SOME OPEN PROBLEMS ON MULTIPLE ERGODIC AVERAGES

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1. Some useful notions

To ease our exposition we collect some notions that are frequently used in subsequent sections.

1.1. **Recurrence.** The next notions are used to describe multiple recurrence properties of a single sequence:

Definition 1.1. We say that the sequence of integers $(a(n))$ is

- *good for ℓ-recurrence of powers* if for every $k_1, \dots, k_\ell \in \mathbb{Z}$, every invertible measure preserving system $(X, \mathcal{B}, \mu, T)$, and set $A \in \mathcal{B}$ with $\mu(A) > 0$, we have
$$\mu(A \cap T^{-k_1 a(n)} A \cap \cdots \cap T^{-k_\ell a(n)} A) > 0$$
for some $n \in \mathbb{N}$ with $a(n) \neq 0$.[1]
- *good for multiple recurrence of powers* if it is good for ℓ-recurrence of powers for every $\ell \in \mathbb{N}$.
- *good for ℓ-recurrence of commuting transformations* if for every probability space $(X, \mathcal{X}, \mu)$, commuting invertible measure preserving transformations $T_1, \dots, T_\ell \colon X \to X$, and set $A \in \mathcal{B}$ with $\mu(A) > 0$, we have
$$\mu(A \cap T_1^{-a(n)} A \cap \cdots \cap T_\ell^{-a(n)} A) > 0$$
for some $n \in \mathbb{N}$ with $a(n) \neq 0$.
- *good for multiple recurrence of commuting transformations* if it is good for ℓ-recurrence of commuting transformations for every $\ell \in \mathbb{N}$.

The fact that the sequence (n) is good for multiple recurrence of powers corresponds to the multiple recurrence result of H. Furstenberg [14], and the fact that it is good for multiple recurrence of commuting transformations corresponds to the multidimensional extension of this result of H. Furstenberg and Y. Katznelson [16]. Further examples of sequences that are good for multiple recurrence of commuting transformations include: integer polynomials with zero constant term [3], the shifted primes ([22] for powers and [5] in general), integer polynomials with zero constant term evaluated at the shifted primes ([22] for powers and [9] in general), several generalized polynomial sequences [6], and some random sequences of zero density [12]. The sequence $([n^c])$, where $c \in \mathbb{R}$ is positive, is known to be good for multiple recurrence of powers [13] (see also [8]), but if $c \in \mathbb{R} \setminus \mathbb{Q}$ is greater than 1, it is not known whether it is good

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[1] We remark, that for this and subsequent statements, the existence of a single $n \in \mathbb{N}$ for which the multiple intersection has positive measure, forces the existence of infinitely many $n \in \mathbb{N}$ with the same property (but not necessarily for a set of $n \in \mathbb{N}$ with positive upper density, although more often than not this is also true).

for multiple recurrence of commuting transformations. Examples of sequences that are good for ℓ-recurrence of powers but are not good for $(\ell+1)$-recurrence of powers can be found in [10].

Let us also mention at this point some obstructions to recurrence. One can check that if the sequence $(a(n))$ is good for 1-recurrence, then the equation $a(n) \equiv 0 \mod r$ has a solution for every $r \in \mathbb{N}$; as a consequence, the sequences (p_n), (n^2+1), (2^n), are not good for 1-recurrence. More generally, if the sequence $(a(n))$ is good for 1-recurrence, then for every $\alpha \in \mathbb{R}$ and $\varepsilon > 0$ the inequality $\|a(n)\alpha\| \leq \varepsilon$ has a solution (letting $\alpha = 1/r$ gives the previous congruence condition); as a consequence, the sequence $([\sqrt{5}n+2])$ is not good for 1-recurrence (try $\alpha = 1/\sqrt{5}$ and $\varepsilon = 1/10$). This obstruction also implies that if a sequence $(a(n))$ is lacunary, meaning, $\liminf_{n\to\infty} a(n+1)/a(n) > 1$, then it is not good for 1-recurrence, since it is well known that in this case there exist an irrational α and a positive ε_0 such that $\|a(n)\alpha\| \geq \varepsilon_0$ for every $n \in \mathbb{N}$.

Furthermore, if a sequence $(a(n))$ is good for 2-recurrence of powers, then for every $\alpha \in \mathbb{R}$ and $\varepsilon > 0$ the inequality $\left\|a(n)^2\alpha\right\| \leq \varepsilon$ has a solution (this is implicit in [15] Section 9.1, and is proved in detail in [10]). It follows that if one forms a sequence by putting the elements of the set $\{n \in \mathbb{N}\colon \left\|n^2\sqrt{2}\right\| \in [1/4, 1/2]\}$ in increasing order, then this sequence is not going to be good for 2-recurrence of powers (one can show that it is going to be good for 1-recurrence [10]). And there are also other more restrictive obstructions, if the sequence $(a(n))$ is good for 2-recurrence of powers, then for every $\alpha, \beta \in \mathbb{R}$ and $\varepsilon > 0$ the inequality $\|[a(n)\alpha]a(n)\beta\| \leq \varepsilon$ has a solution. More generally, for ℓ-recurrence of powers, one can state further obstructions by using higher degree polynomials with zero constant term, or more complicated generalized polynomials.

The next notions are used to describe multiple recurrence properties of a collection of sequences:

Definition 1.2. We say that the collection of sequences of integers $\{(a_1(n)), \ldots, (a_\ell(n))\}$ is

- *good for ℓ-recurrence of a single transformation* if for every invertible measure preserving system $(X, \mathcal{B}, \mu, T)$, and set $A \in \mathcal{B}$ with $\mu(A) > 0$, we have
$$\mu(A \cap T^{-a_1(n)}A \cap \cdots \cap T^{-a_\ell(n)}A) > 0$$
for some $n \in \mathbb{N}$ with $a_i(n) \neq 0$ for $i = 1, \ldots, \ell$.
- *good for ℓ-recurrence of commuting transformations* if for every probability space $(X, \mathcal{X}, \mu)$, commuting invertible measure preserving transformations $T_1, \ldots, T_\ell\colon X \to X$, and set $A \in \mathcal{B}$ with $\mu(A) > 0$, we have
$$\mu(A \cap T_1^{-a_1(n)}A \cap \cdots \cap T_\ell^{-a_\ell(n)}A) > 0$$
for some $n \in \mathbb{N}$ with $a_i(n) \neq 0$ for $i = 1, \ldots, \ell$.

Examples of collections of ℓ sequences that are known to be good for ℓ-recurrence of commuting transformations include collections of: integer polynomials with zero constant term [3] (for arbitrary integer polynomials necessary and sufficient conditions where given in [4]) and integer polynomials with zero constant term evaluated at the shifted primes [9]. For every $\ell \in \mathbb{N}$, the collection $\{([n^{c_1}]), \ldots, ([n^{c_\ell}])\}$, where $c_1, \ldots, c_\ell \in \mathbb{R} \setminus \mathbb{Z}$, is known to be good for ℓ-recurrence of a single transformation [8], but it is not known whether it is always good for ℓ-recurrence of commuting transformations.

The correspondence principle of Furstenberg enables us to deduce statements in density Ramsey theory from multiple recurrence statements in ergodic theory. Using this principle and some elementary arguments one is able to reformulate the previous ergodic notions to purely combinatorial ones:

Theorem. *The sequence of integers $(a(n))$ is*

- *good for ℓ-recurrence of powers if and only if for every $k_1, \ldots, k_\ell \in \mathbb{Z}$, every set $E \subset \mathbb{Z}$ with $\bar{d}(E) > 0$ contains patterns of the form*
$$\{m, m + k_1 a(n), \ldots, m + k_\ell a(n)\}$$
for some $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ with $a(n) \neq 0$.
- *good for ℓ-recurrence of commuting transformations if and only if for every $d \in \mathbb{N}$ and $\mathbf{v}_1, \ldots, \mathbf{v}_\ell \in \mathbb{Z}^d$, every set $E \subset \mathbb{Z}^d$ with $\bar{d}(E) > 0$ contains patterns of the form*
$$\{\mathbf{m}, \mathbf{m} + a(n)\mathbf{v}_1, \ldots, \mathbf{m} + a(n)\mathbf{v}_\ell\}$$
for some $\mathbf{m} \in \mathbb{Z}^d$ and $n \in \mathbb{N}$ with $a(n) \neq 0$.

For instance, the fact that the sequence (n) is good for multiple recurrence of powers corresponds to the theorem of Szemerédi on arithmetic progressions.

Theorem. *The collection of sequences of integers $\{a_1(n), \ldots, a_\ell(n)\}$ is*

- *good for ℓ-recurrence of a single transformation if and only if every set $E \subset \mathbb{Z}$ with $\bar{d}(E) > 0$ contains patterns of the form*
$$\{m, m + a_1(n), \ldots, m + a_\ell(n)\}$$
for some $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ with $a_i(n) \neq 0$ for $i = 1, \ldots, \ell$.
- *good for ℓ-recurrence of commuting transformations if and only if for every $\mathbf{v}_1, \ldots, \mathbf{v}_\ell \in \mathbb{Z}^d$, every set $E \subset \mathbb{Z}^d$ with $\bar{d}(E) > 0$ contains patterns of the form*
$$\{\mathbf{m}, \mathbf{m} + a_1(n)\mathbf{v}_1, \ldots, \mathbf{m} + a_\ell(n)\mathbf{v}_\ell\}$$
for some $\mathbf{m} \in \mathbb{Z}^d$ and $n \in \mathbb{N}$ with $a_i(n) \neq 0$ for $i = 1, \ldots, \ell$.

Let us also remark that the previous notions admit equivalent uniform versions that are often useful for applications. For instance, one can prove the following (see [2] for an argument that works for polynomials and [11] for an argument that works for general sequences):

Theorem. *The collection of sequences of integers $\{a_1(n), \ldots, a_\ell(n)\}$ is good for ℓ-recurrence of a single transformation if and only if*

(i) For every $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon) > 0$ and $N_0 = N_0(\varepsilon)$, such that for every $N \geq N_0$ and integer set $E \subset [-N, N]$ with $|E| \geq \varepsilon N$, we have
$$|E \cap (E - a_1(n)) \cap \ldots \cap (E - a_\ell(n))| \geq \delta N$$
for some $n \in [1, N_0]$.

(ii) For every $\varepsilon > 0$ there exist $\gamma = \gamma(\varepsilon) > 0$ and $N_1 = N_1(\varepsilon)$, such that for every invertible measure preserving system $(X, \mathcal{B}, \mu, T)$ and $A \in \mathcal{B}$ with $\mu(A) \geq \varepsilon$, we have that
$$\mu(A \cap T^{-a_1(n)} A \cap \ldots \cap T^{-a_\ell(n)} A) \geq \gamma$$
for some $n \in [1, N_1]$.

1.2. **Convergence.** The next notion is used to describe multiple convergence properties of a single sequence:

Definition 1.3. We say that the sequence of integers $(a(n))$ is

- *good for ℓ-convergence of powers (of a single transformation)* if for every $k_1, \ldots, k_\ell \in \mathbb{Z}$, invertible measure preserving system $(X, \mathcal{X}, \mu, T)$, and functions $f_1, \ldots, f_\ell \in L^\infty(\mu)$, the averages
$$\frac{1}{N}\sum_{n=1}^{N} T^{k_1 a(n)} f_1 \cdot \ldots \cdot T^{k_\ell a(n)} f_\ell$$
converge in the mean.
- *good for multiple convergence of powers (of a single transformation)* if it is good for ℓ-convergence of powers for every $\ell \in \mathbb{N}$.
- *good for ℓ-convergence of commuting transformations* if for every probability space $(X, \mathcal{X}, \mu)$, commuting invertible measure preserving transformations $T_1, \ldots, T_\ell \colon X \to X$, and functions $f_1, \ldots, f_\ell \in L^\infty(\mu)$, the averages
$$\frac{1}{N}\sum_{n=1}^{N} T_1^{a(n)} f_1 \cdot \ldots \cdot T_\ell^{a(n)} f_\ell$$
converge in the mean.
- *good for multiple convergence of commuting transformations* if it is good for ℓ-convergence of commuting transformations for every $\ell \in \mathbb{N}$.

Examples of sequences that are good for multiple convergence of commuting transformations include: the sequence (n) [20] (see also [21, 1, 17]), the sequence of primes [9], and some random sequences of zero density [12]. Additional examples of sequences that are known to be good for multiple convergence of powers include: sequences given by integer polynomials [18, 19], integer polynomials evaluated at the primes [22] (see also [9]), and sequences of the form $([n^a])$ where $a > 0$ [8]. Examples of sequences that are good for ℓ-convergence of powers but are not good for $(\ell+1)$-convergence of powers can be found in [10].

The next notion is used to describe multiple convergence properties of a collection of sequences:

Definition 1.4. We say that the collection of sequences of integers $\{(a_1(n)), \ldots, (a_\ell(n))\}$ is

- *good for ℓ-convergence of a single transformation* if for every invertible measure preserving system $(X, \mathcal{X}, \mu, T)$, and functions $f_1, \ldots, f_\ell \in L^\infty(\mu)$, the averages
$$\frac{1}{N}\sum_{n=1}^{N} T^{a_1(n)} f_1 \cdot \ldots \cdot T^{a_\ell(n)} f_\ell$$
converge in the mean.
- *good for ℓ-convergence of commuting transformations* if for every probability space $(X, \mathcal{X}, \mu)$, commuting invertible measure preserving transformations $T_1, \ldots, T_\ell \colon X \to X$, and functions $f_1, \ldots, f_\ell \in L^\infty(\mu)$, the averages
$$\frac{1}{N}\sum_{n=1}^{N} T_1^{a_1(n)} f_1 \cdot \ldots \cdot T_\ell^{a_\ell(n)} f_\ell$$

converge in the mean.

Examples of collections of sequences (not coming from multiples of the same sequence) that are known to be good for ℓ-convergence of commuting transformations include: collections of integer polynomials with distinct degrees [7] or any such collection of polynomials evaluated at the primes [9], and collections of sequences of the form $\{(n), (a_n(\omega))\}$ where $(a_n(\omega))$ is a well chosen random sequence of zero density [12]. Additional examples of collections of sequences that are good for ℓ-convergence of a single transformation include: arbitrary collections of integer polynomials, or integer polynomials evaluated at the primes [22] (see also [9]), and the collection $\{([n^{c_1}]), \ldots, ([n^{c_\ell}])\}$, where $c_1, \ldots, c_\ell \in \mathbb{R} \setminus \mathbb{Z}$ are positive [8].

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