JOINT ERGODICITY OF SEQUENCES - AN EXPOSITION

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Abstract. A collection of integer sequences is jointly ergodic if for every ergodic measure preserving system the multiple ergodic averages, with iterates given by this collection of sequences, converge in the mean to the product of the integrals of the functions involved. Convenient necessary and sufficient conditions for joint ergodicity were given in [11] and this exposition uses a simplified version of the argument in [11] in order to recover its main results under somewhat stronger assumptions. The argument we give is rather short and avoids deep tools from ergodic theory. The main result can be used to prove new ergodic theorems and give vast simplifications of older results that depended on deep machinery from ergodic theory.

Dedicated to the memory of Dimitris Gatzouras

1. Introduction

The polynomial Szemerédi theorem of Bergelson and Leibman [1] states that if Λ is a set of integers with positive upper density and $p_1, \ldots, p_\ell \in \mathbb{Z}[t]$ are polynomials with zero constant term, then there exist $m, n \in \mathbb{N}$ such that

$$m, m + p_1(n), \ldots, m + p_\ell(n) \in \Lambda.$$

This generalizes the theorem of Szemerédi [28] on arithmetic progressions that corresponds to the case where $p_1(n) = n, p_2(n) = 2n, \ldots, p_\ell(n) = \ell n$. The proof of Bergelson and Leibman uses ergodic theory and up to this day it is the only proof that covers the full generality of this result. Using the correspondence principle of Furstenberg [14, 15] it turns out that it suffices to verify the following: For every measure preserving system $(X, \mathcal{X}, \mu, T)$ and set $A \in \mathcal{X}$ with positive measure, there exists $n \in \mathbb{N}$ such that

$$\mu(A \cap T^{-p_1(n)}A \cap \cdots \cap T^{-p_\ell(n)}A) > 0.$$

The proof of this multiple recurrence property proceeds by analyzing the limiting behavior in $L^2(\mu)$ of the following multiple ergodic averages (see our notation for averages in Section 1.1)

$$\mathbb{E}_{n \in [N]} T^{p_1(n)} f_1 \cdot \ldots \cdot T^{p_\ell(n)} f_\ell. \tag{1}$$

Finding an explicit formula for this limit for all polynomials is still an unresolved problem, but in some cases the limit takes a particularly simple form, namely, it is the product of the integrals of the functions $f_1, \ldots, f_\ell$. Due to congruence obstructions this can only be the case for totally ergodic systems, which is the reason why we are particularly interested in this class of systems. The prototypical result was established by Furstenberg and Weiss [16] and states that in a totally ergodic system for every $f, g \in L^\infty(\mu)$ we have

$$\lim_{N\to\infty} \mathbb{E}_{n\in[N]} T^n f \cdot T^{n^2} g = \int f \, d\mu \cdot \int g \, d\mu \tag{2}$$

in $L^2(\mu)$. The proof of this result is rather involved and its most difficult component is the analysis of a special class of two step distal systems, called Conze-Lesigne systems (introduced in [7]), that control the limiting behavior of these averages. Conze-Lesigne systems are particular examples of systems with nilpotent structure, a concept that has played an important role in subsequent developments in the field. By combining the

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Host-Kra theory of characteristic factors [17] and equidistribution results on nilmanifolds from [22], the author and Kra extended in [13] the result of Furstenberg and Weiss by showing that in a totally ergodic system if the polynomials $p_1,\dots,p_\ell \in \mathbb{Z}[t]$ are rationally independent,[1] then for all $f_1,\dots,f_\ell \in L^\infty(\mu)$ we have

$$\lim_{N\to\infty} \mathbb{E}_{n\in[N]}\, T^{p_1(n)}f_1\cdot\ldots\cdot T^{p_\ell(n)}f_\ell = \int f_1\, d\mu \cdots \int f_\ell\, d\mu \tag{3}$$

in $L^2(\mu)$. If the polynomials are rationally dependent, then easy examples of totally ergodic circle rotations show that the previous limit formula fails. In fact, when $p_1(n) = n, p_2(n) = 2n, \dots, p_\ell(n) = \ell n$, the limit can be computed using the results in [17, 30, 31] and it turns out that it genuinely depends on the $(\ell-1)$-step nilsystems that are factors of the original system. So in order to obtain an explicit limit formula for the averages (1) for dependent polynomials, the use of deep structural results from ergodic theory and equidistribution results on nilmanifolds seems unavoidable. This is not the case though for rationally independent polynomials, and it has been a tantalizing open problem for quite a while to get an "elementary" proof for the limit formulas (2) and (3). The main purpose of this note is to reproduce a simplified version of an argument from [11] that accomplishes this goal. Moreover, as in [11], our main result (Theorem 2.1) gives a rather general statement that applies to a variety of sequences, not just polynomials, and this allows to prove some new convergence results and establish some conjectures. We record a few examples from recent literature in Section 2.3. As in [11], our argument was motivated by techniques of Peluse [24] and Peluse and Prendiville [26] of finitary nature that were originally devised to give quantitative estimates for special cases of the polynomial Szemerédi theorem.

1.1. **Definitions and notation.** With $\mathbb{N}$ we denote the set of positive integers and with $\mathbb{Z}_+$ the set of non-negative integers. For $t \in \mathbb{R}$ we let $e(t) := e^{2\pi i t}$. With $\mathbb{T}$ we denote the one dimensional torus and we often identify it with $\mathbb{R}/\mathbb{Z}$ or with $[0,1)$. With $\Re(z)$ we denote the real part of the complex number z. For $N \in \mathbb{N}$ we let $[N] := \{1,\dots,N\}$. If $a\colon \mathbb{N}^s \to \mathbb{C}$ is a bounded sequence for some $s \in \mathbb{N}$ and A is a non-empty finite subset of $\mathbb{N}^s$, we let $\mathbb{E}_{n\in A}\, a(n) := \frac{1}{|A|}\sum_{n\in A} a(n)$.

2. Main results

2.1. **Definitions.** In order to facilitate our exposition we reproduce some definitions from [11].

Definition. We say that the collection of sequences $a_1,\dots,a_\ell\colon \mathbb{N}\to\mathbb{Z}$ is *jointly ergodic for the ergodic system* $(X,\mathcal{X},\mu,T)$, if for all $f_1,\dots,f_\ell \in L^\infty(\mu)$ we have

$$\lim_{N\to\infty} \mathbb{E}_{n\in[N]}\, T^{a_1(n)}f_1\cdot\ldots\cdot T^{a_\ell(n)}f_\ell = \int f_1\, d\mu \cdot\ldots\cdot \int f_\ell\, d\mu \tag{4}$$

in $L^2(\mu)$.

If a collection of sequences is jointly ergodic for every ergodic system, then an ergodic decomposition argument shows that the limit formula (4) holds for every system $(X,\mathcal{X},\mu,T)$ (not necessarily ergodic), if we use in place of the integrals $\int f_i\, d\mu$ the conditional expectations $\mathbb{E}(f_i|\mathcal{I}(T))$ ($\mathcal{I}(T)$ is the σ-algebra of T-invariant sets). This implies that the following strong multiple recurrence property holds

$$\lim_{N\to\infty} \mathbb{E}_{n\in[N]}\, \mu(A\cap T^{-a_1(n)}A\cap\cdots\cap T^{-a_\ell(n)}A) \geq (\mu(A))^{\ell+1},$$

[1] A collection of integer polynomials is *rationally independent*, if every non-trivial linear combination of the polynomials is non-constant.

for every system $(X, \mathcal{X}, \mu, T)$ and every $A \in \mathcal{X}$. It is then a consequence of the correspondence principle of Furstenberg [15] that every set of integers with positive upper density contains patterns of the form $m, m + a_1(n), \ldots, m + a_\ell(n)$, for some $m, n \in \mathbb{N}$.

Definition. If $(X, \mathcal{X}, \mu, T)$ is a system we defined its *spectrum* as follows

$$\operatorname{Spec}(T) := \{t \in [0, 1) \colon Tf = e(t)\, f \text{ for some non-zero } f \in L^2(\mu)\}.$$

For the definition of the seminorms $\|\!|\cdot|\!\|_s$ we refer the reader to Section 3.2.

Definition. We say that the collection of sequences $a_1, \ldots, a_\ell \colon \mathbb{N} \to \mathbb{Z}$ is:

(i) *good for seminorm estimates for the system* $(X, \mathcal{X}, \mu, T)$, if there exists $s \in \mathbb{N}$ such that if $f_1, \ldots, f_\ell \in L^\infty(\mu)$ and $\|\!|f_i|\!\|_s = 0$ for some $i \in \{1, \ldots, \ell\}$, then

$$\lim_{N\to\infty} \mathbb{E}_{n\in[N]}\, T^{a_1(n)} f_1 \cdot \ldots \cdot T^{a_\ell(n)} f_\ell = 0$$

in $L^2(\mu)$;

(ii) *good for equidistribution on* $S \subset [0, 1)$, if for all $t_1, \ldots, t_\ell \in S$, not all of them 0, we have

$$\lim_{N\to\infty} \mathbb{E}_{n\in[N]}\, e(a_1(n)t_1 + \cdots + a_\ell(n)t_\ell) = 0. \tag{5}$$

It is known [18, 23] that if $p_1, \ldots, p_\ell \colon \mathbb{N} \to \mathbb{Z}$ are polynomials with pairwise non-constant differences, then they are good for seminorm estimates for every ergodic system. They are also good for equidistribution for all totally ergodic systems if and only if the polynomials are rationally independent; this follows easily from a well known equidistribution result of Weyl. If $c_1, \ldots, c_\ell$ are positive distinct non-integers, then it can be shown [9] that the collection of sequences $[n^{c_1}], \ldots, [n^{c_k}]$ is good for seminorm estimates and good for equidistribution for all ergodic systems.

2.2. **Main result.** We are now ready to state our main result (in applications we are going to use it for $S = [0, 1)$ and $S = ([0, 1) \setminus \mathbb{Q}) \cup \{0\}$).

Theorem 2.1. *Let S be a susbet of $[0, 1)$ with countable complement in $[0, 1)$. The collection of sequences $a_1, \ldots, a_\ell \colon \mathbb{N} \to \mathbb{Z}$ is jointly ergodic for all systems with spectrum in S if and only if it is good for seminorm estimates and equidistribution for these systems.*

Remarks. • The necessity of the conditions is easy to establish, the interesting part is the sufficiency.

• Theorem 1.1 in [11] uses somewhat weaker assumptions. The stronger assumption we use here allows to simplify the proof in [11].

• Theorem 1.4 in [11] shows that under weaker equidistribution hypothesis, which are satisfied by collections of rationally independent integer polynomials, the rational Kronecker factor controls the limiting behavior of the associated multiple ergodic averages. One can deduce this result from Theorem 2.1 as in [11, Section 5].

In order to facilitate understanding, we are going to first prove Theorem 2.1 for $\ell = 2$ in Section 4 and then explain the necessary changes needed for the proof of the general case in Section 5.

Since for totally ergodic systems $(X, \mathcal{X}, \mu, T)$ we have $\operatorname{Spec}(T) \subset ([0, 1) \setminus \mathbb{Q}) \cup \{0\}$, an immediate consequence of Theorem 2.1 (for $S = ([0, 1) \setminus \mathbb{Q}) \cup \{0\}$) is the following result:

Corollary 2.2. *The collection of sequences $a_1, \ldots, a_\ell \colon \mathbb{N} \to \mathbb{Z}$ is jointly ergodic for all totally ergodic systems if and only if it is good for seminorm estimates for all totally ergodic systems and* (5) *holds for all $t_1, \ldots, t_\ell \in [0, 1)$ that are either irrational or zero but not all of them zero.*

This applies to collections of rationally independent polynomials $p_1, \ldots, p_\ell \in \mathbb{Z}[t]$, hence we recover the limit formulas (2) and (3).

2.3. **Applications.** Theorem 2.1 can be used to give significantly simpler proofs of results in [3, 9, 13, 16, 20] (the parts that correspond to joint ergodicity properties). But it also gives access to convergence results not previously known. The main reason why Theorem 2.1 is advantageous for these applications, is that it enables us to bypass some difficult and often inaccessible equidistribution results on nilmanifolds that need to be established in order to use the Host-Kra theory of characteristic factors. We record a few instances of these applications below. We remark that in all these cases the most difficult component is to verify the good seminorm property; verifying the needed good equidistribution property is usually a simple matter.

Theorem 2.1 was used in [12] to prove the following joint ergodicity result for sequences given by fractional powers of primes.

Theorem 2.3 ([12]). *Let $c_1, \dots, c_\ell$ be distinct positive non-integers. Then the collection of sequences $[p_n^{c_1}], \dots, [p_n^{c_\ell}]$ is jointly ergodic for every ergodic system.*

Previously this was only known for $\ell = 1$ and for $\ell = 2$ it was not even known for nilsystems or weakly mixing systems.

Another very interesting application of Theorem 2.1 was recently obtained by Tsinas [29] who verified a conjecture of the author from [9] (see also [10, Problem 23]).

Theorem 2.4 ([29]). *Let $a_1, \dots, a_\ell \colon [1, \infty) \to \mathbb{R}$ be functions from a Hardy field[2] that have polynomial growth. Then the collection of sequences $[a_1(n)], \dots, [a_\ell(n)]$ is jointly ergodic for all ergodic systems if whenever $a(t)$ is a non-trivial linear combination of the functions $a_1, \dots, a_\ell$ we have*

$$\lim_{t\to\infty} \frac{|a(t) - p(t)|}{\log t} = \infty$$

for all polynomials $p \in \mathbb{Z}[t]$.[3]

Previously this was known for $\ell = 1$ (it follows easily from [6]) and for general ℓ partial progress was made in [3, 9, 20, 27].

Theorem 2.1 was also used in [11] in order to address a problem of Bergelson, Moreira, and Richter [4, Conjecture 6.1]. It establishes an extension of the limit formula (3) that covers iterates given by polynomials with fractional powers.

Theorem 2.5. *Let $a_1, \dots, a_\ell \colon \mathbb{R}_+ \to \mathbb{R}$ be linearly independent functions of the form $\sum_{i=1}^k \alpha_i t^{c_i}$ where $\alpha_1, \dots, \alpha_k \in \mathbb{Q}$ and $c_1, \dots, c_k \in (0, +\infty)$. Then the collection of sequences $[a_1(n)], \dots, [a_\ell(n)]$ is jointly ergodic for all totally ergodic systems.*

Lastly, Theorem 2.1 was recently extended by Best and Moragues [5] to a large class of countable Abelian group actions, and this extension was subsequently used by Donoso, Koutsogiannis, and Sun [8] to prove joint ergodicity results for commuting transformations with polynomial iterates under some ergodicity assumptions.

3. Background

3.1. **Measure preserving systems.** A *measure preserving system*, or simply *a system*, is a quadruple $(X, \mathcal{X}, \mu, T)$ where $(X, \mathcal{X}, \mu)$ is a Lebesgue probability space and $T \colon X \to X$ is an invertible, measurable, measure preserving transformation. Throughout, for $n \in \mathbb{N}$ we denote by T^n the composition $T \circ \cdots \circ T$ (n times) and let $T^{-n} := (T^n)^{-1}$ and $T^0 := \mathrm{id}_X$. Also, for $f \in L^2(\mu)$ and $n \in \mathbb{Z}$ we denote by $T^n f$ the function $f \circ T^n$.

[2] This class includes all linear combinations of the functions $t^a (\log t)^b (\log\log t)^c$, $a, b, c \in \mathbb{R}$, and more generally, all functions defined on some half-line $[c, \infty)$ using a finite combination of the symbols $+, -, \times, :, \log, \exp$, operating on the real variable t and on real constants.

[3] This condition is close to being necessary, in the sense that if it fails for some non-linear p, then the collection of sequences $a_1, \dots, a_\ell$ is not going to be jointly ergodic for some ergodic rotation on the ℓ-dimensional torus.

We say that the system $(X, \mathcal{X}, \mu, T)$ is *ergodic* if the only functions $f \in L^2(\mu)$ that satisfy $Tf = f$ are the constant ones. It is *totally ergodic* if $(X, \mathcal{X}, \mu, T^d)$ is ergodic for every $d \in \mathbb{N}$, or equivalently, if the system is ergodic and $\text{Spec}(T) \subset ([0,1) \setminus \mathbb{Q}) \cup \{0\}$.

A function $f \in L^2(\mu)$ is an *eigenfunction* of the system if $Tf = e(\alpha)f$ for some $\alpha \in \mathbb{R}$. We denote with $\mathcal{E}(T)$ the set of all eigenfunctions of the system with unit modulus.

3.2. **Gowers-Host-Kra seminorms.** Throughout, we use the following notation:

Definition. Let $(X, \mathcal{X}, \mu, T)$ be a system and $f \in L^\infty(\mu)$. If $\underline{n} = (n_1, \ldots, n_s) \in \mathbb{Z}^s$, $\underline{n}' = (n'_1, \ldots, n'_s) \in \mathbb{Z}^s$, $\epsilon = (\epsilon_1, \ldots, \epsilon_s) \in \{0,1\}^s$, and $z \in \mathbb{C}$, we let

(i) $\epsilon \cdot \underline{n} := \epsilon_1 n_1 + \cdots + \epsilon_s n_s$;
(ii) $|\underline{n}| := |n_1| + \cdots + |n_s|$;
(iii) $\mathcal{C}^l z := z$ if l is even and $\mathcal{C}^l z = \overline{z}$ if l is odd;
(iv) $\Delta_n f := T^n f \cdot \overline{f}$, $n \in \mathbb{Z}$;
(v) $\Delta_{\underline{n}} f := \Delta_{n_1} \cdots \Delta_{n_s} f = \prod_{\epsilon \in \{0,1\}^s} \mathcal{C}^{|\epsilon|} T^{\epsilon \cdot \underline{n}} f$.

For instance, we have

$$\Delta_{(n_1,n_2)} f = f \cdot T^{n_1}\overline{f} \cdot T^{n_2}\overline{f} \cdot T^{n_1+n_2} f, \quad n_1, n_2 \in \mathbb{Z}.$$

Given an ergodic system $(X, \mathcal{X}, \mu, T)$ we will make extensive use of the seminorms $\|\!| \cdot |\!\|_s$, $s \in \mathbb{N}$, on $L^\infty(\mu)$, that were introduced in [17]. They are often refereed to as *Gowers-Host-Kra seminorms*, or *uniformity seminorms*, and are defined inductively for $f \in L^\infty(\mu)$ as follows:

$$\|\!| f |\!\|_1 := \Big| \int f \, d\mu \Big|,$$

and for $s \in \mathbb{Z}_+$ we let

$$\|\!| f |\!\|_{s+1}^{2^{s+1}} := \lim_{N \to \infty} \mathbb{E}_{n \in [N]} \|\!| \Delta_n f |\!\|_s^{2^s}. \tag{6}$$

For instance, we have

$$\|\!| f |\!\|_2^4 = \lim_{N \to \infty} \mathbb{E}_{n \in [N]} \Big| \int \overline{f} \cdot T^n f \, d\mu \Big|^2.$$

An application of the mean ergodic theorem shows that

$$\|\!| f |\!\|_2^4 = \lim_{N_1 \to \infty} \mathbb{E}_{n_1 \in [N_1]} \lim_{N_2 \to \infty} \mathbb{E}_{n_2 \in [N_2]} \int f \cdot T^{n_1}\overline{f} \cdot T^{n_2}\overline{f} \cdot T^{n_1+n_2} f \, d\mu. \tag{7}$$

Likewise, by successive applications of the mean ergodic theorem, it can be shown that the limit in (6) exists and for $f \in L^\infty(\mu)$ and $s \in \mathbb{Z}_+$ we have that (see [17] or [19, Chapter 8])

$$\|\!| f |\!\|_s^{2^s} = \lim_{N_1 \to \infty} \cdots \lim_{N_s \to \infty} \mathbb{E}_{n_1 \in [N_1]} \cdots \mathbb{E}_{n_s \in [N_s]} \int \Delta_{(n_1, \ldots, n_s)} f \, d\mu. \tag{8}$$

For $s' \in [s]$ it can be shown that we can take any s' of the iterative limits to be simultaneous limits (i.e. average over $[N]^{s'}$ and let $N \to \infty$) without changing the value of the limit. This was originally proved in [17] and for a much simpler proof see [2]. Taking $s' = s$ gives

$$\|\!| f |\!\|_s^{2^s} = \lim_{N \to \infty} \mathbb{E}_{\underline{n} \in [N]^s} \int \Delta_{\underline{n}} f \, d\mu. \tag{9}$$

For $s \geq 2$ taking $s' = s - 1$ and using the mean ergodic theorem gives

$$\|\!| f |\!\|_s^{2^s} = \lim_{N \to \infty} \mathbb{E}_{\underline{n} \in [N]^{s-1}} \Big| \int \Delta_{\underline{n}} f \, d\mu \Big|^2. \tag{10}$$

Lastly, for $s \geq 3$ taking $s' = s - 2$ gives

$$\|\!| f |\!\|_s^{2^s} = \lim_{N \to \infty} \mathbb{E}_{\underline{n} \in [N]^{s-2}} \|\!| \Delta_{\underline{n}} f |\!\|_2^4. \tag{11}$$

3.3. **Soft inverse theorems.** Recall that if $(X,\mathcal{X},\mu,T)$ is a system, with $\mathcal{E}(T)$ we denote the set of its eigenfunctions with unit modulus.

Proposition 3.1. *Let $(X,\mathcal{X},\mu,T)$ be an ergodic system and $f\in L^\infty(\mu)$ be 1-bounded. Then*
$$\|\!|f|\!\|_2^4 \le \sup_{\chi\in\mathcal{E}(T)} \Re\Big(\int f\cdot\chi\,d\mu\Big).$$

Proof. Let $\mathcal{K}(T)$ be the closed subspace of $L^2(\mu)$ spanned by all eigenfunctions of the system. It is not hard to prove (see for example [19, Chapter 8, Theorem 1]) that
$$\|\!|f|\!\|_2 = \|\!|\tilde f|\!\|_2$$
where $\tilde f := \mathbb{E}(f|\mathcal{K}(T))$. Since the system is ergodic and the underlying probability space is Lebesgue, the subspace $\mathcal{K}(T)$ has an orthonormal basis of eigenfunctions of modulus one, say $(\chi_j)_{j\in\mathbb{N}}$. Then $\tilde f=\sum_{j=1}^\infty c_j\,\chi_j$ where
$$c_j := \int \tilde f\cdot\overline{\chi}_j\,d\mu = \int f\cdot\overline{\chi}_j\,d\mu,\quad j\in\mathbb{N}.$$
We have
$$\|\!|\tilde f|\!\|_2^4 = \sum_{j=1}^\infty |c_j|^4 \le \sup_{j\in\mathbb{N}}(|c_j|^2)\sum_{j=1}^\infty |c_j|^2 = \sup_{j\in\mathbb{N}}(|c_j|^2)\,\|f\|_{L^2(\mu)}^2 \le \sup_{j\in\mathbb{N}}\Big|\int f\cdot\overline{\chi}_j\,d\mu\Big|,$$
where the first identity follows by orthonormality and direct computation using (7), the second identity follows by the Parseval identity, and the last estimate holds since all functions involved are 1-bounded. The result now follows since the set $\mathcal{E}(T)$ is invariant under multiplication by unit modulus constants. □

Proposition 3.2. *Let $(X,\mathcal{X},\mu,T)$ be an ergodic system and $f\in L^\infty(\mu)$ be such that $\|\!|f|\!\|_{s+2}>0$ for some $s\in\mathbb{Z}_+$.*
 (i) *If $s=0$, then there exists $\chi\in\mathcal{E}(T)$ such that $\Re\big(\int f\cdot\chi\,d\mu\big)>0$.*
 (ii) *If $s\ge 1$, then there exist $\chi_{\underline{n}}\in\mathcal{E}(T)$, $\underline{n}\in\mathbb{N}^s$, such that*
$$\liminf_{N\to\infty}\mathbb{E}_{\underline{n}\in[N]^s}\Re\Big(\int \Delta_{\underline{n}}f\cdot\chi_{\underline{n}}\,d\mu\Big)>0.$$

Proof. If $s=0$, then the conclusion follows immediately from Proposition 3.1.

Suppose that $s\ge 1$. By (11) we have that
$$\lim_{N\to\infty}\mathbb{E}_{\underline{n}\in[N]^s}\|\!|\Delta_{\underline{n}}f|\!\|_2^4>0.$$
Using Proposition 3.1 we deduce that
$$\liminf_{N\to\infty}\mathbb{E}_{\underline{n}\in[N]^s}\sup_{\chi\in\mathcal{E}(T)}\Re\Big(\int \Delta_{\underline{n}}f\cdot\chi\,d\mu\Big)>0.$$
This immediately implies the asserted estimate. □

3.4. **Gowers-Cauchy-Schwarz estimates.** We will use the following variant of the so called Gowers-Cauchy-Schwarz inequality:

Lemma 3.3. *Let $(X,\mathcal{X},\mu,T)$ be a system, for $s\in\mathbb{N}$ let $f_\epsilon\in L^\infty(\mu)$, $\epsilon\in\{0,1\}^s$, be 1-bounded functions, and $g_{\underline{n}}\in L^\infty(\mu)$, $\underline{n}\in\mathbb{N}^s$. Let also $\underline{1}:=(1,\ldots,1)$. Then for every $N\in\mathbb{N}$ we have*
$$\Big|\mathbb{E}_{\underline{n}\in[N]^s}\int\prod_{\epsilon\in\{0,1\}^s}T^{\epsilon\cdot\underline{n}}f_\epsilon\cdot g_{\underline{n}}\,d\mu\Big|^{2^s}\le \mathbb{E}_{\underline{n},\underline{n}'\in[N]^s}\int\Delta_{\underline{n}-\underline{n}'}f_{\underline{1}}\cdot T^{-|\underline{n}|}g_{\underline{n},\underline{n}'}\,d\mu,$$
where for every $\underline{n},\underline{n}'\in\mathbb{N}^s$ the function $g_{\underline{n},\underline{n}'}$ is equal to a product of 2^s functions that belong to the set $\{g_{\underline{n}},\overline{g}_{\underline{n}},\underline{n}\in\mathbb{N}^s\}$.

Proof. For notational simplicity we give the details only for $s = 2$. The general case can be proved in a similar manner by successively applying the Cauchy-Schwarz inequality with respect to the variables $n_s, \dots, n_1$, exactly as we do below for $s = 2$. We have that

$$\Big|\mathbb{E}_{n_1,n_2\in[N]} \int f_0 \cdot T^{n_1} f_1 \cdot T^{n_2} f_2 \cdot T^{n_1+n_2} f_3 \cdot g_{n_1,n_2}\, d\mu\Big|^2$$

is bounded by (we use that f_0, f_1 are 1-bounded)

$$\mathbb{E}_{n_1\in[N]} \int \Big|\mathbb{E}_{n_2\in[N]} T^{n_2} f_2 \cdot T^{n_1+n_2} f_3 \cdot g_{n_1,n_2}\Big|^2 d\mu.$$

After expanding the square we find that this expression is equal to

$$\mathbb{E}_{n_1\in[N]} \int \mathbb{E}_{n_2,n_2'\in[N]} T^{n_2} f_2 \cdot T^{n_2'} \overline{f}_2 \cdot T^{n_1+n_2} f_3 \cdot T^{n_1+n_2'} \overline{f}_3 \cdot g_{n_1,n_2} \cdot \overline{g_{n_1,n_2'}}\, d\mu.$$

After composing with T^{-n_2}, exchanging $\mathbb{E}_{n_1\in[N]}$ with $\mathbb{E}_{n_2,n_2'\in[N]}$, using the Cauchy-Schwarz inequality, and that f_2 is 1-bounded, we get that the square of the last expression is bounded by

$$\mathbb{E}_{n_2,n_2'\in[N]} \int \Big|\mathbb{E}_{n_1\in[N]} T^{n_1} f_3 \cdot T^{n_1+n_2'-n_2} \overline{f}_3 \cdot T^{-n_2}(g_{n_1,n_2} \cdot \overline{g}_{n_1,n_2'})\Big|^2 d\mu.$$

As before, we expand the square, and compose with T^{-n_1}. We arrive at the expression

$$\mathbb{E}_{n_1,n_2,n_1',n_2'\in[N]}$$
$$\int f_3 \cdot T^{n_1'-n_1}\overline{f_3} \cdot T^{n_2'-n_2}\overline{f_3} \cdot T^{n_1'+n_2'-n_1-n_2} f_3 \cdot T^{-n_1-n_2}(g_{n_1,n_2} \cdot \overline{g}_{n_1,n_2'} \cdot \overline{g}_{n_1',n_2} \cdot g_{n_1',n_2'})\, d\mu,$$

which is equal to the right hand side of the asserted estimate when $s = 2$ (for $\underline{n} := (n_1, n_2)$, $\underline{n}' := (n_1', n_2')$). Combining the previous two estimates gives the asserted bound for $s = 2$. □

4. Proof of the main result for $\ell = 2$

The goal of this section is to give a proof for the sufficiency of the conditions in Theorem 2.1 for $\ell = 2$ (the necessity is simple). It suffices to prove the following:

Theorem 4.1. *Let S be a susbet of $[0, 1)$ with countable complement in $[0, 1)$. Suppose that the sequences $a, b\colon \mathbb{N} \to \mathbb{Z}$ are good for equidistribution on S and seminorm estimates for the system $(X, \mathcal{X}, \mu, T)$ with $\mathrm{Spec}(T) \subset S$. Then for all $f, g \in L^\infty(\mu)$ we have*

$$\lim_{N\to\infty} \mathbb{E}_{n\in[N]}\, T^{a(n)} f \cdot T^{b(n)} g = \int f\, d\mu \cdot \int g\, d\mu \tag{12}$$

in $L^2(\mu)$.

The proof of Theorem 2.1 for general ℓ is similar to the case $\ell = 2$ but involves an additional induction and is notationally more complicated; we describe the modifications needed to get the more general statement in Section 5.

4.1. Preparation.

In order to ease the exposition of the proof of Theorem 4.1 we use this subsection to gather some preparatory results. We are going to complete the proof of Theorem 4.1 in Section 4.2.

4.1.1. *The case where g is an eigenfunction.* We are going to make essential use of the good equidistribution assumption for the sequences $a, b\colon \mathbb{N} \to \mathbb{Z}$ to prove the next result.

Proposition 4.2. *Theorem 4.1 holds if g is an eigenfunction of the system.*

Proof. If f is constant, then the conclusion easily follows from our equidistribution assumption. Thus, it suffices to show that if $\int f\, d\mu = 0$ and $\chi \in \mathcal{E}(T)$, then

$$\lim_{N\to\infty} \mathbb{E}_{n\in[N]}\, T^{a(n)} f \cdot T^{b(n)}\chi = 0$$

in $L^2(\mu)$.

Suppose that the eigenvalue of χ is $e(\alpha)$ for some $\alpha \in \mathrm{Spec}(T)$. Then $T^{b(n)}\chi = e(b(n)\alpha)\,\chi$, so it suffices to show that

$$\lim_{N\to\infty} \mathbb{E}_{n\in[N]}\, c_n\, T^{a(n)} f = 0 \tag{13}$$

in $L^2(\mu)$ where $c_n := e(b(n)\alpha)$, $n \in \mathbb{N}$. To this end, we invoke the theorem of Herglotz (see for example [21, Section 7.6]) for the positive definite sequence $a(n) := \int \overline{f}\cdot T^n f\, d\mu$, $n \in \mathbb{Z}$. It gives that there exists a positive bounded measure σ on $\mathbb{T}$ (thought of as $[0, 1)$) such that

$$\int \overline{f}\cdot T^n f\, d\mu = \int e(nt)\, d\sigma(t), \quad n \in \mathbb{Z}. \tag{14}$$

Note that σ does not have a point mass on 0 because f has integral 0, or on any other number on the complement of $\mathrm{Spec}(T)$ (we leave these standard facts as an exercise for the reader). A simple computation that uses (14) shows that

$$\left\|\mathbb{E}_{n\in[N]}\, c_n\, T^{a(n)} f\right\|_{L^2(\mu)} = \left\|\mathbb{E}_{n\in[N]}\, c_n\, e(a(n)t)\right\|_{L^2(\sigma)}, \quad N \in \mathbb{N}.$$

Using this identity, the bounded convergence theorem, and the fact that the bounded measure σ does not have point masses on 0 and on the countable set $[0, 1) \setminus S$ (since it is contained on the complement of $\mathrm{Spec}(T)$), we get that (13) would follow if we show that for every non-zero $t \in S$ we have

$$\lim_{N\to\infty} \mathbb{E}_{n\in[N]}\, e(a(n)t + b(n)\alpha) = 0.$$

Since $\alpha \in \mathrm{Spec}(T) \subset S$, this follows from our assumption that the pair of sequences $a, b\colon \mathbb{N} \to \mathbb{Z}$ is good for equidistribution on S. □

4.1.2. *Positivity for g implies positivity for an averaged function $\tilde{g}$.* Our next goal is to show that if the positivity property (15) below holds, then it also holds when we replace g with an averaged function $\tilde{g}$. This is a simple but crucial observation because uniformity properties of $\tilde{g}$ are easier to analyse than those of g.

Proposition 4.3. *Let $(X, \mathcal{X}, \mu, T)$ be a system and $f, g \in L^\infty(\mu)$ be such that*

$$\limsup_{N\to\infty} \left\|\mathbb{E}_{n\in[N]}\, T^{a(n)} f \cdot T^{b(n)} g\right\|_{L^2(\mu)} > 0. \tag{15}$$

Then there exist $N_k \to \infty$ and 1-bounded $g_k \in L^\infty(\mu)$, $k \in \mathbb{N}$, such that for

$$\tilde{g} := \lim_{k\to\infty} \mathbb{E}_{n\in[N_k]}\, T^{-b(n)} g_k \cdot T^{a(n)-b(n)}\overline{f}, \tag{16}$$

where the limit is a weak limit (note that then $\tilde{g} \in L^\infty(\mu)$), we have

$$\limsup_{N\to\infty} \left\|\mathbb{E}_{n\in[N]}\, T^{a(n)} f \cdot T^{b(n)} \tilde{g}\right\|_{L^2(\mu)} > 0. \tag{17}$$

Proof. We can assume that both f and g are 1-bounded. For fixed $f \in L^\infty(\mu)$ we let $\mathcal{C} = \mathcal{C}(f)$ be the $L^2(\mu)$ closure of all linear combinations of all subsequential weak-limits of sequences of the form $\mathbb{E}_{n\in[N]} T^{-b(n)}g_N \cdot T^{-b(n)+a(n)}\overline{f}$, where $g_N \in L^\infty(\mu)$, $N \in \mathbb{N}$, are 1-bounded functions.

We first claim that if g is orthogonal to the subspace $\mathcal{C}$, then

$$\lim_{N\to\infty} \mathbb{E}_{n\in[N]} T^{a(n)}f \cdot T^{b(n)}g = 0$$

in $L^2(\mu)$. Indeed, if this is not the case, then there exist $a > 0$ and $N_k \to \infty$ such that

$$\Big\|\mathbb{E}_{n\in[N_k]} T^{a(n)}f \cdot T^{b(n)}g\Big\|_{L^2(\mu)} \geq a, \quad k \in \mathbb{N}.$$

If we define the 1-bounded functions $g_k := \mathbb{E}_{n\in[N_k]} T^{a(n)}f \cdot T^{b(n)}g$, $k \in \mathbb{N}$, we deduce that

$$\mathbb{E}_{n\in[N_k]} \int \overline{g}_k \cdot T^{a(n)}f \cdot T^{b(n)}g\, d\mu \geq a^2, \quad k \in \mathbb{N}. \tag{18}$$

By passing to a subsequence, we can assume that the sequence of 1-bounded functions $\mathbb{E}_{n\in[N_k]} T^{-b(n)}g_k \cdot T^{a(n)-b(n)}\overline{f}$, $k \in \mathbb{N}$, converges in the weak topology of $L^2(\mu)$ to a function $h \in \mathcal{C}$. Then composing with $T^{-b(n)}$ in (18) we deduce that $\int g \cdot \overline{h}\, d\mu \neq 0$, contradicting our assumption that g is orthogonal to the subspace $\mathcal{C}$. This proves our claim.

From the previous claim we conclude that

$$\lim_{N\to\infty} \mathbb{E}_{n\in[N]} T^{a(n)}f \cdot T^{b(n)}(g - \mathbb{E}(g|\mathcal{C})) = 0$$

in $L^2(\mu)$, where $\mathbb{E}(g|\mathcal{C})$ denotes the orthogonal projection of g onto the closed subspace $\mathcal{C}$. Hence, if (15) holds, then

$$\limsup_{N\to\infty} \Big\|\mathbb{E}_{n\in[N]} T^{a(n)}f \cdot T^{b(n)}\mathbb{E}(g|\mathcal{C})\Big\|_{L^2(\mu)} > 0.$$

Using the definition of $\mathcal{C}$ and an approximation argument, we get that there exist $N_k \to \infty$ and 1-bounded functions $g_k \in L^\infty(\mu)$, $k \in \mathbb{N}$, such that for $\tilde{g}$ as in (16) we have that (17) holds. Lastly, since f and g_k, $k \in \mathbb{N}$, are 1-bounded functions, the same holds for $\tilde{g}$. This completes the proof. □

4.1.3. *Seminorms of averaged functions.* Our next goal is to use Lemma 3.3 in order to show that if the uniformity seminorm of an average of functions is positive, then some positiveness property holds for iterated differences of the individual functions. Note that we do not impose any assumptions on the sequences $a, b\colon \mathbb{N} \to \mathbb{Z}$ here.

Proposition 4.4. *Let $(X, \mathcal{X}, \mu, T)$ be an ergodic system, and $\tilde{g} \in L^\infty(\mu)$ be as in (16) and satisfy $\|\tilde{g}\|_{s+2} > 0$ for some $s \in \mathbb{Z}_+$.*

(i) *If $s = 0$, then there exists $\chi \in \mathcal{E}(T)$ such that*

$$\limsup_{k\to\infty} \mathbb{E}_{n\in[N_k]} \Re\Big(\int g_k \cdot T^{a(n)}\overline{f} \cdot T^{b(n)}\chi\, d\mu\Big) > 0.$$

(ii) *If $s \geq 1$, then there exist $\chi_{\underline{n},\underline{n}'} \in \mathcal{E}(T)$, $\underline{n}, \underline{n}' \in \mathbb{N}^s$, such that*

$$\liminf_{N\to\infty} \mathbb{E}_{\underline{n},\underline{n}'\in[N]^s} \limsup_{k\to\infty} \mathbb{E}_{n\in[N_k]} \Re\Big(\int (\Delta_{\underline{n}-\underline{n}'}g_k) \cdot T^{a(n)}(\Delta_{\underline{n}-\underline{n}'}\overline{f}) \cdot T^{b(n)}\chi_{\underline{n},\underline{n}'}\, d\mu\Big) > 0.$$

Remark. The key point is that positivity properties of expressions involving $\Delta_{\underline{n}}\tilde{g}$, $\underline{n} \in \mathbb{N}^{s+2}$, imply positivity properties of expressions involving $\Delta_{\underline{n}}f$, $\underline{n} \in \mathbb{N}^s$.

Proof. Suppose that $s \geq 1$, the argument is similar if $s = 0$. Proposition 3.2 gives that there exist $\chi_{\underline{n}} \in \mathcal{E}(T)$, $\underline{n} \in \mathbb{N}^s$, such that

$$\liminf_{N\to\infty} \mathbb{E}_{\underline{n}\in[N]^s} \Re\Big(\int \Delta_{\underline{n}}\tilde{g} \cdot \chi_{\underline{n}}\, d\mu\Big) > 0.$$

Since $\Delta_{\underline{n}}\tilde{g} = \prod_{\epsilon\in\{0,1\}^s} \mathcal{C}^{|\epsilon|}T^{\epsilon\cdot\underline{n}}\tilde{g}$, $\underline{n}\in\mathbb{N}^s$, and $\tilde{g} = \lim_{k\to\infty}\mathbb{E}_{n\in[N_k]} f_{k,n}$ (the limit is a weak limit) where

$$f_{k,n} := T^{-b(n)}g_k \cdot T^{a(n)-b(n)}\overline{f}, \quad k,n\in\mathbb{N}, \tag{19}$$

we deduce that

$$\liminf_{N\to\infty}\lim_{k\to\infty}\mathbb{E}_{n\in[N_k]}\Re\Big(\mathbb{E}_{\underline{n}\in[N]^s}\int\prod_{\epsilon\in\{0,1\}^s\setminus\{\underline{1}\}}\mathcal{C}^{|\epsilon|}T^{\epsilon\cdot\underline{n}}\tilde{g}\cdot T^{n_1+\cdots+n_s}f_{k,n}\cdot\chi_{\underline{n}}\,d\mu\Big) > 0.$$

For fixed $k,n\in\mathbb{N}$, we apply Lemma 3.3 with $f_{\underline{1}} := f_{k,n}$, $f_\epsilon := \mathcal{C}^{|\epsilon|}f$ for $\epsilon\in\{0,1\}^s\setminus\underline{1}$, and $g_{\underline{n}} := \chi_{\underline{n}}$, $\underline{n}\in\mathbb{N}^s$, and deduce that

$$\liminf_{N\to\infty}\limsup_{k\to\infty}\mathbb{E}_{n\in[N_k]}\mathbb{E}_{\underline{n},\underline{n}'\in[N]^s}\int\Delta_{\underline{n}-\underline{n}'}f_{k,n}\cdot\chi_{\underline{n},\underline{n}'}\,d\mu > 0$$

for some $\chi_{\underline{n},\underline{n}'}\in\mathcal{E}(T)$, $\underline{n},\underline{n}'\in\mathbb{N}^s$ (we used that $\mathcal{E}(T)$ is closed under products and composition with iterates of T). Note that $\Delta_{\underline{n}}(w\cdot z) = \Delta_{\underline{n}}(w)\cdot\Delta_{\underline{n}}(z)$ and $\Delta_{\underline{n}}(T^k w) = T^k\Delta_{\underline{n}}(w)$ for all $w,z\in L^\infty(\mu)$ and $k\in\mathbb{N}$, $\underline{n}\in\mathbb{Z}^s$. Using this, equation (19), and keeping in mind that the limsup of a sum is at most the sum of the limsups, the asserted estimate follows from the last one after composing each function inside the integral with $T^{b(n)}$. □

4.2. **Proof of Theorem 4.1.** We are now ready to prove Theorem 4.1.

4.2.1. *Reduction to a degree lowering property.* Since the sequences $a,b\colon\mathbb{N}\to\mathbb{Z}$ are good for seminorms estimates for the system $(X,\mathcal{X},\mu,T)$, there exists $s\in\mathbb{Z}_+$ such that the seminorms $\|\!|\cdot|\!\|_{s+2}$ control the averages (12), in the sense that if $f,g\in L^\infty(\mu)$ are such that $\|\!|f|\!\|_{s+2}=0$ or $\|\!|g|\!\|_{s+2}=0$, then

$$\lim_{N\to\infty}\mathbb{E}_{n\in[N]}T^{a(n)}f\cdot T^{b(n)}g = 0 \tag{20}$$

in $L^2(\mu)$. Our goal is to show that a similar property holds with $s-1$ in place of s. Namely, using terminology from [26], we are going to establish the following *"degree lowering property"*:

Proposition 4.5. *Let S be a susbet of $[0,1)$ with countable complement in $[0,1)$. Let $a,b\colon\mathbb{N}\to\mathbb{Z}$ be good for equidistribution for S and $(X,\mathcal{X},\mu,T)$ be an ergodic system with spectrum in S. If for some $s\in\mathbb{Z}_+$ the seminorms $\|\!|\cdot|\!\|_{s+2}$ control the averages* (12)*, then also the seminorms $\|\!|\cdot|\!\|_{s+1}$ control the averages* (12).

This "degree lowering property" is the heart of the proof of Theorem 4.1. Iterating this property $s+1$ times we deduce that the seminorms $\|\!|\cdot|\!\|_1$ control the averages (12). Since $\|\!|f|\!\|_1 = |\int f\,d\mu|$, this proves Theorem 4.1.[4] So we get the following:

Proposition 4.6. *In order to verify Theorem 4.1 it suffices to verify Proposition 4.5.*

4.2.2. *Proof of Proposition 4.5.* We work under the assumption of Proposition 4.5 and our aim is to show that if $f\in L^\infty(\mu)$ satisfies $\|\!|f|\!\|_{s+1}=0$, then (20) holds (similarly we show that if $g\in L^\infty(\mu)$ satisfies $\|\!|g|\!\|_{s+1}=0$, then (20) holds). Equivalently, it suffices to show that if (15) holds, then $\|\!|f|\!\|_{s+1}>0$.

Suppose that $s\geq 1$, the argument is similar if $s=0$ (in this case the conclusion is that $\int f\,d\mu\neq 0$). Using (15) and Proposition 4.3, we deduce that

$$\limsup_{N\to\infty}\Big\|\mathbb{E}_{n\in[N]}T^{a(n)}f\cdot T^{b(n)}\tilde{g}\Big\|_{L^2(\mu)} > 0,$$

where

$$\tilde{g} := \lim_{k\to\infty}\mathbb{E}_{n\in[N_k]}T^{-b(n)}g_k\cdot T^{a(n)-b(n)}\overline{f}, \tag{21}$$

[4] We use here that Theorem 4.1 holds if f or g is constant, which follows from Proposition 4.2.

for some sequence of integers $N_k \to \infty$ and 1-bounded functions $g_k \in L^\infty(\mu)$, $k \in \mathbb{N}$, where the limit is a weak limit. Since, by assumption, the seminorms $\lVert\!| \cdot |\!\rVert_{s+2}$ control the averages in (20) we get that

$$\lVert\!| \tilde{g} |\!\rVert_{s+2} > 0. \tag{22}$$

Using Proposition 4.4 we deduce that

$$\liminf_{N\to\infty} \mathbb{E}_{\underline{n},\underline{n}'\in[N]^s} \limsup_{k\to\infty} \mathbb{E}_{n\in[N_k]} \Re\Big(\int (\Delta_{\underline{n}-\underline{n}'} g_k) \cdot T^{a(n)}(\Delta_{\underline{n}-\underline{n}'} \overline{f}) \cdot T^{b(n)}\chi_{\underline{n},\underline{n}'}\, d\mu \Big) > 0$$

for some $\chi_{\underline{n},\underline{n}'} \in \mathcal{E}(T)$, $\underline{n},\underline{n}' \in \mathbb{N}^s$. Using the Cauchy-Schwarz inequality we get

$$\liminf_{N\to\infty} \mathbb{E}_{\underline{n},\underline{n}'\in[N]^s} \limsup_{k\to\infty} \Big\lVert \mathbb{E}_{n\in[N_k]}\, T^{a(n)}(\Delta_{\underline{n}-\underline{n}'} \overline{f}) \cdot T^{b(n)}\chi_{\underline{n},\underline{n}'} \Big\rVert_{L^2(\mu)} > 0.$$

The advantage now is that since $\chi_{\underline{n},\underline{n}'} \in \mathcal{E}(T)$, $\underline{n},\underline{n}' \in \mathbb{N}^d$, the average over n is much simpler to analyse than the original one in Theorem 4.1. In fact, using Proposition 4.2 we get that it converges in $L^2(\mu)$ to the product of the integrals of the individual functions. We deduce that

$$\liminf_{N\to\infty} \mathbb{E}_{\underline{n},\underline{n}'\in[N]^s} \Big| \int \Delta_{\underline{n}-\underline{n}'} f\, d\mu \cdot \int \chi_{\underline{n},\underline{n}'}\, d\mu \Big| > 0$$

and as a consequence

$$\liminf_{N\to\infty} \mathbb{E}_{\underline{n},\underline{n}'\in[N]^s} \Big| \int \Delta_{\underline{n}-\underline{n}'} f\, d\mu \Big|^2 > 0.$$

Since $|\int \Delta_{\underline{n}} f\, d\mu|$ remains the same if we change the sign of some of the coordinates of $\underline{n}$, we deduce using a simple computation that

$$\liminf_{N\to\infty} \mathbb{E}_{\underline{n}\in[N]^s} \prod_{j=1}^{s} \big(1 - \frac{n_j}{N}\big) \cdot \Big| \int \Delta_{\underline{n}} f\, d\mu \Big|^2 > 0.$$

It follows that

$$\lim_{N\to\infty} \mathbb{E}_{\underline{n}\in[N]^s} \Big| \int \Delta_{\underline{n}} f\, d\mu \Big|^2 > 0$$

(the limit exists by (10)). Hence, by (10) we have that

$$\lVert\!| f |\!\rVert_{s+1} > 0$$

as required. This concludes the proof of Proposition 4.5 and by Proposition 4.6 the proof of Theorem 4.1.

5. Proof of the main result for general ℓ

We now give a summary of the proof of Theorem 2.1 for general ℓ, the reader should find it easy to fill in the missing details.

Let S be a susbet of $[0,1)$ with countable complement in $[0,1)$. Suppose that collection of sequences $a_1,\ldots,a_\ell\colon \mathbb{N}\to\mathbb{Z}$ is good for seminorm estimates and equidistribution for the ergodic system $(X,\mathcal{X},\mu,T)$ with specrum in S. Our goal is to show that for all $f_1,\ldots,f_\ell \in L^\infty(\mu)$ we have

$$\lim_{N\to\infty} \mathbb{E}_{n\in[N]}\, T^{a_1(n)} f_1 \cdot \ldots \cdot T^{a_\ell(n)} f_\ell = \int f_1\, d\mu \cdot \ldots \cdot \int f_\ell\, d\mu \quad \text{in} \quad L^2(\mu). \tag{23}$$

Our proof will deviate slightly from the argument given in the case $\ell = 2$, because a statement analogous to Proposition 4.2 cannot be proved directly when only one of the functions is in $\mathcal{E}(T)$ (the theorem of Herglotz is no longer applicable). As a substitute for this we are going to use an induction that we describe next.

We consider $\ell \geq 2$ fixed and we are going to show the following property by finite induction on $m \in \{1,\ldots,\ell\}$:

(P_m) If $f_j \in \mathcal{E}(T)$ for at least $\ell - m$ values of $j \in [\ell]$, then (23) holds.

If we show this, then taking $m=\ell$ gives that (23) holds for all functions $f_1,\dots,f_\ell\in L^\infty(\mu)$ (and as a consequence Theorem 2.1 holds).

For $m=1$ we can show that (P_1) holds as in Proposition 4.2 using the good equidistribution assumption of the sequences $a_1,\dots,a_\ell$ and the theorem of Herglotz.

Suppose now that property (P_{m-1}) holds for some $m\in\{2,\dots\ell\}$. We are going to show that property (P_m) holds. To this end, we assume, without loss of generality, that $f_j\in\mathcal{E}(T)$ for $j=m+1,\dots,\ell$, and we are going to show that (23) holds by employing a degree lowering argument, similar to the one we used in the previous section. More precisely, our plan is to show that if for some $s\in\mathbb{Z}_+$ the seminorms $\|\!|\cdot|\!\|_{s+2}$ control the averages in (23), in the sense that if $\|\!|f_i|\!\|_{s+2}=0$ for some $i\in\{1,\dots,\ell\}$ and $f_j\in\mathcal{E}(T)$ for $j=m+1,\dots,\ell$, we have that the averages in (23) converge to 0, then the seminorms $\|\!|\cdot|\!\|_{s+1}$ also control these averages. Since, by our good seminorm assumption, the seminorms $\|\!|\cdot|\!\|_{s+2}$ control the averages (23) for some $s\in\mathbb{N}$, iterating this degree lowering property $s+1$ times we deduce that the seminorms $\|\!|\cdot|\!\|_1$ control the averages in (23), and this easily implies that property (P_m) holds.

So suppose that

$$\limsup_{N\to\infty}\Big\|\mathbb{E}_{n\in[N]}\,T^{a_1(n)}f_1\cdot\ldots\cdot T^{a_\ell(n)}f_\ell\Big\|_{L^2(\mu)}>0 \tag{24}$$

for some 1-bounded functions $f_1,\dots,f_\ell\in L^\infty(\mu)$ with $f_j\in\mathcal{E}(T)$ for $j=m+1,\dots,\ell$. Then arguing as in the proof of Proposition 4.3 we get that (24) continues to hold if in place of the function f_1 we use the function $\tilde{f}_1$ defined by

$$\tilde{f}_1:=\lim_{k\to\infty}\mathbb{E}_{n\in[N_k]}\,T^{-a_1(n)}g_k\cdot\prod_{j=2}^{\ell}T^{a_j(n)-a_1(n)}\overline{f}_j, \tag{25}$$

for some $N_k\to\infty$ and 1-bounded functions $g_k\in L^\infty(\mu)$, $k\in\mathbb{N}$, where the limit is a weak limit (note that then $\tilde{f}_1\in L^\infty(\mu)$). Since, by our assumption, the seminorms $\|\!|\cdot|\!\|_{s+2}$ control the averages (23) we deduce that $\|\!|\tilde{f}_1|\!\|_{s+2}>0$. As in the proof of Proposition 4.4 we get for $s\geq 1$ that there exist $\chi_{\underline{n},\underline{n}'}\in\mathcal{E}(T)$, $\underline{n},\underline{n}'\in\mathbb{N}^s$, such that

$$\liminf_{N\to\infty}\mathbb{E}_{\underline{n},\underline{n}'\in[N]^s}\ \limsup_{k\to\infty}\mathbb{E}_{n\in[N_k]}\,\Re\Big(\int(\Delta_{\underline{n}-\underline{n}'}g_k)\cdot T^{a_1(n)}\chi_{\underline{n},\underline{n}'}\cdot\prod_{j=2}^{\ell}T^{a_j(n)}(\Delta_{\underline{n}-\underline{n}'}\overline{f}_j)\,d\mu\Big)>0,$$

and a somewhat simpler statement for $s=0$ that can be dealt in a similar fashion. The advantage now is that since for all $\underline{n},\underline{n}'\in\mathbb{N}^s$ we have $\chi_{\underline{n},\underline{n}'}\in\mathcal{E}(T)$ and $\Delta_{\underline{n}-\underline{n}'}\overline{f}_j\in\mathcal{E}(T)$ for $j=m+1,\dots,\ell$, property (P_{m-1}) applies and gives that the average over the variable n converges in $L^2(\mu)$ to the product of the integrals of the corresponding functions. We then deduce as in the proof of Proposition 4.6 that

$$\|\!|f_j|\!\|_{s+1}>0\quad\text{for}\quad j=2,\dots\ell.$$

Furthermore, since $l\geq 2$ we can apply the same argument for the second position instead of the first and deduce in a similar fashion that $\|\!|f_1|\!\|_{s+1}>0$. We conclude that the seminorms $\|\!|\cdot|\!\|_{s+1}$ control the averages (23). This shows that property (P_m) holds and concludes the proof of the induction and the proof of Theorem 2.1.

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