space (M, g) is locally symmetric ([L4]).

Observe that other sharp inequalities can be directly derived from the above three:

$$\ell \leq H, \; h \leq H^2, \text{ or } 4\delta \leq H^2 \; .$$

Let m be the only θ-invariant probability measure on SM such that, if $\tilde{m}$ denotes the isometry-invariant extension of m to $\widetilde{SM}$, the projection of $\tilde{m}$ by $\tau = \{\tau_x, x \in \widetilde{M}\}$ on $\partial\widetilde{M}$ belongs to the ω-class ([L2], see also [H2], [Ka3]). Then h/ℓ is the Kolmogorov-Sinai entropy of the system $(SM, m; \theta_1)$ and also the Hausdorff dimension of the measure class ω. The first relation follows from the variational principle for the geodesic flow. The proof of the other two relations is based on an integral formula satisfied by the measure $\bar{\omega}$.

(c) Measure rigidity. The question again arises as to whether measure classes at infinity can coincide with ω only when the space (M, g) is locally symmetric. In dimension 2, the curvature is constant if and only if the measure classes ω and λ coincide ([K2], [L1]) or if and only if the measure classes ω and ν coincide ([L3], [H3]). Observe that this problem makes sense for other objects such as graphs. There are examples of finite graphs that are neither homogeneous nor bipartite, but such that some pair of natural measures at infinity has the same negligible sets [Ls].

4 Invariant foliations

Recall that the distribution E^{ss} is continuous in TSM and that it admits integral manifolds W^{ss} defined by

$$W^{ss}(v) = \{w : \lim_{t \to +\infty} d(\theta_t v, \theta_t w) = 0\}$$

(see [A]).

The W^{ss} form a continuous foliation with smooth leaves and there is a natural metric on the leaves, lifted from the metric g on M through the canonical projection. Let Δ^{ss} be the Laplace-Beltrami operator along the leaves W^{ss}. Then, for any continuous function f on SM, which is C^2 along the W^{ss} leaves, $\int \Delta^{ss} f \, d\bar{\nu} = 0$.

The measure $\bar{\nu}$ is — up to multiplication by a constant factor — the unique measure with that property (the proof of this uses results from [Ka2] and [BM]). The measure $\bar{\nu}/\bar{\nu}(SM)$ can also be seen as the limit of averages on large spheres in $\widetilde{SM}$ ([Kn]). In particular $H = \frac{\int B d\bar{\nu}}{\bar{\nu}(SM)}$.

(a) Stable foliation. The manifolds W^s given by $W^s(v) = \bigcup_{t\in\mathbf{R}} \theta_t W^{ss}(v)$ form a continuous foliation, with smooth leaves and with $TW^s = E^s$. Consider again the metric on the leaves lifted from the metric g on M, and let Δ^s be the corresponding Laplace-Beltrami operator. The measure $\bar{\omega}$ is — up to multiplication by a constant factor — the unique measure on SM satisfying $\int \Delta^s f \, d\bar{\omega} = 0$ for any continuous f, which is C^2 along the W^s leaves ([G]).

For a continuous function f on SM write $\tilde{f}$ for the continuous function on $\widetilde{M} \times \partial\widetilde{M}$ given by

$$\tilde{f}(x,\xi) = f \cdot \pi(x, \tau_x^{-1}\xi) .$$

Then for $t > 0$, there is a function $Q_t f$ on SM such that:

$$\widetilde{Q_t f}\ (x,\xi) = \int p(t,x,y)\ \tilde{f}(y,\xi)\ d\mathrm{vol}\ (y) .$$

The operator Q_t is the leafwise heat operator $Q_t = \exp\ t\ \Delta^s$.

There is a Hölder norm $|\ |$ on functions on SM with the following property: there are $C > 0$ and $\chi < 1$ such that for all $t > 0$ any function f on SM:

$$\left| Q_t\ f - \frac{\int f d\bar{\omega}}{\mathrm{vol}\ M} \right| < c\ \chi^t\ |f|$$

([L5]).

From this follow asymptotic properties of the Brownian motion on $\widetilde{M}$ and a decomposition theorem for closed regular leafwise 1-forms ([L6]). As a consequence define for $s \in \mathbf{R}$ the function $\varphi(s)$ by

$$\varphi(s) = \lim_{t\to\infty} \frac{1}{t} \log \max_{(x,\xi)} \int p(t,x,y)\ k^s(x,y,\xi)\ d\mathrm{vol}\ (y) .$$

The function φ is convex and analytic in a neighborhood of 0. The space (M,g) is locally symmetric if and only if $\varphi(s) = as(s-1)$ for some constant a (in fact a is then the common value of ℓ^2, H^2, h, or 4δ) or if and only if we have

$$2\ \varphi'(0) + \varphi''(0) = 0 .$$

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