# Stable Big Bang Formation for Einstein's Equations: The Complete Sub-Critical Regime 

Grigorios Fournodavlos,* Igor Rodnianski, ${ }^{\dagger}$ Jared Speck ${ }^{\ddagger}$


#### Abstract

For $(t, x) \in(0, \infty) \times \mathbb{T}^{\mathcal{D}}$, the generalized Kasner solutions (which we refer to as Kasner solutions for short) are a family of explicit solutions to various Einstein-matter systems that, exceptional cases aside, start out smooth but then develop a Big Bang singularity as $t \downarrow 0$, i.e., a singularity along an entire spacelike hypersurface, where various curvature scalars blow up monotonically. The family is parameterized by the Kasner exponents $\widetilde{q}_{1}, \cdots, \widetilde{q}_{\mathcal{D}} \in \mathbb{R}$, which satisfy two algebraic constraints. There are heuristics in the mathematical physics literature, going back more than 50 years, suggesting that the Big Bang formation should be dynamically stable, that is, stable under perturbations of the Kasner initial data, given say at $\{t=1\}$, as long as the exponents are "sub-critical" in the following sense: $\max _{\substack{I, J, B=1, \ldots, \mathfrak{D} \\ I<J}}\left\{\widetilde{q}_{I}+\widetilde{q}_{J}-\widetilde{q}_{B}\right\}<1$. Previous works have rigorously shown the dynamic stability of the Kasner Big Bang singularity under stronger assumptions: 1) the Einstein-scalar field system with $\mathfrak{D}=3$ and $\widetilde{q}_{1} \approx \widetilde{q}_{2} \approx \widetilde{q}_{3} \approx 1 / 3$, which corresponds to the stability of the FLRW solution's Big Bang or 2) the Einstein-vacuum equations for $\mathfrak{D} \geq 38$ with $\max _{I=1, \cdots, \mathfrak{D}}\left|\widetilde{q}_{I}\right|<1 / 6$. In this paper, we prove that the Kasner singularity is dynamically stable for all sub-critical Kasner exponents, thereby justifying the heuristics in the literature in the full regime where stable monotonic-type curvature blowup is expected. We treat in detail the $1+\mathfrak{D}$-dimensional Einstein-scalar field system for all $\mathfrak{D} \geq 3$ and the $1+\mathfrak{D}$-dimensional Einstein-vacuum equations for $\mathfrak{D} \geq 10$; both of these systems feature non-empty sets of sub-critical Kasner solutions. Moreover, for the Einstein-vacuum equations in $1+3$ dimensions, where instabilities are in general expected, we prove that all singular Kasner solutions have dynamically stable Big Bangs under polarized $U(1)$-symmetric perturbations of their initial data. Our results hold for open sets of initial data in Sobolev spaces without symmetry, apart from our work on polarized $U(1)$-symmetric solutions.

Our proof relies on a new formulation of Einstein's equations that privileges the role of scalar functions over tensorial objects: we use a constant-mean-curvature foliation, and the unknowns are the scalar field, the lapse, the components of the spatial connection and second fundamental form relative to a FermiWalker transported spatial orthonormal frame, and the components of the orthonormal frame vectors with respect to a transported spatial coordinate system. In this formulation, the PDE evolution system for the structure coefficients of the orthonormal frame approximately diagonalizes in a way that sharply reveals the significance of the Kasner exponent sub-criticality condition for the dynamic stability of the flow: the condition leads to the time-integrability of many terms in the equations, at least at the low derivative levels. At the high derivative levels, the solutions that we study can be much more singular with respect to $t$, and to handle this difficulty, we use $t$-weighted high order energies, and we control nonlinear error terms by exploiting monotonicity induced by the $t$-weights and interpolating between the singularity-strength of the solution's low order and high order derivatives. Finally, we note that our formulation of Einstein's equations highlights the quantities that might generate instabilities outside of the sub-critical regime.


Keywords: Big Bang, constant mean curvature, curvature singularity, Fermi-Walker transport, geodesically incomplete, Hawking's theorem, Kasner solutions, maximal globally hyperbolic development, singularity theorem, stable blowup, transported spatial coordinates

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## 1 Introduction

Our main results in this paper are proofs of stable Big Bang formation (i.e., curvature blowup along an entire spacelike hypersurface) for cosmologica ${ }^{1}$ solutions to the Cauchy problem for the Einstein-vacuum and Einstein-scalar field systems. All of our results hold for open sets of solutions without symmetry, except for our results on polarized $U(1)$-symmetric solutions to the Einstein-vacuum equations in $1+3$ dimensions. We assume that initial data are given on the manifold $\Sigma_{1}=\mathbb{T}^{\mathfrak{D}}:=[-\pi, \pi]^{\mathfrak{D}}$ (with the endpoints identified),

[^1]where $\mathfrak{D} \geq 3$ is the number of spatial dimensions. Later on, we provide a precise description of which kinds of data our results apply to and how the value of $\mathfrak{D}$ is tied to the data. As we will explain, our results are sharp in the sense that they rigorously confirm the dynamic stability of the singularity formation in the entire regime where heuristics in the literature have suggested it might occur. In particular, our results significantly extend the prior results [58, 59, 60, which yield stable Big Bang formation for open sets of solutions without symmetry. We refer readers to Theorem 1.5 for a rough version of our main results and to Theorems 6.1 and 6.5 for precise statements.

The sharpness of our results is possible because we have developed a new analytic framework for constant mean curvature (CMC) foliations in which we study the components of various spatial tensors relative to an orthonormal "spatial frame," obtained by Fermi-Walker transport, as well as the connection coefficients and structure coefficients of the frame. We refer readers to Sect. 2 for the precise details behind the gauge and the corresponding formulation of Einstein's equations that we use to derive estimates. We also refer to Sect. 1.8 for an overview of the proof. Our framework allows us to precisely and efficiently detect the terms in the equations that are integrable-in-time up to the singularity, which is key to understanding the stability of the blowup. Our framework also pinpoints the terms in the equations that might generate instabilities in other regimes; see Remark 1.2,

### 1.1 The Cauchy problem for the Einstein-scalar field equations

### 1.1.1 The Einstein-scalar field equations

Relative to arbitrary coordinates, the Einstein-scalar field equations can be expressed as

$$
\begin{align*}
\mathbf{R i c}_{\mu \nu} & =\partial_{\mu} \psi \partial_{\nu} \psi  \tag{1.1a}\\
\square_{\mathbf{g}} \psi & =0 \tag{1.1b}
\end{align*}
$$

In 1.1a-1.1b and throughout, Ric is the Ricci curvature of the spacetime metric $\mathbf{g}$ (which has signature $(-,+,+, \cdots,+)), \square_{\mathbf{g}}:=\left(\mathbf{g}^{-1}\right)^{\alpha \beta} \mathbf{D}_{\alpha} \mathbf{D}_{\beta}$ is the covariant wave operator of $\mathbf{g}, \mathbf{D}$ is the Levi-Civita connection of $\mathbf{g}$, and $\psi$ is the scalar field. Note that in the special case $\psi \equiv 0$, the system is equivalent to the Einsteinvacuum equations.

### 1.1.2 The initial value problem formulation and the initial data

It is well-known that the system 1.1a 1.1 b has an initial value problem formulation in which sufficiently regular initial data give rise to unique solutions. An initial data set for the system is defined to be $\left(\Sigma_{1}, \stackrel{\circ}{g}, \stackrel{\circ}{k}, \dot{\psi}, \stackrel{\circ}{\phi}\right)$, where $\stackrel{\circ}{g}$ is a Riemannian metric on the manifold $\Sigma_{1}$ (in this paper, we assume that $\Sigma_{1}:=\mathbb{T}^{\mathfrak{D}}$ ), $\grave{k}$ is a symmetric two-tensor, and $\dot{\psi}, \dot{\phi}$ are a pair of scalar functions. It is well-known that admissible data must verify the Hamiltonian and momentum constraint equations, which are respectively:

$$
\begin{align*}
& \stackrel{\circ}{R}-|\stackrel{\circ}{k}|^{2}+(\operatorname{tr} \stackrel{\circ}{k})^{2}=\dot{\phi}^{2}+|\nabla \circ \stackrel{\circ}{\psi}|^{2},  \tag{1.2a}\\
& \operatorname{div} \grave{k}-\stackrel{\circ}{\nabla} \operatorname{tr} \stackrel{\circ}{k}=-\dot{\phi} \stackrel{\circ}{\nabla} \dot{\psi}, \tag{1.2b}
\end{align*}
$$

where $\stackrel{\circ}{\nabla}$ is the Levi-Civita connection of $\stackrel{\circ}{g}$ (with respect to which all covariant spatial operators along $\Sigma_{1}$ are defined) and $\stackrel{\circ}{R}$ is the scalar curvature of $\stackrel{\circ}{g}$.

### 1.1.3 Globally hyperbolic developments

A globally hyperbolic development of the data, which can be thought of as a solution to the initial value problem, is a triplet $(\mathcal{M}, \mathbf{g}, \psi)$ and an embedding $i: \Sigma_{1} \rightarrow \mathcal{M}$ such that

- $\mathcal{M}$ is a $1+\mathfrak{D}$-dimensional spacetime manifold
- $\mathbf{g}$ is a Lorentzian metric on $\mathcal{M}$ and $\psi$ is a scalar function on $\mathcal{M}$ that together solve the equations (1.1a)-1.1b
- $i\left(\Sigma_{1}\right)$ is a Cauchy hypersurfac $\overbrace{}^{2}$ in $(\mathcal{M}, \mathbf{g})$

[^2]- The pullbacks of the first and second fundamental forms of the image surface $i\left(\Sigma_{1}\right)$ (see Sect. 2 for our sign conventions for the second fundamental form) are equal to $\stackrel{\circ}{g}, \stackrel{\circ}{k}$ respectively, and the pullbacks of the initial values of the scalar field $\psi$ and its derivative with respect to the future unit normal to $i\left(\Sigma_{1}\right)$ are equal to $\dot{\psi}, \dot{\phi}$ respectively.

The fundamental work [16] of Choquet-Bruhat and Geroch shows that for sufficiently regular initial data verifying the constraints, there is a unique (up to diffeomorphism) maximal classical globally hyperbolic development $\left(\mathcal{M}_{M a x}, \mathbf{g}_{M a x}, \psi_{M a x}\right)$ (and a corresponding embedding $i_{M a x}$ that we will suppress), which we refer to as the "maximal development" for short. Roughly, the solution furnished by [16] is the largest possible classical solution to $1.1 \mathrm{a}-1.1 \mathrm{~b}$ that is uniquely determined by the initial data; we refer the reader to [55] for detailed discussion of the maximal development. Although it is of philosophical importance to know that the maximal development exists and is unique, the results of [16] do not reveal much about its structure. Our goal in this article is to fully understand its structure for open sets of solutions that exhibit curvature blowup.

### 1.2 Connections with the Hawking-Penrose singularity theorems

The celebrated "singularity theorems" 33, 34, 46] of Hawking and Penrose show that there exist large sets of regular initial data for the Einstein equations such that the corresponding solutions eventually break down in the sense that the spacetime is causally geodesically incomplete. The results apply to any matter model verifying the strong energy condition, including the scalar field model and the vacuum. Although these works are of immense philosophical importance in general relativity and have had a great impact on the direction of the field, they are limited in that their proofs are by contradiction and do not provide any information about the nature of the breakdown, aside from geodesic incompleteness. Through various telling examples, it is known that different kinds of breakdown are possible. A particularly sinister scenario is found in the Taub-NUT and Kerr spacetimes, where the breakdown is not caused by any singularity in the metric (including its higher derivatives), but rather is caused by the development of a Cauchy horizon, across which the solution can be smoothly extended in more than one way, signifying the failure of determinism past the maximal globally hyperbolic development of the data. In the opposite direction, the Strong Cosmic Censorshin ${ }^{3}$ conjecture suggests that, "generically," the maximal globally hyperbolic development of the data is inextendible, roughly due to the formation of some kind of singularity ${ }_{4}^{4}$ Confirming some version of the Strong Cosmic Censorship hypothesis, at least in a perturbative regime around explicit solutions, turns out to be extremely difficult, due to the strength of the nonlinearities in the system and the possibility of complicated dynamics near singularities. Moreover, it is important to appreciate that regularity considerations are of crucial importance when defining what is meant by the "Strong Cosmic Censorship hypothesis;" thanks to the remarkable work [27] on the $C^{0}$-stability of the Kerr Cauchy horizon ${ }^{5}$ we now know that the $C^{0}$ formulation of the Strong Cosmic Censorship hypothesis is not generically true. More precisely, in [27], it was shown that for an open set of near-Kerr solutions, the metric can be continuously extended beyond the Cauchy horizon. However, it is conceivable that these metrics generically do not enjoy any additional regularity and in particular that they cannot even be extended past the Cauchy horizon as weak solutions to Einstein's equations. It therefore remains possible that a revised version of the Strong Cosmic Censorship hypothesis is true, in which "generically, there is breakdown at the boundary of the maximal development," where "breakdown" is defined to be any loss of regularity that is sufficiently strong to prevent one from extending the solution as a weak solution to Einstein's equations. There are works in spherical symmetry that support this possibility, notably [25, 26, 44], where [44] has the compelling feature that it is a large data result. More precisely, 44 proves that for an open and dense set of two-ended asymptotically flat initial data for the Einstein-Maxwell-(real)-scalar-field system in spherical symmetry, the maximal development is $C^{2}$-inextendible.

[^3]
### 1.3 Beyond the Hawking-Penrose singularity theorems

In the wake of the Hawking-Penrose singularity theorems, there have been many works devoted towards understanding the precise cause of the geodesic incompleteness. An interesting type of breakdown that has received extensive attention - rigorous and otherwise - over the past half-century is the "Kasner-like scenario," which concerns solutions whose metrics $\mathbf{g}$ are asymptotic to

$$
\begin{equation*}
\mathbf{g}_{\text {Limiting }}(t, x)=-d t \otimes d t+\sum_{I=1}^{\mathfrak{D}} t^{2 q_{I}(x)} \theta^{I}(x) \otimes \theta^{I}(x), \quad \theta^{I}(x)=\theta_{a}^{I}(x) d x^{a} \tag{1.3}
\end{equation*}
$$

as $t \downarrow 0$ (i.e., towards the singularity), inspired by the Kasner solutions themselves, which we discuss in Sect. 1.4 . It is important to note that the metrics $\mathbf{g}_{\text {Limiting }}$ are not generally solutions to Einstein's equations. However, in the special case that $\theta^{I}=d x^{I}$ and the $\left\{q_{I}\right\}_{I=1, \cdots, \mathfrak{D}}$ are constants satisfying two algebraic constraints (see (1.7)), $\mathbf{g}_{\text {Limiting }}$ is a solution, known as a Kasner solution in the vacuum case and a "generalized Kasner solution" in the presence of matter (for short, we sometimes refer to all such solutions simply as "Kasner solutions"). The Kasner solutions are spatially homogeneous and, exceptional cases aside, exhibit monotonic Big Bang formation (i.e., monotonic blowup of the spacetime Kretschmann scalar Riem ${ }^{\alpha \mu \beta \nu}$ Riem $_{\alpha \mu \beta \nu}$ along a spacelike hypersurface) as $t \downarrow 0$, as do the metrics $\mathbf{g}_{\text {Limiting }}$. We stress that Big Bang formation is consistent with the assertions of a $C^{2}$-inextendibility formulation of the Strong Cosmic Censorship hypothesis. We also note that in the remainder of the paper, we often denote the (constant) Kasner exponents by $\left\{\widetilde{q}_{I}\right\}_{I=1, \cdots, \mathfrak{D}}$, where the tilde emphasizes that they are associated to a "background Kasner solution."

A standout question, then, is: besides the explicit Kasner solutions (which we describe in Sect. 1.4), are there any other solutions to Einstein's equations - in particular ones with spatial dependence - that are asymptotic to a metric of the form $\mathbf{g}_{\text {Limiting }}$ and thus exhibit monotonic-type Big Bang formation? In an influential paper [40], the authors gave heuristic arguments suggesting that in the vacuum case in $1+3$ dimensions, solutions that are asymptotic to a metric of the form $\mathbf{g}_{\text {Limiting }}$ should be non-generic (in particular, unstable). In the subsequent work [12, still in the vacuum case in $1+3$ dimensions, it was suggested that "generically" (the meaning of "generic" was not rigorously defined), solutions that exhibit Big Bang formation "should" - unlike the Kasner solutions - be highly oscillatory in time as the singularity is approached. The alleged oscillatory behavior is sometimes referred to as the "Mixmaster scenario," where the terminology goes back to Misner's important paper [45] on oscillatory solutions with Bianchi IX symmetry. The oscillations are one of several features that have been conjectured to hold for "most" $1+3$-dimensional Einstein-vacuum solutions that have regions with incomplete timelike geodesics. This picture has come to be known, somewhat imprecisely, as the "BKL" conjecture. As of present, the only rigorous ${ }^{6}$ evidence to support this scenario concerns only spatially homogeneous solutions. The first important result in this direction was provided by Ringström, who proved 50 that for various matter models, solutions with Bianchi IX symmetry generically exhibit oscillatory behavior towards their singularity. His paper was followed by various works, including [10, 14, 30, 35, 42, 48. We also again highlight that, whatever one's interpretation of the BKL conjecture, the work [27] shows that some of its basic qualitative assertions are false for Einstein-vacuum solutions corresponding to near-Kerr black hole initial data. For example, the Cauchy horizons in [27] are null, which is at odds with the BKL prediction that incompleteness should be tied to some kind of blowup along a spacelike hypersurface.

Later on, there were further heuristic works suggesting that if the Einstein equations are coupled to a scalar field [11] or a stiff fluid ${ }^{7}$ [8, or if one considers the Einstein-vacuum equations in $1+\mathfrak{D}$ dimensions with $\mathfrak{D} \geq 10$ [29], then the oscillations can be silenced, leading back to the Kasner-like scenario. More precisely, there "should" exist open sets of initial data whose solutions exhibit monotonic Big Bang formation. The essence of these works is that the following "sub-criticality condition" (which we sometimes refer to as a "stability condition") for the Kasner exponents $\left\{q_{I}(x)\right\}_{I=1, \cdots, \mathfrak{D}}$ might be sufficient to ensure the existence of sets of solutions - containing all the gravitational degrees of freedom - that have Kasner-like Big Bang

[^4]singularities:
\[

$$
\begin{equation*}
\max _{\substack{I, J, B=1, \cdots, \mathfrak{D} \\ I<J}}\left\{q_{I}(x)+q_{J}(x)-q_{B}(x)\right\}<1 . \tag{1.4}
\end{equation*}
$$

\]

The condition $\sqrt{1.4}$ is centra $]^{8}$ to our main results, and we will discuss its implications in detail below.
We highlight that, due to the constraints 1.7 , for the Einstein-vacuum equations in $1+\mathfrak{D}$ dimensions, the condition $\sqrt[1.4]{ }$ can be satisfied only if $\mathfrak{D} \geq 10$; this algebraic fact was first observed in [29]. The papers described in the previous paragraph, which were in favor of the dynamic stability of the Big Bang, were based on heuristic justifications of the claim that the condition (1.4) "should" lead to asymptotically velocity term dominated (AVTD) behavior in perturbed solutions. Roughly, AVTD behavior for a solution is such that in Einstein's equations, the spatial derivative terms become negligible compared to the time derivative terms as the singularity is approached. Put differently, AVTD behavior is such that the solution becomes asymptotic to a truncated version of Einstein's equations in which all spatial derivative terms are thrown away. Since the truncated equations are ODEs at each fixed spatial point $x$, one could say that AVTD solutions are asymptotically $x$-parameterized ODE solutions. As we will later explain, the condition (1.4) (see also 1.8) suggests that for perturbations of the Kasner solution, the Ricci tensor of the perturbed spatial metric, which we denote by Ric, should satisfy, for some $\sigma>0,|R i c| \lesssim t^{-2+\sigma}$ as $t \downarrow 0$. It turns out that, when available, this bound leads to the time-integrability of various terms in Einstein's equations. In turn, the time-integrability is key to proving the AVTD nature of perturbations of Kasner solutions and for controlling the dynamics up to the singularity. In Sect. 1.6 we provide a more detailed explanation of the significance of the bound $|R i c| \lesssim t^{-2+\sigma}$ for the proofs of our main results.

Clearly, any rigorous justification of the above circle of ideas requires, at a minimum, the construction of a gauge relative to which the AVTD behavior can be exhibited. In the present paper, we introduce a general gauge + framework for proving stable singularity formation for "Kasner-like" solutions with spatial dependence and for proving the AVTD behavior. As in previous works on stable Big Bang formation 58, 59, 60, 61, we rely on constant mean curvature foliations in which the level sets of the time function $t$ have mean curvaturf ${ }^{9}$ equal to $-\frac{1}{\mathfrak{D} t}$, and we control the lapse $n:=\left[-\left(\mathbf{g}^{-1}\right)^{\alpha \beta} \partial_{\alpha} t \partial_{\beta} t\right]^{-1 / 2}$ via elliptic estimates. The main new idea in our paper lies in our approach to controlling the dynamic "spatia ${ }^{10}$ tensorfields": we construct a gauge for Einstein's equations in which the main dynamical unknowns are the components of various spatial tensorfields relative to an orthonormal "spatial frame" $\left\{e_{I}\right\}_{I=1, \cdots, \mathfrak{D}}$, obtained by Fermi-Walker transport (see equation 2.8) and Remark 2.1), as well as the connection coefficients $\gamma_{I J B}:=\mathbf{g}\left(\mathbf{D}_{e_{I}} e_{J}, e_{B}\right)$. One of our key observations is: as a consequence of the special structure of Einstein's equations and the Fermi-Walker transport equation $\sqrt{2.8}$, the frame is one degree more differentiable than naive estimates suggest. More precisely, the transport equation 2.23a, which is an equivalent formulation of 2.8), suggests that the frame vectorfield components $\left\{e_{I}^{i}\right\}_{I, i=1, \cdots, \mathcal{D}}$ are only as regular as the second fundamental form $k$ of $\Sigma_{t}$. However, our gauge allows us to prove that in fact, the connection coefficients $\left\{\gamma_{I J B}\right\}_{I, J, B=1, \cdots, \mathfrak{D}}$ of the frame enjoy the same Sobolev regularity as the components $\left\{k_{I J}\right\}_{I, J=1, \cdots, \mathfrak{D}}$ where $k_{I J}:=k\left(e_{I}, e_{J}\right)=k_{c d} e_{I}^{c} e_{J}^{d}$; this signifies a gain of one derivative for the frame. Roughly, the gain in regularity stems from the fact that $\left\{\gamma_{I J B}\right\}_{I, J, B=1, \cdots, \mathfrak{D}}$ and $\left\{k_{I J}\right\}_{I, J=1, \cdots, \mathfrak{D}}$ satisfy a system of wave-like equations (coupled to $n$ and the scalar field) that allow us to propagate the Sobolev regularity of their initial data. We refer readers to Lemma 5.19 for a differential version of the basic energy identity that we use to obtain the desired regularity for $\gamma$ and $k$.

A second key observation is that the structure coefficients of the frame, namely $\left.{ }^{11}\right] \mathbf{g}\left(\left[e_{I}, e_{J}\right], e_{B}\right)=\gamma_{I J B}+$ $\gamma_{J B I}$, satisfy an evolution equation system (see Proposition 5.7) that is diagonal up to quadratic error terms, and such that the strength of the main linear terms in the equations is controlled by the Kasner stability condition (1.8). More precisely, we have $\partial_{t}\left(\gamma_{I J B}+\gamma_{J B I}\right)=-\frac{\left(\widetilde{q}_{\underline{I}}+\widetilde{q}_{J}-\widetilde{q}_{\underline{B}}\right)}{t}\left(\gamma_{\underline{I J B}}+\gamma_{\underline{J B I}}\right)+\cdots$, where here and throughout the paper, we do not sum over repeated underlined indices. From this equation and the

[^5]condition (1.8), we are able to prove that there exists a constant $q<1$ such that
\[

$$
\begin{equation*}
\max _{\substack{I, J, B=1, \cdots, \mathfrak{D} \\ I<J}} t^{q}\left|\gamma_{I J B}+\gamma_{J B I}\right| \lesssim d a t a, \quad(t, x) \in(0,1] \times \mathbb{T}^{\mathfrak{D}} \tag{1.5}
\end{equation*}
$$

\]

where "data" denotes a small term that is controlled by the size of the perturbation of the initial data from the Kasner data (in particular, "data" vanishes for Kasner solutions). The estimate (1.5) leads to the timeintegrability of many terms in the evolution equations, allows us to rigorously justify the aforementioned spatial Ricci curvature bound ${ }^{12}|R i c| \lesssim t^{-2+\sigma}$, and allows us to prove the AVTD behavior of perturbations of any Kasner solution with exponents verifying (1.8).
Remark 1.1 (A basis of structure coefficient functions). The antisymmetry property ${ }^{13} \gamma_{I J B}+\gamma_{J B I}=$ $-\left(\gamma_{J I B}+\gamma_{I B J}\right)$, which follows from 2.20 , implies that $\left\{\gamma_{I J B}+\gamma_{J B I} \mid 1 \leq I, J, B \leq \mathfrak{D}, I<J\right\}$ forms a basis for the structure coefficient functions. This explains the condition $I<J$ on LHS (1.5). We use this simple fact throughout the article without always explicitly mentioning it.
Remark 1.2 (Sharply identifying possible obstructions to stability: Three distinct indices). Recall that we only have to consider structure coefficients with $I<J$ (see Remark 1.1) and that (aside from the trivial case of a single non-zero Kasner exponent equal to unity) we have $\max _{I=1, \cdots, \mathfrak{D}}\left|\widetilde{q}_{I}\right|<1$ (see Remark 1.3 ). It follows that when $I<J$, unless all three indices are distinct, two of the terms in the sum $q_{I}+q_{J}-q_{B}$ must cancel each other, leaving us with a single term $q_{\text {survivor }}$ satisfying $\left|q_{\text {survivor }}\right|<1$. Recalling also the evolution equation $\partial_{t}\left(\gamma_{I J B}+\gamma_{J B I}\right)=-\frac{\left(\widetilde{q}_{\underline{I}}+\widetilde{q}_{\underline{J}}-\widetilde{q}_{\underline{B}}\right)}{t}\left(\gamma_{\underline{I J B}}+\gamma_{\underline{J B I}}\right)+\cdots$ mentioned above, we see that when $I<J$, unless all three indices are distinct, the structure coefficient $\gamma_{I J B}+\gamma_{J B I}$ is expected to behave (modulo the error terms "...") like $t^{-q_{\text {survivor }} . ~ I n ~ p a r t i c u l a r, ~ m o d u l o ~ t h e ~ e f f e c t ~ o f ~ t h e ~ e r r o r ~ t e r m s ~ " . . ., " ~ s u c h ~ s t r u c t u r e ~}$ coefficients are integrable with respect to $t$ near $t=0$ and are compatible with our proof of the stability of the Big Bang. Thus, for perturbations of Kasner solutions, the only structure coefficients $\gamma_{I J B}+\gamma_{J B I}$ (with $I<J)$ that in principle could serve as an obstruction to stable Big Bang formation are those such that the sum $\widetilde{q}_{I}+\widetilde{q}_{J}-\widetilde{q}_{B}$ is greater than 1 , and this is possible only when all three indices are distinct; the stability condition 1.8 is the assumption that this obstruction is absent.

Finally, we highlight that our framework also extends to some symmetric sub-regimes of regimes where Mixmaster-related instabilities might generally occur, such as in the vacuum case in $1+3$ dimensions. More precisely, one does not truly need the condition (1.8) to prove monotonic-type Big Bang formation; our approach works as long as one can prove the estimate (for some constant $q<1$ ). The point is that by imposing symmetries on solutions, one can eliminate some of the gravitational degrees of freedom in the problem, and it can become possible to prove the estimate 1.5 even if the condition 1.8 fails. Roughly, this is sometimes possible because symmetries can force some of the structure coefficients to vanish. For example, in this paper, we treat in detail the case of polarized $U(1)$-symmetric solutions to the $1+3$-dimensional Einstein-vacuum equations, and under symmetric perturbations, we prove the stability of the Big Bang for all Kasner solutions - not just ones that satisfy (1.8). In the next section, we precisely describe the models that we treat in detail. Moreover, in Sect. 1.9 , we describe other contexts in which our methods are potentially applicable.

### 1.4 The models

Our main results yield stable curvature blowup for a subset of the family of generalized Kasner solutions on $(0, \infty) \times \mathbb{T}^{\mathfrak{D}}$, which can be expressed as follows:

$$
\begin{equation*}
\widetilde{\mathbf{g}}=-d t \otimes d t+\widetilde{g}, \quad \widetilde{g}:=\sum_{I=1, \cdots, \mathfrak{D}} t^{2 \widetilde{q}_{I}} d x^{I} \otimes d x^{I}, \quad \widetilde{\psi}=\widetilde{B} \log t \tag{1.6}
\end{equation*}
$$

The Kasner exponents $\left\{\widetilde{q}_{I}\right\}_{I=1, \cdots, \mathfrak{D}}$ and $\widetilde{B}$ are constants constrained by the following two algebraic equations:

$$
\begin{equation*}
\sum_{I=1}^{\mathfrak{D}} \widetilde{q}_{I}=1, \quad \quad \sum_{I=1}^{\mathfrak{D}} \widetilde{q}_{I}^{2}=1-\widetilde{B}^{2} \tag{1.7}
\end{equation*}
$$

[^6]The equations in (1.7) are consequences of two other equations: i) the mean curvature condition $\operatorname{tr} \widetilde{k}=-\frac{1}{t}$ (which we discuss in more detail later), where $\widetilde{k}$ is the second fundamental form of $\Sigma_{t}$ with respect to $\widetilde{\mathbf{g}}$, and ii) the Hamiltonian constraint 1.2 a . One can check that under the above assumptions, the tensorfields $(\widetilde{\mathbf{g}}, \widetilde{\psi})$ are solutions to the $1+\mathfrak{D}$-dimensional Einstein-scalar field equations 1.1a 1.1 b .

Our main results come in two flavors. In the first case, we make no symmetry assumptions on the initial data, and our results yield the dynamic stability of the Kasner Big Bang singularity whenever the exponents of the background Kasner solution themselves verify the sub-criticality condition (which we also refer to as the "stability condition"), in which case it reads:

$$
\begin{equation*}
\max _{\substack{I, J, B=1, \cdots, \mathfrak{D} \\ I<J}}\left\{\widetilde{q}_{I}+\widetilde{q}_{J}-\widetilde{q}_{B}\right\}<1 \tag{1.8}
\end{equation*}
$$

Since our results imply that the final Kasner exponents of the perturbed singular solution are close to those of the background (see $\sqrt{6.6}$ ), and since $\sqrt{1.8}$ ) is an open condition, our perturbed Kasner-like solutions will satisfy the original condition $(1.4)$ as well.

Remark 1.3. The Kasner constraints (1.7) imply that, aside from the trivial case in which one of the $\widetilde{q}_{I}$ is equal to 1 and the others vanish (in which case the Kasner spacetime metric is flat), we must have $\max _{I=1, \cdots, \mathfrak{D}}\left|\widetilde{q}_{I}\right|<1$. Thus, assuming the Kasner exponent constraints, we could replace 1.8 with the condition

$$
\begin{equation*}
\max _{\substack{I, J, B=1, \cdots, \mathfrak{D} \\ I \neq J \neq B \neq I}}\left\{\widetilde{q}_{I}+\widetilde{q}_{J}-\widetilde{q}_{B}\right\}<1 . \tag{1.9}
\end{equation*}
$$

In stating our main results, we prefer to refer to the condition 1.8 because the case $I=B$ explicitly indicates that $\widetilde{q}_{J}<1$ for $J=2, \cdots, \mathfrak{D}$, while the case $I=1$ with $J=B$ explicitly indicates that $\widetilde{q}_{1}<1$ too.

In the second case, we consider polarized $U(1)$-symmetric solutions to the Einstein-vacuum equations in $1+3$ dimensions and prove stable Big Bang formation for symmetric perturbations of any Kasner solution (with exponents verifying the constraints $(1.7), \widetilde{B}=0$ ). We emphasize that for polarized $U(1)$-symmetric solutions, the spatial connection coefficients featuring three distinct indices automatically vanish (see Lemma 5.11 for a proof and Remark 1.2 for a discussion of the relevance of this fact), which leads to a simple proof of (1.5) (see the end of the proof of Proposition 5.26).

We will now describe these two setups in more detail.

### 1.4.1 Regimes with no symmetry assumptions on the perturbed initial data

Under the following assumptions, our results yield the stability of the Kasner Big Bang singularity for nonempty sets of background Kasner solutions:

1. The Einstein-vacuum equations (i.e., $\psi=0$ ) for $\mathfrak{D} \geq 10$.
2. The Einstein-scalar field equations for $\mathfrak{D} \geq 3$.

As we have stressed, without symmetry, we require that the background Kasner exponents satisfy the stability condition $\sqrt{1.8}$ ), which, for example, for any $\mathfrak{D} \geq 3$, is satisfied when all Kasner exponents are positive (which can be achieved in the presence of a non-zero scalar field, i.e., $\widetilde{B} \neq 0$ ). Also, as it was observed in [29], in vacuum $(\widetilde{B}=0)$, the condition $(\sqrt{1.8})$ is non-empty for $\mathfrak{D} \geq 10$, while for $\mathfrak{D} \leq 9,(1.8)$ is algebraically impossible, given the constraints 1.7 .

### 1.4.2 The definition of the polarized $U(1)$-symmetry class

Our discussion in this section refers to polarized $U(1)$-symmetric solutions to the Einstein-vacuum equations (i.e., $\psi \equiv 0$ ) on $I \times \mathbb{T}^{3}$, where $I$ is an interval of time. This symmetry class is defined as follows:

1. Polarized $U(1)$-symmetric initial data. There exists a non-degenerate ${ }^{14}$ hypersurface-orthogonal, spacelike Killing vectorfield $\bar{X}$ on $\Sigma_{1} \simeq \mathbb{T}^{3}$ with $\mathbb{T}^{1}$ orbits such that $\mathcal{L} \overline{\bar{X}} \stackrel{\circ}{g}=\mathcal{L}_{\bar{X}} \stackrel{\circ}{k}=0$, where $\mathcal{L}$ is the Lie derivative operator. Moreover, the second fundamental form of $\Sigma_{1}$ satisfies $\grave{k}(\bar{X}, \bar{Y})=0$ for every $\Sigma_{1}$-tangent vectorfield $\bar{Y}$ such that $\stackrel{\circ}{g}(\bar{X}, \bar{Y})=0$. We can construct coordinates $\left\{^{15}\left\{x^{i}\right\}_{i=1,2,3}\right.$ on $\Sigma_{1}$ such that all coordinate components of $\stackrel{\circ}{g}$ and $\stackrel{\circ}{k}$ are independent of $x^{3}$ and such that $\bar{X}=\partial_{3}$, i.e. $\stackrel{\circ}{g}_{13}=\stackrel{\circ}{g}_{23}=\stackrel{\circ}{k}_{13}=\stackrel{\circ}{k}_{23} \equiv 0$; see the discussion in [37, Section 2].
2. Polarized $U(1)$-symmetric solutions. Einstein-vacuum spacetimes that arise from such data contain a non-degenerate, hypersurface-orthogonal, spacelike Killing vectorfield $X$, such that $\left.X\right|_{\Sigma_{1}}=\bar{X}$. In fact, relative to appropriately constructed CMC-transported spatial coordinates, we have $X=\partial_{3}$; see Lemma 2.3.

One can easily check that in $1+3$ spacetime dimensions in the vacuum case, the condition 1.8 is violated by all Kasner solutions, i.e., by all Kasner exponents satisfying 1.7 ) with $\widetilde{B}=0$. Indeed, the algebraic relations (1.7) imply that at least one Kasner exponent must be negative and that

$$
\begin{equation*}
\max _{\substack{I, J, B=1,2,3 \\ I<J}}\left\{\widetilde{q}_{I}+\widetilde{q}_{J}-\widetilde{q}_{B}\right\} \geq 1-2 \min _{B=1,2,3}\left\{\widetilde{q}_{B}\right\}>1 \tag{1.10}
\end{equation*}
$$

Hence, in $1+3$ spacetime dimensions in the vacuum case, without symmetries or other additional assumptions, the Kasner singularity might not be stable under perturbations of the Kasner initial data on $\Sigma_{1}$. However, we show that within the class of polarized $U(1)$-symmetric solutions, the Kasner singularity is in fact stable. There are both heuristic and analytic reasons for this phenomenon, which we discuss in Sections 1.6 and 1.8 .

### 1.5 Rough version of the main theorem

Given a "background" generalized Kasner solution (1.6), within the regimes described in Sect. 1.4 we perturb its initial data on $\Sigma_{1}=\{t=1\}$ and study the corresponding maximal development in the past of $\Sigma_{1}$. As in the previous works of the last two authors [58, 59, 60, 61, in order to synchronize the singularity along $\{t=0\}$, we use a constant mean curvature (CMC) foliation that is realized by the level sets $\Sigma_{t}$ of a time function $t \in(0,1]$; as we describe below, this gauge features an elliptic PDE, which involves an infinite speed of propagation, allowing for a synchronization of the singularity. Relative to "transported" spatial coordinates $\left\{x^{i}\right\}_{i=1, \cdots, \mathfrak{D}}$, which by definition are constant along the integral curves of the future-directed unit normal to $\Sigma_{t}$, the perturbed spacetime metric takes the form (see also 2.37 ) in the $U(1)$-symmetric polarized case):

$$
\begin{equation*}
\mathbf{g}=-n^{2} d t \otimes d t+g_{c d} d x^{c} \otimes d x^{d}, \quad n=\left[-\left(\mathbf{g}^{-1}\right)^{\alpha \beta} \partial_{\alpha} t \partial_{\beta} t\right]^{-\frac{1}{2}} \tag{1.11}
\end{equation*}
$$

where $g$ is the first fundamental form of $\Sigma_{t}$ (i.e., the Riemannian metric on $\Sigma_{t}$ induced by $\mathbf{g}$ ) and $n>0$ is the lapse of the $\Sigma_{t}$ foliation. The CMC condition reads

$$
\begin{equation*}
\operatorname{tr} k=-\frac{1}{t} \tag{1.12}
\end{equation*}
$$

where $k$ is the second fundamental form of $\Sigma_{t}$. We emphasize that $\sqrt{1.12}$ is the gauge condition tied to the infinite speed of propagation, since it implies an elliptic equation for $n$ (see 2.25 ).

Remark 1.4 (Initial CMC slice). The condition $\sqrt{1.12}$ presupposes that the data on the initial Cauchy hypersurface $\Sigma_{1}$ have constant mean curvature $\left.\operatorname{tr} k\right|_{\Sigma_{1}}=-1$. Such an assumption can be made without loss of generality for solutions that start out close to background Kasner solutions. The reason is that for near-Kasner data (not necessarily CMC data), one can first use the standard wave coordinate gauge to solve Einstein's equations in a neighborhood of $\Sigma_{1}$, and then prove the existence of a CMC slice in that neighborhood with the desired properties; see [59, Proposition 14.4] and [9, Theorem 4.2].

[^7]Polarized $U(1)$-symmetric case. In the polarized $U(1)$-symmetric vacuum case with $\mathfrak{D}=3$, our setup will be such that $x^{3}$ corresponds to the symmetry. In particular, relative to the transported spatial coordinates $\left\{x^{i}\right\}_{i=1,2,3}, n,\left\{g_{i j}\right\}_{i, j=1,2,3}$, and $\left\{k_{i j}\right\}_{i, j=1,2,3}$ will not depend on $x^{3}$. Moreover, $\partial_{3}$ will be a hypersurfaceorthogonal Killing vectorfield, everywhere defined in the past of $\Sigma_{1}$ and with positive norm away from the singularity; see Lemma 2.3 .

We now state a first, rough version of our main stability results. See Theorems 6.1 and 6.5 for precise statements.

Theorem 1.5 (Stable Big Bang formation (Rough version)). In $1+\mathfrak{D}$ spacetime dimensions, consider an explicit generalized "background" Kasner solution (1.6) whose Kasner exponents satisfy the condition (1.8), which is possible for $\mathfrak{D} \geq 3$ in the presence of a scalar field and for $\mathfrak{D} \geq 10$ in vacuum (i.e., with $\widetilde{B}=0$ in (1.6). These background solutions are dynamically stable under perturbations - without symmetry of their initial data near their Big Bang singularities, as solutions to the Einstein-scalar field equations in the case $\mathfrak{D} \geq 3$, and, when $\widetilde{B}=0$, as solutions to the Einstein-vacuum equations in the case $\mathfrak{D} \geq 10$. Moreover, in $1+3$ spacetime dimensions, all Kasner solutions (with $\widetilde{B}=0$ ) are dynamically stable solutions to the Einstein-vacuum equations under perturbations - with polarized U(1)-symmetry - near their Big Bang singularities, even though they all violate the condition (1.8).

More precisely, under the above assumptions, sufficiently regular perturbations (i.e., perturbations belonging to suitably high order Sobolev spaces) of the Kasner initial data on $\Sigma_{1}$ give rise to maximal developments that terminate in a Big Bang singularity to the past. In particular, the spacetime solutions in the past of $\Sigma_{1}$ are foliated by spacelike hypersurfaces $\Sigma_{t}$ that are equal to the level sets of a time function $t$ verifying the CMC condition $\operatorname{tr} k=-t^{-1}$, and the perturbed Kretschmann scalars $\mathbf{R i e m}{ }^{\alpha \mu \beta \nu} \mathbf{R i e m}_{\alpha \mu \beta \nu}$ blow up like $t^{-4}$ as $t \downarrow 0$. Finally, the perturbed solutions exhibit AVTD behavior (see just below equation (1.4) for further discussion of the notion of "AVTD") as the singularity is approached, and various $t$-rescaled solution variables have regular limits as $t \downarrow 0$.

### 1.6 Background on "Kasner-like behavior:" Heuristics

We now aim to provide further background on our main results. In Sect.1.7, we will discuss prior works in the literature. Many of those works concern solutions that exhibit "Kasner-like behavior," a concept that we now discuss. We do not attempt to ascribe rigorous meaning to this terminology; rather, we will highlight some properties that are meant to capture the idea that a metric with spatial dependence is "blowing up in a manner similar to the Kasner solutions." We find the discussion in [29, 40] instructive, where the spacetime metric, to leading order near $t=0$, is assumed to take the form

$$
\begin{equation*}
\mathbf{g}=-d t \otimes d t+g, \quad g \cong \sum_{I=1}^{\mathfrak{D}} t^{2 q_{I}(x)} \theta^{I}(x) \otimes \theta^{I}(x), \quad \theta^{I}=\theta_{a}^{I}(x) d x^{a} \tag{1.13}
\end{equation*}
$$

where " $\cong$ " means "asymptotic to as $t \downarrow 0$," and the scalar functions $\left\{q_{I}(x)\right\}_{I=1, \cdots, \mathfrak{D}}$ satisfy the following (vacuum) analogs of 1.7):

$$
\begin{equation*}
\sum_{I=1}^{\mathfrak{D}} q_{I}(x)=\sum_{I=1}^{\mathfrak{D}} q_{I}^{2}(x)=1 \tag{1.14}
\end{equation*}
$$

Note that in 1.13 , the one-forms ${ }^{16}\left\{t^{q_{\underline{I}}(x)} \theta^{\underline{I}}(x)\right\}_{I=1, \cdots, \mathfrak{D}}$ "exactly represent the Kasner-like directions". Moreover, although the metric components may vary in $x$, they are all monotonic in $t$ at fixed $x$. We stress that our discussion here is heuristic in the sense that metrics of the form 1.13 are not generally solutions to Einstein's equations, though they might approximate actual solutions.

Let $\left\{k^{I}{ }_{J}\right\}_{I, J=1, \ldots, \mathfrak{D}}$ denote ${ }^{17}$ the components of the type $\binom{1}{1}$ second fundamental form of $\Sigma_{t}$ with respect to the co-frame $\left\{\theta^{I}(x)\right\}_{I=1, \cdots, \mathfrak{D}}$ and its basis-dua ${ }^{18}$ frame. Standard computations yield that for metrics of

[^8]the form (1.13), we have $k^{I}{ }_{J} \sim t^{-1}$. On the other hand, in coordinates such that the lapse $\left|\mathbf{g}\left(\partial_{t}, \partial_{t}\right)\right|^{1 / 2}$ is equal to 1 (as on RHS 1.13 ), the components $k^{I}{ }_{J}$ satisfy the evolution equations
\[

$$
\begin{equation*}
\partial_{t} k_{J}^{I}-\operatorname{tr} k k_{J}^{I}=R i c_{J}^{I}-\mathbf{R i c}_{J}^{I} \tag{1.15}
\end{equation*}
$$

\]

where $\operatorname{Ric}^{I}{ }_{J}$ denotes a component of the type $\binom{1}{1}$ Ricci curvature of $g$ with respect to the co-frame $\left\{\theta^{I}(x)\right\}_{I=1, \cdots, \mathcal{D}}$ and its basis-dual frame, and similarly for $\mathbf{R i c}^{I}{ }_{J}$.

## Heuristic criterion for Kasner-like behavior

- If Ric $=0$ (e.g., if the metric $\mathbf{g}$ from 1.13 was already known to be a solution to the Einstein-vacuum equations), then the leading order behavior $k^{I}{ }_{J} \sim t^{-1}$ can easily be derived directly from (1.15) if i) one knew that $\operatorname{tr} k=-t^{-1}+\mathcal{O}\left(t^{-1+\sigma}\right)$ for some $\sigma>0$, and $\mathbf{i i}$ ) one could prove the following pointwise estimate for $t$ larger than but close to 0 :

$$
\begin{equation*}
\max _{I, J=1, \cdots, \mathfrak{D}}\left|\operatorname{Ric}_{J}^{I}\right| \lesssim t^{-2+\sigma} . \tag{1.16}
\end{equation*}
$$

In our main results, we impose the condition $\operatorname{tr} k=-t^{-1}$ by using constant mean curvature foliations. This gauge is not compatible with the ansatz 1.13 because it generally requires the lapse $\left[-\left(\mathbf{g}^{-1}\right)^{\alpha \beta} \partial_{\alpha} t \partial_{\beta} t\right]^{-\frac{1}{2}}$ to be different from unity. For convenience, we will downplay this issue in the present discussion ${ }^{19}$ In the presence of matter, the same conclusions $k^{I}{ }_{J} \sim t^{-1}$ hold if one can also show that ${ }^{20} \max _{I, J=1, \cdots, \mathfrak{D}}\left|\mathbf{R i c}_{J}^{I}\right| \lesssim t^{-2+\sigma}$.
Remark 1.6 (Remarks on our use of time-dependent orthonormal frames). We make the following remarks:

- The above discussion of heuristics referred to the components of tensorfields with respect to the timeindependent co-frame $\left\{\theta^{I}(x)\right\}_{I=1, \cdots, \mathfrak{D}}$ and its basis-dual frame. In particular, $\left\{\theta^{I}(x)\right\}_{I=1, \cdots, \mathfrak{D}}$ is not g-orthonormal. However, for perturbations of Kasner solutions, there is no reason to believe that there generally exists a time-independent co-frame in which the perturbed metric takes the form 1.13). Hence, we again stress that our approach is based on deriving estimates for the components of tensorfields relative to an orthonormal spatial frame $\left\{e_{I}(t, x)\right\}_{I=1, \cdots, \mathfrak{D}}$ obtained by Fermi-Walker transport, and that our use of an orthonormal frame is crucial so that we can exploit the approximately diagonal nature of the structure coefficient evolution equations (see Sect. 1.8.3).
- In particular, in our main results, we will prove an analog of 1.16 for the components of Ric relative to an orthonormal frame; see Remark 1.10 . Equivalently, instead of $\sqrt{1.16}$ ), our main results will rely on a proof of the bound

$$
\begin{equation*}
|R i c| \lesssim t^{-2+\sigma} \tag{1.17}
\end{equation*}
$$

where LHS 1.17 denotes the usual invariant pointwise norm of the spatial Ricci tensor.

- We also highlight that we are able to close our estimates without showing that the metric is asymptotic to a metric of the form 1.13 . In fact, we close the proof with only very weak information about the orthonormal frame $\left\{e_{I}(t, x)\right\}_{I=1, \cdots, \mathfrak{D}}$ and co-frame $\left\{\omega^{I}(t, x)\right\}_{I=1, \cdots, \mathfrak{D}}$ : we prove only that their coordinate components $\left\{e_{I}^{i}\right\}_{I, i=1, \cdots, \mathfrak{D}}$ and $\left\{\omega_{i}^{I}(t, x)\right\}_{I, i=1, \cdots, \mathfrak{D}}$ are bounded in magnitude by $\lesssim t^{-q}$ for some $q \in(0,1)$ depending on the background Kasner exponents; see also Remark 6.2
- Despite the previous comment, for the solutions under study, we are able to prove the existence of "final Kasner exponents" $\left\{q_{I}^{(\infty)}(x)\right\}_{I=1, \cdots, \mathfrak{D}}$ as the singularity is approached; see Proposition 6.6.

Conditions for the validity of the heuristic criterion 1.16 for metrics of the form 1.13

[^9]- A computation using 1.13 shows that in the absence of special algebraic structure, we typically hav ${ }^{21}$

$$
\begin{equation*}
\max _{I, J=1, \cdots, \mathfrak{D}}\left|R i c^{I}{ }_{J}\right| \approx \max _{\substack{I, J, B=1, \cdots, \mathfrak{D} \\ I<J}}\left\{t^{2\left(q_{B}-q_{I}-q_{J}\right)}\right\} . \tag{1.18}
\end{equation*}
$$

- In view of 1.18 , we see that the estimate 1.16 holds if

$$
\begin{equation*}
\max _{\substack{I, J, B=1, \cdots, \mathfrak{N} \\ I<J}}\left\{q_{I}+q_{J}-q_{B}\right\}<1 \tag{1.19}
\end{equation*}
$$

- In $1+3$ spacetime dimensions in the vacuum case, where the condition $\sqrt{1.19}$ ) is always violated (see (1.10), one can show that for metrics of the form 1.13 , the estimate 1.16 is valid if the following relation holds, where d denotes the exterior derivative operator:

$$
\begin{equation*}
\theta^{-} \wedge \mathrm{d} \theta^{-}=0 \tag{1.20}
\end{equation*}
$$

where $q_{-}(x)<0$ is th ${ }^{22}$ negative Kasner-like exponent in 1.13 and $\theta^{-}(x)$ is the corresponding oneform, i.e., these quantities are such that the tensor product $\theta^{-}(x) \otimes \theta^{-}(x)$ is multiplied by the factor $t^{2 q_{-}(x)}$. The condition 1.20 eliminates the terms responsible for the worst behavior on RHS 1.18), which, if present, would have been more singular than RHS 1.16).
A geometric interpretation of the condition 1.20 for metrics of the form 1.13

- The Frobenius Theorem states that 1.20 is equivalent to the integrability of the 2-dimensional subspaces $V_{p}^{-}$annihilated by $\theta^{-}$, where for $p \in \mathbb{T}^{3}$,

$$
\begin{equation*}
V_{p}^{-}=\left\{Y \in T_{p} \mathbb{T}^{3}: \theta_{p}^{-}(Y)=0\right\} \tag{1.21}
\end{equation*}
$$

We note that 1.20 is equivalent to the existence of functions $u, v: \mathbb{T}^{3} \rightarrow \mathbb{R}$ such that $\theta^{-}=u d v$.
As we already mentioned in Sect. 1.4 , for the models that we consider in our results without symmetry assumptions, it was already observed in [11, 29] that the condition (1.19) is not vacuous, at least in the sense that there exist generalized Kasner (in particular, spatially homogeneous) solutions whose exponents satisfy it. We also stress that for solutions with $x$-dependence, in the context of the heuristic works [11, 29], the condition 1.19) can be interpreted as an inequality that should be satisfied by the "final Kasner exponents," i.e., the exponents $\left\{q_{I}(x)\right\}_{I=1, \cdots, \mathfrak{D}}$ of the alleged asymptotic form 1.13 ) of an alleged Kasner-like solution. Our main results in fact justify the existence of ( $x$-dependent) Kasner-like solutions with "final Kasner exponents" $\left\{q_{I}(x)\right\}_{I=1, \cdots, \mathfrak{D}}$ verifying the stability condition 1.19 , at least when the data are close to generalized Kasner solutions whose exponents verify the same condition; see Proposition 6.6. Our proof of these facts relies, of course, on the open nature of the condition 1.19 .

The above discussion suggests that in $1+3$ spacetime dimensions in the vacuum case, $x$-dependent Kasnerlike solutions can exist if the "polarization-type" condition 1.20 holds. However, the condition 1.20 refers to the structure of the metric "at the singularity" (i.e., since 1.13 is only supposed to capture the asymptotic structure of the metric, 1.20 is a statement about the structure of the asymptotic behavior of the metric near the singularity), and we are not aware of any "general method" for solutions without symmetry that allows one to ensure the validity of 1.20 via assumptions on the initial data on $\Sigma_{1}$. However, for polarized $U(1)$-symmetric solutions, discussed further below, the condition 1.20 automatically holds.

## Polarized $U(1)$-symmetric metrics of the form 1.13 satisfy 1.20

- Recall that we defined the polarized $U(1)$-symmetry class in Sect.1.4.2. Assume that $\partial_{3}$ is the hypersurface-orthogonal Killing vectorfield with $\mathbb{T}^{1}$-orbits. This will be the case in our study of solutions with symmetry; see Lemma 2.3. In addition, we assume that the leading order expression 1.13 ) of the Kasner-like metrics in question respects the symmetry, ie. $\partial_{3} q_{I}(x)=0, \mathcal{L}_{\partial_{3}} \theta^{I}(x)=0$. The argument for the validity of $\sqrt[1.20]{ }$ is divided in the following two cases depending on the sign of the Kasner exponent associated to the norm of the Killing field $\partial_{3}$ :

[^10]- If $q_{3}$, the Kasner-like exponent corresponding to the direction of symmetry, satisfies $q_{3}<0$, then for a metric of the form $\sqrt{1.131}, 1.20$ (with $\theta^{3}$ in the role of $\theta^{-}$) is immediate, since $\theta^{3}$ must be a scalar function multiple of $d x^{3}$.
- Again assume that $\partial_{3}$ is the hypersurface-orthogonal Killing vectorfield with $\mathbb{T}^{1}$-orbits, but now assume that $q_{3}>0$ and $q_{1}<0$, where $q_{3}$ is still the Kasner-like exponent corresponding to the direction of symmetry. Then the subspaces annihilated by the one-form $\theta^{1}=\theta^{-}$, corresponding to $q_{1}$, are 2 dimensional and contain $\partial_{3}$. Let $Y$ be a unit-length vectorfield in the kernel of $\theta^{1}$ that is orthogonal to $\partial_{3}$. Fix a point $p \in \mathbb{T}^{3}$ and consider the integral curve $s \rightarrow a_{p}(s)$ of $Y$ passing through it, normalized by $a_{p}(0)=p$. Then the image of $a_{p}$ times the orbits of $\partial_{3}$, i.e., $\operatorname{Img}\left(a_{p}\right) \times[-\pi, \pi]_{x_{3}}$, is a surface whose tangent planes are exactly the kernel of $\theta^{1}$, since $\partial_{3}$ (being Killing and annihilating $\theta^{1}$ ) commutes with $Y$. Hence, the planes $V_{p}^{-}$are integrable, and by the Frobenius Theorem, this is equivalent to the condition 1.20 .


### 1.7 Related works

Before outlining the main ideas behind our proof of Theorem 1.5, we first describe some prior results on Kasner-like singularities. There are many such results, and we roughly divide them into three categories.

### 1.7.1 Big Bang formation under symmetry assumptions

There are many works that provide a detailed description of stable Big Bang formation, or more generally, spacelike singularity formation with AVTD behavior (e.g., in black hole interiors), for large sets of initial data on a smooth Cauchy hypersurface in a model with sufficient symmetry such that the problem reduces to a system of ODEs or $1+1$-dimensional PDEs. We further divide these results into sub-categories.

The interior of black holes. In Christodoulou's influential works [19, 20] on the spherically symmetric Einstein-scalar field system with large data, it was shown that black holes form and contain spacelike singularities in their interior, where their Kretschmann scalars blow up.

Polarized Gowdy-symmetry. In [23, the authors studied polarized Gowdy-solution ${ }_{[23}^{23}$ to the Einsteinvacuum equations and proved Strong Cosmic Censorship, that is, that for an open and dense set of polarized Gowdy-symmetric initial data on $\mathbb{T}^{3}$ or $\mathbb{S}^{2} \times \mathbb{S}^{1}$, the maximal globally hyperbolic development is inextendible, and causal geodesics are generically inextendible in one direction due to curvature blowup.

Gowdy-symmetry. In [54, Ringström proved a similar result for Gowdy-solutions with spatial topology $\mathbb{T}^{3}$, without the polarization assumption. See also the related works [24, 51, 52] and the survey article [53. The general Gowdy-case turned out to be significantly more difficult to handle in view of a possible phenomenon that was shown to be absent in the polarized case: "spikes." Roughly, spikes are regions where spatial derivatives can become large, i.e., regions where solutions do not exhibit AVTD behavior. For an open and dense set of data in the topology of $C^{\infty}$, Ringström proved that a curvature singularity forms and that the solution exhibits Kasner-like behavior, except for possibly at a finite number of spikes.

Polarized $\mathbb{T}^{2}$-symmetry. In [4], the authors proved the dynamic stability of Kasner Big Bang singularities under $\mathbb{T}^{2}$-symmetric polarized perturbations, for a certain range of Kasner exponents. In [5], the same authors proved a similar result in the presence of an arbitrary cosmological constant $\Lambda \in \mathbb{R}$. There is a relationship between these solutions and the polarized $U(1)$-symmetric solutions that we treat in Theorem 6.5, see Section 1.9.1 for further discussion.

Polarized axi-symmetric initial data. The Schwarzschild black hole singularity is highly unstable, as is shown by the fact that instead of singularities, near-Schwarzschild Kerr solutions have Cauchy horizons inside their black holes, and the metric can be smoothly extended across them. However, the Schwarzschild singularity was recently shown to be stable 1 as a solution to the Einstein-vacuum equations under symmetric perturbations, specifically those perturbations whose solutions exhibit a hypersurface-orthogonal, spacelike, Killing vectorfield $X$ with $\mathbb{T}^{1}$ orbits ${ }^{24}$ This symmetry class is closely related to the polarized $U(1)$-symmetry class that we study in Theorem 6.5, as we now describe. Compared to the $U(1)$-symmetric polarized solutions

[^11]with $\mathbb{T}^{3}$ spatial topology that we study in the present paper, the main difference in [1] is that $X$ degenerates at a 2-dimensional submanifold; since the vectorfield $X$ in [1] is tangent to 2 -spheres, such degeneracies are topologically unavoidable. This can be concretely seen already in the case of the background Schwarzschild metric in classical $(t, r, \theta, \phi)$ coordinates, where $X:=\partial_{\phi}$ is the Killing field, and away from the singularity $\{r=0\}$, its (square) norm $\mathbf{g}_{\text {Schwarzschild }}\left(\partial_{\phi}, \partial_{\phi}\right)=r^{2} \sin ^{2} \theta$ vanishes at exactly $\theta=0, \pi$. Apart from this extra feature of the degenerate Killing vectorfield and the difference in topology $\left(\mathbb{R} \times \mathbb{S}^{2}\right.$ instead of the $\mathbb{T}^{3}$ topology considered here), the stability result of 1 can be seen to correspond ${ }^{25}$ to a special case of our symmetric blowup-results, specifically Theorem 6.5 , with background Kasner exponents $\widetilde{q}_{1}=-\frac{1}{3}$, $\widetilde{q}_{2}=\widetilde{q}_{3}=\frac{2}{3}$. The method of proof introduced in [1] is very much tied to the specific symmetry class, relying on a wave-maps reduction of the Einstein-vacuum equations, and it is therefore not applicable to the non-symmetric solutions that we study in Theorem 6.1. However, it seems that the use of the particular symmetry reduction in [1] allowed for the derivation of more refined asymptotic behaviors for the spatial components of the metric compared to the results we derive in Theorem 6.5

### 1.7.2 The construction of solutions with Big Bang singularities - without a proof of stability

Numerous papers have provided a construction of solutions that exhibit a Kasner-like singularity. Most of these works concern cosmological spacetimes and employed Fuchsian techniques in regimes where the discussion in Sect. 1.6 suggests that one might expect the singularity formation to be dynamically stable.

Gowdy-symmetry. The first result of this type 41] yielded the construction of analytic solutions with Gowdy-symmetry. The analyticity assumption was removed in 49. See also 62 for more general topologies and [3] for a treatment in generalized wave gauges.

Polarized and half-polarized $\mathbb{T}^{2}$-symmetry. Analytic singularities in polarized $\mathbb{T}^{2}$-symmetry class were first constructed in [36]. The analyticity assumption was later removed in [2], where the authors also constructed half-polarized solutions.

Polarized or half-polarized $U(1)$-symmetry. Polarized and half-polarized $U(1)$-symmetric analytic solutions with $\mathbb{T}^{3}$ spatial topology were constructed in 37. More general topologies were later treated in [17]. We note that in these works, the authors defined their notion of polarized and half-polarized solutions at the singularity, i.e., at $t=0$, by eliminating free functions relative to a given ansatz, in the spirit of 1.13 ) and 1.20 .

Einstein-scalar field or stiff fluid. The first construction of singular solutions without symmetries was carried out in [7]. The authors studied the Einstein-scalar field and Einstein-stiff fluid systems and used Fuchsian techniques to construct analytic solutions whose "final Kasner exponents" (see the last point of Remark 1.6 and Proposition 6.6) are all positive.

Sub-critical Einstein-matter systems. In [28], the authors extended the results of [7] by constructing singular, analytic, Kasner-like solutions without symmetries to various Einstein-matter systems and to the Einstein-vacuum equations in $1+\mathfrak{D}$ dimensions with $\mathfrak{D} \geq 10$. As in the present paper, the solution regimes treated in [28] were sub-critical in the sense that the solutions exhibited the crucial bound 1.17] for the spatial Ricci curvature. Roughly, our present work shows that an open set of solutions constructed in [28] are dynamically stable under Sobolev-class perturbations of their initial data near their Big Bangs, at least for the vacuum and scalar field matter model.
$1+3$ vacuum without symmetries. As we alluded to in Sect. 1.3 . Kasner solutions might be unstable under general perturbations without symmetries, unless some kind of condition, such as a polarization condition of the type $(1.20)$, is imposed. Nevertheless, in 39, the author constructed analytic Kasner-like singular solutions without symmetries, demonstrating that such solutions exist, even though they might be unstable. Moreover, in [32, for distinct Kasner exponents, the authors constructed Sobolev-class solutions that exhibit Kasner-like singularities. The solutions do not a priori enjoy any symmetry, but they satisfy the polarization condition 1.20 .

Asymptotically Schwarzschild on a 2-sphere. Finally, we mention the first author's work [31], which, in a Lorentz gauge, yielded the construction of a class of spacetimes that converge to a portion of the Schwarzschild black hole singularity. The construction requires no symmetry or analyticity assumptions.

[^12]While the construction does not yield a full spacelike singular hypersurface, it does provide a spacelike singular 2-sphere.

### 1.7.3 Stable Big Bang formation without symmetry assumptions

The stability of some Kasner solutions towards their Big Bang singularities, without symmetries and for open sets of initial data, was only fairly recently shown by the last two authors. For the scalar field and stiff fluid matter models, the stability of the (isotropic) FLRW solutions with $\mathbb{T}^{3}$ spatial topology (i.e., $\widetilde{q}_{1}=\widetilde{q}_{2}=\widetilde{q}_{3}=\frac{1}{3}$ and $\widetilde{B}=\sqrt{2 / 3}$ ) was shown in [58, [59], while the case of the scalar field matter model with $\mathbb{S}^{3}$ topology was handled in 61]. The Einstein-vacuum equations were handled in 60 under a "moderate anisotropy" assumption on the Kasner exponents, specifically $\max _{I=1, \cdots, \mathfrak{D}}\left|\widetilde{q}_{I}\right|<\frac{1}{6}$, which is possible in $1+\mathfrak{D}$ spacetime dimensions when $\mathfrak{D} \geq 38$. Some aspects of our analysis here are in the spirit of the analysis in [60].

### 1.7.4 Conditional Kasner-like behavior

There are recent results that derive Kasner-like behavior for solutions under assumed bounds on certain key quantities. For example, assuming mainly scale invariant bounds on the Riemann curvature of Hubblenormalized time slices, Lott [43] showed that the corresponding singular solutions converge to Kasner flows in appropriate topologies. Ringström [56, 57] derived sharp results on the geometry of Kasner-like solutions by assuming mainly bounds on the renormalized Weingarten map ${ }^{26}$ Ringström's sharp estimates crucially rely on a frame that asymptotically diagonalizes the normalized Weingarten map. This frame is very different from ours, which is Fermi-propagated from $\Sigma_{1}$. It would be interesting to explore whether a change of frames could yield more refined estimates for the spatial frame and connection coefficients compared to the estimates we obtain in Theorem 6.1. see also Remark 1.9 .

### 1.8 Overview of our proof

Our proofs of Theorems 6.1 and 6.5 are based on deriving estimates for a set of reduced variables that solve an elliptic-hyperbolic PDE system. Here we will summarize the main features of the system and how its structures allow us to prove our main results. We will confine our discussion to sketching proofs of various low order and high order a priori estimates for near-Kasner initial data given on $\Sigma_{1}=\{t=1\}$. In practice, the low order and high order estimates are coupled, and we derive them via a bootstrap argument. The a priori estimates are sufficient to ensure (see Proposition 5.28) that the solution exists on $(0,1] \times \mathbb{T}^{\mathfrak{D}}$, which is the main step in the paper. The proof of curvature blowup and other aspects of the solution are relatively straightforward consequences of the a priori estimates. We will not discuss those results in this section; instead, we refer readers to Sect. 6 for those details.

### 1.8.1 The gauge

We use a constant mean curvature foliation in which, for $t \in(0,1]$, the level sets $\Sigma_{t}$ of the time function $t$ satisfy $\operatorname{tr} k=-\frac{1}{t}$, where $k$ is the second fundamental form of $\Sigma_{t}$. We also use spatial coordinates $\left\{x^{i}\right\}_{i=1, \cdots, \mathfrak{D}}$ that are transported along the unit normals to $\Sigma_{t}$. In this gauge, the spacetime metric satisfies $\mathbf{g}=-n^{2} d t \otimes$ $d t+g_{a b} d x^{a} \otimes d x^{b}$, where $n$ is the lapse and $g$ is the first fundamental form of $\Sigma_{t}$. This setup is the same as in [58, 59, 60]. However, to derive the sharp results of the present paper, we use a crucial additional ingredient: we use Fermi-Walker transport to construct a $\Sigma_{t}$-tangent orthonormal "spatial frame" $\left\{e_{I}\right\}_{I=1, \cdots, \mathfrak{D}}$, which is globally defined in space. When supplemented with $e_{0}:=n^{-1} \partial_{t}$, we obtain an orthonormal spacetime frame. We then formulate Einstein's equations in such a way that the unknowns are $n$, the components $\left\{e_{I}^{i}\right\}_{I, i=1, \cdots, \mathfrak{D}}$ of the orthonormal frame with respect to the transported spatial coordinates, the components $\left\{\omega_{i}^{I}\right\}_{I, i=1, \cdots, \mathfrak{D}}$ of the corresponding dual co-frame with respect to the transported spatial coordinates, the frame components $k_{I J}:=k_{c d} e_{I}^{c} e_{J}^{d}$ of the second fundamental form with respect to the frame, the connection coefficients $\gamma_{I J B}:=g\left(\nabla_{e_{I}} e_{J}, e_{B}\right)$ of the spatial frame (where $\nabla$ is the Levi-Civita connection of $g$ ), the

[^13]timelike unit normal derivative $e_{0} \psi$ of the scalar field, and the spatial frame derivatives $\left\{e_{I} \psi\right\}_{I=1, \cdots, \mathfrak{N}}$ of the scalar field 27 We refer readers to Sect. 2 for the details.

### 1.8.2 The lapse, the dynamic variables, and the "less singular" nature of spatial derivative terms

The lapse $n$ satisfies an elliptic PDE (see (2.25)) with source terms depending on some of the other solution variables, specifically the "dynamic variables" $e_{I}^{i}, \gamma_{I J B}$, and $e_{I} \psi$. Thus, to control $n$, we use elliptic estimates to control it in terms of these dynamic variables. These estimates are rather standard and we will not discuss them in detail here. We simply highlight that it is crucial for our results that the right-hand side of the elliptic lapse PDE depends only the spatial derivatives of various tensorfields, i.e., there are no time derivative terms, the point being that in the problem under study, spatial derivative terms are less singular with respect to $t$ compared to time derivative terms; this is a manifestation of AVTD behavior, which we first mentioned in Sect. 1.3 . We refer readers to Sect. 5.4 for a detailed proof of the lapse estimates. To control the dynamic variables, including $e_{I}^{i}, \omega_{i}^{I}, k_{I J}, \gamma_{I J B}, e_{0} \psi$, and $e_{I} \psi$, we derive "low order" $L^{\infty}$ estimates and "high order" energy estimates based on first-order formulations of the flow; we refer to Proposition 2.2 and Lemma 5.22 for the first-order formulations of the equations. As we explained in the discussion above (1.5), we also crucially rely on the special "diagonal structure" exhibited by the PDE system satisfied by the structure coefficients of the spatial frame. We provide this PDE system in Proposition 5.7, and we will discuss it in more detail in Sect.1.8.3.

### 1.8.3 Approximately diagonal form of the structure coefficient evolution equations

Away from symmetry, to control the $\gamma_{I J B}$ 's, we rely on the crucial observation that the terms

$$
\begin{equation*}
\left\{S_{I J B}:=\gamma_{I J B}+\gamma_{J B I} \mid 1 \leq I, J, B \leq \mathfrak{D}, I<J\right\} \tag{1.22}
\end{equation*}
$$

solve an evolution equation system whose "main linear part" is diagonal with coefficient magnitudes that are smaller than $t^{-1}$, provided the condition $\sqrt{1.8}$ is satisfied by the background Kasner exponents; see equation (5.19) for the precise equation, and equation 1.25 b for an abbreviated version. To caricature, the system is of the form $\dot{S}=\frac{M}{t} \cdot S+\cdots$, where $M$ is a diagonal matrix whose components verify $\left|M_{I J}\right|<1$ when 1.8 , holds. This allows us to prove that under 1.8 , we have $|S| \lesssim t^{-q}$ for some $q<1$. This bound is crucial for the entire proof, as we use it to show that the solutions exhibit AVTD behavior. The variables $S_{I J B}$ in (1.22) are precisely the structure coefficients of the spatial orthonormal frame $\left\{e_{I}\right\}_{I=1, \cdots, \mathfrak{D}}$. Here we note that by the simple identity 5.22 , to control all of the $\gamma_{I J B}$ 's, it suffices to control the structure coefficients.

Moreover, as we highlighted in Remark 1.2, even in cases such that the stability condition 1.8 is violated, only some of the structure coefficients $S_{I J B}$ could possibly serve as an obstruction to proving the desired estimates: those with three distinct indices. That is, our work essentially shows that in regimes where (1.8) is violated (such as the Einstein-vacuum equations in $1+3$ dimensions without symmetries), any instabilities would arise from the combinations $S_{I J B}$ with distinct indices. This observation is precisely what allows us to extend our stable blowup-results to the class of polarized $U(1)$-symmetric Einstein-vacuum solutions in $1+3$ dimensions: by considering a spatial orthonormal frame $\left\{e_{I}\right\}_{I=1,2,3}$ such that $e_{3}=\left(g_{33}\right)^{-\frac{1}{2}} \partial_{3}$ corresponds to the normalized Killing direction (see Lemma 2.4), we can conclude that the spatial connection coefficients with distinct indices are automatically zero (see Lemma 5.11). Hence, the observations described above allow us to sufficiently control the non-zero structure coefficients and prove stable blowup.

We also note that the less singular behavior (than $t^{-1}$ ) of the $\gamma_{I J B}$ 's is consistent with the renormalized second fundamental form frame components $t k_{I J}(t, x)$ having a continuous limit, $\kappa_{I J}^{(\infty)}(x)$, as $t \downarrow 0$, which is the main feature of a Kasner-like singularity (as we described in Sect. 1.6). This is once again a manifestation of AVTD behavior. The eigenvalues of $-\kappa_{I J}^{(\infty)}(x)$ can be viewed as the "final, $x$-dependent" Kasner exponents of the perturbed spacetime; see Proposition 6.6

### 1.8.4 The bootstrap argument and initial discussion of the behavior of the high order energies

In practice, to prove our main results, we rely on a bootstrap argument in which we assume that various low order and high order norms are small (indicating that the solution is near-Kasner) on a time interval ( $\left.T_{\text {Boot }}, 1\right]$;

[^14]see (3.9) for the precise bootstrap assumptions. Then the main task becomes deriving strict improvements of the bootstrap assumptions for near-Kasner initial data, where we remind the reader that the data are given along $\Sigma_{1}=\{t=1\}$. In the rest of Sect. 1.8 , to illustrate the main ideas, we will not explain the full bootstrap argument in detail, but will instead show how the different parts of the analysis consistently fit together. As a starting point, we note that our analysis will eventually show that we have a top-order energy bound of the form
\[

$$
\begin{equation*}
t^{A_{*}+q}\left\|e_{I}^{i}\right\|_{\dot{H}^{N}\left(\Sigma_{t}\right)}, t^{A_{*}+1}\left\|k_{I J}\right\|_{\dot{H}^{N}}\left(\Sigma_{t}\right), t^{A_{*}+1}\left\|\gamma_{I J B}\right\|_{\dot{H}^{N}\left(\Sigma_{t}\right)}, t^{A_{*}+1}\left\|e_{0} \psi\right\|_{\dot{H}^{N}\left(\Sigma_{t}\right)}, t^{A_{*}+1}\left\|e_{I} \psi\right\|_{\dot{H}^{N}\left(\Sigma_{t}\right)} \lesssim d a t a \tag{1.23}
\end{equation*}
$$

\]

where $q$ is as in Sect. 1.8 .3 (see just above 1.24 a ) for further discussion) and $\|\cdot\|_{\dot{H}^{N}}$ is a standard homogeneous Sobolev norm; see Sect. 3 for the details.

We now highlight some crucial aspects of our analysis of the high order energies:
To close the proof and justify the estimate (1.23), we must first choose the parameter $A_{*}$ to be sufficiently large, then choose the "regularity parameter" $N$ to be sufficiently large relative to $A_{*}$, and finally choose data to be sufficiently small, where for the rest of Sect. 1.8 , "data" denotes a small number whose size is controlled by the closeness of the initial data to the Kasner data in a high order Sobolev norm.

### 1.8.5 The behavior of the low order $L^{\infty}$ norms

In this section, we will explain how a high order energy bound of the form $\sqrt{1.23}$ allows us to derive sharp $L^{\infty}$ estimates for the solution variables at the low derivative levels. We already stress that our proof fundamentally requires that we prove much less singular (with respect to $t$ ) estimates at the low derivative levels compared to (1.23; here, we are thinking of 1.23 as a "very singular estimate" in the sense that $A_{*}$ is large. In particular, at the low derivative levels, we must prove estimates for the perturbed $k_{I J}$ and $e_{0} \psi$ variables showing that they are not more singular than their Kasner analogs, which blow up like $t^{-1}$. To keep the presentation short, in most of the rest of Sect. 1.8 we will focus only on the estimates for $e_{I}^{i}, k_{I J}$, and $\gamma_{I J B}$; the estimates for the scalar field can be obtained in a similar fashion. Moreover, we again highlight that we derive control of the connection coefficients at the low derivative levels by relying on the structure coefficients $S_{I J B}:=\gamma_{I J B}+\gamma_{J B I}$ (whereas for the energy estimates at the high derivative levels, we can work directly with the connection coefficients $\gamma_{I J B}$ ). Finally, we note that our discussion here will mainly concern the analysis away from symmetry under the sub-criticality condition 1.8.

Remark 1.7 (No need for a precisely adapted frame). It seems remarkable to us that away from symmetry, for all sub-critical Kasner exponents, we have a lot of freedom in constructing the orthonormal frame. More precisely, in Sect. 5.11, we use the Gram-Schmidt algorithm to construct an initial orthonormal frame that is a perturbation of the spatial coordinate frame $\left\{\partial_{i}\right\}_{i=1, \cdots, \mathfrak{D}}$, and then we propagate this frame using the Fermi-Walker transport equations 2.8 . There is nothing special about our choice of initial data for the frame; any nearby initial data for the orthonormal frame would have worked just as well. In particular, we can close the estimates without using a spatial frame that is adapted to the perturbed Kasner directions, that is, without the frame being aligned with the eigenvectors of the perturbed second fundamental form; see also Remark 6.2. Many previous studies of Kasner-like singularities in fact relied on a frame that is adapted to the eigenvectors of $k$ (see Sect. 1.7 for a list of related works).

Remark 1.8 (The role of $N_{0}$ ). In our main theorem, there appears a parameter $N_{0} \geq 1$ that represents, roughly, the number of derivatives that we sharply control in $\|\cdot\|_{L^{\infty}}$. We are free to choose it at the start of the bootstrap argument. For example, $N_{0}=1$ is permissible. However, the choice of $N_{0}$ will affect the minimal allowable size of $N$ (see Theorem 6.1). $N_{0}$ also captures the amount of regularity that the "limiting renormalized solution variables" enjoy along the Big Bang hypersurface $\Sigma_{0}$ (see Sect. 6.2. . We introduced $N_{0}$ mainly to clarify that for "very smooth" initial data that fall under the scope of our main results, the corresponding limiting solution variables will inherit a quantifiable amount of the smoothness. For convenience, in our heuristic discussion here, we will only discuss the case $N_{0}=1$, i.e., the $L^{\infty}$ estimates at the level of the undifferentiated equations.

To proceed, we let $\widetilde{e}_{I}^{i}(t)$ and $\widetilde{k}_{I J}(t):=\widetilde{k}_{c d}(t) \widetilde{e}_{I}^{c}(t) \widetilde{e}_{J}^{d}(t)$ respectively denote the background Kasner frame components and second fundamental form components; see Sect. 2.3 for the precise definitions. We aim to sketch a proof of the following pointwise estimates for $(t, x) \in(0,1] \times \mathbb{T}^{\mathfrak{D}}$, where in what follows, $q$ and $\sigma$ are fixed constants that satisfy $0<2 \sigma<2 \sigma+\max _{\substack{I, J, B=1, \ldots, \mathfrak{D} \\ I<J}}\left\{\left|\widetilde{q}_{B}\right|, \widetilde{q}_{I}+\widetilde{q}_{J}-\widetilde{q}_{B}\right\}<q<1-2 \sigma$ (such constants exist whenever the sub-criticality condition 1.8 holds):

$$
\begin{align*}
\left|t k_{I J}-t \widetilde{k}_{I J}\right|(t, x) & \lesssim d a t a,  \tag{1.24a}\\
t^{q}\left|S_{I J B}\right|(t, x) & \lesssim d a t a,  \tag{1.24b}\\
t^{q}\left|e_{I}^{i}-\widetilde{e}_{I}^{i}\right|(t, x) & \lesssim d a t a . \tag{1.24c}
\end{align*}
$$

The estimate 1.24 a is sharp and is of particular importance because it is needed to control various "borderline terms" in the energy estimates, as we explain in Sect.1.8.6. Similar remarks apply for the $L^{\infty}$ estimates for $t e_{0} \psi$ (which we do not discuss here). The estimates 1.24 b$)-\sqrt{1.24 \mathrm{c}}$ ) are not quite sharp with respect to powers of $t$, and we have chosen the power $t^{q}$ on LHSs 1.24 b$\left.)-1.24 \mathrm{c}\right)$ so as to allow for the simplest possible analysis. Estimates at the low derivative levels, in the spirit of $1.24 \mathrm{a}-1.24 \mathrm{c})$, are sufficient to allow us to identify the limiting Kasner-like behavior of the perturbed solutions; see Propsition 6.6 for the details. We stress that, although Propsition 6.6 shows that the $t$-weighted scalar functions $t k_{I J}$ and $t e_{0} \psi$ have non-trivial, regular limits as $t \downarrow 0$, we do not obtain (or need!) analogous sharp limits for the frame components or spatial derivative-involving terms; see Remark 6.2.
Remark 1.9 (Refined estimates with a different frame?). It is conceivable that a different choice of orthonormal frame might yield sharper asymptotic estimates for the spatial frame and connection coefficients, as in [57]. However, for solutions without symmetry, to close a bootstrap argument with a refined frame, such as a frame that is adapted to the eigenvectors of $k$, one would have to overcome serious technical difficulties, such as a potential loss of derivatives for the frame. It is an interesting open problem to understand whether such an approach is viable, i.e., whether the entire proof can be carried out using a refined frame. On the other hand, in symmetric regimes, such as polarized $U(1)$-symmetry [1] and polarized $\mathbb{T}^{2}$-symmetry [4, 5], the authors derived sharp asymptotics for various solution variables by using frames that are well-adapted to the different Kasner directions. In those works, the sharp estimates were derived by taking advantage of the special structures found in various symmetry-reduced versions of Einstein's equations.
Remark 1.10 (The crucial bound for the spatial Ricci curvature). Using the estimates $1.24 \mathrm{~b}-1.24 \mathrm{c})$ and similar estimates for the spatial derivatives of $S_{I J B}$, the algebraic identity 5.22 , and the spatial Ricci
 This is a frame component analog of the classic sub-criticality condition 1.16), and in practice, one needs such an estimate to prove 1.24 a .

To sketch the main ideas behind the proofs of $1.24 \mathrm{a}-1.24 \mathrm{c}$, we note that the evolution equations for $k_{I J}-\widetilde{k}_{I J}, S_{I J B}$, and $e_{I}^{i}-\widetilde{e}_{I}^{i}$ can be caricatured as follows (see Propositions 2.2 and 5.7 and Lemma 5.13 for the precise equations):

$$
\begin{align*}
\partial_{t}\left(k_{I J}-\widetilde{k}_{I J}\right)+\frac{1}{t}\left(k_{I J}-\widetilde{k}_{I J}\right) & =e_{I}^{i} \cdot \partial \gamma+\gamma \cdot \gamma+\cdots,  \tag{1.25a}\\
\partial_{t} S_{I J B}+\frac{\left(\widetilde{q}_{\underline{I}}+\widetilde{q}_{\underline{J}}-\widetilde{q}_{\underline{B}}\right)}{t} S_{\underline{I J B}} & =\cdots,  \tag{1.25b}\\
\partial_{t}\left(e_{I}^{i}-\widetilde{e}_{I}^{i}\right)+\frac{\widetilde{q}_{I}}{t}\left(e_{\underline{I}}^{i}-\widetilde{e}_{\underline{I}}^{i}\right) & =\cdots, \tag{1.25c}
\end{align*}
$$

where $\cdot$. denotes similar or simpler error terms that we ignore to simplify the discussion, and we recall that we do not sum over repeated underlined indices.

Remark 1.11 (On the approximately diagonal structure of the evolution equations for the structure coefficients). Note that 1.25 b shows that the $S_{I J B}$ solve an evolution equation system that is approximately diagonal, as we highlighted in Sect.1.8.3.

[^15]Next, we note that the estimates (1.24b) and $\sqrt{1.24 c}$ ) are easy to derive (modulo the omitted terms "...") via integrating factors as a consequence of equations $(1.25 \mathrm{~b})-\sqrt{1.25 c}$ ) and the definition of $q$. In reality, the proofs of $\sqrt{1.25 \mathrm{a}}-\sqrt{1.25 \mathrm{c}}$ must be handled simultaneously, via a bootstrap argument, due to coupling terms, but we will ignore this issue here; see the proof of Proposition 5.26 for the details.

Next, to illustrate the interplay between low order $L^{\infty}$ estimates and high order energy estimates, we will now explain how to derive the bound 1.24 a for $k_{I J}$, assuming the high order energy bound 1.23 and the estimates 1.24 b and 1.24 c . To this end, we must explain how to control the term $e_{I}^{i} \cdot \partial \gamma$ on RHS 1.25 a . This term loses one derivative and must ultimately be handled with the help of energy estimates (which we discuss in Sect. 1.8.6), but as we explain, its $L^{\infty}$ norm is sub-critical with respect to powers of $t$. By this, we mean that the behavior of $e_{I}^{i} \cdot \partial \gamma$ with respect to $t$ is strictly less singular with respect to $t$, as $t \downarrow 0$, compared to the terms on LHS 1.25a (i.e., less singular than $t^{-2}$ ) and thus, near the singularity, it is a negligible error term. To see this, one can use standard Sobolev embedding and interpolation estimates (see Lemmas 4.1 and 4.2) to infer that there is a constant $\delta_{N}>0$ (depending on $N$ ) such that $\delta_{N} \rightarrow 0$ as $N \rightarrow \infty$ and such that the following crucial estimate holds:

$$
\begin{equation*}
\|\partial \gamma\|_{L^{\infty}\left(\Sigma_{t}\right)} \lesssim\|\gamma\|_{L^{\infty}\left(\Sigma_{t}\right)}+\|\gamma\|_{L^{\infty}\left(\Sigma_{t}\right)}^{1-\delta_{N}}\|\gamma\|_{\dot{H}^{N}\left(\Sigma_{t}\right)}^{\delta_{N}} . \tag{1.26}
\end{equation*}
$$

Combining (1.26) with (1.23) and 1.24b, and using the fact that the connection coefficients $\gamma$ are linear combinations of the structure coefficients $S$ (see 5.22$)$, we find that $\|\partial \gamma\|_{L^{\infty}\left(\Sigma_{t}\right)} \lesssim d a t a \times t^{-q}+d a t a . \times$ $t^{-\left(1-\delta_{N}\right) q} t^{-\delta_{N}\left(A_{*}+1\right)}$. Thus, by choosing $N$ sufficiently large, exploiting that $A_{*}$ does not depend on $N$, and that $\delta_{N} \rightarrow 0$ as $N \rightarrow \infty$, we find that

$$
\begin{equation*}
\|\partial \gamma\|_{L^{\infty}\left(\Sigma_{t}\right)} \lesssim d a t a \times t^{-1+\sigma} \tag{1.27}
\end{equation*}
$$

The importance of 1.26 and 1.27 is that they show that when $N$ is large, the singularity strength of $\|\partial \gamma\|_{L^{\infty}\left(\Sigma_{t}\right)}$ is not much worse than the singularity strength of $\|\gamma\|_{L^{\infty}\left(\Sigma_{t}\right)}$, even if $\|\gamma\|_{\dot{H}^{N}\left(\Sigma_{t}\right)}$ obeys a much worse estimate of the form $\|\gamma\|_{\dot{H}^{N}\left(\Sigma_{t}\right)} \lesssim d a t a \times t^{-\left(A_{*}+1\right)}$ (which is the bound afforded by the energy estimate 1.23 ).

Hence, also using 1.24 c , we conclude that $\left\|e_{I}^{i} \cdot \partial \gamma\right\|_{L^{\infty}\left(\Sigma_{t}\right)} \lesssim d a t a \times t^{-2+\sigma}$, i.e., that this term is less singular than $t^{-2}$, as desired. Let us now sketch the proof that these bounds imply the desired estimate 1.24 a . Using these bounds, multiplying the evolution equation 1.25a by $t$ and noting that the resulting LHS is equal to $\partial_{t}\left[t\left(k_{I J}-\widetilde{k}_{I J}\right)\right]$, and then using the fundamental theorem of calculus, we deduce the pointwise bound

$$
\begin{align*}
\left|t k_{I J}-t \widetilde{k}_{I J}\right|(t, x) & \lesssim\left|t k_{I J}-t \widetilde{k}_{I J}\right|(1, x) \\
& +\int_{t}^{1} s\left|e_{I}^{i}\right|(s, x) \cdot|\partial \gamma|(s, x) d s+\int_{t}^{1} s|\gamma|(s, x) \cdot|\gamma|(s, x) d s+\cdots  \tag{1.28}\\
& \lesssim d a t a+d a t a \int_{t}^{1} s^{-1+\sigma} d s+\cdots \lesssim d a t a+\cdots
\end{align*}
$$

which yields the desired bound 1.24a, up to the error terms "..." We close this section by highlighting that in a fully detailed proof of 1.28 , the estimate 1.17 is crucial for obtaining the power $s^{-1+\sigma}$ in the next-to-last inequality in 1.28 .

Remark 1.12 (How large does $N$ need to be?). The following natural question emerges from the above discussion: how large does $N$ need to be for the above scheme to work? The interpolation inequality (1.26) already suggests that the rough estimate $N \sim \sigma^{-1}$ is sufficient to guarantee that $\left\|\partial \gamma_{I J B}\right\|_{L^{\infty}\left(\Sigma_{t}\right)}$ is less singular than $t^{-1}$, given the high order energy bounds 1.23 for $t^{A_{*}+1}\left\|\gamma_{I J B}\right\|_{\dot{H}^{N}\left(\Sigma_{t}\right)}$ and the fact that $\gamma$ satisfies $\left|\gamma_{I J B}\right| \lesssim d a t a \times t^{-1+\sigma}$ and that $\delta_{N} \sim N^{-1}$ (see the proof of Lemma 4.2 for some details on the $N$-dependence of the $\delta_{N}$ that appear in our interpolation estimates). The precise largeness of $N$ needed for this argument to go through depends on $\mathfrak{D}$ and the size of $A_{*}$, which we discuss in Remark 1.13 . We also note that, even in the best case scenario, we would expect the behavior of at least one of the connection coefficient norms $\left\|\gamma_{I J B}\right\|_{L^{\infty}\left(\Sigma_{t}\right)}$ to be at least as singular as $t^{-} \max _{I=1, \ldots, \mathcal{D}} \widetilde{\widetilde{N}}_{I}$. Hence, in view of our choice (3.6) of
the parameters $q, \sigma$, we see that $\sigma$ has order of magnitude at least as small as $1-\max _{I=1, \cdots, \mathfrak{D}} \widetilde{q}_{I}$. In particular, this implies that as the background Kasner exponents tend towards an extreme case, e.g., as $\max _{I=1, \cdots, \mathfrak{D}} \widetilde{q}_{I} \rightarrow 1$ (and hence $\sigma \rightarrow 0$ ), the number of derivatives $N$ we would need to close our bootstrap argument would tend to $\infty$. Similarly, as LHS $(1.8)$ tends towards 1 , our arguments would require $N$ to tend to $\infty$ as well. Moreover, going back to Remark 1.8 , we note that one could use similar interpolation arguments (see Lemmas 4.1 and 4.2) to infer that choosing $N \sim N_{0} \sigma^{-1}$ would be sufficient to guarantee that $\left\|\partial^{N_{0}} \gamma_{I J B}\right\|_{L^{\infty}\left(\Sigma_{t}\right)}$ is less singular than $t^{-1}$.

### 1.8.6 The high order energy estimates

We now explain how we derive our top-order energy estimates for the dynamic variables $e_{I}^{i}, k_{I J}, \gamma_{I J B}, e_{0} \psi$, and $e_{I} \psi$, that is, how we prove 1.23 . We will highlight the role played by the $L^{\infty}$ estimates of Sect. 1.8 .5 . We first commute the evolution equations (recall that in our formulation, all of the evolution equations are first-order) with $\partial^{\iota}$, where $\partial^{\iota}$ is an $N$ th-order differential operator corresponding to repeated differentiation with respect to the transported spatial coordinate vectorfields. We then derive energy identities for solutions to the commuted equations, where we incorporate $t^{A_{*}+1}$-weights into the identities. Below we will explain the analytic role of the weights. The energy identity for the scalar field is standard, and we will not discuss it in detail here; we refer readers to Lemma 5.24 for a differential version of that energy identity. Similar remarks apply for the energy identity for the frame component functions $e_{I}^{i}$.

However, the derivation of the energy identity for the second fundamental form frame components $k_{I J}$ and the connection coefficients $\gamma_{I J B}$ is more subtle, since the identity corresponds to a surprising gain of one derivative for the frame, as we highlighted in Sect.1.3. The identity can be derived using a modification of the approach used in [58, 59, 60. The main difficulty is that the evolution equations 2.22a)-2.22b for $\gamma$ and $k$ do not form a symmetric hyperbolic system, which, at first glance, seems to obstruct the availability of a basic energy identity. However, one can use differentiation by parts and the momentum constraint equation, as well as the special structure of the equations relative to CMC foliations (see 2.26 b ), to replace the problematic terms with source terms that exhibit an allowable amount of regularity. We refer readers to Lemma 5.34 for this top-order energy identity, expressed in differential form.

We will now describe our top-order energy estimates. We will give a simplified, schematic presentation in order to focus on the main ideas. We introduce the top-order energy ${ }^{29}$

$$
\begin{equation*}
\mathbb{E}_{N}(t):=t^{A_{*}+1}\|k\|_{\dot{H}^{N}}+t^{A_{*}+1}\|\gamma\|_{\dot{H}^{N}}+t^{A_{*}+q} \sum_{I, i=1, \cdots, \mathcal{D}}\left\|e_{I}^{i}\right\|_{\dot{H}^{N}}+t^{A_{*}+1}\left\|e_{0} \psi\right\|_{\dot{H}^{N}}+t^{A_{*}+1} \sum_{I=1, \cdots, \mathfrak{D}}\left\|e_{I} \psi\right\|_{\dot{H}^{N}} . \tag{1.29}
\end{equation*}
$$

We will sketch our proof that if $A_{*}$ is chosen to be sufficiently large and then $N$ is chosen to be sufficiently large such that the $L^{\infty}$ estimates of Sect. 1.8.5 hold, then we have the following bound ${ }^{30} \mathbb{E}_{N}(t) \leq C_{N} \times$ data, i.e., the estimate 1.23 holds. To obtain this bound, we combine the energy identities mentioned in the previous paragraphs with elliptic estimates for the lapse, and we use the $L^{\infty}$ estimates from Sect. 1.8 .5 and interpolation to control the nonlinear error terms. This allows us to derive the following energy integral inequality for $t \in(0,1]$ (see Proposition 5.27 for the precise inequalities), where $C_{*}$ is a constant that captures the strength of the borderline terms in the equations and that can be chosen to be independent of $N$ and $A_{*}$ (as long as "data" is small), while $C_{N}>0$ is a large, $N$-dependent constant:

$$
\begin{equation*}
\mathbb{E}_{N}^{2}(t) \leq d a t a^{2}+\left(C_{*}-A_{*}\right) \int_{t}^{1} \frac{\mathbb{E}_{N}^{2}(s)}{s} d s+C_{N} \int_{t}^{1} s^{-1+\sigma} \mathbb{E}_{N}^{2}(s) d s \tag{1.30}
\end{equation*}
$$

The crucial point is that if we choose $A_{*}$ to be larger than $C_{*}$, then the time integral on RHS 1.30 becomes non-positive, and we can discard it. Finally, from 1.30 and Grönwall's lemma, we obtain that $\mathbb{E}_{N}(t) \leq C_{N} \times d a t a$ as desired. This concludes our schematic discussion of the a priori estimates.

Some closing remarks are in order.

[^16]- The negative definite integral $-A_{*} \int_{t}^{1} \frac{\mathbb{E}_{N}^{2}(s)}{s} d s$ on RHS 1.30 arises from our energy identities, specifically from the $t^{A_{*}+1}$ weights that we have incorporated into them. This negative definite integral allows us to absorb the dangerous borderline error integral $C_{*} \int_{t}^{1} \frac{\mathbb{E}_{N}^{2}(s)}{s} d s$, but at the expense of forcing us to work with energies that are very degenerate near $t=0$.
- Above we mentioned the notion of a "borderline term." To handle such terms, we must rely on the sharp $L^{\infty}$ estimates from Sect.1.8.5 for borderline terms, there is "no room" in the $L^{\infty}$ estimates. In the context of energy estimates, borderline terms contribute to the dangerous integral $C_{*} \int_{t}^{1} \frac{\mathbb{E}_{N}^{2}(s)}{s} d s$ on RHS 1.30). One example of a borderline error integral is $\int_{t}^{1} \int_{\Sigma_{s}} s^{2\left(A_{*}+1\right)} \cdot k \cdot \partial^{\iota} \gamma \cdot \partial^{\iota} \gamma d x d s$, where $\partial^{\iota}$ is an $N$ th-order spatial differential operator of the type mentioned earlier. To bound this integral by $C_{*} \int_{t}^{1} \frac{\mathbb{E}_{N}^{2}(s)}{s} d s$, we need to use the sharp estimate $\left\|k_{I J}\right\|_{L^{\infty}\left(\Sigma_{s}\right)} \leq \frac{C_{*}}{s}$ implied by 1.24 a . If, instead of this sharp bound, we only knew that $\left\|k_{I J}\right\|_{L^{\infty}} \leq C_{*} t^{-(1+\epsilon)}$ for some $\epsilon>0$, then on RHS 1.30), we would have an additional error integral of the form $C_{*} \int_{t}^{1} \frac{\mathbb{E}_{N}^{2}(s)}{s^{1+\epsilon}} d s$. By virtue of Grönwall's lemma, this integral would lead to dramatically worse a priori estimates, which would in turn prevent us from closing our bootstrap argument.

Remark 1.13 (The size of $A_{*}$ ). It is possible, in principle, to compute how large $A_{*}$ has to be for the above proof to work; one simply needs to derive an explicit upper bound for the constant $C_{*}$ on RHS $\sqrt{1.30}$. We will provide an outline of how to estimate $C_{*}$ (and thus $A_{*}$ ), although we do not provide an explicit estimate. To shorten the discussion, we will restrict our attention to the Einstein-vacuum equations, i.e., we will assume that $\psi=0$. In short, the constant $C_{*}$ can be controlled by the number of borderline terms in the top-order energy estimates and elliptic estimates and the size of the coefficients in front of these terms. More precisely, the borderline terms in the energy estimates for the connection coefficients and second fundamental form are generated by the terms on RHSs 55.32a, 5.32c, and 5.32e. The main terms driving the size of $C_{*}$ are the top-order ones with coefficients of size $\approx t^{-1}$, e.g., the term $\partial^{l}(n-1) \cdot \widetilde{k}$ on RHS (5.32a) and the terms $n \cdot \widetilde{k} \cdot \partial^{\iota} \gamma$ and $\widetilde{k} \cdot \partial^{\iota} \vec{e} n$ on RHS 5.32 c . These terms lead to borderline error integrals in the energy estimates, such as the integral $\int_{t}^{1} \int_{\Sigma_{s}} s^{2\left(A_{*}+1\right)} \cdot k \cdot \partial^{\iota} \gamma \cdot \partial^{\iota} \gamma d x d s$ mentioned above. In each borderline term, the coefficient of the top-order term, specifically $k$ in the previous integral, can be bounded by $\left(\left|\widetilde{q}_{I}\right|+C \epsilon\right) t^{-1}$ for some $I$, where $\widetilde{q}_{I}$ can be any of the background exponents and $C \epsilon$ can be as small as desired, by taking the initial data on $\Sigma_{1}$ to be sufficiently close to the Kasner data. More precisely, decomposing $k=\widetilde{k}+(k-\widetilde{k})$, we see that the factor $\left|\widetilde{q}_{I}\right| t^{-1}$ is generated by $\widetilde{k}$ (see 2.58$)$, while the factor $C \epsilon t^{-1}$ from is generated by the bound 1.24a). Since $\left|\widetilde{q}_{I}\right|<1$, by counting all borderline terms, we could crudely bound the contribution of these terms to the constant $C_{*}$ by $\leq$ (number of borderline terms) $+C \varepsilon$. The sum of all the corresponding toporder error integrals would then be bounded by $C_{*} \int_{t}^{1} \frac{\mathbb{E}_{N}^{2}(s)}{s} d s$. We also stress that one encounters borderline terms in the elliptic estimates for the top-order derivatives of $n$ (see, for example, the $C_{*}$-involving term on RHS (5.5a) , and that these borderline terms propagate into the top-order energy estimates via terms such as the one $\partial^{\iota}(n-1) \cdot \widetilde{k}$ mentioned above. In particular, these terms affect the size of $C_{*}$. We also note that the arguments given here allow for the possibility that $C_{*}$ might increase with respect to $\mathfrak{D}$. Finally, we note that the estimate for $C_{*}$ sketched here is not necessarily optimal. In fact, in the near-FLRW regime (where all Kasner exponents are nearly equal), the last two authors [58, 59] showed that striking cancellations take place, and $A_{*}$ can in fact be taken very small, i.e., $C_{*}=C \varepsilon$. It is not known to us whether such cancellations exist for perturbations of highly anisotropic background Kasner solutions.

### 1.9 Applicability of the method

### 1.9.1 Polarized $\mathbb{T}^{2}$-symmetry

We already mentioned that Kasner-like singularities have been constructed [2, 36] for the Einstein-vacuum equations in $1+3$ dimensions within the polarized $\mathbb{T}^{2}$-symmetry class. This symmetry class contains vacuum spacetimes with two orthogonal, spacelike, Killing vectorfields $X, Y$ that commute, and it is more general than the polarized Gowdy-class in the sense that the twist constants, which measure the obstruction to the integrability of the 2 -dimensional orthogonal planes to $X, Y$, do not have to vanish. It turns out that polarized $\mathbb{T}^{2}$-symmetric solutions can be can be viewed as special cases of polarized $U(1)$-symmetric solutions
in which one extra symmetry is present. This fact is not immediately apparent in the sense that our definition of polarized $U(1)$-symmetry (recall the discussion in Sect. 1.4.2 requires a spacelike Killing field to be hypersurface-orthogonal, whereas the definition of polarized $\mathbb{T}^{2}$-symmetry does not refer to hypersurface orthogonality. Nevertheless, for polarized $\mathbb{T}^{2}$-symmetric solutions, it is always possible ${ }^{31}$ to construct coordinates such that one of the twist constants vanishes and such that one of the spatial coordinate partial derivative vectorfields associated to the $\mathbb{T}^{2}$-symmetry is Killing and hypersurface-orthogonal; see [4, Section 2.2], where this coordinate Killing vectorfield is denoted by " $\partial_{x}$." Given such coordinates, the corresponding solutions can indeed be viewed as special cases of polarized $U(1)$-symmetric solutions in which one extra symmetry is present. Hence, our results on polarized $U(1)$-symmetric solutions imply, as a special case, that all Kasner solutions are also stable (as solutions to the Einstein-vacuum equations in $1+3$ dimensions) near their Big Bangs under polarized $\mathbb{T}^{2}$-symmetric perturbations. Here, by "stable," we mean that the results of Theorem 6.5 hold for the near-Kasner polarized $\mathbb{T}^{2}$-symmetric solutions, where the hypersurface-orthogonal Killing vectorfield " $\partial_{3}$ " from Theorem 6.5 corresponds to the vectorfield " $\partial_{x}$ " from 4].

We also highlight that recently, an alternate approach has been developed 4 for studying the stability of Kasner Big Bangs under polarized $\mathbb{T}^{2}$-symmetric perturbations, including in the presence of an arbitrary cosmological constant [5]. More precisely, in 4], the authors used an areal time foliation to prove the stability of a subset of the Kasner solutions near their Big Bang singularities under polarized $\mathbb{T}^{2}$-symmetric perturbations, where the notion of "stability" in [4] is more precise compared to our notion of stability in Theorem 6.5. In particular, the approach in 4 yields a sharp description of the behavior of the metric components relative to areal coordinates. It would be interesting to see if the results of [4] - especially the ones concerning the sharp asymptotics - can be extended to the full set of Kasner solutions; this could sharpen our understanding (see Remark 6.2) of the behavior of near-Kasner, polarized, $\mathbb{T}^{2}$-symmetric solutions near their Big Bangs.

### 1.9.2 Potential further applications

Our method could likely be adapted to prove stable Big Bang formation in other models that are not, strictly speaking, covered in the present paper. We mention here some interesting cases.

The stiff-fluid model, for $\mathfrak{D} \geq 3$. This matter model reduces to the scalar field matter model in the case of vanishing vorticity. In [59, stable Big Bang formation was proved in the special case $\mathfrak{D}=3$ for the background FLRW solution, in which $\widetilde{q}_{1}=\widetilde{q}_{2}=\widetilde{q}_{3}=1 / 3$, and the presence of matter is needed to ensure the validity of the Kasner exponent constraints 1.7).

Perturbing around fixed, non-explicit, backgrounds/ solutions with large spatial dependence. The stability problems that we study in detail in this paper concern perturbations of explicit, spatially homogeneous, singularity-forming solutions. However, one could try to use our methods to study perturbations of any of the singular solutions constructed in the works that we mentioned in Sect. 1.7, including solutions with spatial dependence. From an analytical point of view, when dealing with background solutions that exhibit spatial dependence, there is an additional technical difficulty in the derivation of various estimates, since the terms where derivatives hit the background solution will no longer be zero or necessarily small. Nevertheless, our method is still potentially applicable. To simplify the approach, one could consider data with large spatial derivatives given on a hypersurface close to the expected singularity, that is, on $\Sigma_{t_{\text {Data }}}$, with $t_{\text {Data }}$ larger than but close to 0 (where $t_{\text {Data }}$ has to be chosen to be small in a manner that depends on the largeness of the data); the point is that the smallness of the amount of time for which one needs to control the solution can compensate for the largeness of the data. Moreover, by applying this philosophy to the setup of the present paper, one could produce open sets of singularity-forming solutions that have "substantial $x$-dependence."

Black hole interior. There are numerous examples of black hole spacetimes containing a spacelike singularity, such as the classical Oppenheimer-Snyder model of gravitational collapse or the solutions detected by Christodoulou in his aforementioned studies [19, 20] of the spherically symmetric Einstein-scalar field model. For the latter solutions, it would be interesting to see whether Kasner-like blowup holds for perturbations of solutions (in some class other than spherical symmetry, which was handled in [19, 20]). Compared to our work here, the difference in topology might pose additional analytical difficulties. Moreover, one would have to grapple with the question of whether the initial data given only in the interior of a black hole could arise

[^17]as induced data of solutions to the global Cauchy problem.

### 1.10 Paper outline

In Sect. 2 , we introduce our analytic framework, including the reduced solution variables and a formulation of the Einstein-scalar field equations relative to CMC-transported spatial coordinates with a Fermi-Walker transported orthonormal frame. In Sect. 3, we define various norms and introduce our bootstrap assumptions for perturbations of Kasner solutions. Our bootstrap assumptions involve $t$-weighted $L^{\infty}$ norms at the low derivative levels and $t$-weighted Sobolev norms at the high derivative levels, where the $t$-weights are much smaller at the high derivative levels (which corresponds to our allowing for very singular high order derivatives as $t \downarrow 0$ ). In Sect. 4 , we provide standard Sobolev and interpolation estimates that we will use to control various error terms when we derive our main estimates. In Sect. 5, we derive the core estimates at both the low and high derivative levels. These estimates in particular yield a strict improvement of the bootstrap assumptions. Finally, in Sect. 6, we use the estimates of Sect. 5 to prove our main theorems exhibiting the stability of the Kasner Big Bang singularity.

### 1.11 Notation and conventions

In the rest of the paper, we use the following notation and conventions.

- $\left\{x^{i}\right\}_{i=1} \ldots \mathfrak{D}$ denote standard local spatial coordinates on $\mathbb{T}^{\mathfrak{D}}$ that are transported in the sense described in Sect.2.1.1, and $\partial_{i}:=\frac{\partial}{\partial x^{i}}$ denote the corresponding spatial partial derivative vectorfields. The frame $\left\{\partial_{i}\right\}_{i=1, \cdots, \mathfrak{D}}$ extends to a smooth global holonomic frame on $\mathbb{T}^{\mathfrak{D}}$, and by abuse of notation, we denote the globally defined vectorfields by the symbols $\partial_{i}$, even though the coordinate functions are not globally defined.
- Lowercase Latin "spatial" indices such as $a, b, i, j$ range over $\{1, \cdots, \mathfrak{D}\}$ and correspond to the transported spatial coordinates $x^{1}, \cdots, x^{\mathfrak{D}}$ (see Sect.2]. For example, $g_{i j}:=g\left(\partial_{i}, \partial_{j}\right)$. Lowercase Greek "spacetime" indices such as $\alpha, \beta, \mu, \nu$ range over $\{0,1, \cdots, \mathfrak{D}\}$ and usually correspond to the spacetime coordinates $t, x^{1}, \cdots, x^{\mathcal{D}}$, where the " 0 " index corresponds to $t$. For example, $\mathbf{g}_{0 i}=\mathbf{g}_{t i}:=\mathbf{g}\left(\partial_{t}, \partial_{i}\right)$. In a few instances, $\left\{e_{\alpha}\right\}_{\alpha=0, \cdots, \mathcal{D}}$ denotes an orthonormal spacetime frame, i.e., $\mathbf{g}\left(\mathbf{e}_{\alpha}, \mathbf{e}_{\beta}\right)=\mathbf{m}_{\alpha \beta}$, where $\mathbf{m}_{\alpha \beta}:=\operatorname{diag}(-1,1, \cdots, 1)$. Uppercase Latin "spatial frame" indices such as $I, J$ range over $\{1, \cdots, \mathfrak{D}\}$ and, with one exception, correspond to the orthonormal spatial frame $\left\{e_{I}\right\}_{I=1, \cdots, \mathfrak{D}}$ or co-frame $\left\{\omega^{I}\right\}_{I=1, \cdots, \mathfrak{D}}$ (see Sect. 2 . For example, $k_{I J}:=k\left(e_{I}, e_{J}\right)=k_{c d} e_{I}^{c} e_{J}^{d}$. The exception is that for background Kasner tensors, uppercase Latin indices denote their components with respect to the background Kasner orthonormal frame $\left\{\widetilde{e}_{I}\right\}_{I=1, \cdots, \mathfrak{D}}$; see Remark 2.6 for further discussion.
- We use Einstein summation for repeated indices, including frame indices. We stress that no metric is directly involved in contractions involving the frame indices. For example, $k_{I C} \gamma_{C J B}$ stands for $\sum_{C=1}^{\mathfrak{D}} k_{I C} \gamma_{C J B}$, where $\mathfrak{D}$ is the number of spatial dimensions.
- If $\mathbf{X}$ is a vectorfield and $f$ is a scalar function, then $\mathbf{X} f:=\mathbf{X}^{\alpha} \partial_{\alpha} f$ denotes the derivative of $f$ in the direction $\mathbf{X}$.
- $\left\{d x^{i}\right\}_{i=1, \cdots, \mathfrak{D}}$ denotes the globally defined basis-dual co-frame of $\left\{\partial_{i}\right\}_{i=1, \cdots, \mathfrak{D}}$, i.e., $d x^{i}\left(\partial_{j}\right):=\delta_{j}^{i}$, with $\delta_{j}^{i}$ the Kronecker delta.
- No summation of underlined terms. In a handful of key terms that explicitly involve the Kasner exponents, we will not use Einstein summation convention for some of the indices. More precisely, in a given product, whenever there is no summation over a particular index, we indicate this by underlining all instances of that index in the product. For example, there is no summation over the index $I$ in the following expression: $\frac{\widetilde{q}_{\underline{I}}}{t} \gamma_{\underline{I} J B}$.
- If $\mathbf{X}$ and $\mathbf{Y}$ are vectorfields, then $\mathbf{X Y} f:=\mathbf{X}^{\alpha} \partial_{\alpha}\left(\mathbf{Y}^{\beta} \partial_{\beta} f\right)$. Similarly, if $\mathbf{T}$ is a tensorfield and $\mathbf{D}$ denotes the Levi-Civita connection of $\mathbf{g}$, then $\mathbf{D}_{\mathbf{X}} \mathbf{T}:=\mathbf{X}^{\alpha} \mathbf{D}_{\alpha} \mathbf{T}$ and $\mathbf{D}_{\mathbf{X}} \mathbf{D}_{\mathbf{Y}} \mathbf{T}:=\mathbf{X}^{\alpha} \mathbf{D}_{\alpha}\left(\mathbf{Y}^{\beta} \mathbf{D}_{\beta} \mathbf{T}\right)$. In addition, $\mathbf{D}_{\mathbf{X Y}}^{2} \mathbf{T}:=\mathbf{X}^{\alpha} \mathbf{Y}^{\beta} \mathbf{D}_{\alpha} \mathbf{D}_{\beta} \mathbf{T}$. Note that in the latter expression, contractions are taken after covariant differentiation and thus generally, $\mathbf{D}_{\mathbf{X Y}}^{2} \mathbf{T} \neq \mathbf{D}_{\mathbf{X}} \mathbf{D}_{\mathbf{Y}} \mathbf{T}$.
- If $\mathbf{X}$ and $\mathbf{Y}$ are vectorfields, then $\mathbf{g}(\mathbf{X}, \mathbf{Y}):=\mathbf{g}_{\alpha \beta} \mathbf{X}^{\alpha} \mathbf{Y}^{\beta}$. We use similar notation for contractions of higher-order tensorfields against vectorfields. For example,

$$
\boldsymbol{\operatorname { R i e m }}(\mathbf{W}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}):=\operatorname{Riem}_{\alpha \beta \gamma \delta} \mathbf{W}^{\alpha} \mathbf{X}^{\beta} \mathbf{Y}^{\gamma} \mathbf{Z}^{\delta}
$$

- $\iota$ denotes a spatial multi-index. That is, for some positive integer $m, \iota=\left(a_{1}, \cdots, a_{m}\right)$, where $a_{i} \in$ $\{1, \cdots, \mathfrak{D}\}$ for $1 \leq i \leq m$ and $|\iota|:=m$ denotes the length of the index. $\partial^{\iota}:=\partial_{a_{1}} \cdots \partial_{a_{m}}$ denotes the corresponding order $m$ differential operator involving repeated differentiation with respect to the transported spatial coordinate partial derivative vectorfields. $\iota_{1} \cup \iota_{2}=\iota$ means that for some $r$ with $1 \leq r \leq m$, we have $\iota_{1}=\left(a_{i_{1}}, \cdots, a_{i_{r}}\right)$ and $\iota_{2}=\left(a_{i_{r+1}}, \cdots, a_{i_{m}}\right)$, where $\left(i_{1}, \cdots, i_{m}\right)$ is a permutation of $(1, \cdots, m)$ such that $i_{1}<i_{2}<\cdots<i_{r}$ and $i_{r+1}<i_{r+2}<\cdots<i_{m} . \iota_{1} \cup \iota_{2} \cup \iota_{3}=\iota, \iota_{1} \cup \iota_{2} \cup \iota_{3} \cup \iota_{4}=\iota$, etc. have analogous meanings.
Note that our multi-index convention in $\mathfrak{D}$ spatial dimensions differs from the more standard one, in which multi-indices $\alpha$ satisfy $\alpha \in \mathbb{N}^{\mathfrak{D}}$. For instance, in the more standard notation, in 3 spatial dimensions, $\alpha:=(1,1,0)$ corresponds to $\partial^{\alpha}=\partial_{1} \partial_{2}$, whereas with our multi-index notation in 3 spatial dimensions, $\iota=(1,1)$ corresponds to $\partial^{\iota}=\partial_{1} \partial_{1}$.


## Parameters

- $A_{*} \geq 1$ denotes a "time-weight exponent parameter" that is featured in the high order solution norms from Definition 3.1. To close our estimates, we will choose $A_{*}$ to be large enough to overwhelm various universal constants $C_{*}$ (see below). This corresponds to our use of high order energies featuring large powers of $t$, which leads to weak high order energies near $t=0$.
- $0<q<1$ is a constant, fixed throughout the proof, that bounds the crucial quantity

$$
\max _{\substack{I, J, B=1, \cdots, \mathfrak{D} \\ I<J}}\left\{\left|\widetilde{q}_{B}\right|, \widetilde{q}_{I}+\widetilde{q}_{J}-\widetilde{q}_{B}\right\}
$$

- $\sigma>0$ is a small constant, fixed throughout the proof, that we use to simplify the proofs of various estimates that "have room in them."
- $q$ and $\sigma$ are constrained by (3.6).
- $N_{0} \geq 1$ roughly corresponds to the number of derivatives of the solution that we control in $L^{\infty}$ (the precise derivative count depends on the solution variable - see Definition 3.1).
- $N$ denotes the maximum number of times that we commute the equations with spatial derivatives (e.g., $k \in H^{N}\left(\Sigma_{t}\right)$ and $n \in H^{N+1}\left(\Sigma_{t}\right)$ - see Definition 3.1. To close our estimates, we will choose $N$ to be sufficiently large in a (non-explicit) manner that depends on $N_{0}, A_{*}, \mathfrak{D}, q$, and $\sigma$.
- $\delta>0$ is a small $(N, \mathfrak{D})$-dependent parameter that is allowed to vary from line to line and that is generated by the estimates of Lemma 4.1. We use the convention that a sum of two $\delta$ 's is another $\delta$. The only important feature of $\delta$ that we exploit throughout the paper is the following: at fixed $\mathfrak{D}$, we have $\lim _{N \rightarrow \infty} \delta=0$. In particular, if $A_{*}$ is also fixed, then $\lim _{N \rightarrow \infty} A_{*} \delta=0$.
- $\varepsilon$ is a small "bootstrap parameter" that, in our bootstrap argument, bounds the size of the solution norms; see (3.9). The smallness of $\varepsilon$ needed to close the estimates is allowed to depend on the parameters $N, N_{0}, A_{*}, \mathfrak{D}, q$, and $\sigma$.


## Constants

- $C$ denotes a positive constant that is free to vary from line to line. $C$ can depend on $N, N_{0}, A_{*}, \mathfrak{D}, q$, and $\sigma$, but it can be chosen to be independent of all $\varepsilon>0$ that are sufficiently small in the manner described just above.
- $C_{*}$ denotes a positive constant that is free to vary from line to line and that can depend on $\mathfrak{D}$. Like $C$, $C_{*}$ can be chosen to be independent of all $\varepsilon>0$ that are sufficiently small in the manner described just above. However, unlike $C, C_{*}$ can be chosen to be independent of $N, N_{0}$, and $A_{*} . C_{*}$ can be chosen to be independent of $q$, and $\sigma$, but that is less important in the sense that we view $q$ and $\sigma$ to be fixed throughout the article. For example, $1+C N!\varepsilon \leq C_{*}$ while $N!=C$ and $N!/ \sigma=C$, where $C$ and $C_{*}$ are as above.
- We write $v \lesssim w$ to indicate that $v \leq C w$, with $C$ as above.
- We write $v=\mathcal{O}(w)$ to indicate that $|v| \leq C|w|$, with $C$ as above.


### 1.12 Acknowledgments

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## 2 Analytic setup and the formulation of the Einstein-scalar field equations

In this section, we introduce the framework that we will use to study perturbations of Kasner solutions. In particular, we provide the formulation of the Einstein-scalar field equations that we will use to derive estimates.

### 2.1 The reduced equations relative to a CMC-transported orthonormal frame

Our main goal in this section is to prove Proposition 2.2, which provides the formulation of the Einstein-scalar equations that forms the starting point for our analysis. We start by providing some basic constructions.

### 2.1.1 The form of the spacetime metric, the lapse, and the transported spatial coordinates

Relative to CMC-transported spatial coordinates on a slab $(t, x) \in(T, 1] \times \mathbb{T}^{\mathfrak{D}}$, the spacetime metric $\mathbf{g}$ takes the form

$$
\begin{equation*}
\mathbf{g}=-n^{2} d t \otimes d t+g_{a b} d x^{a} \otimes d x^{b} \tag{2.1}
\end{equation*}
$$

where $n$ is the lapse and $g$ is the first fundamental form of the constant-time slice $\Sigma_{t}:=\{(s, x) \in(T, 1] \times$ $\left.\mathbb{T}^{\mathfrak{D}} \mid s=t\right\}$, i.e., $g$ is the Riemannian metric induced by $\mathbf{g}$ on $\Sigma_{t}$. Here and throughout, $t$ is the time function. In Sect.2.1.3, we state our CMC normalization condition for $t$. The spatial coordinates $\left\{x^{i}\right\}_{i=1, \cdots, \mathfrak{N}}$ are said to be "transported" because $n^{-1} \partial_{t} x^{i}=0$, where $n^{-1} \partial_{t}$ is the future-directed unit normal to $\Sigma_{t}$.

### 2.1.2 The orthonormal frame

Our proofs fundamentally rely on expressing Einstein's equations relative to an orthonormal frame:

$$
\begin{equation*}
e_{0}=n^{-1} \partial_{t}, \quad e_{I}=e_{I}^{c} \partial_{c}, \quad I=1, \cdots, \mathfrak{D} \tag{2.2}
\end{equation*}
$$

where $e_{0}$ is the future-directed unit normal to $\Sigma_{t}$ (in particular, $\mathbf{g}\left(e_{0}, e_{0}\right)=-1$ and $\mathbf{g}\left(e_{0}, X\right)=0$ for all $\Sigma_{t}$-tangent vectorfields $X$ ), the "spatial" frame $\left\{e_{I}\right\}_{I=1, \cdots, \mathfrak{D}}$ is $\Sigma_{t}$-tangent and normalized by

$$
\begin{equation*}
g\left(e_{I}, e_{J}\right)=\delta_{I J}, \quad \delta_{I J}=\text { the Kronecker delta } \tag{2.3}
\end{equation*}
$$

and the spatially-globally defined (see Sect.1.11) scalar functions $\left\{e_{I}^{i}\right\}_{i=1, \cdots, \mathfrak{D}}$ in 2.2) are the components of $e_{I}$ relative to the transported spatial coordinates. Just below, we will describe how we construct the
spatial frame. We let $\left\{\omega^{I}\right\}_{I=1, \cdots, \mathfrak{D}}$ denote the corresponding $\Sigma_{t}$-tangent one-forms that are a co-frame for the spatial frame $\left\{e_{I}\right\}_{I=1, \cdots, \mathfrak{D}}$, defined by

$$
\begin{equation*}
\omega^{I}\left(e_{J}\right)=\delta_{J}^{I} \tag{2.4}
\end{equation*}
$$

where $\delta_{J}^{I}$ is the Kronecker delta. Note that $\omega^{I}=\omega_{a}^{I} d x^{a}$, where the spatially-globally defined scalar functions $\left\{\omega_{i}^{I}\right\}_{i=1, \cdots, \mathfrak{D}}$ are the components of $\omega^{I}$ relative to the transported spatial coordinates. Thus, we have

$$
\begin{equation*}
\omega_{a}^{I} e_{J}^{a}=\delta_{J}^{I}, \quad \quad \omega_{j}^{A} e_{A}^{i}=\delta_{j}^{i}, \quad I, J, i, j=1, \cdots, \mathfrak{D} \tag{2.5}
\end{equation*}
$$

Moreover, since $\left\{e_{I}\right\}_{I=1, \cdots, \mathfrak{D}}$ is orthonormal, $\omega^{I}$ is in fact the $g$-dual of $e_{I}$, that is,

$$
\begin{equation*}
\omega_{i}^{I}=g_{i a} e_{I}^{a}, \quad I, i=1, \cdots, \mathfrak{D} \tag{2.6}
\end{equation*}
$$

We also note that from (2.5) and the relation $e_{I}=e_{I}^{c} \partial_{c}$, it follows that

$$
\begin{equation*}
\partial_{i}=\omega_{i}^{C} e_{C}, \quad i=1, \cdots, \mathfrak{D} \tag{2.7}
\end{equation*}
$$

We now describe our construction of a spatial frame. There is freedom in the construction; see Remark 1.7 . In Sect.5.11, we use the Gram-Schmidt process to construct an initial orthonormal spatial frame on $\Sigma_{1}$ that is suitable for proving our main results. Given this frame on $\Sigma_{1}$, we propagate it to slabs of the form $(T, 1] \times \mathbb{T}^{\mathfrak{D}}$ by solving the propagation equations

$$
\begin{equation*}
\mathbf{D}_{e_{0}} e_{I}=n^{-1}\left(e_{I} n\right) e_{0} \tag{2.8}
\end{equation*}
$$

where $\mathbf{D}$ is the Levi-Civita connection of $\mathbf{g}$.
From equation 2.8), it follows that the scalar functions $\left\{e_{I}^{i}\right\}_{I, i=1, \ldots, \mathfrak{D}}$ satisfy a system of transport equations; see 2.23a). It is straightforward to check (for example, with the help of equation (2.14)) that if $\mathbf{g}$ is $C^{1}$ on $(T, 1] \times \mathbb{T}^{\mathfrak{D}}$ and the initial spatial frame on $\Sigma_{1}$ is orthonormal and $C^{1}$, then the frame $\left\{e_{I}\right\}_{I=1, \cdots, \mathcal{D}}$ obtained by propagating the initial frame via the transport equations 2.8 is orthonormal and tangent to $\Sigma_{t}$ for $t \in(T, 1]$. In particular, we have

$$
\begin{equation*}
\mathbf{g}\left(e_{\alpha}, e_{\beta}\right)=\mathbf{m}_{\alpha \beta}, \quad \alpha, \beta=0,1, \cdots, \mathfrak{D} \tag{2.9}
\end{equation*}
$$

where $\mathbf{m}_{\alpha \beta}:=\operatorname{diag}(-1,1, \cdots, 1)$, and

$$
\begin{equation*}
e_{I} t=0, \quad I=1, \cdots, \mathfrak{D} \tag{2.10}
\end{equation*}
$$

Moreover, relative to the orthonormal frame $\left\{e_{\alpha}\right\}_{\alpha=0,1, \cdots, \mathfrak{N}}$, with $\mathbf{m}^{\mu \nu}:=\operatorname{diag}(-1,1, \cdots, 1)$ and $\delta^{I J}$ the Kronecker delta, we have

$$
\begin{equation*}
\mathbf{g}^{-1}=\mathbf{m}^{\gamma \delta} e_{\gamma} \otimes e_{\delta}, \quad g^{-1}=\delta^{C D} e_{C} \otimes e_{D} \tag{2.11}
\end{equation*}
$$

In addition, differentiating 2.9, we find that

$$
\begin{equation*}
\mathbf{g}\left(\mathbf{D}_{e_{\alpha}} e_{\beta}, e_{\gamma}\right)=-\mathbf{g}\left(e_{\beta}, \mathbf{D}_{e_{\alpha}} e_{\gamma}\right) \tag{2.12}
\end{equation*}
$$

which in particular implies that

$$
\begin{equation*}
\mathbf{g}\left(\mathbf{D}_{e_{\alpha}} e_{\beta}, e_{\gamma}\right)=0, \quad \text { if } \beta=\gamma \tag{2.13}
\end{equation*}
$$

We also note the following identity, which is straightforward to verify using the form (2.1) of the metric:

$$
\begin{equation*}
\mathbf{D}_{e_{0}} e_{0}=n^{-1}\left(e_{C} n\right) e_{C} \tag{2.14}
\end{equation*}
$$

Remark 2.1 (Fermi-Walker transport). When the frame initial data are $\Sigma_{1}$-tangent, equation 2.8 is equivalent to the well-known Fermi-Walker transport equation for $e_{I}$ along the integral curves of $e_{0}$ (which, up to re-parametrization, are the same as the integral curves of $\partial_{t}$ ). We remark that the "standard" FermiWalker transport equation is $\mathbf{D}_{e_{0}} e_{I}=n^{-1}\left(e_{I} n\right) e_{0}-\mathbf{g}\left(e_{I}, e_{0}\right) n^{-1}\left(e_{C} n\right) e_{C}$, and that we have omitted the term $-\mathbf{g}\left(e_{I}, e_{0}\right) n^{-1}\left(e_{C} n\right) e_{C}$ from RHS (2.8). This term vanishes in the present context because our frame initial data will verify $\left.\mathbf{g}\left(e_{I}, e_{0}\right)\right|_{\Sigma_{1}}=0$ and $\left.\mathbf{g}\left(e_{I}, e_{J}\right)\right|_{\Sigma_{1}}=\delta_{I J}$, and these orthogonality conditions are propagated by solutions to equation 2.8.

### 2.1.3 The second fundamental form, the CMC condition, and the connection coefficients

Relative to the transported spatial coordinates $\left\{x^{i}\right\}_{i=1, \cdots, \mathfrak{D}}$ on $\Sigma_{t}$, the second fundamental form $k$ of $\Sigma_{t}$ is the $\Sigma_{t}$-tangent tensorfield with components $k_{i j}:=-\mathbf{g}\left(\mathbf{D}_{\partial_{i}} e_{0}, \partial_{j}\right)$. Hence, the components of the second fundamental form with respect to the frame $\left\{e_{I}\right\}_{I=1, \cdots, \mathfrak{D}}$ are

$$
\begin{equation*}
k_{I J}=-\mathbf{g}\left(\mathbf{D}_{e_{I}} e_{0}, e_{J}\right)=k_{J I}, \tag{2.15}
\end{equation*}
$$

where the symmetry property $k_{I J}=k_{J I}$ is a well-known consequence of the torsion-free property of $\mathbf{D}$ and the fact that the commutators $\left[e_{I}, e_{J}\right]$ are $\Sigma_{t}$-tangent (and thus orthogonal to $e_{0}$ ). Note that (2.9), 2.13), and (2.15) imply that

$$
\begin{equation*}
\mathbf{D}_{e_{I}} e_{0}=-k_{I C} e_{C} \tag{2.16}
\end{equation*}
$$

We now normalize the time function $t$ according to the CMC condition

$$
\begin{equation*}
\operatorname{tr} k:=k_{a}^{a}:=\left(g^{-1}\right)^{a b} k_{a b}=k_{A A}=-\frac{1}{t} . \tag{2.17}
\end{equation*}
$$

It is well-known that 2.17 leads to an elliptic equation for the lapse $n$ (see 2.25 ), which means in particular that our gauge involves an infinite speed of propagation.

In our analysis, we also study the spatial connection coefficients of the frame $\left\{e_{I}\right\}_{I=1, \cdots, \mathfrak{D}}$, which are defined by:

$$
\begin{equation*}
\gamma_{I J B}:=\mathbf{g}\left(\mathbf{D}_{e_{I}} e_{J}, e_{B}\right)=g\left(\nabla_{e_{I}} e_{J}, e_{B}\right) \tag{2.18}
\end{equation*}
$$

In (2.18) and throughout, $\nabla$ denotes the Levi-Civita connection of $g$. Note that (2.9), 2.12), 2.15, and (2.18) imply that

$$
\begin{equation*}
\mathbf{D}_{e_{I}} e_{J}=-k_{I J} e_{0}+\gamma_{I J C} e_{C}, \quad \nabla_{e_{I}} e_{J}=\gamma_{I J C} e_{C} \tag{2.19}
\end{equation*}
$$

Finally, by differentiating the relation $\mathbf{g}\left(e_{J}, e_{B}\right)=\delta_{J B}$ with $\mathbf{D}_{e_{I}}$, we deduce the antisymmetry property

$$
\begin{equation*}
\gamma_{I J B}=-\gamma_{I B J} \tag{2.20}
\end{equation*}
$$

### 2.1.4 Curvature tensors

Our sign conventions for the Riemann curvature Riem of $\mathbf{g}$, the Ricci curvature Ric of $\mathbf{g}$, and the scalar curvature $\mathbf{R}$ of $\mathbf{g}$, are as follows relative to the orthonormal frame $\left\{e_{\alpha}\right\}_{\alpha=0,1, \cdots, \mathfrak{D}}$ constructed in Sect. 2.1 .2 , where $\mathbf{m}^{\alpha \beta}$ is as in 2.11):

$$
\begin{align*}
\mathbf{g}\left(\mathbf{D}_{e_{\alpha} e_{\beta}}^{2} e_{\nu}-\mathbf{D}_{e_{\beta} e_{\alpha}}^{2} e_{\nu}, e_{\mu}\right) & :=\mathbf{R i e m}\left(e_{\alpha}, e_{\beta}, e_{\mu}, e_{\nu}\right)  \tag{2.21a}\\
\operatorname{Ric}\left(e_{\alpha}, e_{\beta}\right) & :=\mathbf{m}^{\mu \nu} \operatorname{Riem}\left(e_{\alpha}, e_{\mu}, e_{\beta}, e_{\nu}\right)  \tag{2.21~b}\\
\mathbf{R} & :=\mathbf{m}^{\mu \nu} \operatorname{Ric}\left(e_{\mu}, e_{\nu}\right) \tag{2.21c}
\end{align*}
$$

Our sign conventions for the curvature of tensors of $g$, namely its Riemann curvature Riem, Ricci curvature Ric, and scalar curvature $R$, are analogous to the ones in 2.21a-2.21c .

### 2.1.5 The reduced equations

In the next proposition, we provide the PDEs that we use to study perturbations of generalized Kasner solutions.

Proposition 2.2 (The reduced Einstein-scalar field equations relative to CMC-transported spatial coordinates and a Fermi-Walker transported orthonormal frame). Let $T \in(0,1)$, and let $(\mathbf{g}, \psi)$ respectively be a Lorentzian metric and a scalar function on the manifold $(T, 1] \times \mathbb{T}^{\mathfrak{D}}$. Assume that $(T, 1] \times \mathbb{T}^{\mathfrak{D}}$ is equipped with a CMC time function $t$ and transported spatial coordinates $\left\{x^{i}\right\}_{i=1, \cdots, \mathfrak{D}}$ such that the level sets
$\Sigma_{t}=\{t\} \times \mathbb{T}^{\mathfrak{D}}$ are CMC hypersurfaces normalized by (2.17), as is described in Sects. 2.1.1 2.1.3 (in particular, the $\left\{x^{i}\right\}_{i=1, \cdots, \mathfrak{D}}$ are coordinates on each $\Sigma_{t}$ ). Let $n$ be the lapse, let $e_{0}=n^{-1} \partial_{t}$ be the future-directed normal to $\Sigma_{t}$, let $\left\{e_{I}\right\}_{I=1, \cdots, \mathfrak{D}}$ be the $\Sigma_{t}$-tangent orthonormal spatial frame described in Sects. 2.1.1 2.1.3, and let $\left\{\omega^{I}\right\}_{I=1, \cdots, \mathfrak{D}}$ be the corresponding $\Sigma_{t}$-tangent orthonormal spatial co-frame. Let $\left\{e_{I}^{i}\right\}_{i=1, \cdots, \mathfrak{D}}$ denote the components of $e_{I}$ with respect to the transported spatial coordinates, and similarly for $\left\{\boldsymbol{\omega}_{i}^{I}\right\}_{i=1, \cdots, \mathfrak{D}}$. Let $\left\{k_{I J}\right\}_{I, J=1, \cdots, \mathfrak{D}}$ denote the components of the second fundamental form of $\Sigma_{t}$ with respect to the spatial frame, and let $\left\{\gamma_{I J B}\right\}_{I, J=1, \cdots, B}$ denote the connection coefficients of $\left\{e_{I}\right\}_{I=1, \cdots, \mathfrak{D}}$, as described in Sect. 2.1.3. Then the scalar functions $k_{I J}, \gamma_{I J B}, e_{I}^{i}, \omega_{i}^{I}, n, \psi, I, J, B, i=1, \cdots, \mathfrak{D}$, satisfy al ${ }^{32}$ of the following" "reduced" equations on $(T, 1] \times \mathbb{T}^{\mathfrak{D}}$ if and only if $(\mathbf{g}, \psi)$ are solutions to the Einstein-scalar field equations (1.1a)-1.1b, where the spacetime metric can be expressed in terms of the reduced variables via the formulas $\mathbf{g}=-n^{2} d t \otimes d t+g_{a b} d x^{a} \otimes d x^{b}$ and $g_{i j}=g\left(\partial_{i}, \partial_{j}\right)=\omega_{i}^{A} \omega_{j}^{A}$ :

## Evolution equations for the second fundamental form and connection coefficient components

$$
\begin{align*}
e_{0} k_{I J}= & -n^{-1} e_{I} e_{J} n+e_{C} \gamma_{I J C}-e_{I} \gamma_{C J C}-\frac{1}{t} k_{I J}  \tag{2.22a}\\
& +n^{-1} \gamma_{I J C} e_{C} n-\gamma_{D I C} \gamma_{C J D}-\gamma_{D D C} \gamma_{I J C}-\left(e_{I} \psi\right) e_{J} \psi \\
e_{0} \gamma_{I J B}= & e_{B} k_{I J}-e_{J} k_{B I} \\
& -k_{I C} \gamma_{B J C}-k_{C J} \gamma_{B I C}+k_{I C} \gamma_{J B C}+k_{B C} \gamma_{J I C}+k_{I C} \gamma_{C J B}  \tag{2.22~b}\\
& +n^{-1}\left(e_{B} n\right) k_{I J}-n^{-1}\left(e_{J} n\right) k_{B I} .
\end{align*}
$$

Evolution equations for the frame components and co-frame components

$$
\begin{align*}
e_{0} e_{I}^{i} & =k_{I C} e_{C}^{i}  \tag{2.23a}\\
e_{0} \boldsymbol{\omega}_{i}^{I} & =-k_{I C} \omega_{i}^{C} \tag{2.23b}
\end{align*}
$$

Wave equation for the scalar field

$$
\begin{equation*}
e_{0} e_{0} \psi=e_{C} e_{C} \psi-\frac{1}{t} e_{0} \psi+n^{-1}\left(e_{C} n\right) e_{C} \psi-\gamma_{C C D} e_{D} \psi \tag{2.24}
\end{equation*}
$$

## Elliptic lapse equation

$$
\begin{align*}
e_{C} e_{C}(n-1)-t^{-2}(n-1)= & \gamma_{C C D} e_{D}(n-1)+2 n e_{C} \gamma_{D D C}  \tag{2.25}\\
& -n\left\{\gamma_{C D E} \gamma_{E D C}+\gamma_{C C D} \gamma_{E E D}+\left(e_{C} \psi\right) e_{C} \psi\right\} .
\end{align*}
$$

Hamiltonian and momentum constraint equations

$$
\begin{align*}
2 e_{C} \gamma_{D D C}-\gamma_{C D E} \gamma_{E D C}-\gamma_{C C D} \gamma_{E E D}-k_{C D} k_{C D}+t^{-2} & =\left(e_{0} \psi\right)^{2}+\left(e_{C} \psi\right) e_{C} \psi  \tag{2.26a}\\
e_{C} k_{C I} & =\gamma_{C C D} k_{I D}+\gamma_{C I D} k_{C D}-\left(e_{0} \psi\right) e_{I} \psi \tag{2.26b}
\end{align*}
$$

Finally, we also have the following formula:

$$
\begin{equation*}
\gamma_{I J B}=\frac{1}{2}\left\{\omega_{c}^{B}\left(e_{I} e_{J}^{c}-e_{J} e_{I}^{c}\right)-\omega_{c}^{I}\left(e_{J} e_{B}^{c}-e_{B} e_{J}^{c}\right)+\omega_{c}^{J}\left(e_{B} e_{I}^{c}-e_{I} e_{B}^{c}\right)\right\} . \tag{2.27}
\end{equation*}
$$

Proof. We will prove in detail that solutions to the Einstein-scalar field equations $1.1 \mathrm{a}-1.1 \mathrm{~b}$ yield solutions to the reduced equations stated in Proposition 2.2. The converse can be proved by similar arguments, and we will omit those details.

Proof of 2.22a): We first use 2.8, 2.9, 2.14, and 2.19 to compute the following identity:

$$
\begin{equation*}
\boldsymbol{\operatorname { R i e m }}\left(e_{0}, e_{I}, e_{0}, e_{J}\right)=\mathbf{g}\left(\left(\mathbf{D}_{e_{0} e_{I}}^{2}-\mathbf{D}_{e_{I} e_{0}}^{2}\right) e_{J}, e_{0}\right)=e_{0} k_{I J}-k_{I C} k_{C J}+n^{-1} \nabla_{e_{I} e_{J}}^{2} n \tag{2.28}
\end{equation*}
$$

[^18]We then use Gauss' equation, namely

$$
\begin{equation*}
\operatorname{Riem}\left(e_{C}, e_{I}, e_{D}, e_{J}\right)=\operatorname{Riem}\left(e_{C}, e_{I}, e_{D}, e_{J}\right)+k_{C D} k_{I J}-k_{C J} k_{I D} \tag{2.29}
\end{equation*}
$$

Einstein's field equations 1.1a, and 2.21b to rewrite LHS 2.28) as follows:

$$
\begin{align*}
\operatorname{Riem}\left(e_{0}, e_{I}, e_{0}, e_{J}\right) & =-\mathbf{R i c}\left(e_{I}, e_{J}\right)+\boldsymbol{\operatorname { R i e m }}\left(e_{C}, e_{I}, e_{C}, e_{J}\right) \\
& =-\left(e_{I} \psi\right) e_{J} \psi+\operatorname{Ric}\left(e_{I}, e_{J}\right)+\operatorname{tr} k k_{I J}-k_{I C} k_{J C} \tag{2.30}
\end{align*}
$$

Next, we compute that the frame components of the Ricci tensor of $g$ can be expressed as follows:

$$
\begin{align*}
\operatorname{Ric}\left(e_{I}, e_{J}\right) & =\operatorname{Riem}\left(e_{C}, e_{I}, e_{C}, e_{J}\right)=g\left(\left(\nabla_{e_{C} e_{I}}^{2}-\nabla_{e_{I} e_{C}}^{2}\right) e_{J}, e_{C}\right) \\
& =e_{C} \gamma_{I J C}-e_{I} \gamma_{C J C}-\gamma_{C I D} \gamma_{D J C}-\gamma_{C C D} \gamma_{I J D}+\underbrace{\gamma_{I C D} \gamma_{D J C}+\gamma_{I C D} \gamma_{D C J}}_{0}, \tag{2.31}
\end{align*}
$$

where we notice that the last two products $\gamma_{I C D} \gamma_{D J C}+\gamma_{I C D} \gamma_{D C J}$ cancel, due to the antisymmetry property (2.20). Next, we use 2.18 to deduce the following identity for the factor $\nabla_{e_{I} e_{J}}^{2} n$ on RHS 2.28):

$$
\begin{equation*}
\nabla_{e_{I} e_{J}}^{2} n=e_{I} e_{J} n-\gamma_{I J C} e_{C} n \tag{2.32}
\end{equation*}
$$

The evolution equation $(2.22 \mathrm{a})$ for $k_{I J}$ now follows from combining $(2.28)-(2.32)$ and using the CMC condition (2.17).

Proof of 2.22b): First, we take the $e_{0}$ derivative of 2.18 and use (2.8), 2.9), 2.14 , 2.16), 2.19), the symmetries of the curvature tensor, and the Codazzi equations, namely

$$
\begin{equation*}
(\nabla k)_{I J B}-(\nabla k)_{J I B}=\boldsymbol{\operatorname { R i e m }}\left(e_{I}, e_{J}, e_{0}, e_{B}\right) \tag{2.33}
\end{equation*}
$$

(where throughout this proof, $\nabla k$ denotes the type $\binom{0}{3} \Sigma_{t}$-tangent tensorfield with coordinate components $\left.(\nabla k)_{a b c}=\nabla_{a} k_{b c}\right)$, to compute

$$
\begin{align*}
e_{0} \gamma_{I J B}= & \mathbf{g}\left(\mathbf{D}_{e_{0} e_{I}}^{2} e_{J}, e_{B}\right)+\mathbf{g}\left(\left(\mathbf{D}_{e_{0}} e_{I}^{\alpha}\right) \mathbf{D}_{\alpha} e_{J}, e_{B}\right)+\mathbf{g}\left(\mathbf{D}_{e_{I}} e_{J}, \mathbf{D}_{e_{0}} e_{B}\right) \\
= & \mathbf{R i e m}\left(e_{0}, e_{I}, e_{B}, e_{J}\right)+\mathbf{g}\left(\mathbf{D}_{e_{I} e_{0}}^{2} e_{J}, e_{B}\right)+\mathbf{g}\left(\left(\mathbf{D}_{e_{0}} e_{I}^{\alpha}\right) \mathbf{D}_{\alpha} e_{J}, e_{B}\right)+\mathbf{g}\left(\mathbf{D}_{e_{I}} e_{J}, \mathbf{D}_{e_{0}} e_{B}\right) \\
= & \mathbf{R i e m}\left(e_{0}, e_{I}, e_{B}, e_{J}\right)+\mathbf{g}\left(\mathbf{D}_{e_{I}}\left(\mathbf{D}_{e_{0}} e_{J}\right), e_{B}\right) \\
& +\mathbf{g}\left(\left(\mathbf{D}_{e_{0}} e_{I}^{\alpha}\right) \mathbf{D}_{\alpha} e_{J}, e_{B}\right)-\mathbf{g}\left(\left(\mathbf{D}_{e_{I}} e_{0}^{\alpha}\right) \mathbf{D}_{\alpha} e_{J}, e_{B}\right)+\mathbf{g}\left(\mathbf{D}_{e_{I}} e_{J}, \mathbf{D}_{e_{0}} e_{B}\right) \\
= & \mathbf{R i e m}\left(e_{B}, e_{J}, e_{0}, e_{I}\right)+k_{I C} \gamma_{C J B}-n^{-1}\left(e_{J} n\right) k_{I B}+n^{-1}\left(e_{B} n\right) k_{I J}  \tag{2.34}\\
= & (\nabla k)_{B J I}-(\nabla k)_{J B I}+k_{I C} \gamma_{C J B}-n^{-1}\left(e_{J} n\right) k_{I B}+n^{-1}\left(e_{B} n\right) k_{I J} \\
= & e_{B} k_{I J}-e_{J} k_{I B}-k_{C J} \gamma_{B I C}-k_{I C} \gamma_{B J C}+k_{C B} \gamma_{J I C}+k_{I C} \gamma_{J B C} \\
& +k_{I C} \gamma_{C J B}-n^{-1}\left(e_{J} n\right) k_{I B}+n^{-1}\left(e_{B} n\right) k_{I J},
\end{align*}
$$

which yields the desired evolution equation.
Proofs of $2.23 \mathrm{a}-(2.23 \mathrm{~b})$ : First, we use $2.2,2.2 .8,2.16$, and the torsion-free property of the connection D to compute the following identity:

$$
\begin{equation*}
\left(\partial_{t} e_{I}^{c}\right) \partial_{c}=\left[\partial_{t}, e_{I}^{c} \partial_{c}\right]=\left[\partial_{t}, e_{I}\right]=\mathbf{D}_{\partial_{t}} e_{I}-\mathbf{D}_{e_{I}}\left(n e_{0}\right)=n k_{I C} e_{C}=n k_{I C} e_{C}^{c} \partial_{c} \tag{2.35}
\end{equation*}
$$

Considering the $i$ component of 2.35 relative to the transported spatial coordinates and again using 2.2), we arrive at the desired transport equation 2.23a. Using the relations 2.5, we also deduce 2.23 b as a consequence of 2.23 a .
Proof of 2.25: We simply take the $I J$-trace of equation 2.22a and use the CMC condition 2.17 and the antisymmetry property 2.20 .
Proof of 2.26 a : We first note that for solutions to the Einstein-scalar field equations (1.1a)-1.1b), the Hamiltonian constraint 1.2 a holds along all constant-time hypersurfaces, that is: $R-|k|^{2}+(\operatorname{tr} k)^{2}=$ $\left(e_{0} \psi\right)^{2}+|\nabla \psi|^{2}$. We refer to the discussion surrounding [64, Equation (10.2.30)] for a proof of this standard
fact in the context of the Einstein-vacuum equations, and we note that the arguments given there can be modified in a straightforward fashion to apply to the Einstein-scalar field equations. We also note that the sign convention for $k$ used in 64 is the opposite of the one we use here. From this equation, (2.11), (2.31), the identity $R=\operatorname{Ric}\left(e_{C}, e_{C}\right)$, the antisymmetry property 2.20 , and the CMC condition (2.17), we arrive at 2.26 a .

Proof of 2.26 b : We first note that for solutions to the Einstein-scalar field equations 1.1a)-1.1b), the momentum constraint 1.2 b$)$ holds along all constant-time hypersurfaces, that is: $(\nabla k)_{C I C}-(\nabla k)_{I C C}=$ $-\left(e_{0} \psi\right) e_{I} \psi$. We refer to the discussion surrounding [64, Equation (10.2.28)] for a proof of this standard fact in the context of the Einstein-vacuum equations, and we note that the arguments given there can be modified in a straightforward fashion to apply to the Einstein-scalar field equations. We again note that the sign convention for $k$ used in [64] is the opposite of the one we use here. From this identity and the CMC condition (2.17), we find that $(\nabla k)_{C I C}=-\left(e_{0} \psi\right) e_{I} \psi$. Next, using (2.11), 2.18), and the Leibniz rule for covariant differentiation, we find that $(\nabla k)_{C I C}=e_{C} k_{I C}-\gamma_{C I D} k_{D C}-\gamma_{C C D} k_{I D}$. Combining the above equations, we arrive at 2.26 b .
Proof of 2.27): This identity follows from the Koszul formula

$$
\begin{equation*}
\gamma_{I J B}=\frac{1}{2}\left\{g\left(\left[e_{I}, e_{J}\right], e_{B}\right)-g\left(\left[e_{J}, e_{B}\right], e_{I}\right)+g\left(\left[e_{B}, e_{I}\right], e_{J}\right)\right\} \tag{2.36}
\end{equation*}
$$

and the identity $\left[e_{I}, e_{J}\right]=\omega_{l}^{C}\left(e_{I} e_{J}^{l}-e_{J} e_{I}^{l}\right) e_{C}$.
Proof of 2.24): We first note that 1.1 b and 2.11) imply that $-e_{0} e_{0} \psi+e_{C} e_{C} \psi=-\left(\mathbf{D}_{e_{0}} e_{0}\right)^{\alpha} \mathbf{D}_{\alpha} \psi+$ $\left(\mathbf{D}_{e_{C}} e_{C}\right)^{\alpha} \mathbf{D}_{\alpha} \psi$. From this equation, 2.14, and 2.19), we deduce that $e_{0} e_{0} \psi=e_{C} e_{C} \psi+k_{C C} e_{0} \psi+$ $n^{-1}\left(e_{C} n\right) e_{C} \psi-\gamma_{C C D} e_{D} \psi$. From this equation and the CMC condition $k_{C C}=k_{a}^{a}=-\frac{1}{t}$, we arrive at (2.24).

### 2.2 Polarized $U(1)$-symmetry

As we discussed in Sect. 1.4 .2 , polarized $U(1)$-symmetric initial data on $\mathbb{T}^{3}$ for the Einstein-vacuum equations are such that all coordinate components of $\stackrel{\circ}{g}$ and $\stackrel{\circ}{k}$ are independent of $x^{3}$ and such that $\stackrel{\circ}{g}_{13}=\stackrel{\circ}{g}_{23}=\stackrel{\circ}{k}_{13}=$ $\grave{k}_{23} \equiv 0$; see also the discussion in [37, Section 2].

### 2.2.1 Propagation of symmetry

In this section, we show that for polarized $U(1)$-symmetric initial data for the Einstein-vacuum equations, the corresponding solution to the equations of Proposition 2.2 is such that all solution variables are independent of $x^{3}$ and such that $g_{13}=g_{23}=k_{13}=k_{23} \equiv 0$. Note that this implies that $\left(g^{-1}\right)^{13}=\left(g^{-1}\right)^{23}=k_{3}^{1}=k_{1}^{3}=$ $k^{2}{ }_{3}=k^{3}{ }_{2}=k^{13}=k^{23} \equiv 0$. In particular, relative to CMC-transported spatial coordinates, the corresponding spacetime metric takes the form

$$
\begin{equation*}
\mathbf{g}=-n^{2} d t \otimes d t+\sum_{c, d=1,2} g_{c d} d x^{c} \otimes d x^{d}+g_{33} d x^{3} \otimes d x^{3} \quad n=\left[-\left(\mathbf{g}^{-1}\right)^{\alpha \beta} \partial_{\alpha} t \partial_{\beta} t\right]^{-\frac{1}{2}} \tag{2.37}
\end{equation*}
$$

We provide the main propagation-of-symmetry result in the next lemma, namely Lemma 2.3 The result is standard, so we only provide the main steps in the proof. We refer readers to [15, Chapter XVI.3] for an alternate approach to propagating the polarized $U(1)$-symmetry via a wave map-type reduction of Einstein's equations. Although the approach of [15, Chapter XVI.3] is superior from a geometric point of view, Lemma 2.3 is better adapted to the setup of the present paper because it directly refers to the CMC-transported spatial coordinates, thus allowing us to avoid working with multiple gauges.

Lemma 2.3 (Propagation of polarized $U(1)$-symmetry). Let $T \in(0,1)$, and let $\mathbf{g}$ be a solution to the Einstein-vacuum equations (i.e., 1.1 a with $\psi \equiv 0$ ) on the manifold $(T, 1] \times \mathbb{T}^{3}$. Assume that $(T, 1] \times \mathbb{T}^{3}$ is equipped with a CMC time function $t$ and transported spatial coordinates $\left\{x^{i}\right\}_{i=1,2,3}$ such that the level sets $\Sigma_{t}=\{t\} \times \mathbb{T}^{3}$ are CMC hypersurfaces normalized by 2.17), as is described in Sects. 2.1.1 2.1.3 (in particular, the $\left\{x^{i}\right\}_{i=1,2,3}$ are coordinates on each $\left.\Sigma_{t}\right)$. Assume that relative to the coordinates $\left(t, x^{1}, x^{2}, x^{3}\right)$, the components of $\mathbf{g}$ belong to $C^{2}\left((T, 1] \times \mathbb{T}^{3}\right)$. Let $(\stackrel{\circ}{g}, \stackrel{\circ}{k})$ be the corresponding data on $\Sigma_{1}$, i.e., the first and
second fundamental form of $\Sigma_{1}$. Assume that the data are polarized and $U(1)$-symmetric. More precisely, assume that all coordinate components of $\stackrel{\circ}{g}$ and $\stackrel{\circ}{k}$ are independent of $x^{3}$ and that $\stackrel{\circ}{g}_{13}=\stackrel{\circ}{g}_{23}=\stackrel{\circ}{k}_{13}=\stackrel{\circ}{k}_{23} \equiv 0$. Then relative to the CMC-transported spatial coordinates, $\mathbf{g}$ has the polarized form 2.37), and the lapse n, the coordinate components $\left\{g_{i j}\right\}_{i, j=1,2,3}$, and the coordinate components $\left\{k_{i j}\right\}_{i, j=1,2,3}$ of the second fundamental form of $\Sigma_{t}$ are all independent of $x^{3}$.

Proof. We divide the proof into the following two steps:
Step 1. Propagation of $U(1)$-symmetry. In Sect. 5 , we derive a priori estimates for the system of equations in Proposition 2.2 by commuting the equations with sufficiently many spatial derivatives. Here, to propagate the $U(1)$-symmetry, we consider the case in which the equations are commuted with only a single spatial derivative $\partial_{3}$. More precisely, we consider the equations (5.7), 5.27a, 5.27b, 5.31a), (5.31b), and (5.31c) with $\partial^{t}:=\partial_{3}$ and $P:=0$. It is straightforward to see that these commuted equations form a linear PDE system in the "unknowns" $\mathcal{U}_{3}:=\left\{\partial_{3} n\right\} \cup \mathcal{V}_{3}, \mathcal{V}_{3}:=\left\{\partial_{3} e_{I}^{i}, \partial_{3} \omega_{i}^{I}, \partial_{3} k_{I J}, \partial_{3} \gamma_{I J B}\right\}_{I, J, B, i=1,2,3}$ such that, because we are assuming the existence of a classical $C^{2}$ solution, all coefficients in front of the principal PDE terms are $C^{1}$, all inhomogeneous terms are continuous, and all inhomogeneous terms are precisely linear in $\mathcal{U}_{3}$. Hence, for $t \in(T, 1]$, we can use arguments similar to the ones we use in Sect. 5 to derive the elliptic estimate $\left\|\partial_{3} n\right\|_{L^{2}\left(\Sigma_{t}\right)} \lesssim\left\|\mathcal{V}_{3}\right\|_{L^{2}\left(\Sigma_{t}\right)}$ and the following energy estimate:

$$
\begin{equation*}
\left\|\mathcal{V}_{3}\right\|_{L^{2}\left(\Sigma_{t}\right)} \lesssim\left\|\mathcal{V}_{3}\right\|_{L^{2}\left(\Sigma_{1}\right)}+\int_{s=t}^{1}\left\|\mathcal{V}_{3}\right\|_{L^{2}\left(\Sigma_{s}\right)} d s \tag{2.38}
\end{equation*}
$$

We clarify that on RHS 2.38, the implicit constants depend on the $C^{2}$ norm of the classical solution and thus are allowed to grow as $t \downarrow T$. However, that is not relevant for the proof of the lemma. Next, we note that by assumption, the initial data (on $\Sigma_{1}$ ) of $\mathcal{V}_{3}$ are trivial, i.e., $\left\|\mathcal{V}_{3}\right\|_{L^{2}\left(\Sigma_{1}\right)}=0$. Hence, applying Grönwall's lemma to 2.38 , we conclude that $\left\|\mathcal{V}_{3}\right\|_{L^{2}\left(\Sigma_{t}\right)}=0$ ( and thus $\left\|\partial_{3} n\right\|_{L^{2}\left(\Sigma_{t}\right)}=0=\left\|\mathcal{U}_{3}\right\|_{L^{2}\left(\Sigma_{t}\right)}$ too) for $t \in(T, 1]$. In total, we have shown that the $U(1)$-symmetry of the data is propagated to the entire region $(t, x) \in(T, 1] \times \mathbb{T}^{3}$ of classical existence, and in particular, $\partial_{3}$ is a $\mathbf{g}$-Killing vectorfield.

Step 2. Using the $U(1)$-symmetry to propagate the polarization. We now explain how to propagate the vanishing of $g_{13}=g\left(\partial_{1}, \partial_{3}\right), g_{23}=g\left(\partial_{2}, \partial_{3}\right), k_{13}=-\mathbf{g}\left(\partial_{3}, \mathbf{D}_{\partial_{1}} e_{0}\right)$, and $k_{23}=-\mathbf{g}\left(\partial_{3}, \mathbf{D}_{\partial_{2}} e_{0}\right)$ from the initial hypersurface $\Sigma_{1}$ to the entire region of classical existence. To this end, we first derive PDEs satisfied by these variables relative to the transported spatial coordinates. We will exploit the fact, shown in Step 1, that $\partial_{3}$ is g-Killing. Specifically, the relevant PDEs for this step are the following standard "ADM equations" and momentum constraint equation (see, for example, [59, Proposition 3.1]), where in the remainder of the proof, $i, j=1,2$, and $\Gamma_{b}{ }^{a}{ }_{c}:=\frac{1}{2}\left(g^{-1}\right)^{a d}\left(\partial_{b} g_{d c}+\partial_{c} g_{b d}-\partial_{d} g_{b c}\right)$ are the Christoffel symbols of $g$ relative to the transported spatial coordinates:

$$
\begin{align*}
\partial_{t} g_{i 3} & =-2 n k_{i 3},  \tag{2.39}\\
\partial_{t} \partial_{j} g_{i 3} & =-2 n \partial_{j} k_{i 3}-2\left(\partial_{j} n\right) k_{i 3},  \tag{2.40}\\
\partial_{t} k_{i 3} & =-\nabla_{\partial_{i} \partial_{3}}^{2} n+n \operatorname{Ric}\left(\partial_{i}, \partial_{3}\right)+n \operatorname{tr} k k_{i 3}-2 n\left(g^{-1}\right)^{a b} k_{i a} k_{3 b},  \tag{2.41}\\
\left(g^{-1}\right)^{a b} \nabla_{a} k_{b 3} & =\left(g^{-1}\right)^{a b} \partial_{a} k_{b 3}-\left(g^{-1}\right)^{a b} \Gamma_{a}{ }_{a}{ }_{b} k_{c 3}-\left(g^{-1}\right)^{a b} \Gamma_{a}{ }^{c}{ }_{3} k_{b c}=0 . \tag{2.42}
\end{align*}
$$

We clarify that 2.40 follows from differentiating 2.39 with $\partial_{j}$. Next, using straightforward computations and exploiting that $\partial_{3}$ is $\mathbf{g}$-Killing, we expand the Hessian of $n$ and the spatial Ricci components as follows:

$$
\begin{align*}
-\nabla_{\partial_{i} \partial_{3}}^{2} n= & \Gamma_{i}{ }_{3} \partial_{1} n+\Gamma_{i}{ }_{3}^{2} \partial_{2} n=\frac{1}{2}\left(g^{-1}\right)^{1 a}\left(\partial_{i} g_{a 3}-\partial_{a} g_{i 3}\right) \partial_{1} n+\frac{1}{2}\left(g^{-1}\right)^{2 a}\left(\partial_{i} g_{a 3}-\partial_{a} g_{i 3}\right) \partial_{2} n, \\
= & \frac{1}{2}\left(g^{-1}\right)^{11}\left(\partial_{i} g_{13}-\partial_{1} g_{i 3}\right) \partial_{1} n+\frac{1}{2}\left(g^{-1}\right)^{12}\left(\partial_{i} g_{23}-\partial_{2} g_{i 3}\right) \partial_{1} n+\frac{1}{2}\left(g^{-1}\right)^{13}\left(\partial_{i} g_{33}\right) \partial_{1} n  \tag{2.43}\\
& +\frac{1}{2}\left(g^{-1}\right)^{21}\left(\partial_{i} g_{13}-\partial_{1} g_{i 3}\right) \partial_{2} n+\frac{1}{2}\left(g^{-1}\right)^{22}\left(\partial_{i} g_{23}-\partial_{2} g_{i 3}\right) \partial_{2} n+\frac{1}{2}\left(g^{-1}\right)^{23}\left(\partial_{i} g_{33}\right) \partial_{2} n, \\
\operatorname{Ric}\left(\partial_{i}, \partial_{3}\right)= & \operatorname{Ric}\left(\partial_{3}, \partial_{i}\right)=\partial_{a} \Gamma_{3}{ }^{a}{ }_{i}-\partial_{3} \Gamma_{a}{ }^{a}{ }_{i}+\Gamma_{i}{ }^{b} \Gamma_{b}{ }^{a}{ }_{a}-\Gamma_{i}{ }^{b}{ }_{a} \Gamma_{3}{ }^{a}{ }_{b} \\
= & \frac{1}{2} \partial_{a}\left[\left(g^{-1}\right)^{a b}\left(\partial_{i} g_{3 b}-\partial_{b} g_{i 3}\right)\right]+\frac{1}{4}\left(g^{-1}\right)^{b d}\left(g^{-1}\right)^{a c}\left(\partial_{i} g_{3 c}-\partial_{c} g_{i 3}\right) \partial_{b} g_{a d}  \tag{2.44}\\
& -\frac{1}{4}\left(g^{-1}\right)^{b c}\left(g^{-1}\right)^{a d}\left(\partial_{i} g_{a c}+\partial_{a} g_{i c}-\partial_{c} g_{i a}\right)\left(\partial_{b} g_{3 d}-\partial_{d} g_{3 b}\right) .
\end{align*}
$$

We now note that $2.39-2.42$ can be viewed as a linear PDE system in the "unknowns"

$$
\mathcal{V}:=\left(g_{13}, g_{23}, \partial_{1} g_{13}, \partial_{1} g_{23}, \partial_{2} g_{13}, \partial_{2} g_{23}, k_{13}, k_{23}\right)
$$

such that, because we are assuming the existence of a classical $C^{2}$ solution, all coefficients in front of the principal PDE terms are $C^{1}$, all inhomogeneous terms are continuous, and all inhomogeneous terms are linear in $\mathcal{V}$. To see this, we note that the components $\left(g^{-1}\right)^{13}$ and $\left(g^{-1}\right)^{23}$ of $g^{-1}$ can expressed in terms of $g_{13}$ and $g_{23}$ via the following linear algebraic identity, which follows easily from the identity $g_{a c}\left(g^{-1}\right)^{c b}=\delta_{a}^{b}$ :

$$
\binom{\left(g^{-1}\right)^{13}}{\left(g^{-1}\right)^{23}}=-\left(g^{-1}\right)^{33}\left(\begin{array}{ll}
g_{11} & g_{21}  \tag{2.45}\\
g_{12} & g_{22}
\end{array}\right)^{-1}\binom{g_{13}}{g_{23}}
$$

Note that the positive definiteness of the $3 \times 3$ matrix $\left(g_{a b}\right)_{a, b=1,2,3}$ implies the positive definiteness of $2 \times 2$ sub-block $\left(g_{a b}\right)_{a, b=1,2}$, which in turn implies the invertibility of the matrix on RHS 2.45). To finish the proof of the lemma, we will derive energy estimates showing, in particular, that $\|\mathcal{V}\|_{L^{2}\left(\Sigma_{t}\right)}=0$ for $t \in(T, 1]$. To this end, we first use $2.43-(2.44)$ and the above observations to express $2.39-2.42$ in the following form, where $\mathscr{L}(\mathcal{V})$ schematically denotes terms that are linear ${ }^{34}$ in $\mathcal{V}$ with continuous coefficients:

$$
\begin{align*}
\partial_{t} g_{i 3} & =\mathscr{L}(\mathcal{V})  \tag{2.46}\\
\partial_{t} \partial_{j} g_{i 3} & =-2 n \partial_{j} k_{i 3}+\mathscr{L}(\mathcal{V}),  \tag{2.47}\\
\partial_{t} k_{i 3} & =-\frac{1}{2} \sum_{a, b=1,2} n\left(g^{-1}\right)^{a b} \partial_{a} \partial_{b} g_{i 3}+\frac{1}{2} \sum_{a, b=1,2} n\left(g^{-1}\right)^{a b} \partial_{a} \partial_{i} g_{b 3}+\mathscr{L}(\mathcal{V}),  \tag{2.48}\\
\sum_{a, b=1,2}\left(g^{-1}\right)^{a b} \partial_{a} k_{b 3} & =\mathscr{L}(\mathcal{V}) \tag{2.49}
\end{align*}
$$

We clarify that, for example, we have used 2.45 to soak the term $\frac{1}{2} n\left(g^{-1}\right)^{13} \partial_{1} \partial_{i} g_{33}$ on RHS 2.41) (see the first term on RHS (2.44) into the term $\mathscr{L}(\mathcal{V})$ on RHS 2.48) and to soak the term $\left(g^{-1}\right)^{13} \partial_{1} k_{33}$ from 2.42) into the term $\mathscr{L}(\mathcal{V})$ on RHS (2.49).

To derive energy estimates for solutions to $2.46-2.49$, we will rely on the energy $\mathbb{E}(t) \geq 0$ defined by

$$
\begin{align*}
\mathbb{E}^{2}(t): & =\int_{\Sigma_{t}}\left\{\sum_{a, b, a, b^{\prime}=1,2} \frac{1}{4}\left(g^{-1}\right)^{b b^{\prime}}\left(g^{-1}\right)^{a a^{\prime}}\left(\partial_{b} g_{a 3}\right) \partial_{b^{\prime}} g_{a^{\prime} 3}+\sum_{a, b=1,2}\left(g^{-1}\right)^{a b} k_{a 3} k_{b 3}\right\} d x  \tag{2.50}\\
& +\int_{\Sigma_{t}} \sum_{a, b=1,2}\left(g^{-1}\right)^{a b} g_{a 3} g_{b 3} d x
\end{align*}
$$

Note that the positive definiteness of the $3 \times 3$ matrix $\left(\left(g^{-1}\right)^{a b}\right)_{a, b=1,2,3}$ implies the positive definiteness of $2 \times 2$ sub-block $\left(\left(g^{-1}\right)^{a b}\right)_{a, b=1,2}$. Hence, from definition 2.50) and the definition of $\mathcal{V}$, we find that

$$
\begin{equation*}
\mathbb{E}(t) \approx \sum_{a=1,2}\left\|g_{a 3}\right\|_{H^{1}\left(\Sigma_{t}\right)}+\sum_{a=1,2}\left\|k_{a 3}\right\|_{L^{2}\left(\Sigma_{t}\right)} \approx\|\mathcal{V}\|_{L^{2}\left(\Sigma_{t}\right)} \tag{2.51}
\end{equation*}
$$

Therefore, to show that $\mathcal{V}$ vanishes on the region of classical existence, it suffices to show that $\mathbb{E}(t)=0$ for $t \in(T, 1]$. To this end, we will show that the following estimate holds for $t \in(T, 1]$ :

$$
\begin{equation*}
\left|\frac{d}{d t} \mathbb{E}^{2}(t)\right| \lesssim \mathbb{E}^{2}(t) \tag{2.52}
\end{equation*}
$$

where on RHS 2.52 the implicit constants depend on the $C^{2}$ norm of the classical solution and thus are allowed to grow as $t \downarrow T$. As in Step 1, this is not important for the proof. Since our assumption of polarized initial data implies that $\mathbb{E}(1)=0$, it follows from 2.52 and Grönwall's lemma that $\mathbb{E}(t)=0$ for $t \in(T, 1]$, as desired. In total, we have shown that the polarization condition $\mathcal{V}=0$ is propagated from the data to the entire region $(t, x) \in(T, 1] \times \mathbb{T}^{3}$ of classical existence.

[^19]To complete the proof of the lemma, it remains for us to show 2.52 . To proceed, we differentiate 2.50 under the integral and use equations $2.46-2.48$ to substitute time derivatives with spatial derivatives, thereby arriving at the following identity, where in the rest of the proof, we freely integrate by parts from line to line, and "..." denotes error integrals that can easily be bounded in magnitude by $\lesssim \mathbb{E}^{2}(t)$ by virtue of the Cauchy-Schwarz inequality:

$$
\begin{align*}
\frac{d}{d t} \mathbb{E}^{2}(t)= & \int_{\Sigma_{t}} \sum_{a, b, a^{\prime}, b^{\prime}=1,2}\left(g^{-1}\right)^{b b^{\prime}}\left(g^{-1}\right)^{a a^{\prime}}\left(-n \partial_{b} k_{a 3}\right)\left(\partial_{b^{\prime}} g_{a^{\prime} 3}\right) d x \\
& +\int_{\Sigma_{t}} \sum_{a, b, c, d=1,2}\left(g^{-1}\right)^{a b} k_{b 3}\left(-n\left(g^{-1}\right)^{c d} \partial_{c} \partial_{d} g_{a 3}+n\left(g^{-1}\right)^{c d} \partial_{c} \partial_{a} g_{d 3}\right) d x+\cdots \\
= & (\text { upon integrating by parts on the last line of RHS } 2.53)) \\
& \int_{\Sigma_{t}} \sum_{a, b, c, d=1,2}\left[\partial_{c}\left(n\left(g^{-1}\right)^{a b}\left(g^{-1}\right)^{c d}\right)\right] k_{b 3} \partial_{d} g_{a 3} d x-\int_{\Sigma_{t}} \sum_{a, b, c, d=1,2}\left[\partial_{a}\left(n\left(g^{-1}\right)^{a b}\left(g^{-1}\right)^{c d}\right)\right] k_{b 3} \partial_{c} g_{d 3} d x \\
& -\int_{\Sigma_{t}} \sum_{a, b, c, d=1,2}\left(\left(g^{-1}\right)^{a b} \partial_{a} k_{b 3}\right) n\left(g^{-1}\right)^{c d} \partial_{c} g_{d 3} d x+\cdots \tag{2.54}
\end{align*}
$$

$=($ upon using 2.49 for substitution in the last integral on RHS 2.54 $)$

$$
\begin{equation*}
\int_{\Sigma_{t}} \sum_{a, b, c, d=1,2}\left[\partial_{c}\left(n\left(g^{-1}\right)^{a b}\left(g^{-1}\right)^{c d}\right)\right] k_{b 3} \partial_{d} g_{a 3} d x-\int_{\Sigma_{t}} \sum_{a, b, c, d=1,2}\left[\partial_{a}\left(n\left(g^{-1}\right)^{a b}\left(g^{-1}\right)^{c d}\right)\right] k_{b 3} \partial_{c} g_{d 3}+\cdots \tag{2.55}
\end{equation*}
$$

Since all terms on RHS 2.55 are bounded in magnitude by $\leq C \mathbb{E}^{2}(t)$, we have therefore shown 2.52 and finished the proof of the lemma.

### 2.2.2 The normalized Killing direction in polarized $U(1)$-symmetry in $1+3$ dimensions

In the next lemma, for polarized $U(1)$-symmetric Einstein-vacuum solutions with $\mathfrak{D}=3$ such that the transported coordinate vectorfield $\partial_{3}$ is Killing, we construct an orthonormal spatial frame $e_{1}, e_{2}, e_{3}$ such that $e_{3}$ is everywhere parallel to $\partial_{3}$ and such that $\mathcal{L}_{\partial_{3}} e_{I}=0$ for $I=1,2,3$. We use the lemma in the proof of Theorem 6.5, i.e., in the proof of our symmetric stable blowup-results.

Lemma 2.4 (Normalized Killing direction). Suppose that on $(T, 1] \times \mathbb{T}^{3}, \mathbf{g}$ is a polarized and $U(1)$-symmetric $C^{2}$ metric of the form (2.37), where $\partial_{3}$ is the hypersurface-orthogonal Killing vectorfield and the components of $\mathbf{g}$ are independent of $x^{3}$. Define $E_{3}:=\left(g_{33}\right)^{-\frac{1}{2}} \partial_{3}$, and note that $g\left(E_{3}, E_{3}\right)=1$. Let $\dot{\circ}_{1}, \dot{\circ}_{2}$ be an orthonormal pair on $\Sigma_{1}$ that is orthogonal to $\partial_{3}$ along $\Sigma_{1}$ and that respects the symmetry, that is, $\mathcal{L}_{\partial_{3}} \stackrel{\circ}{e}_{1}=\mathcal{L}_{\partial_{3}} \stackrel{\circ}{2}_{2}=0$, where $\mathcal{L}$ denotes Lie differentiation. In particular, $\left\{\dot{e}_{1}, \dot{e}_{2},\left.E_{3}\right|_{\Sigma_{1}}\right\}$ is an orthonormal frame on $\Sigma_{1}$; we refer to Sect. 5.11 for our construction of such a frame. For $(t, x) \in(T, 1] \times \mathbb{T}^{3}$, let $\left\{e_{I}\right\}_{I=1,2,3}$ be the orthonormal frame on $\Sigma_{t}$ obtained by solving the Fermi-Walker transport equations (2.8) with initial data $\left.e_{1}\right|_{\Sigma_{1}}:=\circ_{1}$, $\left.e_{2}\right|_{\Sigma_{1}}:=\stackrel{\circ}{2}_{2},\left.e_{3}\right|_{\Sigma_{1}}:=\left.E_{3}\right|_{\Sigma_{1}}$. Then on $(T, 1] \times \mathbb{T}^{3}$, we have $E_{3}=e_{3}$, and for $I=1,2,3$, we have $\mathcal{L}_{\partial_{3}} e_{I}=0$. In particular, $\left\{e_{1}, e_{2}, e_{3}=\left(g_{33}\right)^{-\frac{1}{2}} \partial_{3}\right\}$ is an orthonormal frame on $(T, 1] \times \mathbb{T}^{3}$.

Proof. We first show that $\mathbf{D}_{e_{0}} E_{3}=0$ on $(T, 1] \times \mathbb{T}^{3}$. Since $\partial_{3}$ is Killing (in particular, $E_{3} n=0$ ), this guarantees that $E_{3}$ satisfies the propagation equation 2.8 . Moreover, since $\mathbf{D}_{e_{0}} e_{3}=0$ and $e_{3}\left|\Sigma_{1}=E_{3}\right| \Sigma_{1}$, ODE uniqueness then implies that $E_{3}=e_{3}$ on $(T, 1] \times \mathbb{T}^{3}$, as is desired. To show that $\mathbf{D}_{e_{0}} E_{3}=0$, we start by noting that since $\partial_{3}$ is the vectorfield of symmetry and since $\mathbf{g}\left(e_{0}, E_{3}\right)=0$, the $e_{0}$ component of $\mathbf{D}_{e_{0}} E_{3}$ equals:

$$
\begin{equation*}
\mathbf{g}\left(\mathbf{D}_{e_{0}} E_{3}, e_{0}\right)=-\mathbf{g}\left(E_{3}, \mathbf{D}_{e_{0}} e_{0}\right) \stackrel{\sqrt{2.14}}{-}-n^{-1}\left(g_{33}\right)^{-\frac{1}{2}} \partial_{3} n=0 \tag{2.56}
\end{equation*}
$$

Similarly, using that $\mathbf{g}\left(E_{3}, E_{3}\right)=1$, we compute that $\mathbf{g}\left(\mathbf{D}_{e_{0}} E_{3}, E_{3}\right)=\frac{1}{2} e_{0}\left\{\mathbf{g}\left(E_{3}, E_{3}\right)\right\}=0$ and hence the $\partial_{3}$ component of $\mathbf{D}_{e_{0}} E_{3}$ vanishes. Thus, the desired relation $\mathbf{D}_{e_{0}} E_{3}=0$ follows from the vanishing of these two
components as well as the following identities:

$$
\begin{equation*}
\mathbf{g}\left(\mathbf{D}_{e_{0}} E_{3}, \partial_{i}\right)=n^{-1}\left(g_{33}\right)^{-\frac{1}{2}} \frac{1}{2}\left(\partial_{t} \mathbf{g}_{3 i}+\partial_{3} \mathbf{g}_{t i}-\partial_{i} \mathbf{g}_{t 3}\right)=0, \quad i=1,2 \tag{2.57}
\end{equation*}
$$

In obtaining these identities, we have used the identity $\mathbf{g}\left(\mathbf{D}_{\partial_{t}} \partial_{3}, \partial_{i}\right)=\frac{1}{2}\left(\partial_{t} \mathbf{g}_{3 i}+\partial_{3} \mathbf{g}_{t i}-\partial_{i} \mathbf{g}_{t 3}\right)$, the fact that $\partial_{3}$ is orthogonal to the elements of $\left\{\partial_{t}, \partial_{1}, \partial_{2}\right\}$, the fact that $\partial_{t}$ is orthogonal to $\Sigma_{t}$, and the fact that the components of $\mathbf{g}$ are independent of $x^{3}$.

To show that $\mathcal{L}_{\partial_{3}} e_{I}=0$ on $(T, 1] \times \mathbb{T}^{3}$, we commute equation 2.23 a (which is equivalent to equation (2.8)) with $\mathcal{L}_{\partial_{3}}$. We find that for metrics $\mathbf{g}$ satisfying the assumptions of the lemma, the scalar function array $\vec{\phi}:=\left\{\partial_{3} e_{I}^{i}\right\}_{I, i=1,2,3}$ satisfies a system of transport equations of the schematic form $\partial_{t} \vec{\phi}=F \cdot \vec{\phi}$, where $F$ is smooth on $(T, 1] \times \mathbb{T}^{3}$. Moreover, the assumptions of the lemma guarantee that $\left.\vec{\phi}\right|_{\Sigma_{1}}=0$. Hence, from ODE uniqueness, we find that $\vec{\phi} \equiv 0$ on $(T, 1] \times \mathbb{T}^{3}$. We have therefore proved the lemma.

Remark 2.5. Throughout the paper, in our analysis of polarized $U(1)$-symmetric solutions with $\mathfrak{D}=3$, we will always assume that $e_{3}$ is the $\partial_{3}$-parallel frame vectorfield constructed in Lemma 2.4 .

### 2.3 The background Kasner variables

Our main results concern perturbations of the explicit generalized Kasner solutions presented in Sect. 1.4 Straightforward computations imply that the reduced variables (see Sects.2.1.1 2.1.3) of the generalized Kasner solutions can be expressed as follows:

$$
\begin{equation*}
\widetilde{n}:=1, \quad \widetilde{e}_{I}^{i}:=t^{-\widetilde{q}_{\underline{I}}} \delta_{\underline{I}}^{i}, \quad \widetilde{\omega}_{i}^{I}:=t^{\widetilde{q}_{\underline{I}}} \delta_{i}^{\underline{I}}, \quad \widetilde{e}_{I}:=t^{-\widetilde{q}_{\underline{I}}} \delta_{\underline{I}}^{i} \partial_{i}, \quad \widetilde{k}_{I J}:=-\frac{\widetilde{q}_{\underline{I}}}{t} \delta_{\underline{I} J}, \quad \widetilde{\gamma}_{I J B}=0, \quad \widetilde{\psi}=\widetilde{B} \log t \tag{2.58}
\end{equation*}
$$

where $\delta_{I}^{i}$, $\delta_{i}^{I}$, and $\delta_{I J}$ are Kronecker deltas and we recall that repeated underlined indices are not summed.
Remark 2.6 (The components of "tilde-decorated" tensors). Note that as defined in $(2.58), \widetilde{k}_{I J}=\widetilde{k}\left(\widetilde{e}_{I}, \widetilde{e}_{J}\right) \neq$ $\widetilde{k}\left(e_{I}, e_{J}\right)$. Put differently, $\widetilde{k}_{I J}$ denotes a component of $\widetilde{k}$ relative to the background Kasner-orthonormal frame $\left\{\widetilde{e}_{I}\right\}_{I=1, \cdots, \mathfrak{D}}$, rather than the perturbed $g$-orthonormal frame $\left\{e_{I}\right\}_{I=1, \cdots, \mathfrak{D}}$. Similar remarks apply to other "tilde-decorated" tensors. That is, for tilde-decorated tensors, capital Latin indices denote components relative to the background Kasner frame or co-frame, whereas for non-tilde-decorated tensors, capital Latin indices denote components relative to the $g$-orthonormal frame or co-frame.

## 3 Norms, bootstrap assumptions, and key parameters

The proofs of our main theorems rely on a continuity argument for solutions to the reduced equations of Proposition 2.2. We make bootstrap assumptions for the size of various norms of the perturbed solution on a time interval $\left(T_{B o o t}, 1\right]$ for some $T_{B o o t} \in(0,1)$. Then, in Proposition 5.1, we derive a priori estimates for the perturbed solution that imply a strict improvement of the bootstrap assumptions on $\left(T_{B o o t}, 1\right]$; this is the difficult part of the proof. Once we have established a priori estimates, standard arguments yield that the perturbed solution exists on $(0,1] \times \mathbb{T}^{\mathcal{D}}$ and satisfies the a priori estimates on $(0,1]$; see Proposition 5.28 for the details. Based on the existence result and the a priori estimates, the proof of curvature blowup as $t \downarrow 0$ and the derivation of other interesting properties of the solution are relatively straightforward; see Sect. 6 Our bootstrap assumptions are formulated in terms of various norms of the reduced variables along the $\Sigma_{t}$ slices, with well-chosen $t$-weights. Before stating the bootstrap assumptions, we will first define the norms and the key parameters $q, \sigma, A_{*}, N, N_{0}$ that lie at the core of our framework.

### 3.1 Running assumption

In the rest of the paper, it is understood that we are studying general perturbations of a background generalized Kasner solution whose Kasner exponents verify the stability condition 1.8 , or that we are studying polarized $U(1)$-symmetric perturbations of an arbitrary vacuum Kasner solution in $1+3$-dimensions; see Sect.1.4. We will often refrain from explicitly stating this assumption.

### 3.2 Some additional differentiation notation

If $f$ is a scalar function, then $\vec{e} f:=\left\{e_{C} f\right\}_{C=1, \cdots, \mathfrak{D}}$, where $\left\{e_{C}\right\}_{C=1, \cdots, \mathfrak{D}}$ denotes the orthonormal spatial frame. Similarly, $\vec{e} k:=\left\{e_{C} k_{I J}\right\}_{C, I, J=1, \cdots, \mathfrak{D}}, \vec{e} \gamma:=\left\{e_{C} \gamma_{I J B}\right\}_{B, C, I, J=1, \cdots, \mathfrak{D}}, \vec{e} e:=\left\{e_{C} e_{I}^{i}\right\}_{C, I, i=1, \cdots, \mathfrak{D}}$, and $\vec{e} \boldsymbol{\omega}:=\left\{e_{C} \boldsymbol{\omega}_{i}^{I}\right\}_{C, I, i=1, \cdots, \mathfrak{D}}$. Note that in the above expressions, all quantities that are differentiated are scalar functions.

### 3.3 Sobolev norms of the reduced variables

For scalar functions $v$, we define its norm $\|v\|_{L^{2}\left(\Sigma_{t}\right)} \geq 0$ by

$$
\begin{equation*}
\|v\|_{L^{2}\left(\Sigma_{t}\right)}^{2}:=\int_{\Sigma_{t}} v^{2}(t, x) d x \tag{3.1}
\end{equation*}
$$

where $d x:=d x^{1} \cdots d x^{\mathfrak{D}}$ denotes the Euclidean volume form on $\Sigma_{t}$.
We also define standard $H^{M}\left(\Sigma_{t}\right), \dot{H}^{M}\left(\Sigma_{t}\right), W^{M, \infty}\left(\Sigma_{t}\right)$, and $\dot{W}^{M, \infty}\left(\Sigma_{t}\right)$ norms of scalar functions $v$ :

$$
\begin{align*}
\|v\|_{H^{M}\left(\Sigma_{t}\right)}^{2} & :=\sum_{|\iota| \leq M}\left\|\partial^{\iota} v\right\|_{L^{2}\left(\Sigma_{t}\right)}^{2}, & \|v\|_{\dot{H}^{M}\left(\Sigma_{t}\right)}^{2} & :=\sum_{|\iota|=M}\left\|\partial^{\iota} v\right\|_{L^{2}\left(\Sigma_{t}\right)}^{2},  \tag{3.2}\\
\|v\|_{W^{M, \infty}\left(\Sigma_{t}\right)} & :=\sum_{|\iota| \leq M}\left\|\partial^{\iota} v\right\|_{L^{\infty}\left(\Sigma_{t}\right)}, & \|v\|_{\dot{W}^{M, \infty}\left(\Sigma_{t}\right)} & :=\sum_{|\iota|=M}\left\|\partial^{\iota} v\right\|_{L^{\infty}\left(\Sigma_{t}\right)}, \tag{3.3}
\end{align*}
$$

where $\iota$ is a spatial multi-index, $\partial^{\iota}$ is the corresponding operator involving repeated differentiation with respect to the transported spatial coordinate vectorfields $\left\{\partial_{i}\right\}_{i=1, \cdots, \mathfrak{D}}$ (see Sect. 1.11 , and $\|v\|_{L^{\infty}\left(\Sigma_{t}\right)}:=$ ess $\sup _{x \in \mathbb{T}^{\mathfrak{D}}}|v(t, x)|$. As is standard, we write " $L^{\infty}$ " instead of " $W^{0, \infty}$."

If $v$ is a $\Sigma_{t}$-tangent tensorfield, then we define its $L^{2}\left(\Sigma_{t}\right), H^{M}\left(\Sigma_{t}\right), \dot{H}^{M}\left(\Sigma_{t}\right), W^{M, \infty}\left(\Sigma_{t}\right)$, and $\dot{W}^{M, \infty}\left(\Sigma_{t}\right)$ norms in an analogous fashion, but also summing over all "frame indices." More precisely, with the background Kasner variables $\widetilde{k}_{I J}$, etc., as defined in Sect. 2.3 (see in particular Remark 2.6), we define

$$
\begin{array}{r}
\|k-\widetilde{k}\|_{H^{M}\left(\Sigma_{t}\right)}^{2}:=\sum_{I, J=1}^{\mathfrak{D}}\left\|k_{I J}-\widetilde{k}_{I J}\right\|_{H^{M}\left(\Sigma_{t}\right)}^{2}, \quad\|\gamma-\widetilde{\gamma}\|_{H^{M}\left(\Sigma_{t}\right)}^{2}:=\sum_{I, J, B=1}^{\mathfrak{D}}\left\|\gamma_{I J B}-\widetilde{\gamma}_{I J B}\right\|_{H^{M}\left(\Sigma_{t}\right)}^{2}, \\
\|e-\widetilde{e}\|_{H^{M}\left(\Sigma_{t}\right)}^{2}:=\sum_{I, i=1}^{\mathfrak{D}}\left\|e_{I}^{i}-\widetilde{e}_{I}^{i}\right\|_{H^{M}\left(\Sigma_{t}\right)}^{2}, \quad\|\omega-\widetilde{\omega}\|_{H^{M}\left(\Sigma_{t}\right)}^{2}:=\sum_{I, i=1}^{\mathfrak{D}}\left\|\omega_{i}^{I}-\widetilde{\omega}_{i}^{I}\right\|_{H^{M}\left(\Sigma_{t}\right)}^{2},  \tag{3.4}\\
\|\vec{e} n\|_{H^{M}\left(\Sigma_{t}\right)}^{2}:=\sum_{I=1}^{\mathcal{D}}\left\|e_{I} n\right\|_{H^{M}\left(\Sigma_{t}\right)}^{2}, \quad\|\vec{e} \psi\|_{H^{M}\left(\Sigma_{t}\right)}^{2}:=\sum_{I=1}^{\mathfrak{D}}\left\|e_{I} \psi\right\|_{H^{M}\left(\Sigma_{t}\right)}^{2},
\end{array}
$$

and

$$
\begin{array}{r}
\|k-\widetilde{k}\|_{W^{M, \infty}\left(\Sigma_{t}\right)}:=\sum_{I, J=1}^{\mathcal{D}}\left\|k_{I J}-\widetilde{k}_{I J}\right\|_{W^{M, \infty}\left(\Sigma_{t}\right)}, \quad\|\gamma-\widetilde{\gamma}\|_{W^{M, \infty}\left(\Sigma_{t}\right)}:=\sum_{I, J, B=1}^{\mathcal{D}}\left\|\gamma_{I J B}-\widetilde{\gamma}_{I J B}\right\|_{W^{M, \infty}\left(\Sigma_{t}\right)}, \\
\|e-\widetilde{e}\|_{W^{M, \infty}\left(\Sigma_{t}\right)}:=\sum_{I, i=1}^{\mathcal{D}}\left\|e_{I}^{i}-\widetilde{e}_{I}^{i}\right\|_{W^{M, \infty}\left(\Sigma_{t}\right)}, \quad\|\omega-\widetilde{\omega}\|_{W^{M, \infty}\left(\Sigma_{t}\right)}:=\sum_{I, i=1}^{\mathcal{D}}\left\|\omega_{i}^{I}-\widetilde{\omega}_{i}^{I}\right\|_{W^{M, \infty}\left(\Sigma_{t}\right)}, \\
\|\vec{e} n\|_{W^{M, \infty}\left(\Sigma_{t}\right)}:=\sum_{I=1}^{\mathcal{D}}\left\|e_{I} n\right\|_{W^{M, \infty}\left(\Sigma_{t}\right)}, \quad\|\vec{e} \psi\|_{W^{M, \infty}\left(\Sigma_{t}\right)}:=\sum_{I=1}^{\mathcal{D}}\left\|e_{I} \psi\right\|_{W^{M, \infty}\left(\Sigma_{t}\right)}, \tag{3.5}
\end{array}
$$

and similarly for the homogeneous norms (such as $\left.\|k-\widetilde{k}\|_{\dot{H}^{M}\left(\Sigma_{t}\right)}^{2}:=\sum_{I, J=1}^{\mathfrak{D}}\left\|k_{I J}-\widetilde{k}_{I J}\right\|_{\dot{H}^{M}\left(\Sigma_{t}\right)}^{2}\right)$.

### 3.4 Key parameters

We will formulate the bootstrap assumptions using two key parameters, namely $\sigma, q$, which are any two fixed real numbers verifying the following inequalities:

$$
\begin{cases}0<2 \sigma<2 \sigma+\max _{\substack{1 \leq I, J, B \leq \mathfrak{D} \\ I<J<J}}\left\{\left|\widetilde{q}_{B}\right|, \widetilde{q}_{I}+\widetilde{q}_{J}-\widetilde{q}_{B}\right\}<q<1-2 \sigma, & \text { non-symmetric cases, }  \tag{3.6}\\ 0<2 \sigma<2 \sigma+\max \left\{\left|\widetilde{q}_{1}\right|,\left|\widetilde{q}_{2}\right|,\left|\widetilde{q}_{3}\right|\right\}<q<1-2 \sigma, & \text { Polarized } U(1) \text {-symmetric } 1+3 \text { vacuum. }\end{cases}
$$

The set of Kasner exponents for which such parameters $\sigma, q$ exist is non-empty and open in all the models that we consider; see Sect. 1.4.

Next, we introduce the positive-integer-valued parameters $N_{0}, N$, which, roughly speaking, represent the number of derivatives we will use to to control the solution in $L^{\infty}$ at the low orders (i.e., derivative levels approximately equal to $N_{0}$ ) and in $L^{2}$ at the top-orders (i.e., derivative levels approximately equal to $N$ ); we refer to Remark 3.2 for an important remark about the precise number of low order derivatives that we use in our proof. Our choice of $N_{0}$ and $N$ will be related to another parameter, $A_{*}$, which controls the strength of the $t$-weights (which will be of order $t^{A_{*}}$ ) that we use in our high order energies. For our bootstrap argument to close, the parameters will have to satisfy the following inequalities:

$$
\begin{equation*}
N \gg N_{0} \geq 1, \quad A_{*} \gg 1 \tag{3.7}
\end{equation*}
$$

where throughout the paper, we adjust the size of the parameters as necessary. Roughly, we will first choose $A_{*} \geq 1$ to be large enough to dominate various order-unity structural constants (denoted by the symbol " $C_{*}$ " throughout the paper) in the PDEs. We then fix any $N_{0} \geq 1$. We will then choose $N$ to be sufficiently large in a manner that depends on $N_{0}, A_{*}, q$, and $\sigma$ (as well as $\mathfrak{D}$, the number of spatial dimensions). See also Remarks 1.12 and 1.13 for discussion on how to obtain crude estimates for $N$ and $A_{*}$.

Finally, we will use a small parameter $0<\varepsilon \ll 1$ to capture the smallness of the overall norms that measure the closeness of the perturbed solution to the background generalized Kasner metric. Roughly, for our bootstrap argument to close, we will first have to choose the other parameters as described above and then choose $\varepsilon$ to be sufficiently small in a manner that depends on $N, N_{0}, A_{*}, \mathfrak{D}, q$, and $\sigma$.

### 3.5 Definitions of the solution norms

In our bootstrap argument, we will rely on the $t$-weighted norms in the following definition. Roughly, our main theorem shows that all of the norms in the definition remain small throughout the entire interval $t \in(0,1]$ if they are small at $t=1$.

Definition 3.1 (Solution norms). Recall that the parameter $N_{0}$ verifies $N_{0} \geq 1$, that $q, \sigma \in(0,1)$ are the constants fixed in Sect. 3.4, and that we introduced the notation " $\vec{e} f$ " in Sect.3.2. We define the low order norms

$$
\begin{align*}
\mathbb{L}_{(e, \omega)}(t) & :=\max \left\{t^{q}\|e-\widetilde{e}\|_{W^{N_{0}, \infty}\left(\Sigma_{t}\right)}, t^{q}\|\omega-\widetilde{\omega}\|_{W^{N_{0}, \infty}\left(\Sigma_{t}\right)}\right\} \\
\mathbb{L}_{(n)}(t) & :=\max \left\{t^{-\sigma}\|n-1\|_{W^{N_{0}+1, \infty}\left(\Sigma_{t}\right)}, t^{q-\sigma}\|\vec{e} n\|_{W^{N_{0}, \infty}\left(\Sigma_{t}\right)}\right\}, \\
\mathbb{L}_{(\gamma, k)}(t) & :=\max \left\{t^{q}\|\gamma\|_{W^{N_{0}, \infty}\left(\Sigma_{t}\right)}, t\|k-\widetilde{k}\|_{W^{N_{0}+1, \infty}\left(\Sigma_{t}\right)}\right\}  \tag{3.8a}\\
\mathbb{L}_{(\psi)}(t) & :=\max \left\{t^{q}\|\vec{e} \psi\|_{W^{N_{0}, \infty}\left(\Sigma_{t}\right)}, t\left\|e_{0} \psi-\partial_{t} \widetilde{\psi}\right\|_{W^{N_{0}+1, \infty}\left(\Sigma_{t}\right)}\right\}, \\
\mathbb{L}_{(e, \omega, \gamma, k, \psi)}(t) & :=\mathbb{L}_{(e, \omega)}(t)+\mathbb{L}_{(\gamma, k)}(t)+\mathbb{L}_{(\psi)}(t),
\end{align*}
$$

and the high order norms

$$
\begin{align*}
\mathbb{H}_{(e, \omega)}(t) & :=\max \left\{t^{A_{*}+q}\|e\|_{\dot{H}^{N}\left(\Sigma_{t}\right)}, t^{A_{*}+q}\|\omega\|_{\dot{H}^{N}\left(\Sigma_{t}\right)}\right\}, \\
\mathbb{H}_{(n)}(t) & :=\max \left\{t^{A_{*}}\|n\|_{\dot{H}^{N}\left(\Sigma_{t}\right)}, t^{A_{*}+1}\|\vec{e} n\|_{\dot{H}^{N}\left(\Sigma_{t}\right)}\right\}, \\
\mathbb{H}_{(\gamma, k)}(t) & :=\max \left\{t^{A_{*}+1}\|\gamma\|_{\dot{H}^{N}\left(\Sigma_{t}\right)}, t^{A_{*}+1}\|k\|_{\dot{H}^{N}\left(\Sigma_{t}\right)}\right\},  \tag{3.8b}\\
\mathbb{H}_{(\psi)}(t) & :=\max \left\{t^{A_{*}+1}\|\vec{e} \psi\|_{\dot{H}^{N}\left(\Sigma_{t}\right)}, t^{A_{*}+1}\left\|e_{0} \psi\right\|_{\dot{H}^{N}\left(\Sigma_{t}\right)}\right\}, \\
\mathbb{H}_{(e, \omega, \gamma, k, \psi)}(t) & :=\mathbb{H}_{(e, \omega)}(t)+\mathbb{H}_{(\gamma, k)}(t)+\mathbb{H}_{(\psi)}(t) .
\end{align*}
$$

We also find it convenient to define the following "total norm" for the "dynamic" variables (i.e., the nonlapse ${ }^{35}$ variables):

$$
\begin{equation*}
\mathbb{D}(t):=\mathbb{L}_{(e, \omega, \gamma, k, \psi)}(t)+\mathbb{H}_{(e, \omega, \gamma, k, \psi)}(t) \tag{3.8c}
\end{equation*}
$$

Remark 3.2 (Derivative counts involving $N_{0}$ ). Note that the low order norms in 3.8a) yield control over the "kinetic" (i.e., time-derivative-involving) terms $\left\{k_{I J}-\widetilde{k}_{I J}\right\}_{I, J=1, \cdots, \mathfrak{D}}$ and $e_{0} \psi-\partial_{t} \psi$ at one derivative level higher than the remaining terms. This important for our bootstrap argument, more precisely for our derivation of the lower order estimates; see, for example, Lemma 5.10 and the proof of 5.23 a .

### 3.6 Bootstrap assumptions

Our bootstrap assumptions are that there is a "bootstrap time" $T_{\text {Boot }} \in(0,1)$ such that

$$
\begin{equation*}
\mathbb{D}(t)+\mathbb{L}_{(n)}(t)+\mathbb{H}_{(n)}(t) \leq \varepsilon, \quad \forall t \in\left(T_{\text {Boot }}, 1\right] \tag{3.9}
\end{equation*}
$$

In the proof of our main theorem, such a $T_{\text {Boot }} \in(0,1)$ will exist due to our near-Kasner assumptions on the data and Cauchy stability.

## 4 Basic estimates and identities

In this section, we provide some basic inequalities and commutation formulas that we will frequently use in our main estimates, i.e., in Sect. 5 .

### 4.1 Interpolation and product inequalities

In our ensuing analysis, we will control various error terms with the help of the classical interpolation and Sobolev inequalities provided in the next lemma.

Lemma 4.1 (Sobolev interpolation and product inequalities). Let v be a $\Sigma_{t}$-tangent tensorfield, let $M_{1}, M_{2}$ be two non-negative integers, and let $\iota_{1}, \cdots, \iota_{R}$ be spatial multi-indices such that $\sum_{r=1}^{R}\left|\iota_{r}\right|=M_{1}$. Then the following estimates hold, where norms of tensorfields are defined as in Sect.3.3:

$$
\begin{align*}
\|v\|_{\dot{H}^{M_{1}\left(\Sigma_{t}\right)}} & \lesssim\|v\|_{L^{\infty}\left(\Sigma_{t}\right)}^{1-\frac{M_{1}}{2_{2}}}\|v\|_{\dot{H}^{M_{2}\left(\Sigma_{t}\right)}}^{\frac{M_{1}}{M_{2}}} \lesssim\|v\|_{L^{\infty}\left(\Sigma_{t}\right)}+\|v\|_{\dot{H}^{M_{2}\left(\Sigma_{t}\right)}}, \quad \text { for } M_{2} \geq M_{1},  \tag{4.1}\\
\|v\|_{W^{M_{1}, \infty}\left(\Sigma_{t}\right)} & \lesssim\|v\|_{\left.H^{M_{1}+1+\left\lfloor\frac{\mathcal{D}}{2}\right.}\right\rfloor_{\left(\Sigma_{t}\right)}} \lesssim\|v\|_{L^{\infty}\left(\Sigma_{t}\right)}+\|v\|_{\dot{H}^{M_{2}\left(\Sigma_{t}\right)}}, \quad \text { for } M_{2} \geq M_{1}+1+\left\lfloor\left.\frac{\mathfrak{D}}{2} \right\rvert\,,\right.  \tag{4.2}\\
\left\|\partial^{\iota_{1}} v_{1} \cdots \partial^{\iota_{R}} v_{R}\right\|_{L^{2}\left(\Sigma_{t}\right)} & \lesssim \sum_{r=1}^{R}\left\|v_{r}\right\|_{\dot{H}^{M_{1}\left(\Sigma_{t}\right)}} \prod_{s \neq r}\left\|v_{s}\right\|_{L^{\infty}\left(\Sigma_{t}\right)}, \tag{4.3}
\end{align*}
$$

[^20]where $\left\lfloor\frac{\mathfrak{D}}{2}\right\rfloor$ is the integer part of $\frac{\mathfrak{D}}{2}$. Moreover, if $1 \leq R_{0} \leq R$ and $\iota_{1}, \cdots, \iota_{R}$ are spatial multi-indices such that $\sum_{r=1}^{R}\left|\iota_{r}\right|=M_{1}$ and $\left|\iota_{R-R_{0}+1}\right|, \cdots,\left|\iota_{R}\right| \leq M_{1}-1$, then the following product inequality holds:
\[

$$
\begin{align*}
\left\|\partial^{\iota_{1}} v_{1} \cdots \partial^{\iota_{R}} v_{R}\right\|_{L^{2}\left(\Sigma_{t}\right)} \lesssim & \sum_{r=1}^{R-R_{0}}\left(\left\|v_{r}\right\|_{W^{1, \infty}\left(\Sigma_{t}\right)}+\left\|v_{r}\right\|_{\dot{H}^{M_{1}}\left(\Sigma_{t}\right)}\right) \prod_{s \neq r}\left\|v_{s}\right\|_{W^{1, \infty}\left(\Sigma_{t}\right)} \\
& +\sum_{r=R-R_{0}+1}^{R}\left(\left\|v_{r}\right\|_{W^{1, \infty}\left(\Sigma_{t}\right)}+\left\|v_{r}\right\|_{\dot{H}^{M_{1}-1}\left(\Sigma_{t}\right)}\right) \prod_{s \neq r}\left\|v_{s}\right\|_{W^{1, \infty}\left(\Sigma_{t}\right)} \tag{4.4}
\end{align*}
$$
\]

Proof. The first inequality in (4.1) is immediate from Plancherel's identity, Hölder's inequality, and the bound $\|v\|_{L^{2}\left(\Sigma_{t}\right)} \lesssim\|v\|_{L^{\infty}\left(\Sigma_{t}\right)}$ for scalar functions $v$ (which holds because $\mathbb{T}^{\mathfrak{D}}$ is compact). The second inequality in (4.1) follows from the first and Young's inequality. In the case $\Sigma_{t}=\mathbb{R}^{\mathfrak{D}}$, the inequality (4.3) was proved as [55, Lemma 6.16], and the same proof works in the case $\Sigma_{t}=\mathbb{T}^{\mathfrak{D}}$. The first inequality in (4.2) is standard Sobolev embedding, while the second inequality in 4.2 follows from applying 4.1) to the homogeneous norms $\dot{H}^{M_{1}^{\prime}}\left(\Sigma_{t}\right)$ of $v$, for every $M_{1}^{\prime} \leq M_{1}+1+\left\lfloor\frac{\mathfrak{D}}{2}\right\rfloor$. To derive 4.4 , we first note that either all derivatives act on one of the terms $v_{1}, \cdots, v_{R-R_{0}}$, say $v_{1}$, or there exist at least two factors having at least one derivative, say $v_{1}, v_{a}$, where $a>1$. Then setting $u_{1}:=\partial v_{1}$ in the first case or $u_{1}:=\partial v_{1}, u_{a}:=\partial v_{a}$ in the second case, we apply 4.3 and 4.1 to the product, where we view $u_{1}$ and $u_{a}$ to be terms in the product that are hit with one fewer derivative than $v_{1}$ and $v_{a}$. This yields the desired estimate.

As an immediate application of Lemma 4.1, we provide the next lemma, which yields control of the reduced solution variables at orders slightly higher than $N_{0}$. The price we pay is that the estimates are slightly (when $N$ is large) more singular with respect to powers of $t$ compared to the very-low-order estimates. Nevertheless, a small increase in the singularity strength is allowable for treating error terms that are sub-critical with respect to powers of $t$.

Lemma $4.2\left(L^{\infty}\right.$ control at slightly higher orders than $N_{0}$ - with only a mild increase in singularity strength for large $N)$. Assume that the bootstrap assumptions (3.9) hold. There exists a constant $\delta=\delta(N, \mathfrak{D})$ (which is free to vary from line to line) such that $\delta \rightarrow 0$ as $N \rightarrow \infty$ and such that if $N \geq N_{0}+4+\left\lfloor\frac{\mathfrak{D}}{2}\right\rfloor$, then the following estimates hold for $t \in\left(T_{\text {Boot }}, 1\right]$ :

$$
\begin{align*}
\|e-\widetilde{e}\|_{W^{N_{0}+2, \infty}\left(\Sigma_{t}\right)}+\|\omega-\widetilde{\omega}\|_{W^{N_{0}+2, \infty}\left(\Sigma_{t}\right)} & \lesssim t^{-q-\delta A_{*}}\left\{\mathbb{L}_{(e, \omega)}(t)+\mathbb{H}_{(e, \omega)}(t)\right\},  \tag{4.5}\\
\|\gamma-\widetilde{\gamma}\|_{W^{N_{0}+2, \infty}\left(\Sigma_{t}\right)} & \lesssim t^{-q-\delta A_{*}}\left\{\mathbb{L}_{(\gamma, k)}(t)+\mathbb{H}_{(\gamma, k)}(t)\right\},  \tag{4.6}\\
\|k-\widetilde{k}\|_{W^{N_{0}+2, \infty}\left(\Sigma_{t}\right)} & \lesssim t^{-1-\delta A_{*}}\left\{\mathbb{L}_{(\gamma, k)}(t)+\mathbb{H}_{(\gamma, k)}(t)\right\},  \tag{4.7}\\
\|n-1\|_{W^{N_{0}+3, \infty}\left(\Sigma_{t}\right)}+t^{q}\|\vec{e} n\|_{W^{N_{0}+2, \infty}\left(\Sigma_{t}\right)} & \lesssim t^{\sigma-\delta A_{*}}\left\{\mathbb{L}_{(n)}(t)+\mathbb{H}_{(n)}(t)\right\},  \tag{4.8}\\
t^{q}\|\vec{e} \psi\|_{W^{N_{0}+2, \infty}\left(\Sigma_{t}\right)}+t\left\|\partial_{t} \psi\right\|_{W^{N_{0}+2, \infty}\left(\Sigma_{t}\right)} & \lesssim t^{-\delta A_{*}}\left\{\mathbb{L}_{(\psi)}(t)+\mathbb{H}_{(\psi)}(t)\right\} . \tag{4.9}
\end{align*}
$$

Proof. The argument for all inequalities is essentially the same, so we only prove 4.8). Using first 4.2) and then 4.1), we find that for $N \geq N_{0}+4+\left\lfloor\frac{\mathfrak{D}}{2}\right\rfloor$, we have

$$
\begin{aligned}
& \left.\|n-1\|_{W^{N_{0}+3, \infty}\left(\Sigma_{t}\right)} \lesssim\|n-1\|_{L^{\infty}\left(\Sigma_{t}\right)}+\|n-1\|_{\dot{H}^{N_{0}+4+}\left\lfloor\frac{\mathcal{D}}{2}\right.}\right\rfloor_{\left(\Sigma_{t}\right)} \\
& \lesssim\|n-1\|_{L^{\infty}\left(\Sigma_{t}\right)}+\|n-1\|_{L^{\infty}\left(\Sigma_{t}\right)}^{1-\widetilde{\mathcal{S}}}\|n-1\|_{\dot{H}^{N}\left(\Sigma_{t}\right)}^{\widetilde{\mathcal{S}}} \\
& \leq t^{\sigma} \mathbb{L}_{(n)}(t)+\left(t^{\sigma} \mathbb{L}_{(n)}(t)\right)^{1-\widetilde{\delta}}\left(t^{-A_{*}} \mathbb{H}_{(n)}(t)\right)^{\widetilde{\delta}} \\
& =t^{\sigma} \mathbb{L}_{(n)}(t)+t^{\sigma-\left(A_{*}+\sigma\right)} \widetilde{\delta}_{\mathbb{L}_{(n)}^{1-\widetilde{\delta}}}^{1}(t) \mathbb{H}_{(n)}^{\widetilde{\delta}}(t) \\
& \lesssim t^{\sigma-\delta A_{*}}\left\{\mathbb{L}_{(n)}(t)+\mathbb{H}_{(n)}(t)\right\},
\end{aligned}
$$

where $\widetilde{\delta}:=\frac{N_{0}+4+\left\lfloor\frac{D}{2}\right\rfloor}{N} \leq 1$, and for the last inequality, we used Young's inequality and set $\delta:=\frac{A_{*}+\sigma}{A_{*}} \widetilde{\delta}$. It is clear that $\delta \rightarrow 0$, as $N \rightarrow \infty$, at a rate that is independent of how large $A_{*} \geq 1$ is. This yields 4.8) for the term $\|n-1\|_{W^{N_{0}+3, \infty}\left(\Sigma_{t}\right)}$. The estimate for the term $t^{q}\|\overrightarrow{e n}\|_{W^{N_{0}+2, \infty}\left(\Sigma_{t}\right)}$ would then follow from the Leibniz rule, the estimate for the term $\|n-1\|_{W^{N_{0}+3, \infty}\left(\Sigma_{t}\right)}$, and the estimate 4.5) for the term $\|e-\widetilde{e}\|_{W^{N_{0}+2, \infty}\left(\Sigma_{t}\right)}$
(which for purposes of exposition we assume to have already been proved). We clarify that by this argument, the value of $\delta$ corresponding to the estimate for $t^{q}\|\vec{e} n\|_{W^{N_{0}+2, \infty}\left(\Sigma_{t}\right)}$ might be larger than the value of $\delta$ for $\|n-1\|_{W^{N_{0}+3, \infty}\left(\Sigma_{t}\right)}$, but nevertheless, all " $\delta$ 's" tend to 0 as $N \rightarrow \infty$.

Remark 4.3 ( $\delta$ can vary from line to line). In the rest of the paper, $\delta=\delta(N, \mathfrak{D})$ denotes a small positive constant that is free to vary from line to line, but that always has the property that $\delta \rightarrow 0$ as $N \rightarrow \infty$ (as in Lemma 4.2. In particular, we sometimes express the sum of two $\delta$ 's as another $\delta$.

Remark 4.4 (Smallness of $\delta A_{*}$ ). Later in the paper, when we use Lemma 4.2 to derive estimates for the solution, we will always assume (sometimes without explicitly mentioning it) that $\delta A_{*}$ is as small as we need it to be. In particular, we assume that it is small enough such that $\delta A_{*}<\sigma$ so that, for example, $t^{2 \sigma-\delta A_{*}} \leq t^{\sigma}$ for $t \in(0,1]$. At fixed $A_{*}$, the desired smallness can be ensured by choosing $N$ to be sufficiently large.

Remark 4.5 (Large interpolation constants are not an obstacle to stability). The implicit constants in the interpolation inequalities of Lemmas 4.1 and 4.2 depend on $M_{1}, M_{2}, N_{0}, N$, and the number of spatial dimensions $\mathfrak{D}$. One might worry, especially when taking $N$ sufficiently large to make $\delta$ small, that the constants in the elliptic and energy estimates, corresponding to the terms that we treat using these inequalities, can be quite large. While the constants " $C$ " can in fact be large, largeness does not obstruct the proofs of our results. The reason is that we only apply these inequalities to handle two kinds of error terms: i) error terms that are sub-critical with respect to powers of $t$, for which the largeness of $C$ is admissible within the context of our Grönwall estimates; and ii) critical "borderline" products with one factor that yields a smallness factor of $\varepsilon$, so that the effective coefficient $C \varepsilon$ can be made as small as one wants by choosing the bootstrap parameter $\varepsilon$ to be small (which is possible for initial data on $\Sigma_{1}$ that are sufficiently close to the Kasner data). In particular, in our estimates, the implicit constants in Lemmas 4.1 and 4.2 do not affect the size of the important constants " $C_{*}$ " (see Sect. 1.11 for our conventions for constants " $C_{*}$ ") or the value of the parameter $A_{*}$. See also Sect. 5.3 for further discussion of borderline and below-borderline terms.

### 4.2 Two simple commutation formulas

To derive estimates for the solution's derivatives, we will repeatedly commute the reduced equations with the transported spatial coordinate partial derivative vectorfields $\left\{\partial_{i}\right\}_{i=1, \cdots, \mathfrak{D}}$, and we will use the following commutation relation to uncover the structure of various error terms (see Section 1.11 for our conventions on multi-indices):

$$
\begin{equation*}
\left[\partial^{\iota}, e_{I}\right] v=\sum_{\iota_{1} \cup \iota_{2}=\iota,\left|\iota_{2}\right|<|\iota|}\left(\partial^{\iota_{1}} e_{I}^{c}\right) \partial^{\iota_{2}} \partial_{c} v . \tag{4.10}
\end{equation*}
$$

The identity 4.10 follows easily from expanding $e_{I}=e_{I}^{c} \partial_{c}$.
We will also use the following commutation identity:

$$
\begin{equation*}
\left[\partial_{t}, e_{I}\right]=n k_{I C} e_{C}^{c} \partial_{c} \tag{4.11}
\end{equation*}
$$

which we derived in 2.35 .

## 5 Main estimates

Our main goal in this section is to establish Proposition 5.1. which forms the analytical cornerstone of the paper. The proposition provides a priori estimates for perturbations of the Kasner background solution and in particular yields improvements of the bootstrap assumptions when the data are sufficiently near-Kasner. We also highlight that for near-Kasner data, the a priori estimates and standard arguments collectively imply that the solution exists on the entire half-slab $(0,1] \times \mathbb{T}^{\mathfrak{D}}$ and enjoys the quantitative properties afforded by the a priori estimates; see Proposition 5.28 for those details.

### 5.1 Statement of the main a priori estimates

In the next proposition, we state our main a priori estimates. The proof is located in Sect.5.9. In the sections that precede it, we will establish a series of preliminary identities and estimates for $n, \gamma, k$, the frame $\left\{e_{I}\right\}_{I=1, \cdots, \mathfrak{D}}$, and the co-frame $\left\{\omega^{I}\right\}_{I=1, \cdots, \mathfrak{D}}$. The proof of the proposition essentially amounts to combining the preliminary results.

Proposition 5.1 (The main a priori estimates). Let $\left(n, k_{I J}, \gamma_{I J B}, e_{I}^{i}, \omega_{i}^{I}, \psi\right)_{I, J, B, i=1, \cdots, \mathfrak{D}}$ be a solution to the reduced equations of Proposition 2.2 on $\left(T_{\text {Boot }}, 1\right] \times \mathbb{T}^{\mathfrak{D}}$. Recall that $\mathbb{D}(t)$ is the total norm of the dynamic variables and that $\mathbb{L}_{(n)}(t)$ and $\mathbb{H}_{(n)}(t)$ are norms of the lapse (see Definition 3.1). Let ${ }^{\circ}$ denote the initial value of the total norm of the dynamic variables:

$$
\begin{equation*}
\stackrel{\circ}{\epsilon}:=\mathbb{D}(1)=\mathbb{L}_{(e, \omega)}(1)+\mathbb{L}_{(\gamma, k)}(1)+\mathbb{L}_{(\psi)}(1)+\mathbb{H}_{(e, \omega)}(1)+\mathbb{H}_{(\gamma, k)}(1)+\mathbb{H}_{(\psi)}(1) \tag{5.1}
\end{equation*}
$$

Assume that the bootstrap assumptions (3.9) hold for $t \in\left(T_{B o o t}, 1\right]$. If $A_{*}$ is sufficiently large and $N_{0} \geq 1$, then there exists a constant $C_{N, N_{0}, A_{*}, \mathfrak{D}, q, \sigma}>0$ such that if $N$ is sufficiently large in a manner that depends on $N_{0}, A_{*}, \mathfrak{D}, q$, and $\sigma$, and if $\varepsilon$ is sufficiently small (in a manner that depends on $N, N_{0}, A_{*}, \mathfrak{D}, q$, and $\sigma$ ), then the following estimate holds for $t \in\left(T_{\text {Boot }}, 1\right]$ :

$$
\begin{equation*}
\mathbb{D}(t)+\mathbb{L}_{(n)}(t)+\mathbb{H}_{(n)}(t) \leq C_{N, N_{0}, A_{*}, \mathfrak{D}, q, \sigma} \stackrel{\circ}{\epsilon} . \tag{5.2}
\end{equation*}
$$

In particular, if $C_{N, N_{0}, A_{*}, \mathfrak{D}, q, \sigma} \stackrel{\circ}{\epsilon}<\varepsilon$, then (5.2) yields a strict improvement of the bootstrap assumptions (3.9).

### 5.2 Schematic notation

We will use schematic notation to simplify the presentation of various formulas when the precise structure of the terms is not important. $\partial$ denotes an arbitrary partial derivative with respect to one of the transported spatial coordinate vectorfields. $k$ denotes an arbitrary element of the array $\left(k_{I J}\right)_{I, J=1, \cdots, \mathfrak{D}}$ of components of the second fundamental form with respect to the orthonormal frame. $\partial^{\iota} k$ denotes an arbitrary element of the array $\left(\partial^{\iota} k_{I J}\right)_{I, J=1, \cdots, \mathfrak{D}}$. Similarly, $\gamma$ denotes an arbitrary element of the array $\left(\gamma_{I J B}\right)_{I, J, B=1, \cdots, \mathfrak{D}}$ and $\partial^{\iota} \gamma$ denotes an arbitrary element of the array $\left(\partial^{\iota} \gamma_{I J B}\right)_{I, J, B=1, \cdots, \mathfrak{D}}$. e denotes an arbitrary element of the array $\left(e_{I}^{i}\right)_{I, i=1, \cdots, \mathfrak{D}}$, while $\omega$ denotes an arbitrary element of the array $\left(\omega_{i}^{I}\right)_{I, i=1, \cdots, \mathfrak{D}}$. If $f$ is a scalar function, $\vec{e} f$ denotes the array $\left(e_{I} f\right)_{I=1, \cdots, \mathfrak{D}}$.

As an example, with the help of the notation from Sect.1.11. we can express the commutator $\partial^{l}\left(n e_{C} \gamma_{I J C}\right)-$ $n e_{C} \partial^{\iota} \gamma_{I J C}$ in the following schematic form: $\sum_{\iota_{1} \cup_{\iota_{2}} \cup \iota_{3}=\iota,\left|\iota_{3}\right|<|\iota|} \partial^{\iota_{1}} n \cdot \partial^{\iota_{2}} e \cdot \partial \partial^{\iota_{3}} \gamma$. We remark that we use schematic notation only when the overall signs and precise numerical coefficients in front of the terms is not important. Thus, when using schematic notation for terms, we do not account for their overall signs or precise numerical coefficients.

### 5.3 Borderline terms vs. Junk terms

In our top-order energy estimates, we encounter some delicate error terms that cannot be treated by Grönwall's lemma uniformly in $T_{B o o t} \in(0,1)$. That is, if treated crudely, these terms would prevent us from deriving an energy estimate that would lead to an improvement of our bootstrap assumptions. We described one example of such a term at the end of Sect. 1.8 .6 . Let us revisit this issue in more detail. In our top-order energy estimates, we encounter "borderline" error integrands with the following strength:

$$
\begin{equation*}
\frac{1}{t} t^{2 A_{*}+2} \partial^{\iota} \gamma \cdot \partial^{\iota} \gamma, \quad \frac{1}{t} t^{2 A_{*}+2} \partial^{\iota} k \cdot \partial^{\iota} k, \quad \frac{1}{t} \cdot t^{2 A_{*}+2} \partial^{\iota} \gamma \cdot \partial^{\iota}\left(e_{I} n\right) \tag{5.3}
\end{equation*}
$$

The difficulty is that the integrands in (5.3) are more singular than the energy density itself due to the factors of $\frac{1}{t}$. To handle these error terms, we exploit the following crucial fact, which we must justify in our analysis:

In the energy identities, the coefficients of all of the borderline terms can be bounded by a uniform constant $C_{*}$, independent of $A_{*}$ and $N$, as long as the bootstrap parameter $\varepsilon$ is sufficiently small (in a manner that is allowed to depend on $N$ and $A_{*}$ ). Such terms contribute to the $C_{*}$-multiplied integrals on the right-hand side of the energy inequalities of Proposition 5.27.

We refer readers to Remark 5.18 for further comments on our use of the terminology "borderline."
At this point, the role of the $t^{2 A_{*}+2}$ weights in our energy identities emerges: the weights also generate borderline terms (roughly, when the $\partial_{t}$ derivative falls on the weights in the energy identities) of the same strength as those in (5.3), but unlike the terms in (5.3), the error terms generated by the weights have a favorable sign towards the singularity with an overall coefficient that is proportional to $A_{*}$. These terms contribute to the favorable $-A_{*}$-multiplied integrals on the right-hand side of the energy inequalities of Proposition 5.27. Thus, if $A_{*}$ is chosen sufficiently large, the overall coefficient $C_{*}-A_{*}$ of the borderline terms becomes negative, and in our energy estimates, the corresponding integral has a "good sign" and can be discarded. We again stress that for this argument to work, it is crucial that $C_{*}$ can be chosen to be independent of $A_{*}, N$, at least when $\varepsilon$ is small.

On the other hand, there are many terms in the energy estimates that are "junk" in the sense that they can be bounded by our norms times a factor of strength $C t^{-1+\sigma}$. Although " $C$ " is allowed to depend on $A_{*}, N$, and other parameters (cf. Remark 4.5 regarding the size of the constants $C$ in the interpolation constants), such terms do not pose any difficulty in the a priori energy estimates. The reason is that $C t^{-1+\sigma}$ is integrable in time near $t=0$ and thus, in the context of Grönwall's lemma, the factor $C t^{-1+\sigma}$ causes only finite growth of our energies, which is perfectly compatible with our bootstrap argument and our proof of stability.

Remark 5.2 ("Border" and "Junk" notation). To help the reader navigate the energy estimates, in our ensuing analysis, we label error terms that generate borderline (in the sense above) error terms with the superscript "Border," and we label error terms that generate junk (in the sense above) error terms with the superscript "Junk." See, for example the terms $t^{P-1} \mathfrak{K}_{I J}^{(\text {Border; } ; L)}$ and $t^{P} \mathfrak{K}_{I J}^{(J u n k ; \iota)}$ on RHS 5.31a).

We sometimes use similar notation to distinguish between "borderline terms" and "junk terms" in our pointwise estimates; see, however, Remark 5.18 .

### 5.4 Control of the lapse $n$ in terms of the dynamic solution variables

Our main goal in this subsection is to prove the following proposition, which yields control of the lapse in terms of the remaining "dynamic" solution variables. This is a preliminary step in our derivation of a priori estimates for all solution variables. The proof of the proposition relies on elliptic estimates and the bootstrap assumptions $\sqrt{3.9}$ and is located in Sect. 5.4.4. Before proving the proposition, we first establish some preliminary identities and estimates.

Proposition 5.3 (Estimates for the lapse in terms of the dynamic solution variables). Recall that $\mathbb{L}_{(n)}(t)$, $\mathbb{H}_{(n)}(t), \mathbb{H}_{(\gamma, k)}(t)$, and $\mathbb{D}(t)$ are norms from Definition 3.1. Under the assumptions of Proposition 5.1. there exists a constant $C_{*}>0$ independent of $N, N_{0}$, and $A_{*}$ and a constant $C=C_{N, N_{0}, A_{*}, \mathfrak{D}, q, \sigma}>0$ such that if $N_{0} \geq 1$ and $N$ is sufficiently large in a manner that depends on $N_{0}, A_{*}, \mathfrak{D}, q$, and $\sigma$, and if $\varepsilon$ is sufficiently small (in a manner that depends on $N, N_{0}, A_{*}, \mathfrak{D}, q$, and $\sigma$ ), then the following estimates hold for $t \in\left(T_{\text {Boot }}, 1\right]:$

$$
\begin{equation*}
\|n-1\|_{W^{N_{0}+1, \infty}\left(\Sigma_{t}\right)}+t^{q}\|\vec{e} n\|_{W^{N_{0}, \infty}\left(\Sigma_{t}\right)} \leq C t^{\sigma} \mathbb{D}(t) \tag{5.4}
\end{equation*}
$$

Moreover, if $\iota$ is any spatial multi-index with $|\iota|=N$, then we have

$$
\begin{align*}
t^{A_{*}+1}\left\|\partial^{\iota} \vec{e} n\right\|_{L^{2}\left(\Sigma_{t}\right)}+t^{A_{*}}\left\|\partial^{\iota} n\right\|_{L^{2}\left(\Sigma_{t}\right)} & \leq C_{*} t^{A_{*}+1}\left\|\partial^{\iota} \gamma\right\|_{L^{2}\left(\Sigma_{t}\right)}+C t^{\sigma} \mathbb{D}(t)  \tag{5.5a}\\
t^{A_{*}+1}\|\vec{e} n\|_{\dot{H}^{N}\left(\Sigma_{t}\right)}+t^{A_{*}}\|n\|_{\dot{H}^{N}\left(\Sigma_{t}\right)} & \leq C_{*} \mathbb{H}_{(\gamma, k)}(t)+C t^{\sigma} \mathbb{D}(t) \tag{5.5b}
\end{align*}
$$

Finally, the lapse norms are bounded by the dynamic variable norm:

$$
\begin{equation*}
\mathbb{L}_{(n)}(t)+\mathbb{H}_{(n)}(t) \leq C \mathbb{D}(t) \tag{5.6}
\end{equation*}
$$

### 5.4.1 Equations for controlling the lapse

We start by deriving the elliptic equations satisfied by the derivatives of the lapse.
Lemma 5.4 (The commuted lapse equation). For solutions $n$ to the lapse equation (2.25) and spatial coordinate multi-indices $\iota$ with $|\iota| \leq N$, the following equation holds:

$$
\begin{equation*}
e_{C} \partial^{\iota} e_{C}(n-1)-t^{-2} \partial^{\iota}(n-1)=2 n e_{D} \partial^{\iota} \gamma_{C C D}+\mathfrak{N}^{(\iota)} \tag{5.7}
\end{equation*}
$$

where

$$
\begin{align*}
\mathfrak{N}^{(\iota)}:= & \sum_{\iota_{1} \cup \iota_{2}=\iota,\left|\iota_{2}\right|<|\iota|} \partial^{\iota_{1}} e \cdot \partial \partial^{\iota_{2}} \vec{e} n \\
& +\sum_{\iota_{1} \cup \iota_{2} \cup \iota_{3}=\iota,\left|\iota_{3}\right|<|\iota|} \partial^{\iota_{1}} n \cdot \partial^{\iota_{2}} e \cdot \partial \partial^{\iota_{3}} \gamma+\sum_{\iota_{1} \cup \iota_{2}=\iota} \partial^{\iota_{1}} \gamma \cdot \partial^{\iota_{2}} \overrightarrow{e n}  \tag{5.8}\\
& +\sum_{\iota_{1} \cup \iota_{2} \cup \iota_{3}=\iota} \partial^{\iota_{1} n \cdot \partial^{\iota_{2}} \gamma \cdot \partial^{\iota_{3}} \gamma+\sum_{\iota_{1} \cup \iota_{2} \cup \iota_{3}=\iota} \partial^{\iota_{1}} n \cdot \partial^{\iota_{2}} \vec{e} \psi \cdot \partial^{\iota_{3}} \vec{e} \psi .}
\end{align*}
$$

Proof. (5.7) follows from differentiating (2.25 with $\partial^{\iota}$ and using the commutation formula 4.10 and the Leibniz rule.

### 5.4.2 A standard elliptic identity

In the next lemma, we provide a standard elliptic identity for the lapse. We will use the identity to establish $L^{2}$ control of the lapse at the top order.

Lemma 5.5 (Elliptic identity for $n$ ). Let $\iota$ be a spatial coordinate multi-index with $1 \leq|\iota| \leq N$. For solutions to equation (5.7), the following identity holds:

$$
\begin{align*}
& t^{2 A_{*}+2}\left(\partial^{\iota} e_{C} n\right) \partial^{\iota} e_{C} n+t^{2 A_{*}}\left(\partial^{\iota} n\right)^{2} \\
&=2 n\left(t^{A_{*}+1} \partial^{\iota} e_{D} n\right)\left(t^{A_{*}+1} \partial^{\iota} \gamma_{C C D}\right)-\left(t^{A_{*}} \partial^{\iota} n\right)\left(t^{A+2} \mathfrak{N}^{(\iota)}\right)+t^{2 A_{*}+2} \mathfrak{R}^{(\iota)}  \tag{5.9}\\
&+\partial_{c}\left\{t^{2 A_{*}+2} e_{C}^{c}\left(\partial^{\iota} e_{C} n\right) \partial^{\iota} n\right\}-\partial_{c}\left\{2 t^{2 A_{*}+2} n e_{D}^{c}\left(\partial^{\iota} n\right) \partial^{\iota} \gamma_{C C D}\right\}, \\
& \mathfrak{R}^{(\iota)}:=\left(\partial^{\iota} e_{C} n\right)\left(\left[\partial^{\iota}, e_{C}\right]\right) n-2 n\left(\left[\partial^{\iota}, e_{D}\right] n\right) \partial^{\iota} \gamma_{C C D} \\
&-\left(\partial_{c} e_{C}^{c}\right)\left(\partial^{\iota} e_{C} n\right) \partial^{\iota} n+\left\{\partial_{c}\left(2 n e_{D}^{c}\right)\right\}\left(\partial^{\iota} n\right) \partial^{\iota} \gamma_{C C D} . \tag{5.10}
\end{align*}
$$

Proof. We first multiply (5.7) with $-\partial^{\iota} n$ and differentiate by parts in the top-order terms after expanding $e_{C}=e_{C}^{c} \partial_{c}$ and $e_{D}=e_{D}^{c} \partial_{c}$ to obtain the following identity:

$$
\begin{align*}
- & \partial_{c}\left\{e_{C}^{c}\left(\partial^{\iota} e_{C} n\right) \partial^{\iota} n\right\}+\left(\partial_{c} e_{C}^{c}\right)\left(\partial^{\iota} e_{C} n\right) \partial^{\iota} n+\left(\partial^{\iota} e_{C} n\right)\left[e_{C}, \partial^{\iota}\right] n+\left(\partial^{\iota} e_{C} n\right) \partial^{\iota} e_{C} n+t^{-2}\left(\partial^{\iota} n\right)^{2} \\
= & -2 \partial_{c}\left\{n\left(\partial^{\iota} n\right) e_{D}^{c} \partial^{\iota} \gamma_{C C D}\right\}+\left\{\partial_{c}\left(2 n e_{D}^{c}\right)\right\}\left(\partial^{\iota} n\right) \partial^{\iota} \gamma_{C C D}+2 n\left(\left[e_{D}, \partial^{\iota}\right] n\right) \partial^{\iota} \gamma_{C C D}  \tag{5.11}\\
& +2 n\left(\partial^{\iota} e_{D} n\right) \partial^{\iota} \gamma_{C C D}-\left(\partial^{\iota} n\right) \mathfrak{N}^{(\iota)}
\end{align*}
$$

Multiplying (5.11) by $t^{2 A_{*}+2}$ and rearranging the terms, we arrive at the desired identity (5.9).

### 5.4.3 Control of the error terms in the top-order commuted lapse equation

In the next lemma, we derive $L^{2}$-control of the error terms in the top-order commuted lapse equation.
Lemma 5.6 ( $L^{2}$-control of the error terms in the top-order commuted lapse equation). Let $\mathfrak{N}^{(\iota)}$ and $\mathfrak{R}^{(\iota)}$ denote the lapse equation error terms defined respectively in (5.8) and 5.10) (these terms appear on the righthand side of (5.9). Under the assumptions of Proposition 5.3, there exists a constant $C=C_{N, N_{0}, A_{*}, \mathfrak{D}, q, \sigma}>0$ such that the following estimates hold for $t \in\left(T_{\text {Boot }}, 1\right]$ :

$$
\begin{align*}
& t^{A_{*}+2} \sum_{|\iota|=N}\left\|\mathfrak{N}^{(\iota)}\right\|_{L^{2}\left(\Sigma_{t}\right)} \leq C \varepsilon t^{2 \sigma} \mathbb{D}(t)  \tag{5.12}\\
& \quad t^{2 A_{*}+2} \int_{\Sigma_{t}}\left|\mathfrak{R}^{(\iota)}\right| d x \leq C \varepsilon t^{2 \sigma} \mathbb{D}(t)\left\{t^{A_{*}}\left\|\partial^{\iota} n\right\|_{L^{2}\left(\Sigma_{t}\right)}+t^{A_{*}+1}\left\|\partial^{\iota} \overrightarrow{e n}\right\|_{L^{2}\left(\Sigma_{t}\right)}+\mathbb{D}(t)\right\}, \quad|\iota|=N \tag{5.13}
\end{align*}
$$

Proof. Using the inequalities provided by Lemma 4.1, it is straightforward to estimate every product term in the expressions (5.8) and 5.10 by accounting for the control afforded by our bootstrap assumptions (3.9) and taking into account the powers of $t$ featured in the solution norms of Definition 3.1. We provide the
details for two representative terms. First, using Lemma 4.1 and Definition 3.1, we see that the following term in $\mathfrak{N}^{(\iota)}$ (i.e., the third sum on RHS (5.8)) satisfies:

$$
\begin{equation*}
\left\|\sum_{\iota_{1} \cup \iota_{2}=\iota} \partial^{\iota_{1}} \gamma \cdot \partial^{\iota_{2}} \vec{e} n\right\|_{L^{2}\left(\Sigma_{t}\right)} \lesssim\|\gamma\|_{L^{\infty}\left(\Sigma_{t}\right)}\|\overrightarrow{e n}\|_{\dot{H}^{N}\left(\Sigma_{t}\right)}+\|\overrightarrow{e n}\|_{L^{\infty}\left(\Sigma_{t}\right)}\|\gamma\|_{\dot{H}^{N}\left(\Sigma_{t}\right)} \lesssim \varepsilon t^{-A-1-q} \mathbb{D}(t) \tag{5.14}
\end{equation*}
$$

The factor of $\varepsilon$ on RHS (5.14) comes from the bootstrap assumptions $\sqrt{3.9}$ and the fact that the LHS is quadratic with respect to quantities that vanish for the background Kasner solution. Hence, multiplying (5.14) with $t^{A+2}$ and recalling our assumptions (3.6) on the parameters $q$, $\sigma$, we conclude that the resulting term is $\leq$ RHS (5.12) as desired.

We now give a second example, this time for the $L^{1}\left(\Sigma_{t}\right)$-type inequality (5.13). Specifically, we bound a term in $\mathfrak{R}^{(\iota)}$ (the second term on RHS 5.10) as follows by using 4.10, Lemma 4.1. Definition 3.1, and the bootstrap assumptions (3.9) (which in particular imply that $\|n\|_{L^{\infty}\left(\Sigma_{t}\right)} \leq 2$ ):

$$
\begin{align*}
& \int_{\Sigma_{t}}\left|2 n\left(\left[\partial^{\iota}, e_{D}\right] n\right)\left(\partial^{\iota} \gamma_{C C D}\right)\right| d x \\
& \lesssim\|n\|_{L^{\infty}\left(\Sigma_{t}\right)} \sum_{\iota_{1} \cup \iota_{2}=\iota,\left|\iota_{2}\right|<|\iota|}\left\|\left(\partial^{\iota_{1}} e_{I}^{c}\right)\left(\partial^{\iota_{2}} \partial_{c} n\right)\right\|_{L^{2}\left(\Sigma_{t}\right)}\left\|\partial^{\iota} \gamma_{C C D}\right\|_{L^{2}\left(\Sigma_{t}\right)} \\
& \lesssim\left\{\|e-\widetilde{e}\|_{W^{1, \infty}\left(\Sigma_{t}\right)}\|n-1\|_{W^{1, \infty}\left(\Sigma_{t}\right)}+\|e-\widetilde{e}\|_{W^{1, \infty}\left(\Sigma_{t}\right)}\|n\|_{\dot{H}^{N}\left(\Sigma_{t}\right)}+\|n-1\|_{W^{1, \infty}\left(\Sigma_{t}\right)}\|e-\widetilde{e}\|_{\dot{H}^{N}\left(\Sigma_{t}\right)}\right\}\|\gamma\|_{\dot{H}^{N}\left(\Sigma_{t}\right)} \\
& \leq C \varepsilon\left(t^{-A_{*}-1-q+\sigma}+t^{-2 A_{*}-1-q}+t^{-2 A_{*}-q+\sigma}\right) \mathbb{D}^{2}(t) . \tag{5.15}
\end{align*}
$$

Multiplying (5.15) by $t^{2 A_{*}+2}$ and using the inequality (3.6), we deduce that the resulting term is $\leq$ RHS 5.13 ) as desired.

The remaining terms that need to be bounded can be handled with similar arguments, and we omit the details.

### 5.4.4 Proof of Proposition 5.3

Throughout this proof, we will silently assume that $N$ is large enough such that we can use the smallness of $\delta A_{*}$ described in Remark 4.4
Proof of (5.4). First, for $|\iota| \leq N_{0}+1$, we use 5.7) to solve for $e_{C} e_{C} \partial^{\iota}(n-1)-t^{-2} \partial^{\iota}(n-1)$ and then bound the resulting terms in $L^{\infty}$ using the bootstrap assumptions and Lemma 4.2, in particular bounding all terms involving $n-1$ and its derivatives by $\lesssim t^{\sigma-\delta A_{*}}$, which yields the following pointwise estimate for $|\iota| \leq N_{0}+1$ (see Remark 4.3):

$$
\begin{align*}
\left|e_{C} e_{C} \partial^{\iota}(n-1)-t^{-2} \partial^{\iota}(n-1)\right| & \lesssim\left|2 n e_{D}^{c} \partial_{c} \partial^{\iota} \gamma_{C C D}+\mathfrak{N}^{(\iota)}+e_{C}^{c} \partial_{c}\left\{\left[e_{C}^{d}, \partial^{\iota}\right] \partial_{d}(n-1)\right\}\right|  \tag{5.16}\\
& \lesssim t^{-2 q-\delta A_{*}} \mathbb{D}(t) .
\end{align*}
$$

From (5.16) and the maximum principle, noting that $e_{C} e_{C} \partial^{\iota}(n-1) \leq 0(\geq 0)$ at the maxima (minima) of $\partial^{\iota}(n-1)$ in $\Sigma_{t}$, and using the inequalities in (3.6), we find that $\left\|t^{-2}(n-1)\right\|_{W^{N_{0}+1, \infty}\left(\Sigma_{t}\right)} \lesssim t^{-2+\sigma} \mathbb{D}(t)$. Multiplying the latter inequality by $t^{2}$, we arrive at the desired estimate (5.4) for the first term $\|n-1\|_{W^{N_{0}+1, \infty}\left(\Sigma_{t}\right)}$ on the LHS. To complete the proof of (5.4), we must show that $t^{q}\|\vec{e} n\|_{W^{N_{0}, \infty}\left(\Sigma_{t}\right)} \lesssim t^{\sigma} \mathbb{D}(t)$. Since $e_{I} n=e_{I}^{c} \partial_{c} n$, the desired estimate is a simple consequence of the already obtained bound $\|n-1\|_{W^{N_{0}+1, \infty}\left(\Sigma_{t}\right)} \lesssim t^{\sigma} \mathbb{D}(t)$ and the estimate $t^{q}\|\vec{e}\|_{W^{N_{0}, \infty}\left(\Sigma_{t}\right)} \lesssim 1$, which follows from the bootstrap assumptions, the definition of the background Kasner scalar functions $\widetilde{e}_{I}^{i}$ given in (2.58), and the inequalities in (3.6).
Proof of 5.5 a - 5.5 b . We will show that there are constants $C_{*}>0$ and $C>0$, as in the statement of the proposition, such that for each spatial multi-index $\iota$ with $|\iota|=N$, we have

$$
\begin{align*}
t^{2 A_{*}+2}\left\|\partial^{\iota} \vec{e} n\right\|_{L^{2}\left(\Sigma_{t}\right)}^{2}+t^{2 A_{*}}\left\|\partial^{\iota} n\right\|_{L^{2}\left(\Sigma_{t}\right)}^{2} \leq & \frac{1}{2} t^{2 A_{*}+2}\left\|\partial^{\iota} \vec{e} n\right\|_{L^{2}\left(\Sigma_{t}\right)}^{2}+\frac{1}{2} t^{2 A_{*}}\left\|\partial^{\iota} n\right\|_{L^{2}\left(\Sigma_{t}\right)}^{2}  \tag{5.17}\\
& +C_{*} t^{2 A_{*}+2}\left\|\partial^{\iota} \gamma\right\|_{L^{2}\left(\Sigma_{t}\right)}^{2}+C \varepsilon t^{2 \sigma} \mathbb{D}^{2}(t)
\end{align*}
$$

Once we have proved (5.17), we absorb the first two terms on RHS 5.17) back into the left, at the expense of doubling the constants in front of the remaining terms. Afterwards, taking the square root, we conclude (5.5a). We then sum the square of (5.5a) over all $\iota$ with $|\iota|=N$ and take the square root, thereby concluding, in view of Definition 3.1, the desired estimate 5.5 b .

It remains for us to prove (5.17). We integrate equation (5.9) over $\mathbb{T}^{\mathfrak{D}}$ with respect to $d x$, note that the integrals of the last two terms on RHS $\sqrt{5.9}$ vanish, use the Cauchy-Schwarz inequality for integrals, and use the estimate $\|n\|_{L^{\infty}\left(\Sigma_{t}\right)} \leq 2$ (which follows from the bootstrap assumptions) to obtain

$$
\begin{align*}
t^{2 A_{*}+2}\left\|\partial^{\iota} \vec{e} n\right\|_{L^{2}\left(\Sigma_{t}\right)}^{2}+t^{2 A_{*}}\left\|\partial^{\iota} n\right\|_{L^{2}\left(\Sigma_{t}\right)}^{2} \leq & C_{*}\left\|t^{A_{*}+1} \partial^{\iota} \vec{e} n\right\|_{L^{2}\left(\Sigma_{t}\right)}\left\|t^{A_{*}+1} \partial^{\iota} \gamma\right\|_{L^{2}\left(\Sigma_{t}\right)} \\
& +\left\|t^{A_{*}} \partial^{\iota} n\right\|_{L^{2}\left(\Sigma_{t}\right)}\left\|t^{A+2} \mathfrak{N}^{(\iota)}\right\|_{L^{2}\left(\Sigma_{t}\right)}+\int_{\Sigma_{t}} t^{2 A_{*}+2}\left|\mathfrak{R}^{(\iota)}\right| d x \tag{5.18}
\end{align*}
$$

From (5.18), the error estimates in Lemma 5.6. our bootstrap assumptions (3.9), Young's inequality, and Definition 3.1, we conclude when $\varepsilon$ is sufficiently small, the desired bound 5.17 holds (for a different $C_{*}$, which is nevertheless independent of $A_{*}, N_{0}$, and $N$ ).

Proof of (5.6). The estimate 5.6 follows easily from Definition 3.1 and the estimates (5.4-5.5b).

### 5.5 Preliminary identities and inequalities for $k, \gamma, e$, and $\omega$

In this section, we derive preliminary low order and high order identities and inequalities for $\gamma, k, e$, and $\omega$ by using the evolution equations $2.22 \mathrm{a}-2.22 \mathrm{~b}$ and $2.23 \mathrm{a}-2.23 \mathrm{~b}$, as well as the key evolution equations for the structure coefficients provided by Proposition 5.7. Roughly, we control the inhomogeneous terms in their evolution equations in terms of our solution norms, and we derive differential versions of our energy identities. In Sects. 5.7 5.9, we will combine these preliminary results with related ones for the lapse and scalar field to derive our main a priori estimates, i.e., to prove Proposition 5.1.

### 5.5.1 The key evolution equation verified by the structure coefficients

To control the connection coefficients $\gamma_{I J B}$ at the low derivative levels, we will use the following proposition, which provides evolution equations for the structure coefficients $\gamma_{I J B}+\gamma_{J B I}$ of the orthonormal spatial frame $\left\{e_{I}\right\}_{I=1, \cdots, \mathfrak{D}}$. Although its proof is simple, the proposition is of profound significance for our main results. As we mentioned in Sect. 1.8 , the main virtues of the proposition are: it shows that up to error terms, the evolution equation system for the structure coefficients is diagonal, and it shows that the strength of the main linear terms driving the dynamics is controlled by the Kasner stability condition (1.8). The connection coefficients themselves can be controlled in terms of the structure coefficients via the identity (5.22).

Proposition 5.7 (The key evolution equations for the structure coefficients of the orthonormal frame). For solutions to the equations of Proposition 2.2, the structure coefficients of the orthonormal frame $\left\{e_{I}\right\}_{I=1, \cdots, \mathfrak{D}}$, namely $\gamma_{I J B}+\gamma_{J B I}$ with $I<J$ (see Remark 1.1), verify the following evolution equations, whose left-hand sides exhibit a diagonal structure, where the Kasner background scalars $\left\{\widetilde{e}_{I}^{i}\right\}_{I, i=1, \cdots, \mathfrak{D}}$ and $\left\{\widetilde{k}_{I J}\right\}_{I, J=1, \cdots, \mathfrak{D}}$ are defined in (see also Remark 2.6) and we recall that we do not sum underlined repeated indices:

$$
\begin{align*}
& \partial_{t}\left(\gamma_{I J B}+\gamma_{J B I}\right)+\frac{\left(\widetilde{q}_{\underline{I}}+\widetilde{q}_{\underline{J}}-\widetilde{q}_{\underline{B}}\right)}{t}\left(\gamma_{\underline{I J B}}+\gamma_{\underline{J B I}}\right) \\
& =(n-1)\left\{k_{I C} \gamma_{C J B}-k_{C I} \gamma_{B J C}-k_{J C} \gamma_{B I C}+k_{C I} \gamma_{J B C}+k_{B C} \gamma_{J I C}\right\} \\
& \quad+(n-1)\left\{k_{J C} \gamma_{C B I}-k_{C J} \gamma_{I B C}-k_{B C} \gamma_{I J C}+k_{C J} \gamma_{B I C}+k_{I C} \gamma_{B J C}\right\} \\
& \quad+\left(k_{I C}-\widetilde{k}_{I C}\right) \gamma_{C J B}-\left(k_{C I}-\widetilde{k}_{C I}\right) \gamma_{B J C}-\left(k_{J C}-\widetilde{k}_{J C}\right) \gamma_{B I C}+\left(k_{C I}-\widetilde{k}_{C I}\right) \gamma_{J B C}+\left(k_{B C}-\widetilde{k}_{B C}\right) \gamma_{J I C} \\
& \quad+\left(k_{J C}-\widetilde{k}_{J C}\right) \gamma_{C B I}-\left(k_{C J}-\widetilde{k}_{C J}\right) \gamma_{I B C}-\left(k_{B C}-\widetilde{k}_{B C}\right) \gamma_{I J C}+\left(k_{C J}-\widetilde{k}_{C J}\right) \gamma_{B I C}+\left(k_{I C}-\widetilde{k}_{I C}\right) \gamma_{B J C} \\
& \quad+n\left(e_{I}^{c}-\widetilde{e}_{I}^{c}\right) \partial_{c} k_{B J}-n\left(e_{J}^{c}-\widetilde{e}_{J}^{c}\right) \partial_{c} k_{B I}+n \widetilde{e}_{I}^{c} \partial_{c} k_{B J}-n \widetilde{e}_{J}^{c} \partial_{c} k_{B I}+\left(e_{I} n\right) k_{B J}-\left(e_{J} n\right) k_{B I} . \tag{5.19}
\end{align*}
$$

Moreover, for spatial coordinate multi-indices $\iota$ with $|\iota| \leq N_{0}$, the following evolution equation holds:

$$
\begin{equation*}
\partial_{t}\left[t^{q} \partial^{\iota}\left(\gamma_{I J B}+\gamma_{J B I}\right)\right]=\left\{q-\left(\widetilde{q}_{\underline{I}}+\widetilde{q}_{\underline{J}}-\widetilde{q}_{\underline{B}}\right)\right\} t^{q-1} \partial^{\iota}\left(\gamma_{\underline{I J B}}+\gamma_{\underline{J B I}}\right)+t^{q} \mathfrak{S}_{I J B}^{(\text {Border } ; \iota)}+t^{q} \mathfrak{S}_{I J B}^{(J u n k ; \iota)}, \tag{5.20}
\end{equation*}
$$

where

$$
\begin{align*}
\mathfrak{S}_{I J B}^{(\text {Border } ; \iota)}:= & \sum_{\iota_{1} \cup \iota_{2}=\iota} \partial^{\iota_{1}}(k-\widetilde{k}) \cdot \partial^{\iota_{2}} \gamma+\sum_{\iota_{1} \cup \iota_{2}=\iota} n \cdot \partial^{\iota_{1}}(e-\widetilde{e}) \cdot \partial \partial^{\iota_{2}} k,  \tag{5.21a}\\
\mathfrak{S}_{I J B}^{(J \text { unk } k \iota)}:= & \sum_{\iota_{1} \cup \iota_{2} \cup \iota_{3}=\iota} \partial^{\iota_{1}}(n-1) \cdot \partial^{\iota_{2}} k \cdot \partial^{\iota_{3}} \gamma+\sum_{\iota_{1} \cup \iota_{2}=\iota} \partial^{\iota_{1}} n \cdot \widetilde{e} \cdot \partial \partial^{\iota_{2}} k \\
& +\sum_{\iota_{1} \cup \iota_{2} \cup \iota_{3}=\iota, 1 \leq\left|\iota_{1}\right|} \partial^{\iota_{1}} n \cdot \partial^{\iota_{2}}(e-\widetilde{e}) \cdot \partial \partial^{\iota_{3}} k+\sum_{\iota_{1} \cup \iota_{2}=\iota} \partial^{\iota_{1}} \vec{e} n \cdot \partial^{\iota_{2}} k . \tag{5.21b}
\end{align*}
$$

Finally, the scalar function $\gamma_{I J B}$ can be expressed as a linear combination of three structure coefficients:

$$
\begin{equation*}
\gamma_{I J B}=\frac{1}{2}\left\{\gamma_{I J B}+\gamma_{J B I}\right\}+\frac{1}{2}\left\{\gamma_{B J I}+\gamma_{J I B}\right\}+\frac{1}{2}\left\{\gamma_{B I J}+\gamma_{I J B}\right\} \tag{5.22}
\end{equation*}
$$

Remark 5.8 (Connection between equation (5.19) and the stability condition 1.8 ). If we were to ignore the terms on RHS (5.19), then equation (5.19) would allow us to conclude that $\left|\gamma_{I J B}+\gamma_{J B I}\right| \lesssim t^{-\left(\widetilde{q}_{I}+\widetilde{q}_{J}-\widetilde{q}_{B}\right)}$. This makes the significance of the stability condition (1.8) for equation $\sqrt{5.19}$ clear: under this condition, the quantity $\max _{I, J, B=1, \cdots, \mathfrak{D}}^{I<J}\left|\gamma_{I J B}+\gamma_{J B I}\right|$ is integrable in $t$ near 0 , and by 5.22 (cf. Remark 1.1), $\max _{I, J, B=1, \cdots, \mathfrak{D}}\left|\gamma_{I J B}\right|$ is also integrable in $t$. In our ensuing analysis, we will in fact control the terms on RHS (5.19) and show that $\max _{I=1, \cdots, \mathfrak{D}}\left|\gamma_{I J B}\right|$ is integrable, which is a crucial step in our proof of stable blowup.
Remark 5.9. Interestingly, if we were to try to control the $\gamma_{I J B}$ 's at the low derivative levels by using the formula 2.27 and separately controlling each of the factors $e_{I}^{i}, \omega_{i}^{I}$, then we would not be able to close our estimates for the full range of Kasner exponents verifying the stability condition (1.8). In fact, since RHS 2.27 is cubic in $e_{I}^{i}, \boldsymbol{\omega}_{i}^{I}$ and their derivatives, the crudest version of that approach would yield only $\left|\gamma_{I J B}\right| \lesssim t^{-3 q}$, which, when $q$ is near 1 , is far too singular for proving stability. Moreover, the evolution equation 2.22 b for the $\gamma_{I J B}$ 's is not diagonal at the linear level and thus, a crude treatment based only on this equation would lead to far too singular estimates $S^{36}$ for the connection coefficients at the lower derivative levels. Thus, the diagonal structure revealed by Proposition 5.7 is essential to our overall argument.

Proof of Proposition 5.7. Equations (5.19) follow from the evolution equation 2.22 b , the definition of the background Kasner scalar functions in (2.58), the antisymmetry property 2.20), and straightforward algebraic computations. 5.20 then follows from differentiating 5.19 with $\partial^{\iota}$, applying the product rule, multiplying both sides of the resulting identity by $t^{q}$, and then commuting the factor of $t^{q}$ under the operator $\partial_{t}$ on the LHS and accounting for the commutator $\left[t^{q}, \partial_{t}\right]$.
(5.22) is an immediate consequence of the Koszul formula for an orthonormal frame and the antisymmetry property 2.20 .

### 5.5.2 Pointwise estimates for the error terms in the structure coefficient evolution equations

In the next lemma, we derive pointwise estimates at the low derivative levels for the error terms from Proposition 5.7

Lemma 5.10 (Pointwise estimates for the error terms in the structure coefficient evolution equations at orders $\leq N_{0}$ ). Assume that the bootstrap assumptions (3.9) hold. There exists a constant $C=C_{N, N_{0}, A_{*}, \mathfrak{D}, q, \sigma}>0$ such that if $N_{0} \geq 1$ and $N$ is sufficiently large in a manner that depends on $N_{0}, A_{*}, \mathfrak{D}, q$, and $\sigma$, and if $\varepsilon$ is sufficiently small (in a manner that depends on $N, N_{0}, A_{*}, \mathfrak{D}, q$, and $\sigma$ ), then the following pointwise estimates hold on $\left(T_{B o o t}, 1\right] \times \mathbb{T}^{\mathfrak{D}}$ for the error terms $\mathfrak{S}_{I J B}^{(\text {Border } ; \iota)}$ and $\mathfrak{S}_{I J B}^{(\text {Junk; })}$ defined in 5.21a)-5.21b):

$$
\begin{align*}
t^{q} \sum_{|\iota| \leq N_{0}} \sum_{I, J, B=1, \cdots, \mathfrak{D}}\left|\mathfrak{S}_{I J B}^{(\text {Border } ; \iota)}\right|(t, x) \leq & C \varepsilon t^{q-1} \sum_{|\iota| \leq N_{0}} \sum_{I, J, B=1, \cdots, \mathfrak{D}}\left|\partial^{\iota} \gamma_{I J B}\right|(t, x) \\
& +C \varepsilon t^{q-1} \sum_{|\iota| \leq N_{0}} \sum_{I, i=1, \cdots, \mathfrak{D}}\left|\partial^{\iota}\left(e_{I}^{i}-\widetilde{e}_{I}^{i}\right)\right|(t, x), \tag{5.23a}
\end{align*}
$$

[^21]\[

$$
\begin{equation*}
t^{q} \sum_{|\iota| \leq N_{0}} \sum_{I, J, B=1, \cdots, \mathfrak{D}}\left|\mathfrak{S}_{I J B}^{(\text {Junk } ; \iota)}\right|(t, x) \leq C t^{-1+\sigma} \mathbb{D}(t) \tag{5.23b}
\end{equation*}
$$

\]

Proof. Based on equations $5.21 \mathrm{a}-5.21 \mathrm{~b}$, the estimates $5.23 \mathrm{a}-5.23 \mathrm{~b}$ follow as straightforward consequences of (2.58), the inequalities in (3.6), Definition 3.1, the bootstrap assumptions (3.9), and the already derived low order estimates (5.4) for $n$. Note in particular that we have used the fact that the low order norm (3.8a) controls $k-\widetilde{k}$ at derivative levels $\leq N_{0}+1$ (see Remark 3.2; for example, for $|\iota| \leq N_{0}$, this allows us to pointwise bound the magnitude of the sum $\sum_{\iota_{1} \iota_{2}=\iota} n \cdot \partial^{\iota_{1}}(e-\widetilde{e}) \cdot \partial \partial^{\iota_{2}} k$ on RHS 5.21a) by $\lesssim \varepsilon t^{-1} \sum_{|\iota| \leq N_{0}}\left|\partial^{\iota}(e-\widetilde{e})\right|(t, x)$.

### 5.5.3 Absence of certain structure coefficients in polarized $U(1)$-symmetry

In the next lemma, we show that for polarized $U(1)$-symmetric metrics with $\mathfrak{D}=3$, relative to an orthonormal spatial frame of the type provided by Lemma 2.4 all structure coefficients with three distinct indices vanish. As we explained in Remark 1.2 , this vanishing is crucial for the proof of our main results in the case of the Einstein-vacuum equations in $1+3$ dimensions under polarized $U(1)$-symmetry.

Lemma 5.11 (The vanishing of key variables in polarized $U(1)$-symmetry). Suppose that $\mathfrak{D}=3$ and that g is a polarized $U(1)$-symmetric metric satisfying the hypotheses and conclusions of Lemma 2.4. Moreover, let $\left\{e_{1}, e_{2}, e_{3}\right\}$ be an orthonormal spatial frame satisfying the hypotheses and conclusions of Lemma 2.4. In particular, $e_{3}=\left(g_{33}\right)^{-\frac{1}{2}} \partial_{3}$ and $\mathcal{L}_{\partial_{3}} e_{I}=0$ for $I=1,2,3$, where $\partial_{3}$ is the hypersurface-orthogonal Killing vectorfield. Then the following spatial connection coefficients vanish:

$$
\begin{equation*}
\gamma_{123}=\gamma_{231}=\gamma_{312}=0 \tag{5.24}
\end{equation*}
$$

Moreover, under the same assumptions,

$$
\gamma_{I J B}+\gamma_{J B I}= \begin{cases}0, & \text { if } I=J  \tag{5.25}\\ 0, & \text { if } I, J, B \text { are distinct. }\end{cases}
$$

Proof. Under the assumptions and conclusions of Lemma $2.4, \partial_{3}$ is parallel to $e_{3}$ and orthogonal to $\partial_{1}$ and $\partial_{2}$, and we have $e_{1}^{3}=e_{2}^{3}=e_{3}^{1}=e_{3}^{2}=\omega_{3}^{1}=\omega_{3}^{2}=\omega_{1}^{3}=\omega_{2}^{3}=e_{3} e_{I}^{i}=0$. Hence, using 2.27 we compute

$$
\begin{aligned}
\gamma_{123} & =\frac{1}{2}\left\{\omega_{l}^{3}\left(e_{1} e_{2}^{l}-e_{2} e_{1}^{l}\right)-\omega_{l}^{1}\left(e_{2} e_{3}^{l}-e_{3} e_{2}^{l}\right)+\omega_{l}^{2}\left(e_{3} e_{1}^{l}-e_{1} e_{3}^{l}\right)\right\}=0 \\
\gamma_{231} & =\frac{1}{2}\left\{\omega_{l}^{1}\left(e_{2} e_{3}^{l}-e_{3} e_{2}^{l}\right)-\omega_{l}^{2}\left(e_{3} e_{1}^{l}-e_{1} e_{3}^{l}\right)+\omega_{l}^{3}\left(e_{1} e_{2}^{l}-e_{2} e_{1}^{l}\right)\right\}=0 \\
\gamma_{312} & =\frac{1}{2}\left\{\omega_{l}^{2}\left(e_{3} e_{1}^{l}-e_{1} e_{3}^{l}\right)-\omega_{l}^{3}\left(e_{1} e_{2}^{l}-e_{2} e_{1}^{l}\right)+\omega_{l}^{1}\left(e_{2} e_{3}^{l}-e_{3} e_{2}^{l}\right)\right\}=0
\end{aligned}
$$

which yields 5.24 . A more conceptual justification of the above computations is that in the present setting, $g\left(\left[e_{I}, e_{J}\right], e_{B}\right)=0$ whenever $I, J, B$ are distinct indices; using this fact and the Koszul formula, we conclude that $\gamma_{I J B}=0$ whenever $I, J, B$ are distinct indices, as desired.
(5.25) follows from (5.24) and the antisymmetry property 2.20).

Remark 5.12 (The role of polarized $U(1)$-symmetry). In proving our stable Big Bang formation results for the Einstein-vacuum equations in $1+3$ dimensions, there is precisely one way in which our polarized $U(1)$-symmetry assumption is important for our analysis: it allows us to use the results of Lemma 5.11 , see also Remark 1.2 and the end of the proof of Proposition 5.26 . Put differently, if we were to assume the conclusions $5.24-5$ of the lemma, then the rest of our proof of stable Big Bang formation would go through.

### 5.5.4 Commuted evolution equations for $e$ and $\omega$

In this section, we provide the evolution equations that we will use to control the scalar functions $\left\{e_{I}^{i}\right\}_{I, i=1, \cdots, \mathfrak{D}}$ and $\left\{\boldsymbol{\omega}_{i}^{I}\right\}_{I, i=1, \cdots, \mathfrak{D}}$ as well as their derivatives.

Lemma 5.13 (Evolution equations for $\left\{e_{I}^{i}\right\}_{I, i=1, \cdots, \mathfrak{D}},\left\{\omega_{i}^{I}\right\}_{I, i=1, \cdots, \mathfrak{D}}$, and their derivatives). The evolution equations 2.23a-2.23b can be rewritten as follows, where the Kasner background scalars $\left\{\widetilde{e}_{I}^{i}\right\}_{I, i=1, \cdots, \mathfrak{D}}$, $\left\{\widetilde{\omega}_{i}^{I}\right\}_{I, i=1, \cdots, \mathfrak{D}}$, and $\left\{\widetilde{k}_{I J}\right\}_{I, J=1, \cdots, \mathfrak{D}}$ are defined in (see also Remark 2.58), and we recall that we do not sum over repeated underlined indices:

$$
\begin{align*}
\partial_{t}\left(e_{I}^{i}-\widetilde{e}_{I}^{i}\right)+\frac{\widetilde{q}_{\underline{I}}}{t}\left(e_{\underline{I}}^{i}-\widetilde{e}_{\underline{I}}^{i}\right)= & (n-1) k_{I C}\left(e_{C}^{i}-\widetilde{e}_{C}^{i}\right)+\left(k_{I C}-\widetilde{k}_{I C}\right)\left(e_{C}^{i}-\widetilde{e}_{C}^{i}\right)  \tag{5.26a}\\
& +(n-1) \widetilde{k}_{I C} \widetilde{e}_{C}^{i}+n\left(k_{I C}-\widetilde{k}_{I C}\right) \widetilde{e}_{C}^{i} \\
\partial_{t}\left(\omega_{i}^{I}-\widetilde{\omega}_{i}^{I}\right)-\frac{\widetilde{q}_{I}}{t}\left(\omega_{i}^{I}-\widetilde{\omega}_{i}^{I}\right)= & -(n-1) k_{I C}\left(\omega_{i}^{C}-\widetilde{\omega}_{i}^{C}\right)-\left(k_{I C}-\widetilde{k}_{I C}\right)\left(\omega_{i}^{C}-\widetilde{\omega}_{i}^{C}\right)  \tag{5.26b}\\
& -(n-1) \widetilde{k}_{I C} \widetilde{\omega}_{i}^{C}-n\left(k_{I C}-\widetilde{k}_{I C}\right) \widetilde{\omega}_{i}^{C}
\end{align*}
$$

Moreover, let $\iota$ be a spatial multi-index with $|\iota| \leq N$ and let $P \geq 0$ be a real number. Then the following equations hold:

$$
\begin{align*}
\partial_{t}\left[t^{P} \partial^{\iota}\left(e_{I}^{i}-\widetilde{e}_{I}^{i}\right)\right] & =\left(P-\widetilde{q}_{\underline{I}}\right) t^{P-1} \partial^{\iota}\left(e_{\underline{I}}^{i}-\widetilde{e}_{\underline{I}}^{i}\right)+t^{P} \mathfrak{E}_{I}^{i ;(\text { Border } ; \iota)}+t^{P} \mathfrak{E}_{I}^{i ;(\text { Junk } ; \iota)},  \tag{5.27a}\\
\partial_{t}\left[t^{P} \partial^{\iota}\left(\omega_{i}^{I}-\widetilde{\omega}_{i}^{I}\right)\right] & =\left(P+\widetilde{q}_{\underline{I}}\right) t^{P-1} \partial^{\iota}\left(\omega_{i}^{I}-\widetilde{\omega}_{i}^{I}\right)+t^{P} \mathfrak{O}_{I}^{i ;(\text { Border } ; \iota)}+t^{P} \mathfrak{V}_{I}^{i ;(\text { Junk } ; \iota)} \tag{5.27b}
\end{align*}
$$

where

$$
\begin{align*}
\mathfrak{E}_{I}^{i ;(\text { Border } ; \iota)} & :=\sum_{\iota_{1} \cup \iota_{2} \cup \iota_{3}=\iota} \partial^{\iota_{1}}(n-1) \cdot \partial^{\iota_{2}} k \cdot \partial^{\iota_{3}}(e-\widetilde{e})+\sum_{\iota_{1} \cup \iota_{2}=\iota} \partial^{\iota_{1}}(k-\widetilde{k}) \cdot \partial^{\iota_{2}}(e-\widetilde{e}),  \tag{5.28a}\\
\mathfrak{E}_{I}^{i ;(\text { Junk } ; \iota)}: & =\partial^{\iota}(n-1) \cdot \widetilde{k} \cdot \widetilde{e}+\sum_{\iota_{1} \cup \iota_{2}=\iota} \partial^{\iota_{1}} n \cdot \partial^{\iota_{2}}(k-\widetilde{k}) \cdot \widetilde{e}  \tag{5.28b}\\
\mathfrak{O}_{I}^{i ;(\text { Border } ; \iota)} & :=\sum_{\iota_{1} \cup \iota_{2} \cup \iota_{3}=\iota} \partial^{\iota_{1}}(n-1) \cdot \partial^{\iota_{2}} k \cdot \partial^{\iota_{3}}(\omega-\widetilde{\omega})+\sum_{\iota_{1} \cup \iota_{2}=\iota} \partial^{\iota_{1}}(k-\widetilde{k}) \cdot \partial^{\iota_{2}}(\omega-\widetilde{\omega}),  \tag{5.28c}\\
\mathfrak{O}_{I}^{i ;(\text { Junk } ; \iota)} & :=\partial^{\iota}(n-1) \cdot \widetilde{k} \cdot \widetilde{\omega}+\sum_{\iota_{1} \cup \iota_{2}=\iota} \partial^{\iota_{1}} n \cdot \partial^{\iota_{2}}(k-\widetilde{k}) \cdot \widetilde{\omega} \tag{5.28~d}
\end{align*}
$$

Proof. $5.26 \mathrm{a}-\sqrt{5.26 \mathrm{~b}}$ follow from equations $2.23 \mathrm{a}-\sqrt{2.23 \mathrm{~b}}$ and straightforward algebraic computations. (5.27a) - 5.27 b$)$ then follow from differentiating (5.26a)-(5.26b) with $\partial^{\iota}$, using the Leibniz rule, multiplying both sides of the resulting equations by $t^{P}$, and commuting the factors of $t^{P}$ under the operator $\partial_{t}$ on the LHSs and accounting for the commutator $\left[t^{P}, \partial_{t}\right]$.

### 5.5.5 Pointwise estimates for the error terms in the frame component evolution equations

In this section, at the low derivative levels, we derive pointwise estimates for the error terms in the evolution equations of Lemma 5.13 .

Lemma 5.14 (Pointwise estimates for the error terms in the evolution equations for $\partial^{\leq} N_{0}(e-\widetilde{e})$ and $\left.\partial \leq N_{0}(\omega-\widetilde{\omega})\right)$. Assume that the bootstrap assumptions (3.9) hold. There exists a constant $C=C_{N, N_{0}, A_{*}, \mathfrak{D}, q, \sigma}>$ 0 such that if $N_{0} \geq 1$ and $N$ is sufficiently large in a manner that depends on $N_{0}, A_{*}, \mathfrak{D}, q$, and $\sigma$, and if $\varepsilon$ is sufficiently small (in a manner that depends on $N, N_{0}, A_{*}, \mathfrak{D}, q$, and $\sigma$ ), then the error terms $\mathfrak{E}_{I}^{i ;(\text { Border } ; \iota)}$, $\mathfrak{E}_{I}^{i ;(\text { Junk } ; \iota)}$, $\mathfrak{D}_{I}^{i ;(\text { Border } ; \iota)}$, and $\mathfrak{O}_{I}^{i ;(\text { Junk } ; \iota)}$ defined in 5.28 a - 5.28 d$)$ verify the following pointwise estimates for $(t, x) \in\left(T_{\text {Boot }}, 1\right] \times \mathbb{T}^{\mathfrak{D}}$, where the Kasner background scalars $\left\{\widetilde{e}_{I}^{i}\right\}_{I, i=1, \cdots, \mathfrak{D}}$ and $\left\{\widetilde{\omega}_{i}^{I}\right\}_{I, i=1, \cdots, \mathfrak{D}}$ are defined in 2.58):

$$
\begin{align*}
& \sum_{|\iota| \leq N_{0}} \sum_{I, i=1, \cdots, \mathfrak{D}} t^{q}\left|\mathfrak{E}_{I}^{i ;(\text { Border } ; \iota)}\right|(t, x) \leq C \varepsilon t^{q-1} \sum_{|\iota| \leq N_{0}} \sum_{I, i=1, \cdots, \mathfrak{D}}\left|\partial^{\iota}\left(e_{I}^{i}-\widetilde{e}_{I}^{i}\right)\right|(t, x),  \tag{5.29a}\\
& \sum_{|\iota| \leq N_{0}} \sum_{I, i=1, \cdots, \mathfrak{D}} t^{q}\left|\mathfrak{E}_{I}^{i ;(\text { Junk } ; \iota)}\right|(t, x) \leq C t^{-1+\sigma} \mathbb{D}(t),  \tag{5.29b}\\
& \sum_{|\iota| \leq N_{0}} \sum_{I, i=1, \cdots, \mathfrak{D}} t^{q}\left|\mathfrak{O}_{I}^{i ;(\text { Border } ; \iota)}\right|(t, x) \leq C \varepsilon t^{q-1} \sum_{|\iota| \leq N_{0}} \sum_{I, i=1, \cdots, \mathfrak{D}}\left|\partial^{\iota}\left(\omega_{i}^{I}-\widetilde{\omega}_{i}^{I}\right)\right|(t, x), \tag{5.29c}
\end{align*}
$$

$$
\begin{equation*}
\sum_{|\iota| \leq N_{0}} \sum_{I, i=1, \cdots, \mathfrak{D}} t^{q}\left|\mathfrak{D}_{I}^{i ;(J u n k ; \iota)}\right|(t, x) \leq C t^{-1+\sigma} \mathbb{D}(t) \tag{5.29d}
\end{equation*}
$$

Proof. The lemma follows from the expressions 5.28a -5.28 d , the bootstrap assumptions, the definition of the lower order norms $(3.8 \mathrm{a})$, the explicit formulas $(2.58)$, the inequalities in $(3.6)$, and the already derived low order estimates 5.4 for $n{ }^{37}$

### 5.5.6 $\quad L^{2}$-control of the error terms in the top-order commuted frame component evolution equations

In this section, at the top-order derivative level, we derive $L^{2}$ estimates for the error terms in the evolution equations of Lemma 5.13 .

Lemma 5.15 ( $L^{2}$-control of the error terms in the top-order commuted frame component evolution equations). Recall that $\mathbb{H}_{(\gamma, k)}(t), \mathbb{H}_{(e, \omega)}(t)$, and $\mathbb{D}(t)$ are norms from Definition 3.1, and assume that the bootstrap assumptions (3.9) hold. There exists a constant $C=C_{N, N_{0}, A_{*}, \mathfrak{D}, q, \sigma}>0$ such that if $N_{0} \geq 1$ and $N$ is sufficiently large in a manner that depends on $N_{0}, A_{*}, \mathfrak{D}, q$, and $\sigma$, and if $\varepsilon$ is sufficiently small (in a manner that depends on $N, N_{0}, A_{*}, \mathfrak{D}, q$, and $\sigma$ ), then the error terms $\mathfrak{E}_{I}^{i ;(\text { Border } ; \iota)}, \mathfrak{E}_{I}^{i ;(J u n k ; \iota)}, \mathfrak{O}_{I}^{i ;(\text { Border } ; \iota)}$, and $\mathfrak{O}_{I}^{i ;(J u n k ; \iota)}$ defined in (5.28a) 5.28 d verify the following $L^{2}$ estimates for $t \in\left(T_{B o o t}, 1\right]$ :

$$
\begin{align*}
& t^{A_{*}+q} \sqrt{\sum_{|\iota|=N} \sum_{I, i=1, \cdots, \mathfrak{D}}\left\|\mathfrak{E}_{I}^{i ;(\text { Border } ; \iota)}\right\|_{L^{2}\left(\Sigma_{t}\right)}^{2}} \leq C \varepsilon t^{-1} \mathbb{H}_{(\gamma, k)}(t)+C \varepsilon t^{-1} \mathbb{H}_{(e, \omega)}(t)+C t^{-1+\sigma} \mathbb{D}(t)  \tag{5.30a}\\
& t^{A_{*}+q} \sqrt{\sum_{|\iota|=N} \sum_{I, i=1, \cdots, \mathfrak{D}}\left\|\mathfrak{F}_{I}^{i ;(\text { Junk } ; \iota)}\right\|_{L^{2}\left(\Sigma_{t}\right)}^{2}} \leq C t^{-1+\sigma} \mathbb{D}(t)  \tag{5.30b}\\
& t^{A_{*}+q}  \tag{5.30c}\\
& \sum_{|\iota|=N} \sum_{I, i=1, \cdots, \mathfrak{D}}\left\|\mathfrak{O}_{I}^{i ;(\text { Border } ; \iota)}\right\|_{L^{2}\left(\Sigma_{t}\right)}^{2} \leq \varepsilon t^{-1} \mathbb{H}_{(\gamma, k)}(t)+C \varepsilon t^{-1} \mathbb{H}_{(e, \omega)}(t)+C t^{-1+\sigma} \mathbb{D}(t),  \tag{5.30d}\\
& t^{A_{*}+q} \sqrt{\sum_{|\iota|=N} \sum_{I, i=1, \cdots, \mathfrak{D}}\left\|\mathfrak{V}_{I}^{i ;(\text { Junk } ; \iota)}\right\|_{L^{2}\left(\Sigma_{t}\right)}^{2}} \leq C t^{-1+\sigma} \mathbb{D}(t)
\end{align*}
$$

Proof. The lemma follows from the expressions $5.28 \mathrm{a}-5.28 \mathrm{~d}$, the explicit formulas 2.58 , the inequalities in (3.6), Definition 3.1, the bootstrap assumptions, the product inequality 4.3), and the already derived estimates (5.4)-(5.6) for $n$.

### 5.5.7 Commuted equations for $k$ and $\gamma$

In this section, we provide the evolution equations that we will use to control the scalar functions $\left\{k_{I J}\right\}_{I, J=1, \cdots, \mathcal{D}}$ and $\left\{\gamma_{I J B}\right\}_{I, J, B=1, \cdots, \mathfrak{D}}$ as well as their derivatives.

Lemma $5.16\left(\partial^{\iota}\right.$-commuted equations for $\gamma$ and $\left.k\right)$. Let $\iota$ be a spatial multi-index with $|\iota| \leq N$ and let $P \geq 0$ be a real number. For solutions to the equations of Proposition 2.2, the following evolution equations hold, where the Kasner background scalars $\left\{\widetilde{k}_{I J}\right\}_{I, J=1, \cdots, \mathfrak{D}}$ and $\widetilde{\psi}$ are defined in (2.58) (see also Remark 2.6):

$$
\begin{align*}
\partial_{t}\left[t^{P} \partial^{\iota}\left(k_{I J}-\widetilde{k}_{I J}\right)\right]= & (P-1) t^{P-1} \partial^{\iota}\left(k_{I J}-\widetilde{k}_{I J}\right)+t^{P} n e_{C} \partial^{\iota} \gamma_{I J C}-t^{P} n e_{I} \partial^{\iota} \gamma_{C J C}-t^{P} e_{I} \partial^{\iota} e_{J} n \\
& +t^{P-1} \mathfrak{K}_{I J}^{(\text {Border } ; \iota)}+t^{P} \mathfrak{K}_{I J}^{(J u n k ; \iota)}  \tag{5.31a}\\
\partial_{t}\left(t^{P} \partial^{\iota} \gamma_{I J B}\right)= & P t^{P-1} \partial^{\iota} \gamma_{I J B}+t^{P} n e_{B} \partial^{\iota} k_{J I}-t^{P} n e_{J} \partial^{\iota} k_{B I} \\
& +t^{P} \mathfrak{G}_{I J B}^{(\text {Border } ; \iota)}+t^{P} \mathfrak{G}_{I J B}^{(J u n k ; \iota)}  \tag{5.31b}\\
t^{P} e_{C} \partial^{\iota} k_{C I}= & t^{P} \mathfrak{M}_{I}^{(\text {Border } ; \iota)}+t^{P} \mathfrak{M}_{I}^{(J u n k ; \iota)}, \tag{5.31c}
\end{align*}
$$

where

$$
\begin{equation*}
\mathfrak{K}_{I J}^{(\text {Border } ; \iota)}:=\partial^{\iota}(n-1) \cdot \widetilde{k}+\sum_{\iota_{1} \cup \iota_{2}=\iota} \partial^{\iota_{1}}(n-1) \cdot \partial^{\iota_{2}}(k-\widetilde{k}), \tag{5.32a}
\end{equation*}
$$

[^22]\[

$$
\begin{align*}
& \mathfrak{K}_{I J}^{(\text {Junk } ; \iota)}:= \sum_{\iota_{1} \cup \iota_{2}=\iota,\left|\iota_{2}\right|<|\iota|} \partial^{\iota_{1}} e \cdot \partial \partial^{\iota_{2}} \vec{e} n+\sum_{\iota_{1} \cup \iota_{2}=\iota,\left|\iota_{2}\right|<|\iota|} \partial^{\iota_{1}} \gamma \cdot \partial^{\iota_{2}} \overrightarrow{e n}  \tag{5.32b}\\
&+\sum_{\iota_{1} \cup \iota_{2} \cup \iota_{3}=\iota,\left|\iota_{3}\right|<|\iota|} \partial^{\iota_{1}} n \cdot \partial^{\iota_{2}} e \cdot \partial^{\iota_{3}} \partial \gamma+\sum_{v \in\{\gamma, \vec{e} \psi\}} \sum_{\iota_{1} \cup \iota_{2} \cup \iota_{3}=\iota} \partial^{\iota_{1}} n \cdot \partial^{\iota_{2}} v \cdot \partial^{\iota_{3}} v, \\
& \mathfrak{G}_{I J B}^{(\text {Border } ; \iota)}:= n \cdot \widetilde{k} \cdot \partial^{\iota} \gamma+\sum_{\iota_{1} \cup \iota_{2}=\iota} n \partial^{\iota_{1}}(k-\widetilde{k}) \cdot \partial^{\iota_{2}} \gamma+\widetilde{k} \cdot \partial^{\iota} \vec{e} n+\sum_{\iota_{1} \cup \iota_{2}=\iota} \partial^{\iota_{1}}(k-\widetilde{k}) \cdot \partial^{\iota_{2}} \vec{e} n,  \tag{5.32c}\\
& \mathfrak{G}_{I J B}^{(\text {Junk } ; \iota)}:= \sum_{\iota_{1} \cup \iota_{2} \cup \iota_{3}=\iota,\left|\iota_{1}\right| \geq 1} \partial^{\iota_{1}} n \cdot \partial^{\iota_{2}} k \cdot \partial^{\iota_{3}} \gamma+\sum_{\iota_{1} \cup \iota_{2} \cup \iota_{3}=\iota,\left|\iota_{3}\right|<|\iota|} \partial^{\iota_{1}} n \cdot \partial^{\iota_{2}} e \cdot \partial^{\iota_{3}} \partial k,  \tag{5.32d}\\
& \mathfrak{M}_{I}^{(\text {Border } ; \iota)}:=\widetilde{k} \cdot \partial^{\iota} \gamma+\sum_{\iota_{1} \cup \iota_{2}=\iota} \partial^{\iota_{1}}(k-\widetilde{k}) \cdot \partial^{\iota_{2}} \gamma+\partial_{t} \widetilde{\psi} \cdot \partial^{\iota} \vec{e} \psi+\sum_{\iota_{1} \cup \iota_{2}=\iota} \partial^{\iota_{1}}\left(e_{0} \psi-\partial_{t} \widetilde{\psi}\right) \cdot \partial^{\iota_{2}} \vec{e} \psi,  \tag{5.32e}\\
& \mathfrak{M}_{I}^{(\text {Junk } ; \iota)}:= \sum_{\iota_{1} \cup \iota_{2}=\iota,\left|\iota_{2}\right|<|\iota|} \partial^{\iota_{1}} e \cdot \partial \partial^{\iota_{2}} k . \tag{5.32f}
\end{align*}
$$
\]

Proof. Equations 5.31a -5.31b follow from straightforward computations based on first multiplying equations 2.22a)-2.22b) by $n$, using that $\partial_{t}=n e_{0}$, differentiating the resulting equations with $\partial^{\iota}$, applying the Leibniz rule, multiplying both sides of the resulting identities by $t^{P}$, and then commuting the factor of $t^{P}$ under the operator $\partial_{t}$ on the LHSs and accounting for the commutator $\left[t^{P}, \partial_{t}\right]$. Similarly, equation (5.31c) follows from differentiating equation 2.26 b with $\partial^{\iota}$, applying the Leibniz rule, and then multiplying both sides of the resulting identity by $t^{P}$

### 5.5.8 Pointwise estimates for the error terms in the spatial metric evolution equations

In this section, we derive pointwise estimates for the error terms in the equations of Lemma 5.16 that we will later use to control $k-\widetilde{k}$ at derivative levels $\leq N_{0}+1$.
Lemma 5.17 (Pointwise estimates for the error terms in the evolution equations for $\left.\partial^{\leq} N_{0}+1(k-\widetilde{k})\right)$. Recall that $\mathbb{D}(t)$ is a norm from Definition 3.1. Assume that the bootstrap assumptions (3.9) hold. There exists a constant $C=C_{N, N_{0}, A_{*}, \mathfrak{D}, q, \sigma}>0$ such that if $N_{0} \geq 1$ and $N$ is sufficiently large in a manner that depends on $N_{0}, A_{*}, \mathfrak{D}, q$, and $\sigma$, and if $\varepsilon$ is sufficiently small (in a manner that depends on $N, N_{0}, A_{*}, \mathfrak{D}, q$, and $\sigma$ ), then the following pointwise estimates hold for $(t, x) \in\left(T_{\text {Boot }}, 1\right] \times \mathbb{T}^{\mathfrak{D}}$, where $\mathfrak{K}_{I J}^{(\text {Border } ; \iota)}$ and $\mathfrak{K}_{I J}^{(J u n k ; \iota)}$ are defined in 5.32a-5.32b:

$$
\begin{array}{r}
\sum_{|\iota| \leq N_{0}+1} \sum_{I, J=1, \cdots, \mathfrak{D}} t\left|n e_{C} \partial^{\iota} \gamma_{I J C}-n e_{I} \partial^{\iota} \gamma_{C J C}-e_{I} \partial^{\iota} e_{J} n\right|(t, x) \leq C t^{-1+\sigma} \mathbb{D}(t), \\
\sum_{|\iota| \leq N_{0}+1} \sum_{I, J=1, \cdots, \mathfrak{D}}\left|\mathfrak{K}_{I J}^{(\text {Border } ; \iota)}\right|(t, x)+\sum_{|\iota| \leq N_{0}+1} \sum_{I, J=1, \cdots, \mathfrak{D}} t\left|\mathfrak{K}_{I J}^{(\text {Junk } ; \iota)}\right|(t, x) \leq C t^{-1+\sigma} \mathbb{D}(t) . \tag{5.33b}
\end{array}
$$

Proof. The lemma follows from the explicit formulas 2.58, the inequalities in (3.6), the bootstrap assumptions, the interpolation estimates of Lemma 4.2 (see Remark 4.4), and the already derived lower order estimate (5.4) for $n-1$.

Remark 5.18 ("Borderline" sometimes refers only to the top order). Although the term $\mathfrak{K}_{I J}^{(B o r d e r ; \iota)}$ is explicitly labeled as borderline, the estimate 5.33 b reveals that at the lower orders, it is actually below borderline. Thus, our notion of "borderline" corresponds to the behavior of terms with respect to the toporder energy estimates, a context in which $\mathfrak{K}_{I J}^{(\text {Border } ; \iota)}$ is indeed borderline; see the $C_{*}$-involving term on RHS 5.35a in Lemma 5.21 below. Despite the differing "strength" of $\mathfrak{K}_{I J}^{(\text {Border } ; \iota)}$ at the high and low orders, we choose to keep the same notation throughout the paper to make it easier for the reader to follow the overall argument. Similar remarks apply to other terms that are labeled as "borderline".

### 5.5.9 Differential energy identity for the second fundamental form and connection coefficients

We will derive our top-order energy estimates for the second fundamental form and connection coefficients by integrating the differential identity provided by the following lemma.

Lemma 5.19 (Top-order differential energy identity for $\left\{k_{I J}\right\}_{I, J=1, \cdots, \mathfrak{D}}$ and $\left.\left\{\gamma_{I J B}\right\}_{I, J, B=1, \cdots, \mathfrak{D}}\right)$. Let $\iota$ be a top-order spatial multi-index, i.e., $|\iota|=N$. For solutions to the $\partial^{\iota}$-commuted equations (5.31a)-(5.31c) with $P:=A_{*}+1$, the following differential energy identity holds, where the error terms $\mathfrak{K}_{I J}^{(\text {Border } ; \iota)}$, $\mathfrak{K}_{I J}^{(J u n k ; \iota)}$, $\mathfrak{G}_{I J B}^{(\text {Border } ; \iota)}, \mathfrak{G}_{I J B}^{(\text {Junk } ; \iota)}, \mathfrak{M}_{I}^{(\text {Border } ; \iota)}$, and $\mathfrak{M}_{I}^{(\text {Junk } ; \iota)}$ are defined in 5.32a 5.32 f$):$

$$
\begin{align*}
\partial_{t} & \left\{\left(t^{A_{*}+1} \partial^{\iota} k_{I J}\right)\left(t^{A_{*}+1} \partial^{\iota} k_{I J}\right)\right\}+\frac{1}{2} \partial_{t}\left\{\left(t^{A_{*}+1} \partial^{\iota} \gamma_{I J B}\right)\left(t^{A_{*}+1} \partial^{\iota} \gamma_{I J B}\right)\right\} \\
= & \frac{2 A_{*}}{t}\left(t^{A_{*}+1} \partial^{\iota} k_{I J}\right)\left(t^{A_{*}+1} \partial^{\iota} k_{I J}\right)+\frac{\left(A_{*}+1\right)}{t}\left(t^{A_{*}+1} \partial^{\iota} \gamma_{I J B}\right)\left(t^{A_{*}+1} \partial^{\iota} \gamma_{I J B}\right) \\
& +2\left(t^{A_{*}+1} \partial^{\iota} k_{I J}\right)\left(t^{A_{*}} \mathfrak{K}_{I J}^{(\text {Border } ; \iota)}+t^{A_{*}+1} \mathfrak{K}_{I J}^{(\text {Junk } ; \iota)}\right) \\
& +\left(t^{A_{*}+1} \partial^{\iota} \gamma_{I J B}\right)\left(t^{A_{*}+1} \mathfrak{G}_{I J B}^{(\text {Border } ; \iota)}+t^{A_{*}+1} \mathfrak{G}_{I J B}^{(J \text { unk } ; \iota)}\right) \\
& +2\left(t^{A_{*}+1} \partial^{\iota} e_{J} n\right)\left(t^{A_{*}+1} \mathfrak{M}_{J}^{(\text {Border } ; \iota)}+t^{A_{*}+1} \mathfrak{M}_{J}^{(J u n k ; \iota)}\right) \\
& +2 n\left(t^{A_{*}+1} \partial^{\iota} \gamma_{C J C}\right)\left(t^{A_{*}+1} \mathfrak{M}_{J}^{(\text {Border } ; \iota)}+t^{A_{*}+1} \mathfrak{M}_{J}^{(J u n k ; \iota)}\right) \\
& +2\left(\partial_{c} e_{I}^{c}\right)\left(t^{A_{*}+1} \partial^{\iota} e_{J} n\right)\left(t^{A_{*}+1} \partial^{\iota} k_{I J}\right) \\
& +2\left\{\partial_{c}\left(n e_{I}^{c}\right)\right\}\left(t^{A_{*}+1} \partial^{\iota} k_{I J}\right)\left(t^{A_{*}+1} \partial^{\iota} \gamma_{C J C}\right)-2\left\{\partial_{c}\left(n e_{C}^{c}\right)\right\}\left(t^{A_{*}+1} \partial^{\iota} k_{I J}\right)\left(t^{A_{*}+1} \partial^{\iota} \gamma_{I J C}\right) \\
& -2 \partial_{c}\left\{t^{2 A_{*}+2} e_{I}^{c}\left(\partial^{\iota} e_{J} n\right) \partial^{\iota} k_{I J}\right\}-2 \partial_{c}\left\{t^{2 A_{*}+2} e_{I}^{c} n\left(\partial^{\iota} k_{I J}\right) \partial^{\iota} \gamma_{C J C}\right\}+2 \partial_{c}\left\{t^{2 A_{*}+2} n e_{C}^{c}\left(\partial^{\iota} k_{I J}\right) \partial^{\iota} \gamma_{I J C}\right\} . \tag{5.34}
\end{align*}
$$

Proof. The proof is a calculation that, although lengthy, is straightforward; hence, we only explain the main steps. We first note that $\partial^{\iota} \widetilde{k}_{I J}=0$ and thus we can ignore the formal presence of this term on LHS (5.31a). Next, we expand LHS (5.34 using the Leibniz rule. When $\partial_{t}$ falls on $t^{A_{*}+1} \partial^{\iota} k_{I J}$, we plug in 5.31a) with $P:=A_{*}+1$. When $\partial_{t}$ falls on $t^{A_{*}+1} \partial^{\iota} \gamma_{I J B}$, we plug in 5.31b with $P:=A_{*}+1$. We then differentiate the resulting terms by parts. Next, we use the (differentiated) momentum constraint (5.31c) with $P:=A_{*}+1$ to substitute for the terms $t^{A_{*}+1} e_{I} \partial^{\iota} k_{I J}$ in the product $2\left(t^{A_{*}+1} \partial^{\iota} e_{J} n\right) \cdot t^{A_{*}+1} e_{I} \partial^{\iota} k_{I J}$ (which is "present" in the sense that it is needed to cancel a corresponding product obtained from expanding the third-tolast term $-2 \partial_{c}\left\{t^{2 A_{*}+2} e_{I}^{c}\left(\partial^{\iota} e_{J} n\right) \partial^{\iota} k_{I J}\right\}$ on RHS (5.34). Similarly, we use (5.31c with $P:=A_{*}+1$ to substitute for the terms $t^{A_{*}+1} e_{I} \partial^{\iota} k_{I J}$ in the product $2 n t^{A_{*}+1} e_{I} \partial^{\iota} k_{I J} \cdot t^{A_{*}+1} \partial^{\iota} \gamma_{C J C}$ (which is "present" in the sense that it is needed to cancel a corresponding product obtained from expanding the next-to-last term $-2 \partial_{c}\left\{t^{2 A_{*}+2} e_{I}^{c} n\left(\partial^{\iota} k_{I J}\right) \partial^{\iota} \gamma_{C J C}\right\}$ on RHS (5.34)).

Remark 5.20 (Comments tied to the momentum constraint and well-posedness in CMC-transported spatial coordinates). The (differentiated) momentum constraint $\sqrt{5.31 \mathrm{c}}$ plays a crucial role in our proof of Lemma 5.19 without this constraint equation, the corresponding differential energy identity would have featured terms involving one too many derivatives of $k_{I J}$, which in turn would have led to a fatal loss of one derivative in the top-order estimates. An alternate way to overcome the derivative loss is to use spatial harmonic coordinates on each time slice $\Sigma_{t}$, as in [6. However, such coordinates lead to the presence of a non-zero shift vector in the coordinate expression for the spacetime metric, and it is not currently known whether the corresponding error terms are compatible with a proof of stable Big Bang formation. For a proof of local well-posedness for Einstein's equations in CMC-transported spatial coordinates, we refer to [59, Theorem 14.1]. We emphasize that while energy identities such as 5.34) can be used to derive a priori energy estimates for solutions to the nonlinear reduced equations (where by "reduced," we roughly mean gauge-dependent equations in the spirit of the ones stated in Proposition 2.2), the proof of local wellposedness given by [59, Theorem 14.1] relies on a modified system, which can be shown to be equivalent to the nonlinear reduced equations (and hence, by the "if and only if" aspect of Propsition 2.2 equivalent to Einstein's equations too) for initial data that satisfy the constraints and the CMC condition (1.12) at $t=1$. The key advantage of the modified system is that it does not involve constraint equations; this allows one to show that linearized versions of the modified system also enjoy good energy estimates, which is important for the standard iteration/contraction mapping schemes that are used in proofs of local well-posedness for quasilinear equations.

### 5.5.10 Control of the error terms in the top-order commuted spatial metric equations

In this section, at the top-order derivative level, we derive $L^{2}$ estimates for the error terms in the equations of Lemma 5.16

Lemma 5.21 ( $L^{2}$-control of the error terms in the top-order commuted evolution equations for $k$ and $\gamma$ ). Recall that $\mathbb{H}_{(\gamma, k)}, \mathbb{H}_{(\psi)}$, and $\mathbb{D}(t)$ are norms from Definition 3.1, and assume that the bootstrap assumptions (3.9) hold. Recall that the error terms $\mathfrak{K}_{I J}^{(\text {Border } ; \iota)}, \mathfrak{K}_{I J}^{(\text {Junk } ; \iota)}, \mathfrak{G}_{I J B}^{(\text {Border } ; \iota)}, \mathfrak{G}_{I J B}^{(\text {Junk } ; \iota)}, \mathfrak{M}_{I}^{(\text {Border } ; \iota)}$, and $\mathfrak{M}_{I}^{(\text {Junk } ; \iota)}$ are defined in 5.32a -5.32f. There exists a constant $C_{*}>0$ independent of $N, N_{0}$, and $A_{*}$ and a constant $C=C_{N, N_{0}, A_{*}, \mathfrak{D}, q, \sigma}>0$ such that if $N$ is sufficiently large in a manner that depends on $N_{0}, A_{*}, \mathfrak{D}, q$, and $\sigma$, and if $\varepsilon$ is sufficiently small (in a manner that depends on $N, N_{0}, A_{*}, \mathfrak{D}, q$, and $\sigma$ ), then the following estimates hold for $t \in\left(T_{B o o t}, 1\right]$ :

$$
\begin{gather*}
t^{A_{*}} \sqrt{\sum_{|\iota|=N} \sum_{I, J=1, \cdots, \mathfrak{D}}\left\|\mathfrak{K}_{I J}^{(\text {Border } ; \iota)}\right\|_{L^{2}\left(\Sigma_{t}\right)}^{2}} \leq C_{*} t^{-1} \mathbb{H}_{(\gamma, k)}(t)+C t^{-1+\sigma} \mathbb{D}(t),  \tag{5.35a}\\
t^{A_{*}+1} \sqrt{\sum_{|\iota|=N} \sum_{I, J, B=1, \cdots, \mathfrak{D}}\left\|\mathfrak{G}_{I J B}^{(\text {Border } ; \iota)}\right\|_{L^{2}\left(\Sigma_{t}\right)}^{2}} \leq C_{*} t^{-1} \mathbb{H}_{(\gamma, k)}(t)+C t^{-1+\sigma} \mathbb{D}(t),  \tag{5.35b}\\
t^{A_{*}+1} \sqrt{\sum_{|\iota|=N} \sum_{I=1, \cdots, \mathfrak{D}}\left\|\mathfrak{M}_{I}^{(\text {Border } ; \iota)}\right\|_{L^{2}\left(\Sigma_{t}\right)}^{2}} \leq C_{*} t^{-1} \mathbb{H}_{(\gamma, k)}(t)+C_{*} t^{-1} \mathbb{H}_{(\psi)}(t)+C t^{-1+\sigma} \mathbb{D}(t),  \tag{5.35c}\\
t^{A_{*}+1} \sqrt{\sum_{|\iota|=N} \sum_{I, J=1, \cdots, \mathfrak{D}}\left\|\mathfrak{K}_{I J}^{(\text {Junk; })}\right\|_{L^{2}\left(\Sigma_{t}\right)}^{2}} \leq C t^{-1+\sigma} \mathbb{D}(t),  \tag{5.36a}\\
t^{A_{*}+1} \sqrt{\sum_{|\iota|=N I, J, B=1, \cdots, \mathfrak{D}} \sum_{I J B} \| \mathfrak{G}_{I J B}^{\left(\text {Junk;८)} \|_{L^{2}\left(\Sigma_{t}\right)}^{2}\right.} \leq C t^{-1+\sigma} \mathbb{D}(t),}  \tag{5.36b}\\
t^{A_{*}+1} \sqrt{\sum_{|\iota|=N} \sum_{I=1, \cdots, \mathfrak{D}}\left\|\mathfrak{M}_{I}^{(\text {Junk; } ; \iota)}\right\|_{L^{2}\left(\Sigma_{t}\right)}^{2}} \leq C t^{-1+\sigma} \mathbb{D}(t) . \tag{5.36c}
\end{gather*}
$$

Proof. We will give the proofs of 5.35 a and 5.36 a . The remaining estimates can be proved using similar arguments, and we omit the details. To prove (5.35a), we let $\iota$ be any spatial multi-index with $|\iota|=N$. We multiply both sides of 5.32 a by $t^{A_{*}}$ and take the $\|\cdot\|_{L^{2}\left(\Sigma_{t}\right)}$ norm. Using the bootstrap assumptions, the explicit formulas 2.58, the inequalities in (3.6), Definition 3.1, and the product estimate (4.3), we find that $t^{A_{*}\left\|\mathfrak{K}_{I J}^{(\text {Border } ; \iota)}\right\|_{L^{2}\left(\Sigma_{t}\right)} \leq C_{*} t^{A_{*}-1}\left\|\partial^{\iota} n\right\|_{L^{2}\left(\Sigma_{t}\right)}+C \varepsilon t^{A_{*}-1}\|n\|_{\dot{H}^{N}\left(\Sigma_{t}\right)}+C \varepsilon t^{-1+\sigma} \mathbb{D}(t) \text {. We then square this }, ~}$ estimate, sum over all $\iota$ with $|\iota|=N$, sum over all $1 \leq I, J \leq \mathfrak{D}$, and then take the square root. We find that LHS $5.35 \mathrm{a} \leq\left(C_{*}+C \varepsilon\right) t^{A_{*}-1}\|n\|_{\dot{H}^{N}\left(\Sigma_{t}\right)}+C \varepsilon t^{-1+\sigma} \mathbb{D}(t) \leq C_{*} t^{A_{*}-1}\|n\|_{\dot{H}^{N}\left(\Sigma_{t}\right)}+C \varepsilon t^{-1+\sigma} \mathbb{D}(t)$. From this bound and the already derived high order estimate 5.5 b for $n$, we arrive at the desired bound (5.35a).

The estimate 5.36 a can be proved by multiplying equation 5.32 b by $t^{A_{*}+1}$ and combing arguments similar to the ones we used above with the estimates of Lemma 4.1.

### 5.6 Preliminary identities and inequalities for the scalar field $\psi$

This section is an analog of Sect. 5.5 for the scalar field $\psi$. That is, we derive preliminary low order and high order identities and inequalities for $\psi$ by using the wave equation (2.24). In order to avoid the time derivative of $n$ appearing as an error term in the equations (which would unnecessarily complicate our derivation of the main estimates), we treat $e_{0} \psi,\left\{e_{I} \psi\right\}_{I=1, \cdots, \mathfrak{D}}$ as separate variables satisfying a first-order system derived from the wave equation, cf. 61]. Roughly, we bound the inhomogeneous terms in the evolution equations in terms of our solution norms, and we derive an energy identity in differential form. In Sects. 5.75 .9 , we will combine these preliminary results with related ones for $n,\left\{k_{I J}\right\}_{I, J=1, \cdots, \mathfrak{D}},\left\{\gamma_{I J B}\right\}_{I, J, B=1, \cdots, \mathfrak{D}},\left\{e_{I}^{i}\right\}_{I, i=1, \cdots, \mathfrak{D}}$, and $\left\{\omega_{i}^{I}\right\}_{I, i=1, \cdots, \mathfrak{D}}$ to derive our main a priori estimates, i.e., to prove Proposition 5.1 .

### 5.6.1 Commuted evolution equations for $e_{0} \psi$ and $e_{I} \psi$

In this section, we provide the first-order evolution equations that we will use to control the scalar functions $e_{0} \psi$ and $\left\{e_{I} \psi\right\}_{I=1, \cdots, \mathfrak{D}}$ as well as their derivatives.
Lemma 5.22 (The first-order evolution system for $e_{0} \psi,\left\{e_{I} \psi\right\}_{I=1, \cdots, \mathfrak{D}}$, and their derivatives). For solutions to the equations of Proposition 2.2. the $\mathbf{g}$-orthonormal frame derivatives of $\psi$, namely $e_{0} \psi$ and $\left\{e_{I} \psi\right\}_{I=1, \cdots, \mathfrak{\sim}}$, satisfy the following first-order symmetric hyperbolic system. where the Kasner background scalars $\widetilde{\psi}=\widetilde{B} \log t$ and $\left\{\mathcal{k}_{I J}\right\}_{I, J=1, \cdots, \mathfrak{D}}$ are defined in 2.58 (see also Remark 2.6), and we recall that we do not sum over repeated underlined indices:

$$
\begin{align*}
\partial_{t}\left[t\left(e_{0} \psi-\partial_{t} \widetilde{\psi}\right)\right]= & t n e_{C} e_{C} \psi-\operatorname{tn} \gamma_{C C D} e_{D} \psi+t\left(e_{C} n\right) e_{C} \psi-(n-1) \partial_{t} \widetilde{\psi}-(n-1)\left(e_{0} \psi-\partial_{t} \widetilde{\psi}\right)  \tag{5.37a}\\
\partial_{t} e_{I} \psi= & -\frac{\widetilde{q_{I}}}{t} e_{\underline{I}} \psi+n e_{I} e_{0} \psi+(n-1) k_{I C} e_{C} \psi+\left(k_{I C}-\widetilde{k}_{I C}\right) e_{C} \psi+\left(e_{I} n\right) \partial_{t} \widetilde{\psi}  \tag{5.37b}\\
& +\left(e_{I} n\right)\left(e_{0} \psi-\partial_{t} \widetilde{\psi}\right)
\end{align*}
$$

Moreover, if $\iota$ is a spatial coordinate multi-index and $P \geq 0$ is any real number, then the following equations hold:

$$
\begin{align*}
\partial_{t}\left[t^{P} \partial^{\iota}\left(e_{0} \psi-\partial_{t} \widetilde{\psi}\right)\right]= & (P-1)\left[t^{P-1} \partial^{\iota}\left(e_{0} \psi-\partial_{t} \widetilde{\psi}\right)\right]+t^{P} n e_{C} \partial^{\iota} e_{C} \psi  \tag{5.38a}\\
& +t^{P-1} \mathfrak{P}^{(\text {Border } ; \iota)}+t^{P} \mathfrak{P}^{(\text {Junk } ; \iota)} \\
\partial_{t}\left(t^{P} \partial^{\iota} e_{I} \psi\right)= & \left(P-\widetilde{q}_{\underline{I}}\right) t^{P-1} \partial^{\iota} e_{\underline{I}} \psi+t^{P} n e_{I} \partial^{\iota} e_{0} \psi+t^{P} \mathfrak{Q}_{I}^{(\text {Border } ; \iota)}+t^{P} \mathfrak{Q}_{I}^{(\text {Junk } ; \iota)}, \tag{5.38b}
\end{align*}
$$

where

$$
\begin{align*}
\mathfrak{P}^{(\text {Border } ; \iota)}:= & \partial^{\iota}(n-1) \cdot \partial_{t} \tilde{\psi}+\sum_{\iota_{1} \cup \iota_{2}=\iota} \partial^{\iota_{1}}(n-1) \cdot \partial^{\iota_{2}}\left(e_{0} \psi-\partial_{t} \tilde{\psi}\right),  \tag{5.39a}\\
\mathfrak{P}^{(\text {Junk } ; \iota)}:= & \sum_{\iota_{1} \cup \iota_{2} \cup \iota_{3}=\iota,\left|\iota_{3}\right|<|\iota|} \partial^{\iota_{1}} n \cdot \partial^{\iota_{2}} e \cdot \partial \partial^{\iota_{3}} \vec{e} \psi  \tag{5.39b}\\
& +\sum_{\iota_{1} \cup \iota_{2} \cup \iota_{3}=\iota} \partial^{\iota_{1}} n \cdot \partial^{\iota_{2}} \gamma \cdot \partial^{\iota_{3}} \vec{e} \psi+\sum_{\iota_{1} \cup \iota_{2}=\iota} \partial^{\iota_{1}} \vec{e} n \cdot \partial^{\iota_{2}} \vec{e} \psi, \\
\mathfrak{Q}_{I}^{(\text {Border } ; \iota)}:= & \sum_{\iota_{1} \cup \iota_{2}=\iota} \partial^{\iota_{1}}(k-\widetilde{k}) \cdot \partial^{\iota_{2}} \vec{e} \psi+\partial^{\iota} \vec{e} n \cdot \partial_{t} \tilde{\psi}+\sum_{\iota_{1} \cup \iota_{2}=\iota} \partial^{\iota_{1}} \vec{e} n \cdot \partial^{\iota_{2}}\left(e_{0} \psi-\partial_{t} \tilde{\psi}\right)  \tag{5.39c}\\
& +\sum_{\iota_{1} \cup \iota_{2}=\iota,\left|\iota_{2}\right|<|\iota|} \partial^{\iota_{1}} e \cdot \partial \partial^{\iota_{2}} e_{0} \psi, \\
\mathfrak{Q}_{I}^{(\text {Junk } ; \iota \iota} 38:= & \sum_{\iota_{1} \cup \iota_{2} \cup \iota_{3}=\iota,\left|\iota_{3}\right|<|\iota|} \partial^{\iota_{1}}(n-1) \cdot \partial^{\iota_{2}} e \cdot \partial \partial^{\iota_{3}} e_{0} \psi+\sum_{\iota_{1} \cup \iota_{2} \cup \iota_{3}=\iota} \partial^{\iota_{1}}(n-1) \cdot \partial^{\iota_{2}} k \cdot \partial^{\iota_{3}} \vec{e} \psi . \tag{5.39~d}
\end{align*}
$$

Proof. Equation 5.37a follows from multiplying both sides of 2.24), by $n t$, using that $\partial_{t}\left(t \partial_{t} \widetilde{\psi}\right)=0$, and carrying out straightforward algebraic computations. 5.37 b follows from the identity $\partial_{t}=n e_{0}$, the commutation identity 4.11 , and straightforward algebraic computations. (5.38a)-(5.38b) then follow from differentiating (5.37a) (5.37b) with $\partial^{\iota}$, using the Leibniz rule, multiplying both sides of the resulting equations by $t^{P-1}$ and $t^{P}$ respectively, commuting the factors of $t^{P-1}$ and $t^{P}$ under the operator $\partial_{t}$ on the LHSs, and accounting for the commutators $\left[t^{P-1}, \partial_{t}\right]$ and $\left[t^{P}, \partial_{t}\right]$.

### 5.6.2 Pointwise estimates for the error terms in the scalar field evolution equations

In this section, we derive the pointwise estimates for the error terms in the equations of Lemma 5.22 that we will later use to control $e_{0} \psi-\partial_{t} \widetilde{\psi}$ at derivative levels $\leq N_{0}+1$ and $e_{I} \psi$ at derivative levels $\leq N_{0}$.

Lemma 5.23 (Pointwise estimates for the error terms in the evolution equations for $\partial^{\leq N_{0}+1}\left(e_{0} \psi-\partial_{t} \widetilde{\psi}\right)$ and $\left\{\partial^{\leq N_{0}} e_{I} \psi\right\}_{I=1, \cdots, \mathfrak{D})}$. Recall that $\mathbb{D}(t)$ is a norm from Definition 3.1, and assume that the bootstrap assumptions (3.9) hold. Recall that the error terms $\mathfrak{P}^{(\text {Border } ; \iota)}, \mathfrak{Q}_{I}^{(\text {Border } ; \iota)}, \mathfrak{P}^{(J u n k ; \iota)}$, and $\mathfrak{Q}_{I}^{(\text {Junk } ; \iota)}$ are defined
in 5.39a 5.39 d . There exists a constant $C=C_{N, N_{0}, A_{*}, \mathfrak{D}, q, \sigma}>0$ such that if $N$ is sufficiently large in a manner that depends on $N_{0}, A_{*}, \mathfrak{D}, q$, and $\sigma$, and if $\varepsilon$ is sufficiently small (in a manner that depends on $N, N_{0}, A_{*}, \mathfrak{D}, q$, and $\left.\sigma\right)$, then the following pointwise estimates hold for $(t, x) \in\left(T_{B o o t}, 1\right] \times \mathbb{T}^{\mathfrak{D}}$ :

$$
\begin{align*}
\sum_{|c| \leq N_{0}+1} t\left|n e_{C} \partial^{\iota} e_{C} \psi\right|(t, x) \leq & C t^{-1+\sigma} \mathbb{D}(t)  \tag{5.40a}\\
\sum_{|c| \leq N_{0}+1}\left|\mathfrak{P}^{(\text {Border } ; \iota)}\right|(t, x)+\sum_{|\iota| \leq N_{0}+1} t\left|\mathfrak{P}^{(J u n k ; \iota)}\right|(t, x) \leq & C t^{-1+\sigma} \mathbb{D}(t),  \tag{5.40b}\\
\sum_{|\iota| \leq N_{0}} \sum_{I=1, \cdots, \mathfrak{D}} t^{q}\left|n e_{I} \partial^{\iota} e_{0} \psi\right|(t, x) \leq & C \varepsilon \sum_{|\iota| \leq N_{0}} \sum_{I, i=1, \cdots, \mathfrak{D}} t^{q-1}\left|\partial^{\iota}\left(e_{I}^{i}-\widetilde{e}_{I}^{i}\right)\right|(t, x)  \tag{5.40c}\\
& +C t^{-1+\sigma} \mathbb{D}(t), \\
\sum_{|\iota| \leq N_{0}} \sum_{I=1, \cdots, \mathfrak{D}} t^{q}\left|\mathfrak{Q}_{I}^{(\text {Border } ; \iota)}\right|(t, x) \leq & C \varepsilon \sum_{|\iota| \leq N_{0}} \sum_{I=1, \cdot, \mathfrak{D}} t^{q-1}\left|\partial^{\iota} e_{I} \psi\right|(t, x) \\
& +C \varepsilon \sum_{|c| \leq N_{0} I, i=1, \cdots, \mathfrak{D}} t^{q-1}\left|\partial^{\iota}\left(e_{I}^{i}-\widetilde{e}_{I}^{i}\right)\right|(t, x)  \tag{5.40d}\\
& +C t^{-1+\sigma} \mathbb{D}(t),
\end{align*}
$$

Proof. We apply the same arguments we used in the proof of Lemmas 5.10 and Lemma 5.17 taking into account the structure of the terms on RHS (5.39a) -5.39 d ) and the fact that the low order norm (3.8a) controls $e_{0} \psi$ at up to derivative level $N_{0}+1$ (in particular, we use this fact to derive $5.40 \mathrm{~b}-(5.40 \mathrm{c})$ ).

### 5.6.3 Differential energy identity for the scalar field

We will derive our top-order energy estimates for the scalar field by integrating the differential identity provided by the following lemma.

Lemma 5.24 (Top-order differential energy identity for $e_{0} \psi$ and $\left.\left\{e_{I} \psi\right\}_{I=1, \cdots, \mathfrak{D}}\right)$. Let $\iota$ be a top-order spatial multi-index, i.e., $|\iota|=N$. For solutions to the $\partial^{\iota}$-commuted equations $5.38 \mathrm{a}-5.38 \mathrm{~b}$ with $P:=A_{*}+1$, the following differential energy identity holds, where the error terms $\mathfrak{P}^{(\text {Border } ; \iota)}, \mathfrak{Q}_{I}^{(\text {Border } ; \iota)}, \mathfrak{P}^{(J u n k ; \iota)}$, and $\mathfrak{Q}_{I}^{(\text {Junk; })}$ are defined in 5.39a)-5.39d):

$$
\begin{align*}
\partial_{t} & \left\{\left(t^{A_{*}+1} \partial^{\iota} e_{0} \psi\right)^{2}\right\}+\partial_{t}\left\{\left(t^{A_{*}+1} \partial^{\iota} e_{I} \psi\right)\left(t^{A_{*}+1} \partial^{\iota} e_{I} \psi\right)\right\} \\
= & \frac{2 A_{*}}{t}\left(t^{A_{*}+1} \partial^{\iota} e_{0} \psi\right)^{2}+\frac{2\left(A_{*}+1-\widetilde{q}_{I}\right)}{t}\left(t^{A_{*}+1} \partial^{\iota} e_{I} \psi\right)\left(t^{A_{*}+1} \partial^{\iota} e_{I} \psi\right) \\
& +2\left(t^{A_{*}+1} \partial^{\iota} e_{0} \psi\right)\left(t^{A_{*}} \mathfrak{P}^{(\text {Border } ; \iota)}+t^{A_{*}+1} \mathfrak{P}^{(\text {Junk; })}\right)  \tag{5.41}\\
& +2\left(t^{A_{*}+1} \partial^{\iota} e_{I} \psi\right)\left(t^{A_{*}+1} \mathfrak{Q}_{I}^{(\text {Border } ; \iota)}+t^{A_{*}+1} \mathfrak{Q}_{I}^{(\text {Junk; } ;)}\right) \\
& -2\left\{\partial_{c}\left(n e_{C}^{c}\right)\right\}\left(t^{A_{*}+1} \partial^{\iota} e_{C} \psi\right)\left(t^{A_{*}+1} \partial^{\iota} e_{0} \psi\right)+2 t^{A_{*}+2} \partial_{c}\left\{n e_{C}^{c}\left(\partial^{\iota} e_{C} \psi\right)\left(\partial^{\iota} e_{0} \psi\right)\right\}
\end{align*}
$$

Proof. This lemma follows from straightforward calculation, so we only explain the main steps. We first note that $\partial^{\iota} \partial_{t} \widetilde{\psi}=0$ and thus we can ignore the formal presence of this term on LHS 5.38a). Next, we expand LHS (5.41) using the Leibniz rule. When $\partial_{t}$ falls on $t^{A_{*}+1} \partial^{\iota} e_{0} \psi$, we plug in 5.38a with $P:=A_{*}+1$. When $\partial_{t}$ falls on $t^{A_{*}+1} \partial^{t} e_{I} \psi$, we plug in 5.38 b with $P:=A_{*}+1$. Differentiating by parts on the resulting terms, we arrive at the desired identity 5.41).

### 5.6.4 Control of the error terms in the top-order commuted scalar field evolution equations

In this section, at the top-order derivative level, we derive $L^{2}$ estimates for the error terms in the evolution equations of Lemma 5.22 .

Lemma 5.25 ( $L^{2}$-control of the error terms in the top-order commuted scalar field equations). Recall that $\mathbb{H}_{(\gamma, k)}, \mathbb{H}_{(\psi)}$, and $\mathbb{D}(t)$ are norms from Definition 3.1, and assume that the bootstrap assumptions (3.9) hold. Recall that the error terms $\mathfrak{P}^{(\text {Border } ; \iota)}, \mathfrak{Q}_{I}^{(\text {Border } ; /)}, \mathfrak{P}^{(\text {Junk } ; \iota)}$, and $\mathfrak{Q}_{I}^{(\text {Junk } ; \iota)}$ are defined in 5.39 a -5.39d). There exists a constant $C_{*}>0$ independent of $N, N_{0}$, and $A_{*}$ and a constant $C=C_{N, N_{0}, A_{*}, \mathfrak{D}, q, \sigma}>0$ such that if $N$ is sufficiently large in a manner that depends on $N_{0}, A_{*}, \mathfrak{D}, q$, and $\sigma$, and if $\varepsilon$ is sufficiently small (in a manner that depends on $N, N_{0}, A_{*}, \mathfrak{D}, q$, and $\sigma$ ), then the following estimates hold for $t \in\left(T_{B o o t}, 1\right]$ :

$$
\begin{align*}
& t^{A_{*}} \sqrt{\sum_{|\iota|=N}\left\|\mathfrak{P}^{(\text {Border } ; \iota)}\right\|_{L^{2}\left(\Sigma_{t}\right)}^{2}}+t^{A_{*}+1} \sqrt{\sum_{|\iota|=N} \sum_{I=1, \cdots, \mathfrak{D}}\left\|\mathfrak{Q}_{I}^{(\text {Border } ; \iota)}\right\|_{L^{2}\left(\Sigma_{t}\right)}^{2}}  \tag{5.42a}\\
& \leq C_{*} t^{-1} \mathbb{H}_{(\gamma, k)}(t)+C_{*} t^{-1} \mathbb{H}_{(\psi)}(t)+C t^{-1+\sigma} \mathbb{D}(t), \\
& t^{A_{*}+1} \sqrt{\sum_{|\iota|=N}\left\|\mathfrak{P}^{(\text {Junk } ; \iota)}\right\|_{L^{2}\left(\Sigma_{t}\right)}^{2}}+t^{A_{*}+1} \sqrt{\sum_{|\iota|=N} \sum_{I=1, \cdots, \mathfrak{D}}\left\|\mathfrak{Q}_{I}^{(\text {Junk } ; \iota)}\right\|_{L^{2}\left(\Sigma_{t}\right)}^{2}}  \tag{5.42b}\\
& \leq C t^{-1+\sigma} \mathbb{D}(t) .
\end{align*}
$$

Proof. We apply the same arguments that we used in the proof of Lemma 5.21 to the terms on RHSs (5.39a)(5.39d).

### 5.7 Integral inequality for the low order solution norms

In the next proposition, we combine some of the results derived earlier in Sect. 5 to obtain an integral inequality for the low order solution norms. In Sect. 5.8 , we will derive a related integral inequality for the high order solution norms. Then, in Sect.5.9, we will combine the two integral inequalities and carry out the proof of our main a priori estimates.

Proposition 5.26 (Integral inequality for the low order solution norms). Recall that $\mathbb{L}_{(e, \omega, \gamma, k, \psi)}(t)$ is a low order norm and that $\mathbb{D}(t)$ is the total norm of the dynamic solution variables (see Definition 3.1). Under the assumptions of Proposition 5.1, including the bootstrap assumptions (3.9), there exists a constant $C=C_{N, N_{0}, A_{*}, \mathfrak{D}, q, \sigma}>0$ such that if $N_{0} \geq 1$ and $N$ is sufficiently large in a manner that depends on $N_{0}, A_{*}, \mathfrak{D}, q$, and $\sigma$, and if $\varepsilon$ is sufficiently small (in a manner that depends on $N, N_{0}, A_{*}, \mathfrak{D}, q$, and $\sigma$ ), then the following estimate holds for $t \in\left(T_{\text {Boot }}, 1\right]$ :

$$
\begin{equation*}
\mathbb{L}_{(e, \boldsymbol{\omega}, \gamma, k, \psi)}^{2}(t) \leq C \mathbb{L}_{(e, \boldsymbol{\omega}, \gamma, k, \psi)}^{2}(1)+C \int_{t}^{1} s^{-1+\sigma} \mathbb{D}^{2}(s) d s \tag{5.43}
\end{equation*}
$$

Proof. The polarized $U(1)$-symmetric case will require an additional observation, which we provide at the end of the proof.

The proof except for the $U(1)$-symmetric case. Recall 5.22 and Remark 1.1 . We define the scalar function $Q(t, x) \geq 0$ as follows, where the background Kasner scalars are defined in Sect. 2.3 and we suppress the $(t, x)$ arguments on RHS 5.44:

$$
\begin{align*}
Q^{2}=Q^{2}(t, x):= & \sum_{|\iota| \leq N_{0}} \sum_{I, J, B=1, \cdots, \mathfrak{D}}\left[t^{q} \partial^{\iota}\left(\gamma_{I J B}+\gamma_{J B I}\right)\right]^{2}+\sum_{|\iota| \leq N_{0}+1} \sum_{I, J=1, \cdots, \mathfrak{D}}\left[t \partial^{\iota}\left(k_{I J}-\widetilde{k}_{I J}\right)\right]^{2} \\
& +\sum_{|\iota| \leq N_{0}} \sum_{I, i=1, \cdots, \mathfrak{D}}\left[t^{q} \partial^{\iota}\left(e_{I}^{i}-\widetilde{e}_{I}^{i}\right)\right]^{2}+\sum_{|\iota| \leq N_{0}} \sum_{I, i=1, \cdots, \mathfrak{D}}\left[t^{q} \partial^{\iota}\left(\omega_{i}^{I}-\widetilde{\omega}_{i}^{I}\right)\right]^{2}  \tag{5.44}\\
& +\sum_{|\iota| \leq N_{0}+1}\left[t \partial^{\iota}\left(e_{0} \psi-\partial_{t} \widetilde{\psi}\right]^{2}+\sum_{|\iota| \leq N_{0}} \sum_{I=1, \cdots, \mathfrak{D}}\left[t^{q} \partial^{\iota} e_{I} \psi\right]^{2} .\right.
\end{align*}
$$

Throughout the proof, we will silently use the estimates $C^{-1}\|Q\|_{L^{\infty}\left(\Sigma_{t}\right)} \leq \mathbb{L}_{(e, \omega, \gamma, k, \psi)}(t) \leq C\|Q\|_{L^{\infty}\left(\Sigma_{t}\right)}$ and $\|Q\|_{L^{\infty}\left(\Sigma_{t}\right)} \leq C \mathbb{D}(t)$, which follow easily from the definitions of the quantities involved and the identity (5.22). In particular, to prove 5.43), it suffices to derive the following pointwise bound for $Q^{2}(t, x)$ :

$$
\begin{equation*}
Q^{2}(t, x) \lesssim \mathbb{L}_{(e, \omega, \gamma, k, \psi)}^{2}(1)+\int_{t}^{1} s^{-1+\sigma} \mathbb{D}^{2}(s) d s \tag{5.45}
\end{equation*}
$$

To prove (5.45), we will derive the following pointwise bound for $(t, x) \in\left(T_{B o o t}, 1\right] \times \mathbb{T}^{\mathfrak{D}}$ :

$$
\begin{align*}
Q^{2}(t, x) \leq & C \mathbb{L}_{(e, \omega, \gamma, k, \psi)}^{2}(1)+(C \varepsilon-4 \sigma) \int_{t}^{1} s^{-1} \sum_{|\iota| \leq N_{0}} \sum_{I, J, B=1, \cdots, \mathfrak{D}}\left[s^{q} \partial^{\iota}\left(\gamma_{I J B}+\gamma_{J B I}\right)(s, x)\right]^{2} d s \\
& +(C \varepsilon-4 \sigma) \int_{t}^{1} s^{-1} \sum_{|c| \leq N_{0}} \sum_{I=1, \cdots, \mathfrak{D}}\left[s^{q} \partial^{\iota} e_{I} \psi\right]^{2} d s \\
& +(C \varepsilon-4 \sigma) \int_{t}^{1} s^{-1} \sum_{|\iota| \leq N_{0}} \sum_{I, i=1, \cdots, \mathfrak{D}}\left[s^{q} \partial^{\iota}\left(e_{I}^{i}-\widetilde{e}_{I}^{i}\right)(s, x)\right]^{2} d s  \tag{5.46}\\
& +(C \varepsilon-4 \sigma) \int_{t}^{1} s^{-1} \sum_{|\iota| \leq N_{0}} \sum_{I, i=1, \cdots, \mathfrak{D}}\left[s^{q} \partial^{\iota}\left(\omega_{i}^{I}-\widetilde{\omega}_{i}^{I}\right)(s, x)\right]^{2} d s \\
& +C \int_{t}^{1} s^{-1+\sigma} \mathbb{D}^{2}(s) d s .
\end{align*}
$$

Then for $\varepsilon$ sufficiently small, the first four integrals on RHS 5.46) are negative, and we can discard them; the desired bound 5.45 then follows.

It remains for us to prove 5.46). We will show that the following pointwise estimates hold for $(t, x) \in$ $\left(T_{\text {Boot }}, 1\right] \times \mathbb{T}^{\mathfrak{D}}$, where to condense the notation, we omit the arguments $(t, x)$ on the LHSs and the integrand arguments $(s, x)$ on the RHSs:

$$
\begin{align*}
& \sum_{|c| \leq N_{0}} \sum_{\substack{I, J, B=1, \cdots, \mathfrak{D} \\
I<J}}\left[t^{q} \partial^{\iota}\left(\gamma_{I J B}+\gamma_{J B I}\right)\right]^{2} \leq C \mathbb{L}_{(\gamma, k)}^{2}(1) \\
& +(C \varepsilon-4 \sigma) \int_{t}^{1} s^{-1} \sum_{|\iota| \leq N_{0}} \sum_{\substack{I, J, B=1, \cdots, \mathfrak{D} \\
I<J}}\left[s^{q} \partial^{\iota}\left(\gamma_{I J B}+\gamma_{J B I}\right)\right]^{2} d s \\
& +C \varepsilon \int_{t}^{1} s^{-1} \sum_{|\iota| \leq N_{0}} \sum_{I, i=1, \cdots, \mathfrak{D}}\left[s^{q} \partial^{\iota}\left(e_{I}^{i}-\widetilde{e}_{I}^{i}\right)\right]^{2} d s \\
& +C \int_{t}^{1} s^{-1+\sigma} \mathbb{D}^{2}(s) d s,  \tag{5.47}\\
& \sum_{|c| \leq N_{0}+1} \sum_{I, J=1, \cdots, \mathfrak{D}}\left[t \partial^{\iota}\left(k_{I J}-\widetilde{k}_{I J}\right)\right]^{2} \leq C \mathbb{L}_{(\gamma, k)}^{2}(1)  \tag{5.48}\\
& +C \int_{t}^{1} s^{-1+\sigma} \mathbb{D}^{2}(s) d s, \\
& \sum_{|\iota| \leq N_{0}} \sum_{I, i=1, \cdots, \mathfrak{D}}\left[t^{q} \partial^{\iota}\left(e_{I}^{i}-\widetilde{e}_{I}^{i}\right)\right]^{2} \leq C \mathbb{L}_{(e, \omega)}^{2}(1) \\
& +(C \varepsilon-4 \sigma) \int_{t}^{1} s^{-1} \sum_{|\iota| \leq N_{0}} \sum_{I, i=1, \cdots, \mathfrak{D}}\left[s^{q} \partial^{\iota}\left(e_{I}^{i}-\widetilde{e}_{I}^{i}\right)\right]^{2} d s  \tag{5.49}\\
& +C \int_{t}^{1} s^{-1+\sigma} \mathbb{D}^{2}(s) d s, \\
& \sum_{|\iota| \leq N_{0}} \sum_{I, i=1, \cdots, \mathfrak{D}}\left[t^{q} \partial^{\iota}\left(\boldsymbol{\omega}_{i}^{I}-\widetilde{\boldsymbol{\omega}}_{i}^{I}\right)\right]^{2} \leq C \mathbb{L}_{(e, \omega)}^{2}(1) \\
& +(C \varepsilon-4 \sigma) \int_{t}^{1} s^{-1} \sum_{|\iota| \leq N_{0}} \sum_{I, i=1, \cdots, \mathfrak{D}}\left[s^{q} \partial^{\iota}\left(\boldsymbol{\omega}_{i}^{I}-\widetilde{\omega}_{i}^{I}\right)\right]^{2} d s  \tag{5.50}\\
& +C \int_{t}^{1} s^{-1+\sigma} \mathbb{D}^{2}(s) d s,
\end{align*}
$$

$$
\begin{align*}
\sum_{|c| \leq N_{0}+1}\left[t \partial^{\iota}\left(e_{0} \psi-\partial_{t} \widetilde{\psi}\right)\right]^{2} \leq & C \mathbb{L}_{(\psi)}^{2}(1) \\
& +C \int_{t}^{1} s^{-1+\sigma} \mathbb{D}^{2}(s) d s  \tag{5.51}\\
\sum_{|\iota| \leq N_{0}} \sum_{I=1, \cdots, \mathfrak{D}}\left[t^{q} \partial^{\iota} e_{I} \psi\right]^{2} \leq & C \mathbb{L}_{(\psi)}^{2}(1) \\
& +(C \varepsilon-4 \sigma) \int_{t}^{1} s^{-1} \sum_{|\iota| \leq N_{0}} \sum_{I=1, \cdots, \mathfrak{D}}\left[s^{q} \partial^{\iota} e_{I} \psi\right]^{2} d s  \tag{5.52}\\
& +C \varepsilon \int_{t}^{1} s^{-1} \sum_{|\iota| \leq N_{0} I, i=1, \cdots, \mathfrak{D}}\left[s^{q} \partial^{\iota}\left(e_{I}^{i}-\widetilde{e}_{I}^{i}\right)\right]^{2} d s \\
& +C \int_{t}^{1} s^{-1+\sigma} \mathbb{D}^{2}(s) d s
\end{align*}
$$

Then adding (5.47)-(5.52), we arrive at 5.46).
To prove 5.47), we first multiply equation (5.20) by $2\left[t^{q} \partial^{\iota}\left(\gamma_{I J B}+\gamma_{J B I}\right)\right]$ to obtain the evolution equation $\partial_{t}\left\{\left[t^{q} \partial^{\iota}\left(\gamma_{I J B}+\gamma_{J B I}\right)\right]^{2}\right\}=2\left[t^{q} \partial^{\iota}\left(\gamma_{I J B}+\gamma_{J B I}\right)\right] \times$ RHS 5.20 . We then integrate this equation in time over $[t, 1]$ with respect to $d s$, apply the fundamental theorem of calculus, and then sum the resulting identity over all $\iota$ with $|\iota| \leq N_{0}$ and over all $I, J, B=1, \cdots, \mathfrak{D}$ with $I<J$. The left-hand side of the resulting identity is equal to LHS (5.47), while the resulting initial data term (on $\Sigma_{1}$ ) is $\leq$ the term $C \mathbb{L}_{(\gamma, k)}^{2}(1)$ on RHS 5.47). Next, using (3.6), we see that the terms generated by the first term $\left\{q-\left(\widetilde{q}_{\underline{I}}+\widetilde{q}_{\underline{J}}-\widetilde{q}_{\underline{B}}\right)\right\} t^{q-1} \partial^{\iota}\left(\gamma_{\underline{I J B}}+\gamma_{\underline{J B I}}\right)$ on RHS 5.20 are $\leq$ the term $-4 \sigma \int_{t}^{1} s^{-1} \sum_{|c| \leq N_{0}} \sum_{I, J, B=1, \cdots, \mathfrak{D}}\left[s^{q} \partial^{\iota}\left(\gamma_{I J B}+\gamma_{J B I}\right)\right]^{2} d s$ on RHS 5.47) (note that the overall minus sign is correct, since $\left.t \in\left(T_{\text {Boot }}, 1\right]\right)$. Finally, with the help of the identity (5.22), the error term estimates $5.23 \mathrm{a}-5.23 \mathrm{~b}$, and Young's inequality, we see that the terms generated by the remaining terms on RHS 5.20 are $\leq$ the sum of the remaining terms on RHS (5.47) as desired.

The estimate 5.48 follows from a similar argument based on equation 5.31a with $P:=1$ and $|\iota| \leq N_{0}+1$ and the error term estimates $5.33 \mathrm{a}-5.33 \mathrm{~b}$ ).

The estimate (5.49) follows from a similar argument based on equation (5.27a with $P:=q$ and the error term estimates $5.29 \mathrm{a}-5.29 \mathrm{~b}$. The estimate 5.50 can be proved via similar arguments based on equation 5.27b with $P:=q$ and the error term estimates 5.29 c - 5.29 d .

The estimate (5.51) follows from a similar argument based on equation (5.38a) with $P:=1$ and $|\iota| \leq N_{0}+1$ and the error term estimates 5 5.40a) $-5.40 \mathrm{~b})$.

Finally, the estimate 5.52 follows from a similar argument based on equation 5.38 b with $P:=q$ and the error term estimates 5.40 c$)$-5.40e. This completes the proof except in the polarized $U(1)$-symmetric case.
The proof in the polarized $U(1)$-symmetric case. By (5.25), in polarized $U(1)$-symmetry with $\mathfrak{D}=3$, the structure coefficient $\gamma_{I J B}+\gamma_{J B I}$ vanishes unless $I=B \neq J$ (in which case 2.20) implies $\gamma_{I J B}+\gamma_{J B I}=\gamma_{I J B}$ - though this identity is not needed for our results) or $B=J \neq I$. The key point is that for the non-zero structure coefficients, when $I=B$, the factor $\frac{\widetilde{q}_{I}+\widetilde{q}_{J}-\widetilde{q}_{B}}{t}$ on LHS 5.19 reduces to $\frac{\widetilde{q}_{J}}{t}$, and similarly, when $B=J$, the factor $\frac{\widetilde{q}_{\underline{I}}+\widetilde{q}_{J}-\widetilde{q}_{B}}{t}$ on LHS 5.19) reduces to $\frac{\widetilde{q}_{I}}{t}$. Hence, using the definition (3.6 of $q$ in the polarized $U(1)$-symmetric case, we can repeat the proof of (5.47) given above in the non-symmetric case but making the change $\frac{\left(\widetilde{q}_{\underline{I}}+\widetilde{q}_{\underline{J}}-\widetilde{q}_{\underline{B}}\right)}{t} \rightarrow \frac{\widetilde{q}_{J}}{t}$ or $\frac{\left(\widetilde{q}_{\underline{I}}+\widetilde{q}_{\underline{J}}-\widetilde{q}_{\underline{B}}\right)}{t} \rightarrow \frac{\widetilde{q}_{\underline{I}}}{t}$ in the relevant spots - to derive the desired estimates.

### 5.8 Integral inequality for the high order solution norms

In the next proposition, we combine some of the results derived earlier in Sect. 5 to obtain an integral inequality for the high order solution norms.
Proposition 5.27 (Top-order energy integral inequalities). Recall that $\mathbb{H}_{(\gamma, k)}, \mathbb{H}_{(\psi)}, \mathbb{H}_{(e, \omega)}$, and $\mathbb{D}(t)$ are norms from Definition 3.1. Under the assumptions of Proposition 5.1, including the bootstrap assumptions (3.9), there exists a constant $C_{*}>0$ independent of $N, N_{0}$, and $A_{*}$ and a constant $C=C_{N, N_{0}, A_{*}, \mathfrak{D}, q, \sigma}>0$
such that if $N_{0} \geq 1$ and $N$ is sufficiently large in a manner that depends on $N_{0}, A_{*}, \mathfrak{D}, q$, and $\sigma$, and if $\varepsilon$ is sufficiently small (in a manner that depends on $N, N_{0}, A_{*}, \mathfrak{D}, q$, and $\sigma$ ), then the following estimates hold for $t \in\left(T_{\text {Boot }}, 1\right]:$

$$
\begin{align*}
\mathbb{H}_{(\gamma, k)}^{2}(t) \leq & C \mathbb{H}_{(\gamma, k)}^{2}(1) \\
& +\left(C_{*}-A_{*}\right) \int_{t}^{1} s^{-1} \mathbb{H}_{(\gamma, k)}^{2}(s) d s+C_{*} \int_{t}^{1} s^{-1} \mathbb{H}_{(\psi)}^{2}(s) d s+C \int_{t}^{1} s^{-1+\sigma} \mathbb{D}^{2}(s) d s,  \tag{5.53a}\\
\mathbb{H}_{(\psi)}^{2}(t) \leq & C \mathbb{H}_{(\psi)}^{2}(1) \\
& +C_{*} \int_{t}^{1} s^{-1} \mathbb{H}_{(\gamma, k)}^{2}(s) d s+\left(C_{*}-A_{*}\right) \int_{t}^{1} s^{-1} \mathbb{H}_{(\psi)}^{2}(s) d s+C \int_{t}^{1} s^{-1+\sigma} \mathbb{D}^{2}(s) d s,  \tag{5.53b}\\
\mathbb{H}_{(e, \omega)}^{2}(t) \leq & C \mathbb{H}_{(e, \omega)}^{2}(1) \\
& +C_{*} \int_{t}^{1} s^{-1} \mathbb{H}_{(\gamma, k)}^{2}(s)+\left(C_{*}-A_{*}\right) \int_{t}^{1} s^{-1} \mathbb{H}_{(e, \omega)}^{2}(s) d s+C \int_{t}^{1} s^{-1+\sigma} \mathbb{D}^{2}(s) d s \tag{5.53c}
\end{align*}
$$

Proof. We stress that throughout the proof, $C$ and $C_{*}$ denote constants that have the properties stated in Sect. 1.11 , and that these constants can vary from line to line. In particular, the "final" constants appearing in 5.53 a$)-\sqrt{5.53 \mathrm{c}}$ ) do not have to coincide with the constants appearing in the proof.

To proceed, we first integrate the differential energy identity (5.34) over $[t, 1] \times \mathbb{T}^{\mathfrak{D}}$ with respect to $d s d x$, sum the resulting identity over all $\iota$ with $\iota=N$, use 5.5a to control the top-order derivatives of the lapse, use the estimates $\|n-1\|_{W^{1, \infty}\left(\Sigma_{t}\right)} \lesssim t^{\sigma}$ and $\|e\|_{W^{1, \infty}\left(\Sigma_{t}\right)} \lesssim t^{-1+2 \sigma}$ (which are simple consequences of (2.58), the inequalities in (3.6), and the bootstrap assumptions), and use the Cauchy-Schwarz inequality for integrals and sums and Young's inequality to deduce that the following estimate holds for $t \in\left(T_{B o o t}, 1\right]$, where $C_{*}>0$ and $C>0$ are as in the statement of the proposition:

$$
\begin{align*}
& \sum_{|\iota|=N} \sum_{I, J=1, \cdots, \mathfrak{D}} t^{2 A_{*}+2}\left\|\partial^{\iota} k_{I J}\right\|_{L^{2}\left(\Sigma_{s}\right)}^{2}+\frac{1}{2} t^{2 A_{*}+2} \sum_{|\iota|=N} \sum_{I, J, B=1, \cdots, \mathfrak{D}}\left\|\partial^{\iota} \gamma_{I J B}\right\|_{L^{2}\left(\Sigma_{s}\right)}^{2} \\
& \leq C \mathbb{H}_{(\gamma, k)}^{2}(1) \\
& +\left(C_{*}-A_{*}\right) \int_{t}^{1}\left\{\sum_{|\iota|=N} \sum_{I, J=1, \cdots, \mathfrak{D}} s^{2 A_{*}+1}\left\|\partial^{\iota} k_{I J}\right\|_{L^{2}\left(\Sigma_{s}\right)}^{2}+\sum_{|\iota|=N} \sum_{I, J, B=1, \cdots, \mathfrak{D}} s^{2 A_{*}+1}\left\|\partial^{\iota} \gamma_{I J B}\right\|_{L^{2}\left(\Sigma_{s}\right)}^{2}\right\} d s \\
& +\sum_{|c|=N} \sum_{I, J=1, \cdots, \mathfrak{D}} \int_{t}^{1} s^{2 A_{*}+1}\left\|\mathfrak{K}_{I J}^{(\text {Border } ; \ell)}\right\|_{L^{2}\left(\Sigma_{s}\right)}^{2} d s \\
& +\sum_{|\iota|=N} \sum_{I, J, B=1, \cdots, \mathfrak{D}} \int_{t}^{1} s^{2 A_{*}+3}\left\|\mathfrak{G}_{I J B}^{(\text {Border } ; \iota)}\right\|_{L^{2}\left(\Sigma_{s}\right)}^{2} d s \\
& +\sum_{|\iota|=N} \sum_{J=1, \cdots, \mathfrak{D}} \int_{t}^{1} s^{2 A_{*}+3}\left\|\mathfrak{M}_{J}^{(\text {Border } ; \iota)}\right\|_{L^{2}\left(\Sigma_{s}\right)}^{2} d s \\
& +C \sum_{|c|=N} \sum_{I, J=1, \cdots, \mathfrak{D}} \int_{t}^{1}\left\|s^{A_{*}+1} \partial^{\iota} k_{I J}\right\|_{L^{2}\left(\Sigma_{s}\right)}\left\|s^{A_{*}+1} \mathfrak{K}_{I J}^{(J u n k ; \iota)}\right\|_{L^{2}\left(\Sigma_{s}\right)} d s \\
& +C \sum_{|\iota|=N} \sum_{I, J, B=1, \cdots, \mathfrak{D}} \int_{t}^{1}\left\|s^{A_{*}+1} \partial^{\iota} \gamma_{I J B}\right\|_{L^{2}\left(\Sigma_{s}\right)}\left\|s^{A_{*}+1} \mathfrak{G}_{I J B}^{(J u n k ; \ell)}\right\|_{L^{2}\left(\Sigma_{s}\right)} d s \\
& +C \sum_{|\iota|=N} \sum_{I, J, B, E=1, \cdots, \mathfrak{D}} \int_{t}^{1}\left\{\left\|s^{A_{*}+1} \partial^{\iota} \gamma_{I J B}\right\|_{L^{2}\left(\Sigma_{s}\right)}+s^{\sigma} \mathbb{D}(s)\right\}\left\|s^{A_{*}+1} \mathfrak{M}_{E}^{(\text {Junk; })}\right\|_{L^{2}\left(\Sigma_{s}\right)} d s \\
& +C \sum_{|\iota|=N} \sum_{I, J, B, E, F=1, \cdots, \mathfrak{D}} \int_{t}^{1} s^{-1+\sigma}\left\|s^{A_{*}+1} \partial^{\iota} k_{I J}\right\|_{L^{2}\left(\Sigma_{s}\right)}\left\{\left\|s^{A_{*}+1} \partial^{\iota} \gamma_{B E F}\right\|_{L^{2}\left(\Sigma_{s}\right)}+s^{\sigma} \mathbb{D}(s)\right\} d s . \tag{5.54}
\end{align*}
$$

Using Lemma 5.21 , we deduce that the three integrals involving the borderline terms $\mathfrak{K}_{I J}^{(\text {Border } ; \iota)}, \mathfrak{G}_{I J B}^{(\text {Border } ; \iota)}$, and $\mathfrak{M}_{J}^{(\text {Border } ; 1)}$ are

$$
\leq C_{*} \int_{t}^{1} s^{-1}\left\{\mathbb{H}_{(\gamma, k)}^{2}(s)+\mathbb{H}_{(\psi)}^{2}(s)\right\} d s+C \int_{t}^{1} s^{-1+\sigma} \mathbb{D}^{2}(s) d s
$$

and that (in view of Definition 3.1 the three integrals involving the terms $\mathfrak{K}_{I J}^{(J u n k ; \iota)}, \mathfrak{G}_{I J B}^{(J u n k ; \iota)}$, and $\mathfrak{M}_{J}^{(J u n k ; \iota)}$ are $\leq C \int_{t}^{1} s^{-1+\sigma} \mathbb{D}^{2}(s) d s$. Moreover, appealing to Definition 3.1. we see that the integrals

$$
C \sum_{|\iota|=N} \sum_{I, J, B, E, F=1, \cdots, \mathfrak{D}} \int_{t}^{1} s^{-1+\sigma}\left\|s^{A_{*}+1} \partial^{\iota} k_{I J}\right\|_{L^{2}\left(\Sigma_{s}\right)}\left\{\left\|s^{A_{*}+1} \partial^{\iota} \gamma_{B E F}\right\|_{L^{2}\left(\Sigma_{s}\right)}+s^{\sigma} \mathbb{D}(s)\right\} d s
$$

on the last line of RHS 5.54 are $\leq C \int_{t}^{1} s^{-1+\sigma} \mathbb{D}^{2}(s) d s$. From these estimates, we arrive, in view of Definition 3.1, at the desired estimate 5.53a).

The inequality 5.53 b follows from a similar argument based on the scalar field differential energy identity (5.41) and the error term estimates of Lemma 5.25, we omit the details.

To prove 5.53c), we first set $P:=A_{*}+q$ in equation 5.27a and multiply it by $2\left[t^{A_{*}+q} \partial^{\iota}\left(e_{I}^{i}-\widetilde{e}_{I}^{i}\right)\right]$ to deduce

$$
\begin{align*}
\partial_{t}\left\{\left[t^{A_{*}+q} \partial^{\iota}\left(e_{I}^{i}-\widetilde{e}_{I}^{i}\right)\right]^{2}\right\}= & \frac{2\left(A_{*}+q-\widetilde{q}_{\underline{I}}\right)}{t}\left[t^{A_{*}+q} \partial^{\iota}\left(e_{\underline{I}}^{i}-\widetilde{e}_{\underline{I}}^{i}\right)\right]^{2} \\
& +2\left(t^{A_{*}+q} \mathfrak{E}_{\underline{I}}^{i ;(\text { Border } ; \iota)}\right)\left[t^{A_{*}+q} \partial^{\iota}\left(e_{\underline{I}}^{i}-\widetilde{e}_{\underline{I}}^{i}\right)\right]+2\left(t^{A_{*}+q} \mathfrak{E}_{\underline{I}}^{i ;(\text { Junk } ; \iota)}\right)\left[t^{A_{*}+q} \partial^{\iota}\left(e_{\underline{I}}^{\underline{i}}-\widetilde{e}_{\underline{I}}^{\underline{i}}\right)\right] \tag{5.55}
\end{align*}
$$

We then argue as in the proof of (5.53a), but using (5.55) in place of (5.34) and the error term estimates of Lemma 5.15 in place of those of Lemma 5.21. Summing the resulting inequality over $I, i=1, \cdots, \mathfrak{D}$ and also noting that $C \varepsilon \leq C_{*}$, we deduce that the following estimate holds for $t \in\left(T_{B o o t}, 1\right]$ :

$$
\begin{align*}
t^{2\left(A_{*}+q\right)}\|e\|_{\dot{H}^{N}\left(\Sigma_{t}\right)}^{2} \leq & \|e\|_{\dot{H}^{N}\left(\Sigma_{1}\right)}^{2}+C_{*} \int_{t}^{1} s^{-1} \mathbb{H}_{(\gamma, k)}^{2}(s) d s+C_{*} \int_{t}^{1} s^{-1} \mathbb{H}_{(e, \omega)}^{2}(s) d s \\
& -A_{*} \int_{t}^{1} s^{-1}\left\{s^{2\left(A_{*}+q\right)}\|e\|_{\dot{H}^{N}\left(\Sigma_{s}\right)}^{2}\right\} d s+C \int_{t}^{1} s^{-1+\sigma} \mathbb{D}^{2}(s) d s \tag{5.56}
\end{align*}
$$

Next, we note that the one-form components $\left\{\omega_{i}^{I}\right\}_{I, i=1, \cdots, \mathfrak{D}}$ satisfy the same inequality, that is, 5.56 holds with $\omega$ in place of $e$; to see this, one argues as in the proof of (5.56), but using the evolution equation 5.27b with $P:=A_{*}+q$ and the last two error term estimates in Lemma 5.15. Adding this top-order energy inequality for the $\left\{\omega_{i}^{I}\right\}_{I, i=1, \cdots, \mathfrak{D}}$ to the inequality (5.56), and considering the definition (3.8b) of $\mathbb{H}_{(e, \omega)}(t)$, we arrive at the desired estimate 5.53 c . We have therefore proved the proposition.

### 5.9 Proof of Proposition 5.1

We start by adding the integral inequalities 5.43 and 5.53 a -5.53c to obtain, in view of Definition 3.1 and (5.1), the following inequality for $t \in\left(\overline{T_{B o o t}}, 1\right]$, valid under largeness/smallness assumptions on the parameters that we describe just below (and we again stress that constants labeled " $C_{*}$ " - though we allow them to vary from line to line - are always independent of $N_{0}, N$ and $A_{*}$ ):

$$
\begin{equation*}
\mathbb{D}^{2}(t) \leq C \stackrel{\epsilon}{\epsilon}^{2}+\left(C_{*}-A_{*}\right) \int_{t}^{1} s^{-1} \mathbb{H}_{(e, \omega, \gamma, k, \psi)}^{2}(s) d s+C \int_{t}^{1} s^{-1+\sigma} \mathbb{D}^{2}(s) d s \tag{5.57}
\end{equation*}
$$

We now fix $A_{*}$ to be sufficiently large so that the factor $C_{*}-A_{*}$ on RHS (5.57) is negative. For this fixed value of $A_{*}$ and any fixed integer $N_{0} \geq 1$, we choose $N$ to be sufficiently large (in a manner that depends on $N_{0}, A_{*}, \mathfrak{D}, q$, and $\sigma$ ) and then $\varepsilon$ to be sufficiently small (in a manner that depends on $N, N_{0}, A_{*}, \mathfrak{D}, q$, and $\sigma$ ) such that all of the previous estimates proved in the paper hold true. For this fixed value of $A_{*}$, this justifies inequality (5.57). We now note that the negativity of the factor $C_{*}-A_{*}$ ensures that we can discard the first time integral on RHS 5.57), that is, for $t \in\left(T_{B o o t}, 1\right]$, we have $\mathbb{D}^{2}(t) \leq C \stackrel{\epsilon}{ }^{2}+C \int_{t}^{1} s^{-1+\sigma} \mathbb{D}^{2}(s) d s$. From this inequality and Grönwall's lemma, we deduce that $\mathbb{D}^{2}(t) \leq C \stackrel{\epsilon}{\epsilon}^{2}$. From this estimate and (5.6), we conclude the desired bound 5.2 .

### 5.10 Existence of perturbed solutions on the entire half-slab $(0,1] \times \mathbb{T}^{\mathfrak{D}}$

In the next proposition, we use the a priori estimates of Proposition 5.1 and standard local well-posedness/continuation results to show that the perturbed solution exists on $(0,1] \times \mathbb{T}^{\mathfrak{D}}$.
Proposition 5.28 (Existence of perturbed solutions on the entire half-slab $\left.(0,1] \times \mathbb{T}^{\mathfrak{D}}\right) . \operatorname{Let}\left(\Sigma_{1}=\mathbb{T}^{\mathfrak{D}}, \stackrel{\circ}{g}, \stackrel{\circ}{k}, \stackrel{\circ}{\psi}, \stackrel{\circ}{\phi}\right)$ be geometric initial data (see Sect. 1.1) for the Einstein-scalar field equations verifying the constraint equations 1.2a-1.2b and the CMC condition $\operatorname{tr} k=-1$ (see Remark 1.4), and let $\left\{\dot{e}_{I}\right\}_{I=1, \cdots, \mathfrak{D}}$ be the initial orthonormal frame (on $\Sigma_{1}$ ) constructed in Sect. 5.11. Recall that $\mathbb{L}_{(n)}(t), \mathbb{H}_{(n)}(t)$, and $\mathbb{D}(t)$ are norms from Definition 3.1 and that $\stackrel{\circ}{\epsilon}:=\mathbb{D}(1)$. Assume that the following conditions are satisfied:

- $N_{0} \geq 1$.
- $A_{*} \geq 1$ is sufficiently large.
- $N$ is sufficiently large in a manner that depends on $N_{0}, A_{*}, \mathfrak{D}, q$, and $\sigma$.
- The norm $\oplus$ € defined in (5.1) is sufficiently small in a manner that depends on $N, N_{0}, A_{*}, \mathfrak{D}, q$, and $\sigma$.

Then there exists a constant $C_{N, N_{0}, A_{*}, \mathfrak{D}, q, \sigma}>0$ such these data launch a perturbed solution

$$
\left(n, k_{I J}, \gamma_{I J B}, e_{I}^{i}, \omega_{i}^{I}, \psi\right)_{I, J, B, i=1, \cdots, \mathfrak{D}}
$$

to the reduced equations of Proposition 2.2 that exists classically on $(0,1] \times \mathbb{T}^{\mathfrak{D}}$ and satisfies the following estimate for $t \in(0,1]$ :

$$
\begin{equation*}
\mathbb{D}(t)+\mathbb{L}_{(n)}(t)+\mathbb{H}_{(n)}(t) \leq C_{N, N_{0}, A_{*}, \mathfrak{D}, q, \sigma} \stackrel{\circ}{\epsilon} \tag{5.58}
\end{equation*}
$$

Moreover, if we define $g_{i j}$ and $\mathbf{g}$ in terms of the reduced variables by $g_{i j}:=\omega_{i}^{A} \omega_{j}^{A}$ and $\mathbf{g}:=-n^{2} d t \otimes d t+$ $g_{a b} d x^{a} \otimes d x^{b}$ (where $t$ is the CMC time function and $\left\{x^{i}\right\}_{i=1, \cdots, \mathfrak{D}}$ are the transported spatial coordinates), then the tensorfields $(\mathbf{g}, \psi)$ are also classical solutions to the Einstein-scalar field system 1.1a)-1.1b on $(0,1] \times \mathbb{T}^{\mathfrak{D}}$.

Proof. We first fix $N_{0} \geq 1, A_{*}$ sufficiently large, and $N$ sufficiently large such that if the bootstrap smallness parameter $\varepsilon$ is sufficiently small, then all of the estimates proved in the previous subsections hold true on $\left(T_{\text {Boot }}, 1\right] \times \mathbb{T}^{\mathfrak{D}}$, as long as the bootstrap assumption 3.9 holds for $t \in\left(T_{B o o t}, 1\right]$. By standard local wellposedness, if $\AA$ € is sufficiently small and $C$ is sufficiently large, then there exists a maximal time $T_{\text {Max }} \in[0,1)$, such that the solution $(n, k, \gamma, e, \omega, n, \psi)$ to the reduced equations of Proposition 2.2 exists classically for $(t, x) \in\left(T_{\text {Max }}, 1\right] \times \mathbb{T}^{\mathfrak{D}}$ and such that the bootstrap assumptions (3.9) hold with $T_{\text {Boot }}=T_{\text {Max }}$ and $\varepsilon:=C \stackrel{\circ}{\epsilon}$. By enlarging $C$ if necessary, we can assume that $C \geq 2 C_{N, N_{0}, A_{*}, \mathfrak{D}, q, \sigma}$, where $C_{N, N_{0}, A_{*}, \mathfrak{D}, q, \sigma}$ is the constant on RHS (5.2). For the reader's convenience, we now comment on the "standard local well-posedness" mentioned above. Specifically, readers can consult [6 for the main ideas behind the proof of local well-posedness in a similar but distinct elliptic-hyperbolic gauge for Einstein's equations, or [59, Theorem 14.1] for a sketch of a proof of local well-posedness in CMC-transported spatial coordinates; local well-posedness for the equations of Proposition 2.2 can be proved via similar arguments. We emphasize that, as is stated in Proposition 2.2 , solutions to the reduced equations (including the constraints) are also solutions to the Einstein-scalar field system 1.1a 1.1b, where the spacetime metric can be reconstructed from the reduced variables via the equations $\mathbf{g}=-n^{2} d t \otimes d t+g_{a b} d x^{a} \otimes d x^{b}$ and $g_{i j}=g\left(\partial_{i}, \partial_{j}\right)=\omega_{i}^{C} \omega_{j}^{C}$ (see 2.7). Moreover, in view of the norms defined in Definition 3.1, it is a standard result (again, see 6] for the main ideas) that if $\varepsilon$ is sufficiently small, then either i) $T_{M a x}=0$ or ii) $T_{M a x} \in(0,1)$ and the bootstrap assumptions are saturated on the time interval ( $\left.T_{M a x}, 1\right]$, that is,

$$
\begin{equation*}
\sup _{t \in\left(T_{M a x}, 1\right]}\left\{\mathbb{D}(t)+\mathbb{L}_{(n)}(t)+\mathbb{H}_{(n)}(t)\right\}=\varepsilon \tag{5.59}
\end{equation*}
$$

The latter possibility is ruled out by inequality (5.2) when $\stackrel{\circ}{\epsilon}$ is small enough. Thus, $T_{\text {Max }}=0$. In particular, the solution exists classically for $(t, x) \in(0,1] \times \mathbb{T}^{\mathfrak{D}}$, and the estimate 5.58 holds for $t \in(0,1]$.

### 5.11 Construction of the initial orthonormal spatial frame

Thus far, we have not constructed the initial orthonormal spatial frame $\left\{\dot{e}_{I}\right\}_{I=1, \cdots, \mathfrak{D}}$ on $\Sigma_{1}$. To achieve this away from symmetry, we simply apply the Gram-Schmidt process to the transported spatial coordinate vectorfield frame $\left\{\partial_{i}\right\}_{i=1, \cdots, \mathfrak{D}}$. More precisely, with $\stackrel{\circ}{g}$ denoting the Riemannian metric on $\Sigma_{1}$, we set

$$
\begin{array}{rlr}
\stackrel{\circ}{e}_{1} & :=\frac{\partial_{1}}{\sqrt{\circ_{11}}}=\frac{\partial_{1}}{\sqrt{\circ}\left(\partial_{1}, \partial_{1}\right)}, & M=1, \cdots, \mathfrak{D}-1, \\
\stackrel{\circ}{E}_{M+1} & :=\partial_{M+1}-\sum_{L=1, \cdots, M} \underbrace{\stackrel{\circ}{c d} \delta_{M+1}^{c} \stackrel{\circ}{e}_{L}^{d}}_{\stackrel{\circ}{g}\left(\partial_{M+1}, e_{L}^{\circ}\right)} e_{L}^{\circ}, & M=1, \cdots, \mathfrak{D}-1 . \\
\stackrel{\circ}{e}_{M+1} & :=\frac{\stackrel{\circ}{E}_{M+1}}{\sqrt{\stackrel{\circ}{g}_{c d} \stackrel{\circ}{E}_{M+1}^{c} \stackrel{\circ}{E}_{M+1}^{d}}}, & \tag{5.60c}
\end{array}
$$

By construction, for $1 \leq I, J \leq \mathfrak{D}$, we have the desired identity $\stackrel{\circ}{g}\left({ }_{e}{ }_{I},{ }_{e}^{e}\right)=\delta_{I J}$, where $\delta_{I J}$ is the Kronecker delta.

In the polarized $U(1)$-symmetric case with $\mathfrak{D}=3$, we proceed in a similar fashion, but starting with $\grave{e}_{3}:=\frac{\partial_{3}}{\sqrt{\tilde{g}_{33}}}=\frac{\partial_{3}}{\sqrt{\dot{g}\left(\partial_{3}, \partial_{3}\right)}}$. Note that for metrics that are initially polarized and $U(1)$-symmetric in the sense described in Lemma 2.3 , this Gram-Schmidt process leads to an initial frame that respects the $\partial_{3}$ symmetry: $\mathcal{L}_{\partial_{3}} \dot{e}_{I}=0$ for $I=1,2,3$. Hence, Lemma 2.4 ensures that throughout the classical evolution, we have $e_{3}=\frac{\partial_{3}}{\sqrt{g_{33}}}$ and $\mathcal{L}_{\partial_{3}} e_{I}=0$ for $I=1,2,3$.

### 5.12 The near-Kasner smallness condition on the geometric initial data

Before proving our main theorems, we will first define a norm of the "geometric" initial data ( $\Sigma_{1}=$ $\mathbb{T}^{\mathfrak{D}}, \stackrel{\circ}{g}, \stackrel{\circ}{k}, \stackrel{\circ}{\psi}, \stackrel{\circ}{\phi}$ minus the background Kasner data. The smallness of this difference will be sufficient for the validity of our main results. We highlight that the lapse $n$ is not among the geometric initial data; it is a gauge-dependent quantity that can be controlled in terms of the geometric data. Then, in Lemma 5.29 , we show that if the geometric data are sufficiently near-Kasner, then the full data norm $\mathbb{D}(1)+\mathbb{L}_{(n)}(1)+\mathbb{H}_{(n)}(1)$ is small, i.e., we have smallness not only for the geometric data, but also for all of the gauge-dependent quantities such as $n-1, e_{I}^{i}-\widetilde{e}_{I}^{i}, k_{I J}-\widetilde{k}_{I J}$, etc.

To proceed, we let $\left(\Sigma_{1}=\mathbb{T}^{\mathfrak{D}}, \stackrel{\circ}{g}, \stackrel{\circ}{k}, \stackrel{\dot{\psi}}{\boldsymbol{\phi}} \mathrm{\phi}\right)$ be an initial data set, as described in Sect.1.1. Recall that relative to standard coordinates on $\mathbb{T}^{\mathfrak{D}}$, the Kasner background data (on $\Sigma_{1}$ ) have the following components: $\stackrel{\circ}{g}_{i j}^{K A S}:=\delta_{i j}, \stackrel{\circ}{k}_{i j}^{K A S}:=-\widetilde{q}_{\underline{q}} \delta_{\underline{i j}}, \dot{\psi}^{K A S}:=0, \dot{\phi}^{K A S}:=\widetilde{B}$, where $\delta_{i j}$ is the Kronecker delta, we do not sum over repeated underlined indices, and by assumption, the Kasner exponent constraints (1.7) are satisfied. For $N \in \mathbb{N}$, we define the following norm which, relative to the standard coordinates on $\mathbb{T}^{\mathfrak{D}}$, measures the perturbation of the initial data set away from the Kasner background:

$$
\begin{align*}
\stackrel{\circ}{\alpha}=\stackrel{\circ}{\alpha}(N):= & \sum_{i, j=1, \cdots, \mathfrak{D}}\left\|\stackrel{\circ}{g}_{i j}-\delta_{i j}\right\|_{H^{N+1}\left(\mathbb{T}^{\mathfrak{D}}\right)}+\sum_{i, j=1, \cdots, \mathfrak{D}}\left\|\grave{k}_{i j}+\widetilde{q}_{\underline{q}} \delta_{\underline{i j}}\right\|_{H^{N}\left(\mathbb{T}^{\mathfrak{D}}\right)}  \tag{5.61}\\
& +\|\dot{\psi}\|_{H^{N+1}\left(\mathbb{T}^{\mathfrak{D}}\right)}+\|\stackrel{\circ}{\phi}-\widetilde{B}\|_{H^{N}\left(\mathbb{T}^{\mathfrak{D}}\right)} .
\end{align*}
$$

In the next lemma, we show that the norms appearing in the bootstrap assumptions (3.9) are initially small, provided $\propto$ is sufficiently small.

Lemma 5.29 (A near-Kasner smallness condition on the geometric initial data implies smallness of all solution variables along $\left.\Sigma_{1}\right)$. Recall that $\mathbb{D}(t)$ is the total norm of the dynamic variables and that $\mathbb{L}_{(n)}$ and $\mathbb{H}_{(n)}$ are the norms of the lapse (see Definition 3.1). For $N \in \mathbb{N}$, we define

$$
\begin{equation*}
\stackrel{\circ}{\epsilon}=\stackrel{\circ}{\epsilon}(N):=\mathbb{D}(1)+\mathbb{L}_{(n)}(1)+\mathbb{H}_{(n)}(1) \tag{5.62}
\end{equation*}
$$

Let $\dot{\alpha}$ be the norm of the perturbation of the geometric initial data away from the Kasner data, as defined in (5.61). Let $\left\{\dot{\circ}_{I}\right\}_{I=1, \cdots, \mathfrak{D}}$ be the initial orthonormal frame constructed in Sect.5.11, and let the initial lapse $\stackrel{n}{n}:=\left.n\right|_{\Sigma_{1}}$ be the solution to the elliptic PDE 2.25. Fix $N_{0} \geq 1$. There exists a constant $C=C_{N, N_{0}, \mathfrak{D}}>0$ such that if $N$ is sufficiently large in a manner that depends on $N_{0}$ and $\mathfrak{D}$, and if $\stackrel{\circ}{\alpha}$ is sufficiently small, then

$$
\begin{equation*}
\stackrel{\circ}{\epsilon} \leq C \circ \tag{5.63}
\end{equation*}
$$

Sketch of the proof. This is a standard result, so we will only sketch the proof. Throughout, we will assume that $\dot{\alpha}$ is sufficiently small. From (5.61), we see that the $\mathfrak{D} \times \mathfrak{D}$ matrix $\stackrel{\circ}{g}_{i j}$ is equal to the identity matrix up to an error matrix whose components are bounded in the norm $\|\cdot\|_{H^{N+1}\left(\mathbb{T}^{\mathcal{D}}\right)}$ by $\lesssim \dot{\alpha}$. From this fact, the Gram-Schmidt process described in Sect. 5.11, and the standard Sobolev calculus (i.e., estimates of the type appearing in Lemma 4.1), it follows that for $1 \leq I, i \leq \mathfrak{D}$, we have $\left\|\dot{e}_{I}^{i}-\delta_{I}^{i}\right\|_{H^{N+1}\left(\mathbb{T}^{\mathfrak{D}}\right)} \lesssim \stackrel{\circ}{\alpha}$, where $\delta_{I}^{i}$ denotes the Kronecker delta. To complete the proof of (5.63), we must show that when $t=1$, the remaining norms in Definition 3.1 are all $\lesssim \dot{\alpha}$. This can be achieved by working relative to the standard spatial coordinates $\left\{x^{i}\right\}_{i=1, \cdots, \mathfrak{D}}$ on $\Sigma_{1}$ and using the definition of $\stackrel{\alpha}{\alpha}$, the definitions of the quantities appearing in the norms of Definition 3.1, the standard Sobolev calculus, and elliptic estimates for the lapse, similar to the ones we used to prove Proposition 5.3. As one example, we will show that $\left\|\gamma_{I J B}\right\|_{H^{N}\left(\Sigma_{1}\right)} \lesssim \stackrel{\circ}{\alpha}$. First, we note that $\left.\gamma_{I J B}\right|_{\Sigma_{1}}=\stackrel{\circ}{g}_{a b} \stackrel{\circ}{e}_{I}^{c}\left(\partial_{c} \stackrel{\circ}{e}_{J}^{a}\right) \stackrel{\circ}{e}_{B}^{b}+\stackrel{\circ}{e}_{I}^{i} \stackrel{\circ}{\circ}_{J}^{j} \stackrel{\circ}{e}_{B}^{b} \stackrel{\circ}{\Gamma}_{i b j}$, where $\stackrel{\circ}{\Gamma}_{i b j}=\frac{1}{2}\left\{\partial_{i} \stackrel{\circ}{g}_{b j}+\partial_{j} \stackrel{\circ}{g}_{i b}-\partial_{b} \stackrel{\circ}{g}_{i j}\right\}$ are the (lowered) Christoffel symbols of $\stackrel{\circ}{g}$ relative to the spatial coordinates $\left\{x^{i}\right\}_{i=1, \cdots, \mathfrak{D}}$ on $\Sigma_{1}$. Thus, from this expression for $\left.\gamma_{I J B}\right|_{\Sigma_{1}}$, definition 5.61), the estimates $\left\|\dot{e}_{I}^{i}-\delta_{I}^{i}\right\|_{H^{N+1}\left(\mathbb{T}^{\mathcal{D}}\right)} \lesssim \stackrel{\circ}{\alpha}$ and $\left\|\stackrel{\circ}{g}_{i j}-\delta_{i j}\right\|_{H^{N+1}\left(\mathbb{T}^{\mathfrak{P}}\right)} \lesssim \stackrel{\circ}{\alpha}$, and the standard Sobolev calculus, we conclude the desired bound $\left\|\gamma_{I J B}\right\|_{H^{N}\left(\Sigma_{1}\right)} \lesssim \stackrel{\circ}{\alpha}$. This concludes our proof sketch.

## 6 The two stable blowup theorems

In this section, we prove our two main theorems. The derivation of the a priori estimate 5.58 was the difficult part of the proof, and based on this estimate, the proofs of the main results will unfold in a natural fashion.

### 6.1 Statement of the theorems

In this section, we state the two theorems. The proofs are located in Sect.6.4. Before proving the theorems, we will first establish, in separate sections, some of their key aspects. We start by stating our main theorem for solutions without symmetry.

Theorem 6.1 (Precise version of stable Big Bang formation without symmetry assumptions). Let $\widetilde{\mathbf{g}}:=$ $-d t \otimes d t+\sum_{I=1}^{\mathfrak{D}} t^{2 \widetilde{q}_{I}} d x^{I} \otimes d x^{I}, \widetilde{\psi}:=\widetilde{B} \log t$ be an explicit generalized Kasner solution on $(0, \infty) \times \mathbb{T}^{\mathfrak{D}}$, where the constants $\left\{\widetilde{q}_{I}\right\}_{I=1, \cdots, \mathfrak{D}}$ and $\widetilde{B}$ satisfy the algebraic constraints $\sum_{I=1}^{\mathfrak{D}} \widetilde{q}_{I}=1$ and $\sum_{I=1}^{\mathfrak{D}} \widetilde{q}_{I}^{2}=1-\widetilde{B}^{2}$ as well as the following stability condition:

$$
\begin{equation*}
\max _{\substack{I, J, B=1, \cdots, \mathfrak{D} \\ I<J}}\left\{\widetilde{q}_{I}+\widetilde{q}_{J}-\widetilde{q}_{B}\right\}<1 \tag{6.1}
\end{equation*}
$$

Note that in the vacuum case, we have $\widetilde{B}=0$. As we discussed in Sect. 1.4. in the vacuum case, the set of Kasner solutions satisfying the algebraic constraints and the condition (6.1) is non-empty when $\mathfrak{D} \geq 10$, while in the presence of a scalar field, the set of Kasner solutions satisfying the algebraic constraints and the condition 6.1) is non-empty when $\mathfrak{D} \geq 3$. Let $\widetilde{k}_{I J}=-\widetilde{q}_{\underline{I}} \delta_{\underline{I} J} t^{-1}$ be the components of the second fundamental form of $\Sigma_{t}$ relative to the Kasner metric, with respect to the background orthonormal frame vectors $\widetilde{e}_{I}=$ $t^{-\widetilde{q}_{\underline{I}}} \partial_{\underline{I}}$, where we recall that we do not sum repeated underlined indices. Let $\left(\Sigma_{1}=\mathbb{T}^{\mathfrak{D}}, \stackrel{\circ}{g}, \stackrel{\circ}{k}, \stackrel{\circ}{\psi}, \stackrel{\circ}{\phi}\right)$ be geometric initial data (see Sect. 1.1) for the Einstein-scalar field equations verifying the constraint equations (1.2a)1.2b) and the CMC condition $\operatorname{tr} k=-1$ (see Remark 1.4), and let $\left\{\dot{\circ}_{I}\right\}_{I=1, \cdots, \mathfrak{D}}$ be the initial orthonormal frame (on $\Sigma_{1}$ ) constructed in Sect. 5.11. Note that $\dot{\psi}=\dot{\phi}=0$ corresponds to the Einstein-vacuum equations. Let

$$
\begin{align*}
\stackrel{\circ}{\alpha}:= & \sum_{i, j=1, \cdots, \mathfrak{D}}\left\|\stackrel{\circ}{g}_{i j}-\delta_{i j}\right\|_{H^{N+1}\left(\mathbb{T}^{\mathscr{D}}\right)}+\sum_{i, j=1, \cdots, \mathfrak{D}}\left\|\circ_{i j}+\widetilde{q}_{\underline{i}} \delta_{\underline{i}}\right\|_{H^{N}\left(\mathbb{T}^{\mathscr{D}}\right)}  \tag{6.2}\\
& +\|\dot{\psi}\|_{H^{N+1}\left(\mathbb{T}^{\mathscr{D}}\right)}+\|\stackrel{\circ}{\phi}-\widetilde{B}\|_{H^{N}\left(\mathbb{T}^{\mathscr{D}}\right)}
\end{align*}
$$

denote the norm of the perturbation of the geometric initial data away from the Kasner data, as defined in (5.61). Assume that:

- $N_{0} \geq 1$ is a fixed positive integer (we are free to choose $N_{0}$ ).
- $A_{*}$ is sufficiently large in a manner that depends on $\mathfrak{D}$ and the parameters $q$ and $\sigma$ fixed in (3.6).
- $N$ is sufficiently large in a manner that depends on $N_{0}, A_{*}, \mathfrak{D}, q$, and $\sigma$.
- $\stackrel{\propto}{\alpha}$ is sufficiently small in a manner that depends on $N, N_{0}, A_{*}, \mathfrak{D}, q$, and $\sigma$.

Then the following conclusions hold.
Existence and norm estimates on $(0,1] \times \mathbb{T}^{\mathfrak{D}}$. The initial data launch a unique solution

$$
\left(n, k_{I J}, \gamma_{I J B}, e_{I}^{i}, \omega_{i}^{I}, \psi\right)_{I, J, B, i=1, \cdots, \mathfrak{D}}
$$

to the reduced Einstein-scalar field equations of Proposition 2.2 existing on the slab $(t, x) \in(0,1] \times \mathbb{T}^{\mathfrak{D}}$. Moreover, if we define $g_{i j}$ and $\mathbf{g}$ in terms of the reduced variables by $g_{i j}:=\omega_{i}^{A} \omega_{j}^{A}$ and $\mathbf{g}:=-n^{2} d t \otimes d t+$ $g_{a b} d x^{a} \otimes d x^{b}$ (where $t$ is the CMC time function and $\left\{x^{i}\right\}_{i=1, \cdots, \mathfrak{D}}$ are the transported spatial coordinates), then the tensorfields $(\mathbf{g}, \psi)$ are also classical solutions to the Einstein-scalar field system 1.1a -1.1b on $(0,1] \times \mathbb{T}^{\mathfrak{D}}$. In addition, there exists a constant $C=C_{N, N_{0}, A_{*}, \mathfrak{D}, q, \sigma}$ such that the following estimates hold for $t \in(0,1]$ :

$$
\begin{align*}
& \sum_{I, i=1}^{\mathfrak{D}} t^{q}\left\|e_{I}^{i}-\widetilde{e}_{I}^{i}\right\|_{W^{N_{0}, \infty}\left(\Sigma_{t}\right)}+\sum_{I, i=1}^{\mathfrak{D}} t^{q}\left\|\omega_{i}^{I}-\widetilde{\omega}_{i}^{I}\right\|_{W^{N_{0}, \infty}\left(\Sigma_{t}\right)}+\sum_{I, J, B=1}^{\mathfrak{D}} t^{q}\left\|\gamma_{I J B}\right\|_{W^{N_{0}, \infty}\left(\Sigma_{t}\right)} \\
& +\sum_{I, J=1}^{\mathfrak{D}} t\left\|k_{I J}-\widetilde{k}_{I J}\right\|_{W^{N_{0}+1, \infty}\left(\Sigma_{t}\right)}+\sum_{I=1}^{\mathfrak{D}} t^{q}\left\|e_{I} \psi\right\|_{W^{N_{0}, \infty}\left(\Sigma_{t}\right)}+\left\|t \partial_{t} \psi-\widetilde{B}\right\|_{W^{N_{0}+1, \infty}\left(\Sigma_{t}\right)}  \tag{6.3a}\\
& \quad+t^{-\sigma}\|n-1\|_{W^{N_{0}+1, \infty}\left(\Sigma_{t}\right)}+\sum_{I}^{\mathfrak{D}} t^{q-\sigma}\left\|e_{I} n\right\|_{W^{N_{0}, \infty}\left(\Sigma_{t}\right)} \\
& \leq C \stackrel{\circ}{\alpha}, \\
& \sum_{I, i=1}^{\mathfrak{D}} t^{A_{*}+q}\left\|e_{I}^{i}-\widetilde{e}_{I}^{i}\right\|_{\dot{H}^{N}\left(\Sigma_{t}\right)}+\sum_{I, i=1}^{\mathfrak{D}} t^{A_{*}+q}\left\|\omega_{i}^{I}-\widetilde{\omega}_{i}^{I}\right\|_{\dot{H}^{N}\left(\Sigma_{t}\right)}+\sum_{I, J, B=1}^{\mathfrak{D}} t^{A_{*}+1}\left\|\gamma_{I J B}\right\|_{\dot{H}^{N}\left(\Sigma_{t}\right)} \\
& \quad+\sum_{I, J=1}^{\mathfrak{D}} t^{A_{*}+1}\left\|k_{I J}\right\|_{\dot{H}^{N}\left(\Sigma_{t}\right)}+\sum_{I=1}^{\mathfrak{D}} t^{A_{*}+1}\left\|e_{I} \psi\right\|_{\dot{H}^{N}\left(\Sigma_{t}\right)}+t^{A_{*}+1}\left\|\partial_{t} \psi\right\|_{\dot{H}^{N}\left(\Sigma_{t}\right)}  \tag{6.3b}\\
& \quad+t^{A_{*}}\|n\|_{\dot{H}^{N}\left(\Sigma_{t}\right)}+t^{A_{*}+1}\|\overrightarrow{e n}\|_{\dot{H}^{N}\left(\Sigma_{t}\right)}+\sum_{I=1}^{\mathfrak{D}} t^{A_{*}+1}\left\|e_{I} \psi\right\|_{\dot{H}^{N}\left(\Sigma_{t}\right)} \\
& \leq C \stackrel{\circ}{\alpha} .
\end{align*}
$$

Kasner-like behavior. The scalar component functions $\left\{t k_{I J}(t, x)\right\}_{I, J=1, \cdots, \mathfrak{D}}$ of the renormalized second fundamental form of $\Sigma_{t}$ with respect to the $g$-orthonormal frame $\left\{e_{I}(t, x)\right\}_{I=1, \cdots, \mathfrak{D}}$, as well as the renormalized time derivative $t \partial_{t} \psi(t, x)$ of the scalar field, have continuous $W^{N_{0}+1, \infty}\left(\mathbb{T}^{\mathfrak{D}}\right)$ limits, denoted respectively by $\left\{\kappa_{I J}^{(\infty)}(x)\right\}_{I, J=1, \cdots, \mathfrak{D}}$ and $B^{(\infty)}(x)$, as $t \downarrow 0$. Moreover, the following estimates hold for $t \in(0,1]$ :

$$
\begin{align*}
& \sum_{I, J=1, \cdots, \mathfrak{D}}\left\|t k_{I J}(t, \cdot)-\kappa_{I J}^{(\infty)}\right\|_{W^{N_{0}+1, \infty}\left(\mathbb{T}^{\mathcal{P}}\right)} \leq C \circ{ }^{\alpha} t^{\sigma}, \quad\left\|t \partial_{t} \psi(t, \cdot)-B^{(\infty)}\right\|_{W^{N_{0}+1, \infty}\left(\mathbb{T}^{\mathscr{D}}\right)} \leq C \propto \dot{\alpha} t^{\sigma},  \tag{6.4a}\\
& \sum_{I, J=1, \cdots, \mathfrak{D}}\left\|\kappa_{I J}^{(\infty)}+\widetilde{q}_{\underline{I}} \delta_{\underline{I} J}\right\|_{W^{N_{0}+1, \infty}\left(\mathbb{T}^{\mathcal{D}}\right)} \leq C \stackrel{\circ}{\alpha}, \quad\left\|B^{(\infty)}-\widetilde{B}\right\|_{W^{N_{0}+1, \infty}\left(\mathbb{T}^{\mathcal{D}}\right)} \leq C \stackrel{\circ}{\alpha} . \tag{6.4b}
\end{align*}
$$

In addition, for each $x \in \mathbb{T}^{\mathfrak{D}}$, the symmetric $\mathfrak{D} \times \mathfrak{D}$ matrix $\left(-\kappa_{I J}^{(\infty)}(x)\right)_{I, J=1, \cdots, \mathfrak{D}}$ has $\mathfrak{D}$ (possibly repeated) eigenvalues $q_{I}^{(\infty)}(x)-$ which are the "final" Kasner exponents of the perturbed spacetime - that can be ordered such that

$$
\begin{equation*}
q_{1}^{(\infty)}, \cdots, q_{\mathfrak{D}}^{(\infty)} \in C^{0}\left(\mathbb{T}^{\mathfrak{D}}\right) \tag{6.5}
\end{equation*}
$$

[^23]and such that the following estimate holds:
\[

$$
\begin{equation*}
\left\|q_{I}^{(\infty)}-\widetilde{q}_{I}\right\|_{L^{\infty}\left(\mathbb{T}^{\mathfrak{D}}\right)} \leq C \dot{\alpha}^{\frac{1}{\mathfrak{D}}} \tag{6.6}
\end{equation*}
$$

\]

Moreover, the $\left\{q_{I}^{(\infty)}(x)\right\}_{I=1, \cdots, \mathfrak{D}}$ and $B^{(\infty)}(x)$ satisfy the following pointwise algebraic relations:

$$
\begin{equation*}
\sum_{I=1}^{\mathfrak{D}} q_{I}^{(\infty)}(x)=1, \quad \quad \sum_{I=1}^{\mathfrak{D}}\left[q_{I}^{(\infty)}(x)\right]^{2}=1-\left[B^{(\infty)}(x)\right]^{2} \tag{6.7}
\end{equation*}
$$

Curvature blowup. The Kretschmann scalar of $\mathbf{g}$, namely $\mathbf{R i e m}^{\alpha \mu \beta \nu} \mathbf{R i e m}_{\alpha \mu \beta \nu}$, blows up as $t \downarrow 0$, as is evident from the following pointwise estimate, valid for $(t, x) \in(0,1] \times \mathbb{T}^{\mathfrak{D}}$ :

$$
\begin{align*}
\operatorname{Riem}^{\alpha \mu \beta \nu} \boldsymbol{\operatorname { R i e m }}_{\alpha \mu \beta \nu}(t, x)= & 4 t^{-4}\left\{\sum_{I=1}^{\mathfrak{D}}\left[\left(q_{I}^{(\infty)}(x)\right)^{2}-q_{I}^{(\infty)}(x)\right]^{2}+\sum_{1 \leq I<J \leq \mathfrak{D}}\left(q_{I}^{(\infty)}(x)\right)^{2}\left(q_{J}^{(\infty)}(x)\right)^{2}\right\} \\
& +\mathcal{O}\left(\stackrel{\alpha}{ } t^{-4+\sigma}\right)  \tag{6.8}\\
= & 4 t^{-4}\left\{\sum_{I=1}^{\mathfrak{D}}\left[\widetilde{q}_{I}^{2}-\widetilde{q}_{I}\right]^{2}+\sum_{1 \leq I<J \leq \mathfrak{D}} \widetilde{q}_{I}^{2} \widetilde{q}_{J}^{2}\right\}+\mathcal{O}\left({ }_{\alpha}^{\alpha} t^{-4}\right)
\end{align*}
$$

Inextendibility. The spacetime is past-inextendible as a $C^{2}$ Lorentzian manifold.
Remark 6.2 (No regular limit is claimed for the orthonormal frame vectorfields). Despite the convergence of the renormalized component functions $\left\{t k_{I J}(t, x)\right\}_{I, J=1, \cdots, \mathfrak{D}}$, our proof does not yield (or require!) that the component functions $\left\{e_{I}^{i}(t, x)\right\}_{I, i=1, \cdots, \mathscr{D}}$ of the frame vectorfields with respect to the transported spatial coordinates can be rescaled by powers of $t$ so as to have non-trivial, regular limits as $t \downarrow 0$.

Remark 6.3 (In general, no additional regularity is claimed for the final Kasner exponents). Although $\kappa_{I J}^{(\infty)}(x) \in W^{N_{0}+1}\left(\mathbb{T}^{\mathfrak{D}}\right)$ (where we assume $N_{0} \geq 1$ ), in general, the function space $C^{0}\left(\mathbb{T}^{\mathfrak{D}}\right)$ in 6.5 and the norm $\|\cdot\|_{L^{\infty}\left(\mathbb{T}^{\mathfrak{D}}\right)}$ on LHS (6.6) are optimal (in the class of $C^{m}\left(\mathbb{T}^{\mathfrak{D}}\right)$ spaces with $m \in \mathbb{N}$ ), due to the fact that the background solution is allowed to have repeated Kasner exponents. However, by analyzing the dependence of the characteristic polynomial of the matrix $\left(t k_{I J}(t, x)\right)_{I, J=1, \cdots, \mathfrak{D}}$ on the entries $t k_{I J}$, we could show that if $\widetilde{q}_{I} \neq \widetilde{q}_{J}$ for $1 \leq I<J \leq \mathfrak{D}$, then the space $C^{0}\left(\mathbb{T}^{\mathfrak{D}}\right)$ in 6.5 could be replaced with $C^{N_{0}+1}\left(\mathbb{T}^{\mathfrak{D}}\right)$, the norm $\|\cdot\|_{L^{\infty}\left(\mathbb{T}^{\mathfrak{D}}\right)}$ on LHS (6.6) could be replaced with $\|\cdot\|_{W^{N_{0}+1, \infty}\left(\mathbb{T}^{\mathfrak{D}}\right)}$, and the factor $\propto^{\frac{1}{\mathfrak{D}}}$ on RHS (6.6) could be replaced with $\stackrel{\alpha}{\alpha}$. However, we would have to impose another smallness assumption on $\stackrel{\circ}{\alpha}: \stackrel{\circ}{\alpha} \ll \min _{1 \leq I<J \leq \mathfrak{D}}\left|\widetilde{q}_{I}-\widetilde{q}_{J}\right|$. We thank one of the referees for providing helpful comments tied to this issue.

Remark 6.4. The eigenvectors of the symmetric matrix $\left(\kappa_{I J}^{(\infty)}(x)\right)_{I, J=1, \cdots, \mathfrak{D}}$ might fail to be continuous in $x$, for example in the case where the $q_{I}^{(\infty)}(x)$ 's have contact points of infinite order; see 38, Chapter Two, Example 5.3].

We now state our main theorem for polarized $U(1)$-symmetric solutions.
Theorem 6.5 (Precise version of stable Big Bang formation for polarized $U(1)$-symmetric Einstein-vacuum solutions in $1+3$ dimensions). Let $\widetilde{\mathbf{g}}=-d t \otimes d t+t^{2 \widetilde{q}_{1}} d x^{1} \otimes d x^{1}+t^{2 \widetilde{q}_{2}} d x^{2} \otimes d x^{2}+t^{2 \widetilde{q}_{3}} d x^{3} \otimes d x^{3}$ be a "background" Kasner solution on $(0, \infty) \times \mathbb{T}^{3}$ with Kasner exponents satisfying

$$
\begin{equation*}
\sum_{I=1}^{3} \widetilde{q}_{I}=\sum_{I=1}^{3} \widetilde{q}_{I}^{2}=1, \quad \max _{I=1,2,3} \widetilde{q}_{I}<1 \tag{6.9}
\end{equation*}
$$

Let $\widetilde{k}_{I J}=-\widetilde{q}_{\underline{I}} \delta_{\underline{I} J} t^{-1}$ be the components of the second fundamental form of $\Sigma_{t}$ relative to the Kasner metric, with respect to the background orthonormal frame vectors $\widetilde{e}_{I}=t^{-\widetilde{q}_{\underline{I}}} \partial_{\underline{I}}$. Let $\left(\Sigma_{1}=\mathbb{T}^{3}, \stackrel{\circ}{g}, \stackrel{\circ}{k}\right)$ be polarized $U(1)$ symmetric initial data (see Sects. 1.1 and 1.4.2) for the Einstein-vacuum equations verifying the constraint
equations 1.2a-1.2b (for $\psi=0$ ) and the CMC condition $\operatorname{tr} \stackrel{\circ}{k}=-1$ (see Remark 1.4), and such that $\dot{X}:=\partial_{3}$ is the hypersurface-orthogonal Killing vectorfield of the data. Let $\stackrel{\circ}{\alpha}$ be the norm of the perturbation of the initial data away from the Kasner data, as defined in 5.61) (where the scalar field data on RHS (5.61) are vanishing by assumption). Let $\left\{e_{I}\right\}_{I=1,2,3}$ be the $g$-orthonormal frame obtained by constructing the initial orthonormal frame as in Sect. 5.11 and then using Lemma 2.4 to ensure that throughout the evolution, the corresponding frame solution to the Fermi-Walker transport equation (2.8) verifies $e_{3}=\frac{\partial_{3}}{\sqrt{g_{33}}}$ and $\mathcal{L}_{\partial_{3}} e_{I}=0$ for $I=1,2,3$, where $\mathcal{L}$ denotes Lie differentiation. Assume that the parameters $N, N_{0}, A_{*}, q, \sigma, \stackrel{\circ}{\alpha}$ satisfy the assumptions of Theorem 6.1, where in polarized $U(1)$-symmetry, $q, \sigma$ are fixed constants verifying

$$
\begin{equation*}
0<2 \sigma<2 \sigma+\max \left\{\left|\widetilde{q}_{1}\right|,\left|\widetilde{q}_{2}\right|,\left|\widetilde{q}_{3}\right|\right\}<q<1-2 \sigma \tag{6.10}
\end{equation*}
$$

Then the conclusions stated in Theorem 6.1 hold for the solution to the reduced equations of Proposition 2.2 (which also yields a solution to the Einstein-vacuum equations, i.e., 1.1a with $\psi \equiv 0$ ) that arises from the prescribed polarized $U(1)$-symmetric initial data $(\stackrel{\circ}{g}, \stackrel{\circ}{k})$. Moreover, the solution is polarized $U(1)$-symmetric in the sense that relative to the transported spatial coordinates, $\partial_{3}$ is a hypersurface-orthogonal Killing vectorfield of the spacetime metric $\mathbf{g}$, and $\mathbf{g}$ is of the form 2.37.

### 6.2 Limiting functions

In the next proposition, we show that the scalar functions $\left\{t k_{I J}(t, \cdot)\right\}_{I, J=1, \cdots, \mathfrak{D}}$ and $t \partial_{t} \psi(t, \cdot)$ have limits in $W^{N_{0}+1}\left(\mathbb{T}^{d}\right)$, as $t \downarrow 0$. Moreover, the limiting fields obey a limiting Hamiltonian constraint equation and exhibit other "Kasner-like" properties.

Proposition 6.6 (Asymptotic, Kasner-like limits). Under the assumptions and conclusions of Proposition 5.28, the scalar component functions $\left\{k_{I J}(t, x)\right\}_{I, J=1, \cdots, \mathfrak{D}}$ of the renormalized second fundamental form of $\Sigma_{t}$ with respect to the $g$-orthonormal frame $\left\{e_{I}(t, x)\right\}_{I=1, \cdots, \mathfrak{D}}$ and the renormalized scalar field velocity $t_{t} \psi(t, x)$ have continuous limits in $W^{N_{0}+1, \infty}\left(\mathbb{T}^{\mathfrak{D}}\right)$, denoted respectively by $\left\{\kappa_{I J}^{(\infty)}(x)\right\}_{I, J=1, \cdots, \mathfrak{D}}$ and $B^{(\infty)}(x)$, as $t \downarrow 0$. Moreover, the following estimates hold:

$$
\begin{align*}
\sum_{I, J=1, \cdots, \mathfrak{D}}\left\|t k_{I J}(t, \cdot)-\kappa_{I J}^{(\infty)}\right\|_{W^{N_{0}+1, \infty}\left(\mathbb{T}^{\mathcal{D}}\right)} \lesssim \stackrel{\circ}{\epsilon} t^{\sigma}, & \left\|t \partial_{t} \psi(t, \cdot)-B^{(\infty)}\right\|_{W^{N_{0}+1, \infty}\left(\mathbb{T}^{\mathfrak{D})}\right.} \lesssim \stackrel{\circ}{ } t^{\sigma},  \tag{6.11a}\\
\sum_{I, J=1, \cdots, \mathfrak{D}}\left\|\kappa_{I J}^{(\infty)}+\widetilde{q}_{\underline{I}} \delta_{\underline{I} J}\right\|_{W^{N_{0}+1, \infty}\left(\mathbb{T}^{\mathfrak{D})}\right.} \lesssim \stackrel{\circ}{\epsilon}, & \left\|B^{(\infty)}-\widetilde{B}\right\|_{W^{N_{0}+1, \infty}\left(\mathbb{T}^{\mathscr{D})}\right.} \lesssim \stackrel{\circ}{\epsilon} . \tag{6.11b}
\end{align*}
$$

In addition, for each $x \in \mathbb{T}^{\mathfrak{D}}$, the symmetric $\mathfrak{D} \times \mathfrak{D}$ matrix $\left(-\kappa_{I J}^{(\infty)}(x)\right)_{I, J=1, \cdots, \mathfrak{D}}$ has $\mathfrak{D}$ (possibly repeated) eigenvalues $q_{I}^{(\infty)}(x)-$ which are the "final" Kasner exponents of the perturbed spacetime - that can be ordered such that $q_{1}^{(\infty)}, \cdots, q_{\mathfrak{D}}^{(\infty)} \in C^{0}\left(\mathbb{T}^{\mathfrak{D}}\right)$ and such that the following estimate holds:

$$
\begin{equation*}
\sum_{I=1, \cdots, \mathfrak{D}}\left\|q_{I}^{(\infty)}-\widetilde{q}_{I}\right\|_{L^{\infty}\left(\mathbb{T}^{\mathscr{D}}\right)} \lesssim \stackrel{\circ}{\epsilon}^{\frac{1}{\mathcal{D}}} \tag{6.12}
\end{equation*}
$$

Moreover, the $\left\{q_{I}^{(\infty)}(x)\right\}_{I=1, \cdots, \mathfrak{D}}$ and $B^{(\infty)}(x)$ satisfy the following pointwise algebraic relations:

$$
\begin{equation*}
\sum_{I=1}^{\mathfrak{D}} q_{I}^{(\infty)}(x)=1, \quad \sum_{I=1}^{\mathfrak{D}}\left[q_{I}^{(\infty)}(x)\right]^{2}=1-\left[B^{(\infty)}(x)\right]^{2} \tag{6.13}
\end{equation*}
$$

Proof. Let $\left\{t_{n}\right\}_{n=1}^{\infty} \subset(0,1]$ be a decreasing sequence of times such that $\lim _{n \rightarrow \infty} t_{n}=0$. A straightforward modification of the proof of (5.48, based on the evolution equation 5.31a and the estimate 5.58), yields that when $0<a<b \leq 1$, we have $\left\|a k_{I J}(a, \cdot)-b k_{I J}(b, \cdot)\right\|_{W^{N_{0}+1, \infty}\left(\mathbb{T}^{\mathfrak{P}}\right)} \lesssim \stackrel{\circ}{\epsilon} \int_{a}^{b} s^{-1+\sigma} \mathbb{D}(s) d s \lesssim \stackrel{\circ}{\epsilon} b^{\sigma}$. Hence, $\left\{t_{n} k_{I J}\left(t_{n}, \cdot\right)\right\}_{n=1}^{\infty}$ is a Cauchy sequence in $W^{N_{0}+1, \infty}\left(\mathbb{T}^{\mathfrak{D}}\right)$, and its limit, which we denote by $\kappa_{I J}^{(\infty)}$, verifies $\left\|\kappa_{I J}^{(\infty)}-t k_{I J}(t, \cdot)\right\|_{W^{N_{0}+1, \infty}\left(\mathbb{T}^{\mathcal{D}}\right)} \lesssim \stackrel{\circ}{\epsilon} t^{\sigma}$. In particular, $\left\|\kappa_{I J}^{(\infty)}-k_{I J}(1, \cdot)\right\|_{W^{N_{0}+1, \infty}\left(\mathbb{T}^{\mathcal{D}}\right)} \lesssim \stackrel{\circ}{\epsilon}$. Since $\| k_{I J}(1, \cdot)+$ $\widetilde{q}_{\underline{I}} \delta_{\underline{I} J}\left\|_{W^{N_{0}+1, \infty}\left(\mathbb{T}^{\mathfrak{D}}\right)}=\right\| k_{I J}(1, \cdot)-\widetilde{k}_{I J} \|_{W^{N_{0}+1, \infty}\left(\mathbb{T}^{\mathfrak{D}}\right)} \lesssim \stackrel{\circ}{\epsilon}$, we infer from the triangle inequality that $\| \mathrm{\kappa}_{I J}^{(\infty)}+$
$\widetilde{q}_{\underline{I}} \delta_{\underline{I} J} \|_{W^{N_{0}+1, \infty}\left(\mathbb{T}^{\mathcal{D}}\right)} \lesssim \stackrel{\circ}{\epsilon}$. We have therefore proved 6.11a and 6.11b for $\kappa_{I J}^{(\infty)}$. Moreover, the symmetric matrix $\left(\kappa_{I J}^{(\infty)}(x)\right)_{I, J=1, \cdots, \mathfrak{D}}$ is $\mathcal{O}(\stackrel{\circ}{\epsilon})$-close to the diagonal matrix $\operatorname{diag}\left(-\widetilde{q}_{1}, \cdots,-\widetilde{q}_{\mathfrak{D}}\right)$. Thus, at each fixed $x$, $-\left(\mathrm{K}_{I J}^{(\infty)}(x)\right)_{I, J=1, \cdots, \mathfrak{D}}$ is diagonalizable, and by standard perturbation theory (see 63, Theorem 1.4 in Chapter IV]), its (possibly repeated) eigenvalues $q_{I}(x)$ can be ordered such that $q_{1}^{(\infty)}(x), \cdots, q_{\mathfrak{D}}^{(\infty)}(x) \in C^{0}\left(\mathbb{T}^{\mathfrak{D}}\right)$ and such that the following estimate holds:

$$
\begin{equation*}
\sum_{I=1, \cdots, \mathfrak{D}}\left|q_{I}^{(\infty)}(x)-\widetilde{q}_{I}\right| \lesssim \sum_{I, J=1, \cdots, \mathfrak{D}}\left|\kappa_{I J}^{(\infty)}(x)+\widetilde{q}_{\underline{I}} \delta_{\underline{I} J}\right|^{\frac{1}{\mathfrak{D}}} \tag{6.14}
\end{equation*}
$$

From (6.14) and (6.11b), we conclude (6.12).
The convergence results and estimates for $t \partial_{t} \psi$ can be proved in a similar fashion by making straightforward modifications to the proof of (5.51).

To derive the first relation in (6.13), we employ the CMC condition (2.17) (which is equivalent to $-1=$ $\left.t k_{C C}\right)$ and the estimate $\left\|\kappa_{I J}^{(\infty)}-t k_{I J}(t, \cdot)\right\|_{W^{N_{0}+1, \infty}\left(\mathbb{T}^{\mathcal{D}}\right)} \lesssim \stackrel{\circ}{\epsilon} t^{\sigma}$ proved above to deduce the following pointwise estimate:

$$
\begin{equation*}
-1=\operatorname{ttrk}(t, x)=\mathcal{O}\left(\stackrel{\circ}{\epsilon} t^{\sigma}\right)+\operatorname{tr}{ }^{(\infty)}(x)=\mathcal{O}\left(\stackrel{\AA}{\epsilon} t^{\sigma}\right)-\sum_{I=1}^{\mathfrak{D}} q_{I}^{(\infty)}(x) \tag{6.15}
\end{equation*}
$$

where to obtain the last equality, we used that the trace of the $\mathfrak{D} \times \mathfrak{D}$ matrix $\left(\kappa_{I J}^{(\infty)}\right)_{I, J=1, \cdots, \mathfrak{D}}$ is the sum of its eigenvalues $-q_{1}^{(\infty)}, \cdots,-q_{\mathfrak{D}}^{(\infty)}$. Taking the limit $t \downarrow 0$ on RHS 6.15, we obtain the desired relation.

To derive the second relation in (6.13), we multiply the Hamiltonian constraint 2.26a by $t^{2}$ and use the estimate (5.58), the inequalities in (3.6), and the estimates $\left\|\mathrm{\kappa}_{I J}^{(\infty)}-t k_{I J}(t, \cdot)\right\|_{W^{N_{0}+1, \infty}\left(\mathbb{T}^{\mathcal{D}}\right)} \lesssim \stackrel{\circ}{\epsilon} t^{\sigma}$ and $\left\|B^{(\infty)}-t \partial_{t} \psi(t, \cdot)\right\|_{W^{N_{0}+1, \infty}\left(\mathbb{T}^{\mathcal{D}}\right)} \lesssim \AA^{\circ} t^{\sigma}$ noted above to deduce the following pointwise estimate:

$$
\begin{align*}
1= & t^{2} k_{C D}(t, x) k_{C D}(t, x)-t^{2}\left\{2 e_{C} \gamma_{D D C}-\gamma_{C D E} \gamma_{E D C}-\gamma_{C C D} \gamma_{E E D}\right\}(t, x) \\
& +n^{-2}\left[t \partial_{t} \psi(t, x)\right]^{2}+t^{2}\left[e_{C} \psi(t, x)\right] e_{C} \psi(t, x) \\
= & \kappa_{C D}^{(\infty)}(x) \kappa_{C D}^{(\infty)}(x)+\left[B^{(\infty)}(x)\right]^{2}+\mathcal{O}\left(\stackrel{\circ}{ } t^{\sigma}\right)=\sum_{I=1}^{\mathfrak{D}}\left[q_{I}^{(\infty)}(x)\right]^{2}+\left[B^{(\infty)}(x)\right]^{2}+\mathcal{O}\left({ }^{\circ} t^{\sigma}\right) . \tag{6.16}
\end{align*}
$$

The desired second relation in (6.13) now follows from taking the limit $t \downarrow 0$ on RHS (6.16). This completes the proof of the proposition.

### 6.3 Monotonic blowup of curvature

In the following proposition, we show that the Kretschmann scalars of the solutions studied in the present paper blow up like $t^{-4}$.

Proposition 6.7 (Monotonic blow up of the Kretschmann scalar). Under the assumptions and conclusions of Proposition 5.28, the Kretschmann scalar $\mathbf{R i e m}{ }^{\alpha \mu \beta \nu} \mathbf{R i e m}_{\alpha \mu \beta \nu}$ obeys the following pointwise estimate for $(t, x) \in(0,1] \times \mathbb{T}^{\mathfrak{D}}$, where the functions $\left\{q_{I}^{(\infty)}(x)\right\}_{I=1, \cdots, \mathfrak{D}}$ are as in the conclusions of Proposition 6.6.

$$
\begin{align*}
\operatorname{Riem}^{\alpha \mu \beta \nu} \operatorname{Riem}_{\alpha \mu \beta \nu}(t, x)= & 4 t^{-4}\left\{\sum_{I=1}^{\mathfrak{D}}\left[\left(q_{I}^{(\infty)}(x)\right)^{2}-q_{I}^{(\infty)}(x)\right]^{2}+\sum_{1 \leq I<J \leq \mathfrak{D}}\left(q_{I}^{(\infty)}(x)\right)^{2}\left(q_{J}^{(\infty)}(x)\right)^{2}\right\} \\
& +\mathcal{O}\left(\left(\stackrel{\circ}{ } t^{-4+\sigma}\right)\right.  \tag{6.17}\\
= & 4 t^{-4}\left\{\sum_{I=1}^{\mathfrak{D}}\left[\widetilde{q}_{I}^{2}-\widetilde{q}_{I}\right]^{2}+\sum_{1 \leq I<J \leq \mathfrak{D}} \widetilde{q}_{I}^{2} \widetilde{q}_{J}^{2}\right\}+\mathcal{O}\left(\stackrel{\circ}{\epsilon} t^{-4}\right)
\end{align*}
$$

Proof. We first use the standard symmetries and antisymmetries of the Riemann curvature tensor of $\mathbf{g}$ to compute the following identity for its Kretschmann scalar:

$$
\begin{align*}
\operatorname{Riem}^{\alpha \mu \beta \nu} \operatorname{Riem}_{\alpha \mu \beta \nu}= & \boldsymbol{\operatorname { R i e m }}\left(e_{A}, e_{I}, e_{B}, e_{J}\right) \operatorname{Riem}\left(e_{A}, e_{I}, e_{B}, e_{J}\right) \\
& +4 \operatorname{Riem}\left(e_{0}, e_{I}, e_{0}, e_{J}\right) \operatorname{Riem}\left(e_{0}, e_{I}, e_{0}, e_{J}\right)  \tag{6.18}\\
& -4 \operatorname{Riem}\left(e_{A}, e_{I}, e_{0}, e_{J}\right) \operatorname{Riem}\left(e_{A}, e_{I}, e_{0}, e_{J}\right)
\end{align*}
$$

Next, using the Gauss equation (2.29), the estimate (5.58), the inequalities in (3.6), and the convergence estimate 6.11a for $t k_{I J}$, we derive the following pointwise estimate:

$$
\begin{equation*}
t^{2} \boldsymbol{\operatorname { R i e m }}\left(e_{A}, e_{I}, e_{B}, e_{J}\right)=\left(t k_{I J}\right)\left(t k_{A B}\right)-\left(t k_{A J}\right)\left(t k_{B I}\right)+\mathcal{O}(\stackrel{\circ}{\epsilon}) t^{\sigma}=\kappa_{I J}^{(\infty)} \kappa_{A B}^{(\infty)}-\kappa_{A J}^{(\infty)} \kappa_{B I}^{(\infty)}+\mathcal{O}(\stackrel{\circ}{\epsilon}) t^{\sigma} \tag{6.19}
\end{equation*}
$$

Similarly, with the help of the Codazzi equations 2.33 and 2.19, we compute the following pointwise estimate:

$$
\begin{align*}
t^{2} \operatorname{Riem}\left(e_{A}, e_{I}, e_{0}, e_{J}\right) & =t^{2}\left\{e_{A}^{c} \partial_{c} k_{I J}-e_{I}^{c} \partial_{c} k_{A J}-\gamma_{A I B} k_{B J}-\gamma_{A J B} k_{I B}+\gamma_{I A B} k_{B J}+\gamma_{I J B} k_{A B}\right\}  \tag{6.20}\\
& =\mathcal{O}(\stackrel{\circ}{\epsilon}) t^{\sigma}
\end{align*}
$$

Similarly, with the help of 2.28 and 2.22 a , we deduce the following pointwise estimate:

$$
\begin{align*}
t^{2} \operatorname{Riem}\left(e_{0}, e_{I}, e_{0}, e_{J}\right)= & -t k_{I J}-\left(t k_{I C}\right)\left(t k_{C J}\right) \\
& +t^{2}\left\{e_{D}^{c} \partial_{c} \gamma_{I J D}-e_{I}^{c} \partial_{c} \gamma_{D J D}-\gamma_{D I C} \gamma_{C J D}-\gamma_{D D C} \gamma_{I J C}-\left(e_{I} \psi\right) e_{J} \psi\right\}  \tag{6.21}\\
= & -\kappa_{I J}^{(\infty)}-\kappa_{I C}^{(\infty)} \kappa_{C J}^{(\infty)}+\mathcal{O}(\grave{\epsilon}) t^{\sigma}
\end{align*}
$$

Inserting 6.19-6.21 into 6.18, we deduce the following pointwise estimate:

$$
\begin{align*}
& \text { Riem }^{\alpha \mu \beta \nu} \text { Riem }_{\alpha \mu \beta \nu} \\
& =t^{-4}\left\{\left(\kappa_{I J}^{(\infty)} \kappa_{A B}^{(\infty)}-\kappa_{A J}^{(\infty)} \kappa_{B I}^{(\infty)}\right)\left(\kappa_{I J}^{(\infty)} \kappa_{A B}^{(\infty)}-\kappa_{A J}^{(\infty)} \kappa_{B I}^{(\infty)}\right)+4\left(\kappa_{I J}^{(\infty)}+\kappa_{I B}^{(\infty)} \kappa_{B J}^{(\infty)}\right)\left(\kappa_{I J}^{(\infty)}+\kappa_{I C}^{(\infty)} \kappa_{C J}^{(\infty)}\right)\right\}  \tag{6.22}\\
& \quad+\mathcal{O}\left(\stackrel{\circ}{\epsilon} t^{-4+\sigma}\right) .
\end{align*}
$$

Consider now the symmetric matrix $K:=\left(\kappa_{I J}^{(\infty)}\right)_{I, J=1, \cdots, \mathfrak{D}}$, whose eigenvalues are $-q_{I}^{(\infty)}, \cdots,-q_{\mathfrak{D}}^{(\infty)}$. Using that for $m \in \mathbb{N}$, we have $\operatorname{tr}\left(K^{m}\right)=\sum_{I=1}^{\mathfrak{D}}\left[-q_{I}^{(\infty)}\right]^{m}$, we rewrite the expression in braces on RHS 6.22) as follows:

$$
\begin{align*}
& \left(\kappa_{I J}^{(\infty)} \kappa_{A B}^{(\infty)}-\kappa_{A J}^{(\infty)} \kappa_{B I}^{(\infty)}\right)\left(\kappa_{I J}^{(\infty)} \kappa_{A B}^{(\infty)}-\kappa_{A J}^{(\infty)} \kappa_{B I}^{(\infty)}\right)+4\left(\kappa_{I J}^{(\infty)}+\kappa_{I B}^{(\infty)} \kappa_{B J}^{(\infty)}\right)\left(\kappa_{I J}^{(\infty)}+\kappa_{I C}^{(\infty)} \kappa_{C J}^{(\infty)}\right) \\
& =2[\operatorname{tr}(K K)]^{2}+4 \operatorname{tr}(K K)+8 \operatorname{tr}(K K K)+2 \operatorname{tr}(K K K K) \\
& =4\left\{\sum_{I=1}^{\mathfrak{D}}\left[\left(q_{I}^{(\infty)}\right)^{2}-q_{I}^{(\infty)}\right]^{2}+\sum_{1 \leq I<J \leq \mathfrak{D}}\left(q_{I}^{(\infty)}\right)^{2}\left(q_{J}^{(\infty)}\right)^{2}\right\} . \tag{6.23}
\end{align*}
$$

Combining 6.22-6.23, we arrive at the first equality stated in 6.17. To prove the second equality stated in 6.17, we simply use 6.11b to replace all factors of $\kappa_{I J}^{(\infty)}$ on RHS 6.22 with $-\widetilde{q}_{\underline{I}} \delta_{\underline{I} J}$ up to $\mathcal{O}(\stackrel{\circ}{\epsilon})$ error terms (which, in view of the factor of $t^{-4}$ in front of the braces in 6.22), leads to the error term $\mathcal{O}\left({ }^{\circ} t^{-4}\right)$ on RHS 6.17). This completes the proof of the proposition.

### 6.4 Proof of Theorems 6.1 and 6.5

We first prove Theorem 6.1. The conclusions regarding existence and norm estimates, generalized Kasner behavior, and blow up of curvature follow from Propositions $5.28,6.6$ and 6.7 , and the estimate 5.63 . The $C^{2}$-inextendibility is a direct consequence of the curvature blowup.

To prove Theorem 6.5 we simply note that the polarized $U(1)$-symmetric solutions satisfy the same estimates as the solutions from Theorem 6.1. Hence, the same arguments used to prove Theorem 6.1 also yield Theorem 6.5. Finally, we note that the symmetry properties of polarized $U(1)$-symmetric solutions relative to CMC-transported spatial coordinates stated in the conclusions of the theorem are provided by Lemma 2.3.

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[^0]:    *Princeton University, Mathematics Department, Fine Hall, Washington Road, Princeton, NJ 08544-1000, USA, irod@math.princeton.edu.
    ${ }^{\dagger}$ Princeton University, Mathematics Department, Fine Hall, Washington Road, Princeton, NJ 08544-1000, USA, irod@math.princeton.edu.
    $\ddagger$ Vanderbilt University, Mathematics Department, 1326 Stevenson Center, Nashville, TN 37240, USA, jared.speck@vanderbilt.edu

[^1]:    ${ }^{1}$ By "cosmological solutions," we mean ones with compact spatial topology.

[^2]:    ${ }^{2}$ In this paper, $i\left(\Sigma_{1}\right)$ will be a hypersurface of constant time with respect to a CMC time function $t$. To simplify the exposition, we will often slightly abuse notation by suppressing the embedding and identifying $\Sigma_{1}$ with $i\left(\Sigma_{1}\right)=\{t=1\} \subset \mathcal{M}$.

[^3]:    ${ }^{3}$ See [47] for the original formulation and [18, [22] for more modern versions.
    ${ }^{4}$ One even hopes to rule out the possibility of continuing the solution weakly past the boundary of the maximal development since, at least from the PDE point of view, in principle, it might be possible to make sense of weak solutions in a neighborhood of a classical singularity; see the discussion on pg. 13 of 21 .
    ${ }^{5}$ Although the results of [27] are conditional on a quantitative version of the dynamic stability of the exterior region of Kerr, there have been a series of works that seem to be building towards a definitive proof of the stability of the exterior region.

[^4]:    ${ }^{6}$ For a discussion of numerical work on singularity formation in Einstein's equations, see 13 and the references therein.
    ${ }^{7}$ A stiff fluid is such that the speed of sound is equal to the speed of light. It can be viewed as an analog of the scalar field model that allows for non-zero vorticity.

[^5]:    ${ }^{8}$ More precisely, our main results rely on the assumption that the background solution satisfies 1.8 , which is 1.4 in the special case of a generalized Kasner solution.
    ${ }^{9}$ The mean curvature of a constant-time slice $\Sigma_{t}$ is defined to be the trace of its second fundamental form divided by the number of spatial dimensions $\mathfrak{D}$.
    ${ }^{10}$ By spatial tensorfields, we simply mean ones that are tangent to the level sets of the CMC time function $t$.
    ${ }^{11}$ The identity $\mathbf{g}\left(\left[e_{I}, e_{J}\right], e_{B}\right)=\gamma_{I J B}+\gamma_{J B I}$ is a simple consequence of the torsion-free property of the connection $\mathbf{D}$.

[^6]:    ${ }^{12}$ We also need to adequately control the first derivatives of the structure coefficients to obtain the desired bound for the Ricci curvature.
    ${ }^{13}$ This is equivalent to the antisymmetry of the commutator $\left[e_{I}, e_{J}\right]$ with respect to interchanges of $I$ and $J$.

[^7]:    ${ }^{14}$ That is, $\bar{X}$ has no vanishing points.
    ${ }^{15}$ Although the coordinate functions $\left\{x^{i}\right\}_{i=1,2,3}$ are only locally defined, the corresponding partial derivative vectorfield frame $\left\{\partial_{i}\right\}_{i=1,2,3}$ can be extended to a smooth global frame on $\mathbb{T}^{3}$.

[^8]:    ${ }^{16}$ Recall that we do not sum over repeated underlined indices.
    ${ }^{17}$ This notation should not be confused with the notation " $k_{I J}$ " that we use in the bulk of the article, where $k_{I J}:=k_{c d} e_{I}^{c} e_{J}^{d}$ denotes the components of $k$ relative to a Fermi-Walker propagated orthonormal spatial frame.
    ${ }^{18}$ If $\left\{v_{I}\right\}_{I=1, \cdots, \mathcal{D}}$ denotes the basis-dual frame (i.e., $\theta^{I}\left(v_{J}\right)=\delta_{J}^{I}$, where $\delta_{J}^{I}$ is the Kronecker delta), then relative to arbitrary coordinates $\left\{y^{i}\right\}_{i=1, \cdots, \mathfrak{D}}$ on $\mathbb{T}^{\mathcal{D}}$, we have $\theta^{I}=\theta_{c}^{I} d y^{c}, v_{J}=v_{J}^{c} \frac{\partial}{\partial y^{c}}$, and $k^{I}{ }_{J}:=k_{d}^{c}{ }_{d} \theta_{c}^{I} v_{J}^{d}$, where $k_{b}^{a}=\left(g^{-1}\right)^{a c} k_{c b}$.

[^9]:    ${ }^{19}$ Since we derive estimates showing that $|n-1| \lesssim t^{\sigma}$, the non-constant lapse does not affect the heuristic analysis.
    ${ }^{20}$ In our main results, in the case of the scalar field matter model, we will prove (with the help of 1.1a) pointwise estimates showing that $\max _{I, J=1, \cdots, \mathfrak{D}}\left|\boldsymbol{R i c}\left(e_{I}, e_{J}\right)\right| \lesssim t^{-2+\sigma}$, where $\left\{e_{I}(t, x)\right\}_{I=1, \cdots, \mathfrak{D}}$ is an orthonormal spatial frame; this frame component bound is sufficient for the proof of our main results. These technical estimates are in fact derived in the proof of Lemma 5.17 though it might not be immediately apparent from the statement of the lemma.

[^10]:    ${ }^{21}$ Note that the spatial coordinate components $\left\{\operatorname{Ric}{ }_{j}{ }_{j}\right\}_{i, j=1, \cdots, \mathfrak{D}}$ of the type $\binom{1}{1}$ tensor Ric are bounded in magnitude by $\lesssim$ LHS 1.18 and hence the inequality 1.18 would imply the same bound for $\max _{i, j=1, \cdots, \mathfrak{D}}\left|R i c^{i}{ }_{j}\right|$.
    ${ }^{22}$ Using the equations 1.14 , one can show that in the vacuum case with $\mathfrak{D}=3$, aside from the trivial case in which one of the $q_{I}$ is equal to 1 and the others vanish, precisely one of the $q$ 's must be negative.

[^11]:    ${ }^{23}$ Roughly, Gowdy-solutions are such that there exists a pair of spacelike Killing vectorfields $X$ and $Y$ such that the twist constants $\epsilon_{\alpha \beta \gamma \delta} X^{\alpha} Y^{\beta} \mathbf{D}^{\gamma} X^{\delta}$ and $\epsilon_{\alpha \beta \gamma \delta} X^{\alpha} Y^{\beta} \mathbf{D}^{\gamma} Y^{\delta}$ vanish, where $\epsilon$ is the spacetime volume form. Polarized Gowdy-solutions satisfy one additional condition: $X$ and $Y$ are orthogonal.
    ${ }^{24}$ Note that Kerr solutions, although axi-symmetric, do not contain a hypersurface-orthogonal Killing field.

[^12]:    ${ }^{25}$ To see the correspondence, one must re-parametrize the coordinate $r$ to proper time (recall that $r$ is a time function in the Schwarzschild black hole interior, whereas $t$ is a spatial coordinate, in the classical coordinate representation of the Schwarzschild metric).

[^13]:    ${ }^{26}$ The Weingarten map is the second fundamental form in type $\binom{1}{1}$ form, i.e., in the notation of the present paper, the tensorfield with components $k_{j}^{i}:=\left(g^{-1}\right)^{i a} k_{a j}$.

[^14]:    ${ }^{27}$ We never need to estimate $\psi$ itself since only its derivatives appear in the system $1.1 \mathrm{a}-1.1 \mathrm{~b}$.

[^15]:    ${ }^{28}$ The work [1] is concerned with the near-Schwarzschild black hole interior problem, where the symmetry class is called "polarized axi-symmetry." This regime has some analytical commonalities with the polarized $U(1)$-symmetric solutions that we treat in Theorem 6.5 see the end of Sect. 1.7 .1

[^16]:    ${ }^{29}$ Note that as we have defined it, the energy $\mathbb{E}_{N}$ scales linearly with respect to the quantities that it controls. This is a different convention than is usually used in the literature, in which energies are typically defined so as to scale quadratically in the quantities that they control. Similar remarks apply to the energies we use in the proof of Lemma 2.3
    ${ }^{30}$ In practice, we also derive top-order energy estimates for the co-frame components $\left\{\omega_{i}^{I}\right\}_{I, i=1, \cdots, \mathfrak{D} \cdot}$

[^17]:    ${ }^{31} \mathrm{We}$ are grateful to the authors of 4 for pointing this out to us.

[^18]:    ${ }^{32}$ We clarify that some of these equations, such as $2.23 \mathrm{a}-2.23 \mathrm{~b}$ and 2.27 , are independent of the Einstein-scalar field equations and follow from the constructions given in Sects.2.1.1 2.1.3
    ${ }^{33}$ Recall that $e_{0} k_{I J}:=n^{-1} \partial_{t}\left(k_{I J}\right)=n^{-1} \partial_{t}\left(k_{c d} e_{I}^{c} e_{J}^{d}\right), e_{C} \gamma_{I J C}:=e_{C}^{c} \partial_{c}\left(\gamma_{I J C}\right)$, etc.

[^19]:    ${ }^{34}$ Here, we consider, e.g., terms of type $\mathcal{V} \cdot \mathcal{V}$ to be linear in $\mathcal{V}$ with continuous coefficients because the first factor of $\mathcal{V}$ is assumed to be continuous.

[^20]:    ${ }^{35}$ In Sect. 5.4 we will use elliptic estimates to show that the lapse can be controlled in terms of the dynamic variables; see 5.6 .

[^21]:    ${ }^{36}$ However, the structure of equation 2.22 b is sufficient for our top-order energy estimates, which are allowed to be much more singular within the scope of our approach; this explains why in Lemma 5.16 we derive commuted versions of equation 2.22 b to set up our energy estimates for $\gamma$ and $k$.

[^22]:    ${ }^{37}$ Note in particular that we do not use the interpolation inequalities of Lemma 4.2 in this proof.

[^23]:    ${ }^{39}$ Here we are slightly abusing notation by, for example, using the expression $\mathrm{K}_{I J}^{(\infty)}(x)$ to denote the function $x \rightarrow \mathrm{~K}_{I J}^{(\infty)}(x)$.

