SOME OPEN PROBLEMS ON MULTIPLE ERGODIC AVERAGES

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1. Problems related to general sequences

In this section we give a list of problems related to the study of multiple ergodic averages involving iterates given by general sequences and related applications to multiple recurrence results.

1.1. **The structure of multicorrelation sequences.** Given a probability space $(X, \mathcal{X}, \mu)$, commuting invertible measure preserving transformations $T_1, \ldots, T_\ell \colon X \to X$, and functions $f_1, \ldots, f_\ell \in L^\infty(\mu)$, we are interested in determining the structure of the multiple correlation sequences, meaning sequences $\mathcal{C} \colon \mathbb{Z}^\ell \to \mathbb{C}$ defined by the formula

$$\mathcal{C}(n_1, \ldots, n_\ell) := \int f_0 \cdot T_1^{n_1} f_1 \cdot \ldots \cdot T_\ell^{n_\ell} f_\ell \, d\mu. \tag{1}$$

The next result gives a very satisfactory solution to this problem for $\ell = 1$, and is going to serve as our model for possible generalizations. It can be deduced from Herglotz's theorem on positive definite sequences (the sequence $n \mapsto \int \bar{f} \cdot T^n f \, d\mu$ is positive definite) and a polarization identity.

Theorem. *Let $(X, \mathcal{X}, \mu, T)$ be an invertible measure preserving system and $f, g \in L^\infty(\mu)$. Then there exists a complex Borel measure σ on $[0, 1)$, with bounded variation, such that for every $n \in \mathbb{N}$ we have*

$$\int f \cdot T^n g \, d\mu = \int_0^1 e^{2\pi i n t} \, d\sigma(t). \tag{2}$$

Finding a formula analogous to (2), with the multiple correlation sequences (1) in place of the single correlation sequences, is a very natural and important open problem;[1] a satisfactory solution is going to give us new insights and significantly improve our ability to deal with multiple ergodic averages. There are indications that sequences of polynomial nature should replace the "linear" sequences $(e^{2\pi i n t})_{n \in \mathbb{N}}$. The most reasonable candidates at this point seem to be some collection of multivariable nilsequences.[2] For instance, examples of 2-step nilsequences in 2 variables are the sequences $\big(e^{i([m\alpha]n\beta + m\gamma + n\delta)}\big)$ ($[x]$ denotes the integer part of x), where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$.

Date: March 2011.

[1] I am not aware of a place in the literature where this problem is stated explicitly, but it has definitely been in the mind of experts for several years.

[2] If $X = G/\Gamma$ is a k-step nilmanifold, $b_1, \ldots, b_\ell \in G$, $x \in X$, and F is Riemann integrable on X, we call the sequence $(F(b_1^{n_1} \cdots b_\ell^{n_\ell} x))$ a *basic k-step nilsequence with ℓ-variables*. A *k-step nilsequence with ℓ-variables*, is a uniform limit of basic k-step nilsequences with ℓ-variables. As is easily verified, the collection of k-step nilsequences, with the topology of uniform convergence, forms a closed algebra.

Problem 1. *Determine the structure of the multiple correlation sequences $\mathcal{C}(n_1,\ldots,n_\ell)$ defined by* (1)*. Is it true that the building blocks are multi-variable nilsequences? More explicitly, is it true that there exists a complex Borel measure σ on a compact metric space X, of bounded variation, and multi-variable nilsequences $\mathcal{N}_x(n_1,n_2,\ldots,n_\ell)$, $x\in X$, such that $x\mapsto N_x$ is integrable with respect to σ and*

$$\mathcal{C}(n_1,n_2,\ldots,n_\ell)=\int_X \mathcal{N}_x(n_1,n_2,\ldots,n_\ell)\ d\sigma(x)?$$

Some commutativity assumption on the transformations is needed, otherwise simple examples show that nilsequences cannot be the only building blocks[3]. If the answer turns out to be positive, then several well known problems in the area will be solved immediately using some known equidistribution results on nilmanifolds; for instance, mean convergence for the averages $\frac{1}{N}\sum_{n=1}^N T_1^{p_1(n)}f_1\cdot\ldots\cdot T_\ell^{p_\ell(n)}f_\ell$ where $p_1,\ldots,p_\ell\in\mathbb{Z}[t]$, and mean convergence for the averages $\frac{1}{N}\sum_{n=1}^N T_1^{a_n(\omega)}f_1\cdot\ldots\cdot T_\ell^{a_n(\omega)}f_\ell$ where $(a_n(\omega))$ is a random non-lacunary sequence (both problems are stated carefully in subsequent sections).

Even resolving special cases of this problem would be extremely interesting. One particular instance is the following:

Special Case of Problem 1. *Let $(X,\mathcal{X},\mu,T)$ be an invertible ergodic measure preserving system and $f,g,h\in L^\infty(\mu)$. Determine the structure of the multiple correlation sequences $(\mathcal{C}(n))$ defined by*

$$\mathcal{C}(n):=\int f\cdot T^n g\cdot T^{2n}h\ d\mu.$$

Is it true that the building blocks are 2*-step nilsequences?*

In [2] it is shown that for ergodic systems one has the decomposition

$$\mathcal{C}(n)=\mathcal{N}(n)+e(n)$$

where $\mathcal{N}(n)$ is a (single variable) two step nilsequence and $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N|e(n)|=0$. Unfortunately, this result does not provide information on the error term $e(n)$ (other than it converges to 0 in density), and as a consequence it is of little use when one studies sparse subsequences of the sequence $\mathcal{C}(n)$.

1.2. **Necessary and sufficient conditions for ℓ-convergence.** The next result can be deduced from formula (2) and serves as our model for giving usable necessary and sufficient conditions for ℓ-convergence:

Theorem. *If $(a(n))$ is a sequence of integers, then the following statements are equivalent:*
- *The sequence $(a(n))$ is good for* 1*-convergence.*
- *The sequence $(a(n))$ is good for* 1*-convergence for rotations on the circle.*
- *The sequence $\left(\frac{1}{N}\sum_{n=1}^N e^{ia(n)t}\right)$ converges for every $t\in\mathbb{R}$.*

Since a formula that generalizes (2) to multiple correlation sequences is not available, we are unable to prove analogous necessary and sufficient conditions for ℓ-convergence. Nevertheless, inspired by Problem 1 we make the following natural guess:

[3]Let $T,S\colon\mathbb{T}\to\mathbb{T}$ be given by $Tx:=2x$, $Sx:=2x+\alpha$, and $f(x):=e^{-ix}, g(x):=e^{ix}$. Then $\int T^nf\cdot S^ng\ dx=e^{i\cdot(2^n-1)\alpha}$ and one can show that $(e^{i\cdot 2^n\alpha})$ is not a nilsequence for $\alpha\in\mathbb{R}\setminus\mathbb{Q}$.

Problem 2. *If $(a_1(n)), \dots, (a_\ell(n))$ are sequences of integers show that the following statements are equivalent:*

- *The sequences $(a_1(n)), \dots, (a_\ell(n))$ are good for ℓ-convergence of commuting transformations.*
- *The sequences $(a_1(n)), \dots, (a_\ell(n))$ are good for ℓ-convergence of ℓ-step nilsystems.*
- *The sequence $\big(\frac{1}{N}\sum_{n=1}^N \mathcal{N}(a_1(n), \dots, a_\ell(n))\big)$ converges for every ℓ-variable ℓ-step nilsequence $\mathcal{N}$.*

A similar problem was formulated in [5]. Although the equivalence of the second and third condition does not appear to be obvious, we are mainly interested in knowing if the third (or second) conditions implies the first. Even the following very special case is open:

Special Case of Problem 2. *Let $(a(n))$ be a sequence of integers such that the averages $\frac{1}{N}\sum_{n=1}^N \mathcal{N}(a(n))$ converge for every 2-step nilsequence $\mathcal{N}$. Show that for every invertible measure preserving system $(X, \mathcal{X}, \mu, T)$ and functions $f, g \in L^\infty(\mu)$, the averages*

$$\frac{1}{N}\sum_{n=1}^{N} T^{a(n)} f \cdot T^{2a(n)} g$$

converge in the mean.

This is known when $(a(n))$ is strictly increasing and its range has positive density; it follows from Theorem 1.9 in [2].

1.3. **Sufficient conditions for ℓ-recurrence.** Our model result is the next theorem of T. Kamae and M. Mendès-France [10] that gives usable conditions for checking that a sequence is good for 1-recurrence:

Theorem. *Let $(a(n))$ be sequence of integers that satisfies*

- *the sequence $(a(n)\alpha)_{n\in\mathbb{N}}$ is equidistributed in $\mathbb{T}$ for every irrational α, and*
- *the set $\{n \in \mathbb{N} \colon r | a(n)\}$ has positive upper density for every $r \in \mathbb{N}$.*

Then the sequence $(a(n))$ is good for 1-recurrence.

Once again, the proof of this result depends upon knowing identity (2). Since appropriate generalizations are not known for multiple correlation sequences, we are unable to give a similar criterion for ℓ-recurrence when $\ell \geq 2$. To state a conjectural criterion we extend the notion of an irrational rotation on the circle to general connected nilmanifolds: Given a connected nilmanifold $X = G/\Gamma$, an *irrational nilrotation in X* is an element $b \in G$ such that the sequence $(b^n\Gamma)_{n\in\mathbb{N}}$ is equidistributed on X.

Problem 3. *Let $(a(n))$ be a sequence that satisfies:*

- *for every connected ℓ-step nilmanifold X and every irrational nilrotation b in X the sequence $(b^{a(n)}\Gamma)_{n\in\mathbb{N}}$ is equidistributed in X, and*
- *the set $\{n \in \mathbb{N} \colon r | a(n)\}$ has positive upper density for every $r \in \mathbb{N}$.*

Show that the sequence $(a(n))$ is good for ℓ-recurrence of powers.

This problem was first formulated in [7] and in the same article a positive answer was given for sequences with range a set of integers with positive density. The stated conditions are satisfied by any integer polynomial sequence with zero constant term (follows from results in

[13]), the sequence $([n^c])$ for every $c>0$ (it follows from results in [4]), and the sequence of shifted primes (p_n-1) (it follows from results in [9]).

1.4. **Powers of sequences and recurrence.** It is known that if a sequence is good for ℓ-convergence of powers, then its first ℓ powers are good for 1-convergence. More precisely, the following holds (this is implicit in [8] Section 9.1, and is proved in detail in [6]):

Theorem. *If $(a(n))$ is good for ℓ-convergence of powers, then $(a(n)^k)$ is good for 1-convergence for $k=1,\ldots,\ell$.*

It is unclear whether a similar property holds for recurrence.

Problem 4. *If $(a(n))$ is good for ℓ-recurrence of powers, is then $(a(n)^k)$ good for 1-recurrence for $k=1,\ldots,\ell$?*

This problem was first stated in [6] and it is open even when $\ell=2$. It is known that if $(a(n))$ is good for 2-recurrence of powers, then the sequence $(a(n)^2)$ is good for Bohr recurrence, meaning it is good for 1-recurrence for all rotations on tori (see [8] Section 9.1, or [6]). A well known question of Y. Katznelson asks whether a set of Bohr recurrence is necessarily a set of topological recurrence (for background on this question see [3, 11, 14]). Although there exist examples of sets of topological recurrence that are not sets of 1-recurrence [12], all known examples are rather complicated. All these lead one to believe that a possible example showing that the answer to Problem 4 is negative should be complicated.

1.5. **Commuting vs powers of a single transformation.** If a sequence is good for 2-convergence of commuting transformations, then, of course, it is also good for 2-convergence of powers. Interestingly, no example that distinguishes the two notions is known and in fact there may be none:

Problem 5. *Is there a sequence that is good for 2-convergence of powers but it is not good for 2-convergence of commuting transformations?*

The corresponding question for recurrence is also open:

Problem 6. *Is there a sequence that is good for 2-recurrence of powers but it is not good for 2-recurrence of commuting transformations?*

This question was first stated in [1] (Question 8). In [1], V. Bergelson states that the answer is very likely yes.

1.6. **Fast growing sequences.** Despite the successes in dealing with multiple recurrence and convergence problems of sequences that do not grow faster than polynomials, when it comes down to fast growing sequences our knowledge is very limited. Say that a sequence $(a(n))$ of positive integers is *fast growing* if $\lim_{n\to\infty}\log(a(n))/\log n=\infty$ (equivalently, if it is of the form $(n^{b(n)})$ with $b(n)\to\infty$).

Problem 7. *Give an explicit example of a fast growing sequence that is good for multiple recurrence and convergence of powers.*

Even for $\ell = 2$ no such example is known. Several explicit natural examples of fast growing sequences that do not grow exponentially fast should work, for instance, the sequences $([n^{(\log n)^a}])$, $(n^{[(\log n)^b]})$, $([e^{n^c}])$ for $a, b > 0$ and $c \in (0, 1)$. Unfortunately though, it is very hard to work with these sequences, even for issues related to 1-recurrence and 1-convergence. Probably a sequence like $(n^{[\log\log n]})$ is easier to work with. One could also try to construct a (not so explicit) example using random sequences (more on this in another section).

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