On the initial boundary value problem for the Einstein vacuum equations in the maximal gauge

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Abstract

We consider the initial boundary value problem for the Einstein vacuum equations in the maximal gauge, or more generally, in a gauge where the mean curvature of a timelike foliation is fixed near the boundary. We prove the existence of solutions such that the normal to the boundary is tangent to the time slices, the lapse of the induced time coordinate on the boundary is fixed and the main geometric boundary conditions are given by the 1-parameter family of Riemannian conformal metrics on each two-dimensional section. As in the local existence theory of Christodoulou-Klainerman for the Einstein vacuum equations in the maximal gauge, we use as a reduced system the wave equations satisfied by the components of the second fundamental form of the the time foliation. The main difficulty lies in completing the above set of boundary conditions such that the reduced system is well-posed, but still allows for the recovery of the Einstein equations. We solve this problem by imposing the momentum constraint equations on the boundary, suitably modified by quantities vanishing in the maximal gauge setting. To derive energy estimates for the reduced system at time t, we show that all the terms in the flux integrals on the boundary can be either directly controlled by the boundary conditions or they lead to an integral on the two-dimensional section at time t of the boundary. Exploiting again the maximal gauge condition on the boundary, this contribution to the flux integrals can then be absorbed by a careful trace inequality in the interior energy.

1 Introduction

One of the most fundamental properties of the Einstein equations, if not the most, is their hyperbolic nature. This naturally leads to the initial value problem in General Relativity, solved in the work of Choquet-Bruhat [3]: In the vacuum case, given any Riemannian manifold (Σ, h) and covariant symmetric 2-tensor k, satisfying the constraint equations

$$R - |k|^2 + (\mathrm{tr}k)^2 = 0 \tag{1.1}$$

$$d\,\mathrm{tr}k - \mathrm{div}k = 0, \tag{1.2}$$

where R is the scalar curvature of h, $\mathrm{tr}k$ is the trace of k with respect to h, $d\,\mathrm{tr}k$ its differential and $\mathrm{div}k$ its divergence, there exists a maximal[1] globally hyperbolic development $(\mathcal{M}, \mathbf{g})$ of (Σ, h, k), which is unique modulo diffeomorphisms. As a development of the data, it is a solution to the vacuum Einstein equations (EVE):

$$\mathrm{Ric}(\mathbf{g}) = 0, \tag{1.3}$$

such that (Σ, h, k) can be embedded in $\mathcal{M}$ with (h, k) being the first and second fundamental forms of the embedding. Due to the geometric nature of the equations, to prove local well-posedness, one typically makes various gauge choices to derive a *reduced system* of partial differential equations. This reduced system needs to be both hyperbolic in some sense and allow for the recovery of the full Einstein equations.

In this paper, we will consider the initial boundary value problem, that is to say we are interested in constructing a $3+1$ Lorentzian manifold $(\mathcal{M}, \mathbf{g})$ such that $\partial\mathcal{M} = \Sigma \cup \mathcal{T}$, where Σ is a spacelike hypersurface of $\mathcal{M}$ with boundary S, $\mathcal{T}$ is a timelike hypersurface of $\mathcal{M}$ with compact boundary S and $\Sigma \cap \mathcal{T} = S$. Σ can be thought of as the initial hypersurface and thus, we should consider the first and second fundamental forms of Σ as given, while on $\mathcal{T}$, boundary data or boundary conditions will need to be imposed so as to make the problem well-posed. On S, one typically needs compatibility conditions between the initial data and the boundary data.

On top of its intrinsic mathematical appeal, the initial boundary value problem is motivated by

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[1]The construction of a maximal development is due to Choquet-Bruhat and Geroch [4].

- the study of asymptotically Anti-de-Sitter spacetimes, which naturally leads to an initial boundary value problem after conformal rescaling,
- numerical applications, where, for numerical purposes, one typically needs to solve the equations on a finite domain with boundary,
- possible coupling with massive matter of compact support, as for instance in the study of the Einstein-Euler equations, where the exterior region will posess such a timelike boundary [10].

The initial boundary value problem in General Relativity was first solved in the work of Friedrich in the Anti-de-Sitter case [8] and Friedrich and Nagy for the Einstein vacuum equations with timelike boundary [9]. For an extensive review, we refer to [12]. Apart from the Friedrich-Nagy approach, which is based on the Bianchi equations and the construction of a special frame adapted to the boundary, the other well-developed theory for the study of the initial boundary value problem is that of Kreiss-Reula-Sarbach-Winicour [11] based on generalized harmonic coordinates.

In this paper, we prove well-posedness of the initial boundary value problem for the Einstein vacuum equations formulated in the maximal gauge, or more generally, in any gauge where the mean curvature of a timelike foliation is fixed. More precisely, we prove the existence of solutions such that the time slices intersect the boundary orthogonally, the lapse of the induced time coordinate on the boundary is fixed and the main geometric boundary conditions are given by the 1-parameter family of Riemannian conformal metrics on each two-dimensional section. The dynamical variables that we consider are the components of the second fundamental form of the time foliation which satisfy a system of wave equations, as originally identified by Choquet-Bruhat-Ruggeri [5]. In the work of Christodoulou-Klainerman [6], this wave formulation of the equations was exploited to prove local existence of solutions to the Einstein equations in the maximal gauge. In the presence of a timelike boundary, one now needs to provide boundary conditions for the components of the second fundamental form. As is standard for geometric hyperbolic partial differential equations with constraints, the boundary conditions have to be compatible with the constraints.

We identify that these can be chosen as follows:

- With t as the time function, whose level sets are the maximal slices Σ_t,
$$\mathrm{tr}k = 0, \tag{1.4}$$
and $S_t = \Sigma_t \cap \mathcal{T}$, $\mathcal{T}$ being the timelike boundary, prescribing the conformal class of the induced metrics on each S_t implies Dirichlet type boundary conditions for the traceless part of the projection of k on each S_t. This essentially encodes the standard degrees of freedom corresponding to gravitational radiation.
- These boundary conditions are first complemented by the requirement that the slices Σ_t intersect $\mathcal{T}$ orthogonally, by fixing the lapse of the induced time coordinate on $\mathcal{T}$ and by imposing the maximal condition $\mathrm{tr}k = 0$, on $\mathcal{T}$.
- If A, B are indices that correspond to spatial directions tangent to $\mathcal{T}$ and orthogonal to ∂_t, it remains to impose boundary conditions on the trace of k_{AB} (or equivalently on the volume forms of the induced metrics on S_t), as well as on k_{NA}, where N denotes the unit normal to the boundary. For this, we identify a system of boundary conditions which is essentially equivalent to the momentum constraint equations (1.2) in the maximal gauge, see (2.27)-(2.29).

The fact that, with these boundary conditions, on one hand, one can close energy estimates and on the other hand, one can a posteriori recover the Einstein equations, is the main contribution of this paper.

Since the maximal gauge was instrumental in several global in time results for the Einstein equations, such as the monumental work of Christodoulou-Klainerman [6] on the stability of Minkowski space, our results may find applications in the global analysis of solutions in the presence of a timelike boundary. Moreover, let us mention that the BSSN formulation [2], which is heavily used in numerical analysis, is based on a $3+1$ decomposition of the Lorentzian metric and thus its analysis is likely to be closely related to the one we pursue here.

One of the outstanding issues remaining, concerning the initial boundary value problem, is the geometric uniqueness problem of Friedrich [8]. Apart from the AdS case, all other results establishing well-posedness for some formulations of the initial boundary value problem impose certain gauge conditions on the boundary and the boundary data depends on these choices. In particular, given a solution $(\mathcal{M}, \mathbf{g})$ to the Einstein equations with a timelike boundary, different gauge choices will lead to different boundary data, in each of the formulations for which well-posedness is known. On the other hand, if we had been given the different boundary data a priori, we would not know that these lead to the same solution. The situation is thus different from the usual initial value problem for which only isometric data leads to isometric solutions, which one then regards as the same solution. In the AdS case, this problem admits one solution: in [8], Friedrich proved that one can take the conformal metric of the boundary as boundary data, which is a geometric condition independent of any gauge choice.

The work of this paper still requires certain gauge conditions to be fixed, however, our boundary conditions describe at least part of the geometry of the boundary (via the family of conformal metrics).

To state more precisely our main result, let us consider a Lorentzian manifold $(\mathcal{M}, \mathbf{g})$ with a time function t, such that
$$\mathbf{g} = -\Phi^2 dt^2 + g_{ij} dx^i dx^j,$$

where x^1, x^2, x^3 are t-transported coodinates, g, k denote the first and second fundamental forms of the level sets Σ_t of t, satisfying moreover the maximal condition $\mathrm{tr}k = 0$. We assume that we are given initial data (h, k) on Σ_0 and that $\partial\mathcal{M} = \Sigma_0 \cup \mathcal{T}$, where $\mathcal{T}$ is a timelike boundary, which we assume coincides with $\{x^3 = 0\}$. Here, $\mathcal{M}$ admits coordinates (t, x^1, x^2, x^3), where x^3 is assumed to be a boundary defininining function. The induced metric on the boundary is thus given by

$$H = -\Phi^2 dt^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + 2g_{12}dx^1dx^2.$$

The intersection of Σ_t and $\mathcal{T}$ is a spacelike 2-surface, denoted by S_t, with metric

$$q_t = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + 2g_{12}dx^1dx^2.$$

Our boundary conditions will be such that we fix $\Phi = 1$, as well as the conformal classes $[q_t]$ of the 1-parameter family of metrics q_t. Since the cross sections S_t are all diffeomorphic to each other, one can think of this part of the data as a 1-parameter family of conformal metrics on a fixed 2-dimensional manifold S. Moreover, on S_0, compatibility conditions between q_t and h, k will be required. These are introduced below in Subsection 2.1, see (2.33).

With this notation, we prove the following theorem

Theorem 1.1. *Let (Σ, h, k) be a smooth initial data set satisfying the constraint equations* (1.1)-(1.2)*, with Σ a 3-dimensional manifold with compact boundary S and k being traceless $\mathrm{tr}k = 0$ (near the boundary). Consider a smooth 1-parameter family of conformal metrics $[q_t]_{t\in I}$ on S, verifiying the compabitility conditions discussed in Subsection 2.1. Then, there exists a smooth Lorentzian manifold $(\mathcal{M}, \mathbf{g})$ with timelike boundary $\mathcal{T}$, satisfying* (1.3)*, such that $\mathcal{M}$ is foliated by Cauchy hypersurfaces Σ_t, $t \in I$, an embedding of Σ onto Σ_0 such that (h, k) coincides with the first and second fundamental forms of the embedding, and the boundary conditions are verified on $\mathcal{T}$, as introduced above. The time interval of existence, I, depends continuously on the initial and boundary data.*

Remark 1.2. The conditions $\mathrm{tr}k = 0$, $\Phi\big|_{\mathcal{T}} = 1$, are not essential for our overall local existence argument. However, we do need to fix the foliation Σ_t (near the boundary), such that $\mathrm{tr}k, \Phi\big|_{\mathcal{T}}$ are prescribed, sufficiently regular functions.[2]

Remark 1.3. If the data is asymptotically flat and $\mathrm{tr}k = 0$ initially, then one can obtain a solution which is globally foliated by maximal hypersurfaces.

Remark 1.4. The spacetime metric $\mathbf{g}$ is constructed by solving a set of reduced equations in the above gauge, verifying appropriate boundary conditions, see Section 3. Uniqueness for these equations also holds, however, it does not imply the desired geometric uniqueness we would like to have for the EVE.

The proof of Theorem 1.1 is based on deriving energy estimates, near the boundary, for a system of reduced equations, subject to certain boundary conditions, that we set up in Section 2. The energy argument is carried out in Section 3. We should note that even in the case where no timelike boundary is involved, the most naive scheme based on standard energy estimates would fail to close due to loss of derivatives, see Remark 2.6.

Interestingly, for the boundary value problem, even after exploiting all our boundary conditions, the total energy flux of k at the boundary does not a priori have a sign. However, we demonstrate that after a careful use of trace inequalities, the terms which a priori could have the wrong sign can be absorbed in the interior energies. Emphasis is given to one particular boundary term, which is at the level of the main top order energies, and which requires a certain splitting in order to be absorbed in the left hand side of the estimates, see Remark 3.4. Such a manipulation is possible thanks to the maximal condition being valid on the boundary.[3]

In Section 4, we confirm that the solution to the reduced system of equations is in fact a solution to (1.3). First, we derive a system of propagation equations for the Einstein tensor, subject to boundary conditions induced from the ones of the reduced system, which are eligible to an energy estimate. Combining this fact with the vanishing of the Einstein tensor initially and the homogeneity of the induced boundary conditions for the final system of equations, we infer its vanishing everywhere.

For the energy estimates of Section 3, in order to preserve the choice of boundary conditions, we commute the equations only by tangential derivatives and recover the missing normal derivatives from the equations. A similar argument is used in the recovery of the Einstein equations when commuting the equation for $\mathrm{tr}k$ in Section 4.

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[2] Conditions for the existence of spacelike foliations with prescribed mean curvature, in the presence of boundaries, have been established by Bartnik [1].

[3] We adopt $\mathrm{tr}k = 0$ as one of our boundary conditions, see Subsection 2.1.

2 Framework

Our framework of choice is the one used for proving local existence for the EVE in the original Christodoulou-Klainerman stability of Minkowski proof [6]. We include a detailed outline of the whole procedure for the sake of completeness, cf. [6, §10.2]. Moreover, given that our main point of interest is the initial boundary value problem (IBVP), we will focus mostly on controlling the boundary terms arising in the local existence argument, both in the estimates for the reduced equations (Section 3) and in the recovery of the Einstein vacuum equations (Section 4).

Let $(\mathcal{M}^{1+3},\mathbf{g})$ be a Lorentzian manifold with a timelike, $1+2$ dimensional, boundary $\partial\mathcal{M}$. We consider a transported (t,x^1,x^2,x^3) coordinated system, such that the spacetime metric takes the form

$$\mathbf{g}=-\Phi^2dt^2+g=-\Phi^2dt^2+g_{ij}dx^idx^j,\qquad\qquad \Phi=(-\mathbf{g}^{\alpha\beta}\partial_\alpha t\partial_\beta t)^{-\frac{1}{2}}, \tag{2.1}$$

where Φ is the lapse of the foliation $\{t=\text{const.}\}=:\Sigma_t$, while the shift vector field is set to zero. In this framework, the first variation equations read

$$\partial_t g_{ij}=-2\Phi k_{ij},\qquad\qquad k_{ij}:=\mathbf{g}(D_{\partial_i}\partial_j,e_0)=k_{ji},\quad e_0=\Phi^{-1}\partial_t, \tag{2.2}$$

where D is the covariant derivative intrinsic to $\mathbf{g}$. The 2-tensor k_{ij} is the second fundamental form of Σ_t. We also have

$$\partial_t g^{ij}=2\Phi k^{ij}. \tag{2.3}$$

Moreover, the second variation equations read

$$\partial_t k_{ij}=-\nabla_i\nabla_j\Phi+\Phi(R_{ij}+k_{ij}\mathrm{tr}k-2k_i{}^lk_{jl})-\Phi\mathbf{R}_{ij}, \tag{2.4}$$

where ∇, R_{ij} are the covariant connection and Ricci tensor of g, while $\mathbf{R}_{ij}$ is the Ricci tensor of $\mathbf{g}$. Imposing the EVE, the latter vanishes, whereas the former equals [14, (3.4.5)]:

$$\begin{aligned}\mathrm{R}_{ij}(g)&=\partial_a\Gamma^a_{ji}-\partial_j\Gamma^a_{ia}+\Gamma^a_{ab}\Gamma^b_{ji}-\Gamma^a_{jb}\Gamma^b_{ai}\\&=\nabla_a\Gamma^a_{ji}-\nabla_j\Gamma^a_{ia}-\Gamma^a_{ab}\Gamma^b_{ji}+\Gamma^a_{jb}\Gamma^b_{ai}\end{aligned} \tag{2.5}$$

where $\nabla\Gamma$ is interpreted tensorially, e.g.,

$$\nabla_a\Gamma^a_{ji}:=\partial_a\Gamma^a_{ji}+\Gamma^a_{ab}\Gamma^b_{ij}-\Gamma^b_{aj}\Gamma^a_{bi}-\Gamma^b_{ai}\Gamma^a_{jb}.$$

In order to reveal the hyperbolic structure of (2.2)-(2.4), we need to differentiate (2.4) in ∂_t and work with its second order analogue.

Proposition 2.1. *Let* $\mathbf{g}$ *be a Lorentzian metric expressed in the above framework. Then the propagation equation*

$$e_0\mathbf{R}_{ij}=\nabla_i\mathcal{G}_j+\nabla_j\mathcal{G}_i-\nabla_i\nabla_j\mathrm{tr}k,\qquad\qquad \mathcal{G}_i:=\mathbf{R}_{0i}, \tag{2.6}$$

is equivalent to the following wave equation for k_{ij}*:*

$$\begin{aligned}&e_0^2k_{ij}-\Delta_gk_{ij}\\&=\Phi^{-3}\partial_t\Phi\nabla_i\nabla_j\Phi-\Phi^{-2}\nabla_i\nabla_j\partial_t\Phi+\Phi^{-2}\partial_t\Gamma^l_{ij}\partial_l\Phi+e_0(k_{ij}\mathrm{tr}k-2k_i{}^lk_{jl})\\&\quad+\Phi^{-1}k_{ij}\Delta_g\Phi-\Phi^{-1}k_i{}^a\nabla_a\nabla_j\Phi-\Phi^{-1}k_j{}^a\nabla_a\nabla_i\Phi\\&\quad-\Phi^{-1}\nabla^a\Phi(\nabla_jk_{ia}+\nabla_ik_{ja}-2\nabla_ak_{ij})+\Phi^{-1}\mathrm{tr}k\nabla_j\nabla_i\Phi\\&\quad+\Phi^{-1}\nabla_j\Phi(\nabla_i\mathrm{tr}k-\nabla_ak_i{}^a)+\Phi^{-1}\nabla_i\Phi(\nabla_j\mathrm{tr}k-\nabla_ak_j{}^a)\\&\quad-3(k_{ci}R_j{}^c+k_{cj}R_i{}^c)+2\mathrm{tr}kR_{ji}+2g_{ji}R_a{}^ck_c{}^a+(k_{ij}-g_{ji}\mathrm{tr}k)R,\end{aligned} \tag{2.7}$$

for all $i,j=1,2,3$.

Remark 2.2. The operations in (2.7) are covariant, where the ∂_t differentiations are viewed as applications of the Lie derivative operator $\mathcal{L}_{\partial_t}$, e.g. $e_0k_{ij}=\Phi^{-1}\mathcal{L}_{\partial_t}k_{ij}$, see also (2.9).

Remark 2.3. In the gauge $\mathrm{tr}k=0$, many of the terms in (2.7) can be actually dropped. However, these terms would have to be added later in (2.6), when we will verify that a solution to the reduced equations, is actually a solution to the EVE, see Section 4.

Proof. The main ingredient is the derivation of a formula for the time derivative of R_{ij}. For this purpose, we introduce some commutation formulas:

$$\begin{aligned}\partial_t\nabla_aX^b_{ij}&=\partial_t\Big[\partial_aX^b_{ij}+\Gamma^b_{ac}X^c_{ij}-\Gamma^c_{ai}X^b_{cj}-\Gamma^c_{aj}X^b_{ic}\Big]\\&=\nabla_a\partial_tX^b_{ij}+X^c_{ij}\partial_t\Gamma^b_{ac}-X^b_{cj}\partial_t\Gamma^c_{ai}-X^b_{ic}\partial_t\Gamma^c_{ij},\end{aligned} \tag{2.8}$$

for any $(1,2)$ tensor, where

$$\begin{aligned}\partial_t\Gamma^b_{ac} &= \Phi k^{bl}(\partial_a g_{cl}+\partial_c g_{al}-\partial_l g_{ac})+g^{bl}[\partial_l(\Phi k_{ac})-\partial_a(\Phi k_{cl})-\partial_c(\Phi k_{al})]\\ &= 2\Phi k^{bl}\Gamma^m_{ac}g_{ml}+g^{bl}[\partial_l(\Phi k_{ac})-\partial_a(\Phi k_{cl})-\partial_c(\Phi k_{al})]\\ &= g^{bl}\big[\nabla_l(\Phi k_{ac})-\nabla_a(\Phi k_{cl})-\nabla_c(\Phi k_{al})\big] \end{aligned} \tag{2.9}$$

Differentiating (2.5) and utilising (2.8), we find

$$\begin{aligned}\partial_t R_{ij} &= \nabla_a\partial_t\Gamma^a_{ji}+\Gamma^c_{ji}\partial_t\Gamma^a_{ac}-\Gamma^a_{ci}\partial_t\Gamma^c_{aj}-\Gamma^a_{jc}\partial_t\Gamma^c_{ai}\\ &\quad -\nabla_j\partial_t\Gamma^a_{ia}-\Gamma^c_{ia}\partial_t\Gamma^a_{jc}+\Gamma^a_{ca}\partial_t\Gamma^c_{ji}+\Gamma^a_{ic}\partial_t\Gamma^c_{ja}\\ &\quad -\partial_t(\Gamma^a_{ca}\Gamma^c_{ji})+\partial_t(\Gamma^a_{ci}\Gamma^c_{aj})\\ &= \nabla_a\partial_t\Gamma^a_{ji}-\nabla_j\partial_t\Gamma^a_{ia}\end{aligned} \tag{2.10}$$

Taking now the time derivative of (2.4) and employing (2.9)-(2.10), we derive:

$$\begin{aligned}\partial_t(\Phi^{-1}\partial_t k_{ij}) &= \Phi^{-2}\partial_t\Phi\nabla_i\nabla_j\Phi-\Phi^{-1}\nabla_i\nabla_j\partial_t\Phi+\Phi^{-1}\partial_t\Gamma^l_{ij}\partial_l\Phi+\partial_t(k_{ij}\mathrm{tr}k-2k_i{}^lk_{jl})\\ &\quad+\nabla_a\big[\nabla^a(\Phi k_{ij})-\nabla_j(\Phi k_i{}^a)-\nabla_i(\Phi k_j{}^a)\big]\\ &\quad-\nabla_j\big[\nabla^a(\Phi k_{ia})-\nabla_a(\Phi k_i{}^a)-\nabla_i(\Phi k_a{}^a)\big]-\partial_t\mathbf{R}_{ij}\\ &= \Phi^{-2}\partial_t\Phi\nabla_i\nabla_j\Phi-\Phi^{-1}\nabla_i\nabla_j\partial_t\Phi+\Phi^{-1}\partial_t\Gamma^l_{ij}\partial_l\Phi+\partial_t(k_{ij}\mathrm{tr}k-2k_i{}^lk_{jl})\\ &\quad+\Phi\Delta_g k_{ij}+k_{ij}\Delta_g\Phi+2\nabla^a\Phi\nabla_a k_{ij}-k_i{}^a\nabla_a\nabla_j\Phi-k_j{}^a\nabla_a\nabla_i\Phi\\ &\quad-\nabla^a\Phi(\nabla_j k_{ia}+\nabla_i k_{ja})-\nabla_j\Phi\nabla_a k_i{}^a-\nabla_i\Phi\nabla_a k_j{}^a-\Phi\nabla_a\nabla_j k_i{}^a-\Phi\nabla_a\nabla_i k_j{}^a\\ &\quad+\mathrm{tr}k\nabla_j\nabla_i\Phi+\Phi\nabla_j\nabla_i\mathrm{tr}k+\nabla_i\mathrm{tr}k\nabla_j\Phi+\nabla_j\mathrm{tr}k\nabla_i\Phi-\partial_t\mathbf{R}_{ij}\end{aligned} \tag{2.11}$$

Next, we utilise the identity:

$$-\Phi\nabla_a\nabla_j k_i{}^a-\Phi\nabla_a\nabla_i k_j{}^a=-2\Phi R_{aji}{}^ck_c{}^a-\Phi R_j{}^ck_{ic}-\Phi R_i{}^ck_{jc}-\Phi\nabla_j\nabla_a k_i{}^a-\Phi\nabla_i\nabla_a k_j{}^a, \tag{2.12}$$

Note that in 3D the Riemann tensor can be expressed in terms of the Ricci tensor via the identity [14, (3.2.28)]:

$$\begin{aligned} R_{aji}{}^c &= g_{ai}R_j{}^c-\delta_a{}^cR_{ji}-g_{ji}R_a{}^c+\delta_j{}^cR_{ai}-\frac{1}{2}R(g_{ai}\delta_j{}^c-\delta_a{}^cg_{ji}) \\ \Rightarrow\qquad -2\Phi R_{aji}{}^ck_c{}^a &= -2\Phi\Big[k_{ci}R_j{}^c-\mathrm{tr}kR_{ji}-g_{ji}R_a{}^ck_c{}^a+k_j{}^aR_{ai}-\frac{1}{2}R(k_{ij}-g_{ji}\mathrm{tr}k)\Big]\end{aligned} \tag{2.13}$$

Also, from the contracted Gauss and Codazzi equations we have the identities:

$$R-|k|^2+(\mathrm{tr}k)^2=\mathbf{R}+2\Phi^{-2}\mathbf{R}_{tt} \tag{2.14}$$

$$\partial_j\mathrm{tr}k-\nabla^a k_{aj}=\Phi^{-1}\mathbf{R}_{tj}=\mathcal{G}_j,\qquad j=1,2,3. \tag{2.15}$$

Hence, plugging (2.12),(2.13),(2.15) in (2.11), we arrive at the equation:

$$\begin{aligned}&e_0^2k_{ij}-\Delta_g k_{ij}\\ &=\Phi^{-3}\partial_t\Phi\nabla_i\nabla_j\Phi-\Phi^{-2}\nabla_i\nabla_j\partial_t\Phi+\Phi^{-2}\partial_t\Gamma^l_{ij}\partial_l\Phi+e_0(k_{ij}\mathrm{tr}k-2k_i{}^lk_{jl})\\ &\quad+\Phi^{-1}k_{ij}\Delta_g\Phi-\Phi^{-1}k_i{}^a\nabla_a\nabla_j\Phi-\Phi^{-1}k_j{}^a\nabla_a\nabla_i\Phi\\ &\quad-\Phi^{-1}\nabla^a\Phi(\nabla_j k_{ia}+\nabla_i k_{ja}-2\nabla_a k_{ij})+\Phi^{-1}\mathrm{tr}k\nabla_j\nabla_i\Phi-\nabla_j\nabla_i\mathrm{tr}k\\ &\quad+\Phi^{-1}\nabla_j\Phi(\nabla_i\mathrm{tr}k-\nabla_a k_i{}^a)+\Phi^{-1}\nabla_i\Phi(\nabla_j\mathrm{tr}k-\nabla_a k_j{}^a)\\ &\quad-3(k_{ci}R_j{}^c+k_{cj}R_i{}^c)+2\mathrm{tr}kR_{ji}+2g_{ji}R_a{}^ck_c{}^a+(k_{ij}-g_{ji}\mathrm{tr}k)R\\ &\quad+\nabla_j\mathcal{G}_i+\nabla_i\mathcal{G}_j-e_0\mathbf{R}_{ij}\end{aligned} \tag{2.16}$$

This completes the proof of the proposition. □

In the case of study, where $\mathbf{g}$ is a solution to the EVE, under the maximal gauge condition $\mathrm{tr}k=0$, the equation (2.6) holds trivially and hence so does (2.7). Moreover, taking the trace of (2.4) and using (2.14), we obtain the relations:

$$\partial_t\mathrm{tr}k=-\Delta_g\Phi+\Phi[R+(\mathrm{tr}k)^2]-\Phi(\mathbf{R}+\mathbf{R}_{00})=-\Delta_g\Phi+|k|^2\Phi+\Phi\mathbf{R}_{00} \tag{2.17}$$

Since $\mathrm{tr}k$ and spacetime Ricci vanish, (2.17) yields the following elliptic equation for the lapse:

$$\Delta_g\Phi-|k|^2\Phi=0. \tag{2.18}$$

The reduced equations (2.2),(2.7),(2.18) form a closed system for g,k,Φ.

Remark 2.4. A posteriori, having solved the reduced equations, in order to verify the validity of the maximal gauge, we will need to propagate the vanishing of tr$k$. For this purpose, we compute the trace of (2.7), using only (2.2),(2.18):

$$\begin{aligned} &e_0^2\mathrm{tr}k - \Delta_g \mathrm{tr}k \\ =&\, e_0(2k^{ij}k_{ij}) + 2k^{ij}e_0k_{ij} + g^{ij}(e_0^2 k_{ij} - \Delta_g k_{ij}) \\ =&\, e_0(2k^{ij}k_{ij}) + 2k^{ij}e_0k_{ij} - g^{ij}e_0(\Phi^{-1}\nabla_i\nabla_j\Phi) + g^{ij}e_0(k_{ij}\mathrm{tr}k - 2k_i{}^l k_{jl}) \\ &+ 2\mathrm{tr}k|k|^2 - 2\Phi^{-1}k^{ja}\nabla_a\nabla_j\Phi + 4\Phi^{-1}(\nabla^a\Phi)\mathcal{G}_a \\ =&\, e_0[(\mathrm{tr}k)^2] + 4\Phi^{-1}(\nabla^a\Phi)\mathcal{G}_a \end{aligned} \tag{2.19}$$

Remark 2.5. To our knowledge, the reduction of the EVE to a wave equation for k_{ij} was first demonstrated in the literature by Choquet-Bruhat–Ruggeri [5]. In fact, they derived a system for $P_{ij} := k_{ij} - g_{ij}\mathrm{tr}k$, using the gauge choice

$$\Box_{\mathbf{g}} t = 0 \qquad \Longleftrightarrow \qquad \Phi^{-2}\partial_t\Phi = -\mathrm{tr}k,$$

for the t-foliation.

Remark 2.6. The Ricci tensor of g in the RHS of (2.7) contains terms having two spatial derivatives of g. At first glance, this makes the closure of the reduced system more intricate, since (2.2) does not gain a derivative in space. However, we demonstrate below, see (3.32), how to treat these terms in the energy estimates by integrating by parts. Alternatively, these terms could be replaced in the derivation of (2.7), in favour of spacetime Ricci, by using the second variation equations (2.4), involving only $k, \partial_t k, \nabla\nabla\Phi$. In that case, the propagation equation (2.6) would have to be modified accordingly, adding the appropriate combination of zeroth order Ricci terms.

2.1 Boundary data and boundary conditions

We assume that the timelike boundary of $\mathcal{M}$, $\mathcal{T}$, is foliated by the compact surfaces $\partial\Sigma_t := \Sigma_t \cap \mathcal{T}$. Let H denote the induced, $1+2$, Lorentzian metric on the boundary $\mathcal{T}$. For simplicity, we assume that $t\big|_{\mathcal{T}}$ defines a geodesic foliation with respect to the induced metric on the boundary, i.e., H takes the form

$$H := \mathbf{g}\big|_{\mathcal{T}} = -[d(t\big|_{\mathcal{T}})]^2 + q_t = -[d(t\big|_{\mathcal{T}})]^2 + (q_t)_{AB}dx^A dx^B, \qquad A, B = 1, 2. \tag{2.20}$$

Combining this assumption with the boundary condition

$$\Phi = 1, \qquad \text{on } \mathcal{T}, \tag{2.21}$$

we infer that the vector field $\partial_t\big|_{\mathcal{T}}$ remains tangent to the boundary $\mathcal{T}$, $\partial_t\big|_{\mathcal{T}} \in T(\mathcal{T})$, and hence, it coincides with $\partial_{t|\mathcal{T}}$. Indeed, from the form of the metrics (2.1),(2.20) and the definition of the lapse, it follows that the (outward) unit normal to the boundary, $N \perp T(\mathcal{T})$, annihilates t:

$$N(t) = 0, \qquad \text{on } \mathcal{T}, \tag{2.22}$$

which in turn implies that the $\mathbf{g}$-gradient of t is orthogonal to N.

Moreover, we assume that $\partial\Sigma_0$ has a neighbourhood in Σ_0, which is covered by the level sets of a defining function x^3:[4]

$$x^3 = 0: \quad \text{on } \partial\Sigma_0, \qquad x^3 < 0: \quad \text{in } \Sigma_0 \setminus \partial\Sigma_0, \qquad dx^3 \neq 0: \quad \text{on } \partial\Sigma_0. \tag{2.23}$$

Since $\partial_t\big|_{\mathcal{T}}$ is tangent to the boundary, we may complement x^3 with coordinates x^1, x^2, near a fixed point $p \in \partial\Sigma_0$, and propagate these along ∂_t to obtain a coordinate system (t, x^1, x^2, x^3) in a spacetime neighbourhood of $p \in \partial\Sigma_0$. Evidently, for small t, x^3 will remain a defining function of the boundary. Note that by definition, the gradient of x^3, Dx^3, is normal to the boundary. Hence, setting

$$N := \frac{Dx^3}{\sqrt{\mathbf{g}(Dx^3, Dx^3)}}, \qquad Dx^3 := \mathbf{g}^{ij}\partial_i x^3 \partial_j = g^{3j}\partial_j, \quad g^{33}\partial_3 = (g^{33})^{\frac{1}{2}}N - g^{31}\partial_1 - g^{32}\partial_2, \tag{2.24}$$

the vector field N is an extension in $\mathcal{M}$, locally around p, of the outward unit normal to the boundary $\mathcal{T}$.

Remark 2.7. The defining function x^3 is global near the boundary, but more than one coordinate patches x^1, x^2 have to potentially be used, along the level sets of x^3, in order to cover an entire neighbourhood of the boundary $\partial\Sigma_0$. However, for simplicity in the exposition of our overall argument, we will only work with a single patch, projecting the wave equation for k onto this specific frame. Since the wave equation for k (2.7) is tensorial and since the lapse Φ is independent of the choice of coordinates on Σ_t, the whole procedure can then be carried out tensorially in the planes generated by ∂_1 and ∂_2.

[4] This can be for example, the Gaussian parameter in a tubular neighbourhood of $\partial\Sigma_0$.

Boundary data: We evaluate k on the boundary against the adapted frame $\partial_A, \partial_B, N,\ A, B = 1, 2$. The boundary data for the IBVP are given in terms of the 1-parameter family $[q_t]$ of conformal metrics on $\partial\Sigma_t$, see Theorem 1.1, which only determine the values of the traceless part of k along the cross sections $\partial\Sigma_t$ with one index raised:

Lemma 2.8. *Let* $(q_t)_{AB} = \Omega^2[q_t]_{AB}$ *and let*

$$\widehat{k}_A{}^B := k_A{}^B - \frac{1}{2}\delta_A{}^B k_C{}^C. \tag{2.25}$$

Then, it holds

$$\widehat{k}_A{}^B = -\frac{1}{2}[q_t]^{BC}\partial_t[q_t]_{AC} + \frac{1}{4}\delta_A{}^B[q_t]^{DC}\partial_t[q_t]_{DC}, \qquad \text{on } \partial\Sigma_t, \tag{2.26}$$

for all $A, B = 1, 2$.

Proof. We compute on $\partial\Sigma_t$ using the form (2.20) of the induced metric H on the boundary:

$$\begin{aligned} k_{AB} &:= -\frac{1}{2}\partial_t(\Omega^2[q_t]_{AB}), \qquad k_C{}^C = \Omega^{-2}[q_t]^{DC}k_{DC} = -\frac{1}{2}[q_t]^{DC}\partial_t[q_t]_{DC} - 2\Omega^{-1}\partial_t\Omega,\\ k_A{}^B &= \Omega^{-2}[q_t]^{BC}k_{AC} = -\frac{1}{2}\Omega^{-2}[q_t]^{BC}\partial_t(\Omega^2[q_t]_{AC}) = -\frac{1}{2}[q_t]^{BC}\partial_t[q_t]_{AC} - \delta_A{}^B\Omega^{-1}\partial_t\Omega. \end{aligned}$$

Subtracting $\frac{1}{2}\delta_A{}^B$ times the second formula from the third, we notice that the terms involving Ω cancel out, leaving (2.26). □

To the rest of the components of k, we impose boundary conditions that propagate the maximal gauge and the momentum constraint (1.2) on the boundary $\mathcal{T}$:

$$k_{NN} := -\, k_C{}^C, \tag{2.27}$$

$$\begin{aligned} \nabla_N k_{NA} &:= -\,\nabla_B k_A{}^B \\ &= -\,\partial_B(\widehat{k}_A{}^B + \frac{1}{2}\delta_A{}^B k_C{}^C) + \Gamma^C_{AB}(\widehat{k}_C{}^B - \frac{1}{2}\delta_C{}^B k_{NN}) \\ &\quad - \Gamma^B_{BC}(\widehat{k}_A{}^C - \frac{1}{2}\delta_A{}^C k_{NN}) + \chi_A{}^B k_{NB} + \chi_B{}^B k_{NA}, \end{aligned} \tag{2.28}$$

$$\frac{1}{2}(\nabla_N k_{NN} - \nabla_N k_A{}^A) := -\,\nabla_A k_N{}^A, \tag{2.29}$$

where $\chi_{ij} := \mathbf{g}(D_{\partial_i}\partial_j, N)$ is the second fundamental form of the boundary $\mathcal{T}$, while Γ^C_{AB} are Christoffel symbols associated to the induced metric q_t on $\partial\Sigma_t$.

Remark 2.9. The combination of (2.27)-(2.28), imply the validity of the momentum constraint (1.2), projected on ∂_1, ∂_2. However, the last condition (2.29) differs slightly from (1.2), in the normal direction N, since a priori the Neumann type data $N\mathrm{tr}k$ is not known to vanish on the boundary. We found such a modification necessary for the absorption of the boundary terms that arise in the energy estimates for the reduced equation (2.7), see Section 3. Despite this modification, as we show in Section 4, the above boundary conditions are sufficient for the recovery of the EVE from the reduced equations.

Remark 2.10. At first glance, the heavily coupled, mixed Dirichlet-Neumann boundary conditions (2.27)-(2.29) seem to be losing derivatives in an energy argument for (2.7). However, we show that by some careful manipulations, the arising boundary terms can all be absorbed in the main energies and close the estimates, see Proposition 3.3.

2.2 Initial data and compatibility conditions

An initial data set h, k for the EVE on Σ_0, induces the initial data for (2.2) and half of the initial data for (2.7). These are sufficient to determine Φ from (2.18), satisfying the Dirichlet boundary condition (2.21). Then the $\partial_t k$ part of the initial data for (2.7) is fixed by the second variation equations (2.4), such that the EVE are valid initially on Σ_0:

$$\partial_t k_{ij}\big|_{t=0} = -\nabla_i\nabla_j\Phi + \Phi(R_{ij} + k_{ij}\mathrm{tr}k - 2k_i{}^l k_{jl})\big|_{t=0} \qquad \Longleftrightarrow \qquad \mathbf{R}_{ij}\big|_{\Sigma_0} = 0, \tag{2.30}$$

for every $i, j = 1, 2, 3$. Since g, k satisfy the constraints (1.1)-(1.2) initially, combining (2.30) with the Gauss and Codazzi equations (2.14)-(2.15), we also have:

$$\mathbf{R}_{00}\big|_{\Sigma_0} = \mathbf{R}_{0i}\big|_{\Sigma_0} = 0, \qquad i = 1, 2, 3. \tag{2.31}$$

Moreover, k satisfies the maximal gauge $\mathrm{tr}k = 0$ on Σ_0. Then, by taking the trace in (2.30), we arrive at (2.17) for $t = 0$, where by employing (2.31) and the equation (2.18) for Φ, we obtain:

$$\partial_t\mathrm{tr}k\big|_{t=0} = 0. \tag{2.32}$$

The initial conditions (2.30)-(2.32) will be used in Section 4 to verify the EVE and the maximal gauge everywhere.

The initial tensors h, k on Σ_0 must induce tensors on S, which are compatible with the prescribed boundary data. For example, $[h\big|_{\partial\Sigma_0}] = [q_0]$ and $\widehat{k}_A{}^B\big|_S$ satisfying (2.26). The less obvious condition for $\partial_t\widehat{k}_A{}^B$ is given through (2.30):

$$\partial_t\widehat{k}_A{}^B\big|_S = (R_A{}^B - \frac{1}{2}\delta_A{}^B\partial_t k_C{}^C - \nabla_A\nabla^B\Phi)\big|_S, \tag{2.33}$$

where we used the vanishing of tr$k$ on Σ_0 and (2.21). Notice that the LHS is expressed via (2.26) solely in terms of the conformal metric class $[q_0]$ on S and the time derivatives of $[q_t]$ up to order two, evaluated at $t = 0$. Similar relations can be computed to any higher order. Also, note that by virtue of (2.21), the Hessian of Φ above equals:

$$\nabla_A\nabla^B\Phi\big|_S = -\chi_A{}^B N\Phi\big|_S,$$

where $N\Phi$ is determined through the Dirichlet to Neumann map for (2.18).

2.3 The commuted equations and boundary conditions

We find it suitable to evaluate the wave equation (2.7) against $\partial_1, \partial_2, N$, and raise the second index of k_{ij} using the metric, ie. $k_i{}^j = g^{aj}k_{ia}$. The correction terms from this procedure are incorporated in the RHS, which we re-write in the following schematic form:

$$\begin{aligned}(\partial_t^2 - \Phi^2\Delta_g)k_i{}^j =& \{\Phi^{-1}\partial_t\Phi\partial^2\Phi + \Phi^{-1}\partial g\partial_t\Phi\partial\Phi + \partial^2\partial_t\Phi + \partial g\partial\partial_t\Phi + k\partial\Phi\partial\Phi + \Phi\partial k\partial\Phi + \Phi k\partial_t k \\ &+\Phi k^3 + \Phi^{-1}\partial_t\Phi\partial k + \Phi k\partial^2\Phi + \Phi k\partial g\partial\Phi + \Phi^2 k\partial^2 g + \Phi^2 k\partial g\partial g + \Phi^2\partial g\partial k\}_i{}^j,\end{aligned} \tag{2.34}$$

for every $i, j = 1, 2, N$, where each term in the previous RHS represents an algebraic sum of terms of the depicted type, involving contractions and all spatial derivatives $\partial = \partial_1, \partial_2, \partial_3$. The specific indices in these generic terms do not matter for the estimates we derive below.

Commuting (2.34) with r tangential vector fields to the boundary, $\underline{\partial}^r$, $\underline{\partial} = \partial_t, \partial_1, \partial_2$, we have:

$$\begin{aligned}(\partial_t^2 - \Phi^2\Delta_g)\underline{\partial}^r k_i{}^j =& \underline{\partial}^r\{\Phi^{-1}\partial_t\Phi\partial^2\Phi + \Phi^{-1}\partial g\partial_t\Phi\partial\Phi + \partial^2\partial_t\Phi + \partial g\partial\partial_t\Phi + k\partial\Phi\partial\Phi + \Phi\partial k\partial\Phi + \Phi k\partial_t k \\ &+\Phi k^3 + \Phi^{-1}\partial_t\Phi\partial k + \Phi k\partial^2\Phi + \Phi k\partial g\partial\Phi + \Phi^2 k\partial^2 g + \Phi^2 k\partial g\partial g + \Phi^2\partial g\partial k\}_i{}^j \\ &+ \sum_{r_1+r_2+r_3\le r,\, r_3<r}\{\underline{\partial}^{r_1}(\Phi^2)\underline{\partial}^{r_2}g\underline{\partial}^{r_3}\partial^2 k + \underline{\partial}^{r_1}(\Phi^2)\underline{\partial}^{r_2}\partial g\underline{\partial}^{r_3}\partial k\}_i{}^j\end{aligned} \tag{2.35}$$

Notice that $\underline{\partial}^r$ will also hit g, g^{-1} which are involved in the contractions of all indices in the previous RHS, but are omitted. We choose to neglect the terms generated by this process, since they are more regular than the ones we treat in the energy estimates below.

Recall that the boundary data we prescribe imply the Dirichlet boundary conditions (2.26) for $\widehat{k}_A{}^B = k_A{}^B - \frac{1}{2}\delta_A{}^B k_C{}^C$. We modify $\widehat{k}_A{}^B$ such that it has zero Dirichlet boundary data:

$$\widetilde{k}_A{}^B = \widehat{k}_A{}^B - f_A{}^B, \tag{2.36}$$

where $f_A{}^B$ is a smooth extension in $\mathcal{M}$ of $\widehat{k}_A{}^B\big|_{\mathcal{T}}$. Then (2.35) for $\widetilde{k}_A{}^B$ becomes:

$$\begin{aligned}(\partial_t^2 - \Phi^2\Delta_g)\underline{\partial}^r\widetilde{k}_A{}^B =& \underline{\partial}^r\{\Phi^{-1}\partial_t\Phi\partial^2\Phi + \Phi^{-1}\partial g\partial_t\Phi\partial\Phi + \partial^2\partial_t\Phi + \partial g\partial\partial_t\Phi + k\partial\Phi\partial\Phi + \Phi\partial k\partial\Phi + \Phi k\partial_t k \\ &+\Phi k^3 + \Phi^{-1}\partial_t\Phi\partial k + \Phi k\partial^2\Phi + \Phi k\partial g\partial\Phi + \Phi^2 k\partial^2 g + \Phi^2 k\partial g\partial g + \Phi^2\partial g\partial k\}_A{}^B \\ &+ \sum_{r_1+r_2+r_3\le r,\, r_3<r}\{\underline{\partial}^{r_1}(\Phi^2)\underline{\partial}^{r_2}g\underline{\partial}^{r_3}\partial^2 k + \underline{\partial}^{r_1}(\Phi^2)\underline{\partial}^{r_2}\partial g\underline{\partial}^{r_3}\partial k\}_A{}^B \\ &+ \Phi^2\{\partial^2\underline{\partial}^r f + \partial g\partial\underline{\partial}^r f - \Phi^{-2}\partial_t^2\underline{\partial}^r f\}_A{}^B\end{aligned} \tag{2.37}$$

The commuted equations (2.2), (2.3), (2.18) for g_{ij}, g^{ij}, Φ read:

$$\partial_t\partial^l\underline{\partial}^r g_{ij} = -2\partial^l\underline{\partial}^r(\Phi k_{ij}) \tag{2.38}$$

$$\partial_t\partial^l\underline{\partial}^r g^{ij} = 2\partial^l\underline{\partial}^r(\Phi k^{ij}) \tag{2.39}$$

$$\Delta_g\underline{\partial}^r\Phi = \underline{\partial}^r(|k|^2\Phi) + \sum_{r_1+r_2\le r,\, r_2<r}(\underline{\partial}^{r_1}g\underline{\partial}^{r_2}\partial^2\Phi + \underline{\partial}^{r_1}\partial g\underline{\partial}^{r_2}\partial\Phi) \tag{2.40}$$

Note that (2.38), (2.39) contain also transversal vector fields to the boundary, since they are ODEs in t and no boundary conditions are needed to propagate energy estimates.

Remark 2.11. The top order terms $\Phi^2(k\underline{\partial}^r\partial^2 g)_i{}^j$ in the RHSs of (2.35), (2.37), containing $r + 2$ spatial derivatives of g, when for example $\underline{\partial}^r = \partial_1^r$, cannot be directly estimated in L^2 in terms of the energy of the wave operator in the LHS, since (2.38) does not gain a derivative in space. We show how to treat these terms in the energy estimates, using the structure of the equations, in the proof of Proposition 3.3.

We will also use the boundary conditions (2.28)-(2.29), commuted with $\partial_t^{r_2}\underline{\partial}^{r_1}$:

$$N\underline{\partial}^r k_{NA} = -\frac{1}{2}\partial_A\underline{\partial}^r k_C{}^C + \underline{\partial}^r(k\partial g + \partial f)_A + \sum_{r_1+r_2\le r,\, r_2<r} (\underline{\partial}^{r_1} g\underline{\partial}^{r_2}\partial k)_A, \tag{2.41}$$

$$N\underline{\partial}^r k_{NN} - N\underline{\partial}^r k_A{}^A = -2\partial_A\underline{\partial}^r k_N{}^A + \underline{\partial}^r(k\partial g + \partial f) + \sum_{r_1+r_2\le r,\, r_2<r} \underline{\partial}^{r_1} g\underline{\partial}^{r_2}\partial k. \tag{2.42}$$

Remark 2.12. The term in (2.41)-(2.42) containing $\underline{\partial}^r\partial g$ cannot be directly absorbed in L^2 by the energy of the system (2.35), (2.37), via a trace inequality, at top order $r = s$. However, we may use the fact that g gains a derivative in ∂_t to make a trade off and close the energy estimates for k, see the proof of Proposition 3.3.

3 Local existence for the reduced system

Our main goal in this section is to show how to derive energy estimates for the reduced system (2.2),(2.7),(2.18), subject to the boundary conditions (2.21),(2.27),(2.28),(2.29). In the end of this section we outline the steps that upgrade these energy estimates to a Picard iteration argument, hence, proving local existence for the reduced system of equations in the same energy spaces. The fact that the Ricci tensor of a solution to the reduced system vanishes is then demonstrated in the next section, which completes the proof of Theorem 1.1.

3.1 Localization of the problem

First, we argue that the problem can be localized in a neighbourhood of the boundary by realizing the following three steps:

(P1) Consider the solution $\mathbf{g}_1$ to the EVE in the domain of dependence $D(\Sigma_0)$ of the initial hypersurface Σ_0. We may consider a timelike hypersurface $\mathcal{T}_{ind} := \{x^3 = \varepsilon\}$, for some $\varepsilon > 0$.

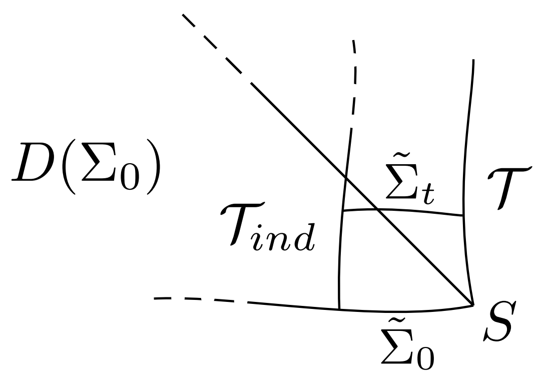

Figure 1: The domain of dependence $\mathcal{D}(\Sigma_0)$.

(P2) Then, we restrict our attention to the region bounded between $\mathcal{T}_{ind}, \mathcal{T}$. In particular, we solve the reduced equations (2.2), (2.7), (2.18), by imposing the boundary conditions (2.21), (2.27), (2.28), (2.29) on both $\mathcal{T}$ and the artificial timelike boundary $\mathcal{T}_{ind}$. We also impose the Dirichlet boundary conditions (2.26) for $\widehat{k}_A{}^B$ on $\mathcal{T}$, see (2.36), and impose arbitrary[5] Dirichlet boundary conditions for $\widehat{k}_A{}^B$ on $\mathcal{T}_{ind}$, everything defined with respect to a maximal foliation $\tilde{\Sigma}_t$, as depicted in Figure 2.

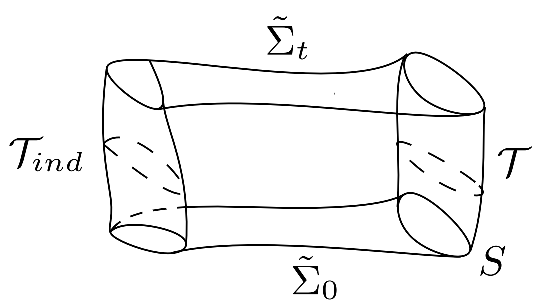

Figure 2: The region between $\mathcal{T}_{ind}, \mathcal{T}$.

[5] It makes no difference for our overall argument what the boundary data of $\widehat{k}_A{}^B$ are on $\mathcal{T}_{ind}$, since we discard part of the solution to the reduced equations near the artificial boundary.

(P3) After having solved the above reduced system of equations, in the region between $\mathcal{T}_{ind}, \mathcal{T}$, and have concluded that it consitutes a solution to the EVE, $\mathbf{g}_2$, see Section 4, we then define our final vacuum Lorentzian manifold by considering the metric

$$\mathbf{g} = \begin{cases} \mathbf{g}_1, & D(\Sigma_0) \\ \mathbf{g}_2, & D(\tilde{\Sigma}_0) \cup D_{free} \end{cases}, \tag{3.1}$$

derived from the two solutions $\mathbf{g}_1, \mathbf{g}_2$ in the union of the three regions depicted in Figure 3. The fact that $\mathbf{g}$ is well-defined follows from the classical geometric uniqueness for the initial value problem in the domain dependence of $\tilde{\Sigma}_0$, which implies that $\mathbf{g}_1, \mathbf{g}_2$ are isometric in $D(\tilde{\Sigma}_0)$.

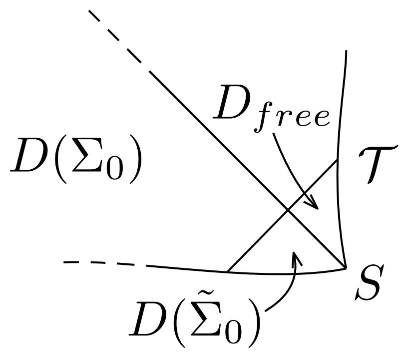

Figure 3: The domain $D(\Sigma_0) \cup D_{free}$ of the resulting solution.

The domain of $\mathbf{g}$ obviously covers a future spacetime neighbourhood of Σ_0. This completes our localisation procedure.

3.2 Energy estimates and Picard iteration

In the rest of this section, we realize the second part (P2), solving the reduced equations in the cylindrical region between $\mathcal{T}_{ind}, \mathcal{T}$. Apart from the energy estimates for k that are needed to prove local existence, using (2.35), (2.37), and the boundary conditions (2.27), (2.41), (2.42), the rest of the argument is for the most part standard, so we refrain from going into too much detail.

Suppose that the boundary values of $\widehat{k}_A{}^B$ on $\mathcal{T}_{ind}$ have been incorporated in the definition (2.36) of $\widetilde{k}_A{}^B$, such that $\widetilde{k}_A{}^B$ has homogeneous Dirichlet boundary data on both $\mathcal{T}, \mathcal{T}_{ind}$:

$$\widetilde{k}_A{}^B = 0, \qquad \text{on } \partial\tilde{\Sigma}_t. \tag{3.2}$$

We will be working with the following energies:

$$E_{total}(t) := E_k(t) + \sum_{r=0}^{s+1}\sum_{i,j=1}^{3}\left[\|\partial_t^r g_{ij}\|^2_{H^{s+1-r}(\tilde{\Sigma}_t)} + \|\partial_t^r g^{ij}\|^2_{H^{s+1-r}(\tilde{\Sigma}_t)}\right] + \sum_{r=0}^{s+1}\|\partial_t^r \Phi\|^2_{H^{s+2-r}(\tilde{\Sigma}_t)}, \tag{3.3}$$

$$\begin{aligned} E_k(t) := &\sum_{r\leq s}\int_{\tilde{\Sigma}_t}\Big(\sum_{A,B}\left[(\partial_t\underline{\partial}^r\widetilde{k}_A{}^B)^2 + \Phi^2|\nabla\underline{\partial}^r\widetilde{k}_A{}^B|^2_g\right] + (\partial_t\underline{\partial}^r k_C{}^C)^2 + \Phi^2|\nabla\underline{\partial}^r k_C{}^C|^2_g \\ &+ 4g^{AB}\left[\partial_t\underline{\partial}^r k_{NA}\partial_t\underline{\partial}^r k_{NB} + \Phi^2 g^{ij}\partial_i\underline{\partial}^r k_{NA}\partial_j\underline{\partial}^r k_{NB}\right] \\ &+ (\partial_t\underline{\partial}^r k_{NN})^2 + \Phi^2|\nabla\underline{\partial}^r k_{NN}|^2_g\Big)\mathrm{vol}_{\tilde{\Sigma}_t}, \end{aligned} \tag{3.4}$$

where $|\nabla u|^2_g = g^{ij}\partial_i u\partial_j u$, $\mathrm{vol}_{\tilde{\Sigma}_t}$ is the intrinsic volume form, and

$$\|u\|^2_{H^r(\tilde{\Sigma}_t)} := \sum_{r_1\leq r}\int_{\tilde{\Sigma}_t}(\partial^{r_1}u)^2\mathrm{vol}_{\tilde{\Sigma}_t}, \tag{3.5}$$

for all possible combinations of $\partial = \partial_1, \partial_2, \partial_3$ and $\underline{\partial} = \partial_t, \partial_1, \partial_2$.

In what follows, we fix an $s \geq 3$, assume a solution exists in the above energy spaces, and make the following bootstrap assumption on the total energy:

$$E_{total}(t) \leq C_0, \qquad t \in [0, T_0], \tag{3.6}$$

for some $0 < T_0 < 1$. The main part of our local existence argument is based on quantifying (3.6) by deriving a priori energy estimates using the differentiated equations (2.38), (2.39), (2.35), (2.37), (2.40), see Proposition 3.3.

We will frequently use the classical Sobolev inequalities

$$\|u\|_{L^\infty(\tilde\Sigma_t)} \lesssim \|u\|_{H^2(\tilde\Sigma_t)}, \qquad \|u\|_{L^4(\tilde\Sigma_t)} \lesssim \|u\|_{H^1(\tilde\Sigma_t)}, \tag{3.7}$$

and the bootstrap assumption (3.6). Since $s\ge 3$, $E_{total}(t)$ controls up to two spatial derivatives of $g,\partial_t\Phi$, up to three spatial derivatives of Φ, and up to one derivative of $k=-\frac12\Phi^{-1}\partial_t g$ in $L^\infty(\tilde\Sigma_t)$. As for the inverse powers of Φ entering the equations (2.35), (2.37), we note that by virtue of the maximum principle and Harnack's inequality, applied to the equation (2.18) subject to the boundary condition $\Phi=1$ on both $\mathcal{T},\mathcal{T}_{ind}$, it follows that

$$0<c\le\Phi\le 1, \tag{3.8}$$

for some $c>0$. Hence, we need only estimate Φ.

Lemma 3.1. *Assume the bootstrap assumption* (3.6) *holds, for a fixed $s\ge3$. Then the energy $E_{total}(t)$ controls the corresponding energies of k that include N derivatives:*

$$\int_{\tilde\Sigma_t}(\partial_t N^{r_2}\underline{\partial}^{r_1}u)^2+\Phi^2|\nabla N^{r_2}\underline{\partial}^{r_1}u|_g^2\mathrm{vol}_{\tilde\Sigma_t}\lesssim E_{total}(t)+\sum_{r=0}^{s+1}\sum_{A,B=1,2}\|\partial_t^r f_A{}^B\|^2_{H^{s+1-r}(\tilde\Sigma_t)}, \tag{3.9}$$

for all $r_1+r_2\le s$, $t\in[0,T_0]$, $u=\widetilde{k}_A{}^B,k_C{}^C,k_{NA},k_{NN}$, $A,B=1,2$, where the implicit constant in $\lesssim$ depends on C_0.

Proof. We start with controlling the RHSs of the wave equations (2.35), (2.37) and their N derivatives in $L^2(\tilde\Sigma_t)$, using the Sobolev inequalities (3.7) and the bootstrap assumption (3.6). More precisely, for $l+r\le s-1$, expressing N relative to ∂, as in (2.24), and using the L^∞ estimate in (3.7), we have

$$\begin{aligned}&\|N^l\underline{\partial}^r\{\Phi^{-1}\partial_t\Phi\partial^2\Phi+\Phi^{-1}\partial g\partial_t\Phi\partial\Phi+\partial^2\partial_t\Phi+\partial g\partial\partial_t\Phi+k\partial\Phi\partial\Phi+\Phi\partial k\partial\Phi+\Phi k^3+\Phi^{-1}\partial_t\Phi\partial k\}\|_{L^2(\tilde\Sigma_t)}\\&\lesssim\sqrt{E_{total}(t)}+\sum_{l'\le l,\,r'\le r}\{\|N^{l'}\underline{\partial}^{r'}k\|_{L^2(\tilde\Sigma_t)}+\|N^{l'}\underline{\partial}^{r'}\partial k\|_{L^2(\tilde\Sigma_t)}\}\end{aligned} \tag{3.10}$$

Using in addition Cauchy-Schwartz and the L^4 estimate in (3.7), we obtain the inequality:

$$\begin{aligned}&\|N^l\underline{\partial}^r\{\Phi k\partial_t k+\Phi k\partial^2\Phi+\Phi k\partial g\partial\Phi+\Phi^2k\partial^2 g+\Phi^2k\partial g\partial g+\Phi^2\partial g\partial k\}\|_{L^2(\tilde\Sigma_t)}\\&\lesssim\sum_{l'\le l,\,r'\le r}\{\|N^{l'}\underline{\partial}^{r'}k\|_{L^2(\tilde\Sigma_t)}+\|N^{l'}\underline{\partial}^{r'}\partial k\|_{L^2(\tilde\Sigma_t)}\}\\&\quad+\sum_{l_1+l_2\le l,\,r_1+r_2\le r}\|N^{l_1}\underline{\partial}^{r_1}k\|_{H^1(\tilde\Sigma_t)}\|N^{l_2}\underline{\partial}^{r_2}\partial_t k\|_{H^1(\tilde\Sigma_t)}+\delta_{l+r=s-1}\|k\|_{H^2(\tilde\Sigma_t)},\end{aligned} \tag{3.11}$$

where $\delta_{l+r=s-1}=0$ for $l+r<s-1$ and $=1$ for $l+r=s-1$. We note that the last term in (3.11) only appears when $l+r=s-1$ and all derivatives $N^l\underline{\partial}^r$ hit the factor $\partial^2 g$ in $\Phi^2k\partial^2 g$. Arguing similarly, it holds

$$\begin{aligned}&\Bigg\|\sum_{r_1+r_2+r_3\le r,\,r_3<r}N^l\{\underline{\partial}^{r_1}(\Phi^2)\underline{\partial}^{r_2}g\underline{\partial}^{r_3}\partial^2k+\underline{\partial}^{r_1}(\Phi^2)\underline{\partial}^{r_2}\partial g\underline{\partial}^{r_3}\partial k\}\\&+N^l\{\Phi^2\partial^2\underline{\partial}^rf+\Phi^2\partial g\partial\underline{\partial}^rf-\Phi^2\partial_t^2\underline{\partial}^rf\}\Bigg\|_{L^2(\tilde\Sigma_t)}\\&\lesssim\sum_{l'\le l,\,r'\le r-1}\{\|N^{l'}\underline{\partial}^{r'}\partial k\|_{L^2(\tilde\Sigma_t)}+\|N^{l'}\underline{\partial}^{r'}\partial^2k\|_{L^2(\tilde\Sigma_t)}\}+\sum_{r'=0}^{s+1}\|\partial_t^{r'}f\|_{H^{s+1-r'}(\tilde\Sigma_t)}\end{aligned} \tag{3.12}$$

Recall the equations (2.35), (2.37). Combining (3.10)-(3.12) we conclude the estimate:

$$\begin{aligned}&\|N^l(\partial_t^2-\Phi^2\Delta_g)\underline{\partial}^ru\|_{L^2(\tilde\Sigma_t)}\\&\lesssim\sum_{l'\le l,\,r'\le r}\{\|N^{l'}\underline{\partial}^{r'}k\|_{L^2(\tilde\Sigma_t)}+\|N^{l'}\underline{\partial}^{r'}\partial k\|_{L^2(\tilde\Sigma_t)}\}+\sum_{l_1+l_2\le l,\,r_1+r_2\le r}\|N^{l_1}\underline{\partial}^{r_1}k\|_{H^1(\tilde\Sigma_t)}\|N^{l_2}\underline{\partial}^{r_2}\partial_tk\|_{H^1(\tilde\Sigma_t)}\\&\quad+\sqrt{E_{total}(t)}+\sum_{r'=0}^{s+1}\|\partial_t^{r'}f\|_{H^{s+1-r'}(\tilde\Sigma_t)}+\delta_{l+r=s-1}\|k\|_{H^2(\tilde\Sigma_t)}+\sum_{l'\le l,\,r'\le r-1}\|N^{l'}\underline{\partial}^{r'}\partial^2k\|_{L^2(\tilde\Sigma_t)},\end{aligned} \tag{3.13}$$

for $u=k_i{}^j,\widetilde{k}_A{}^B$, where we note that for $r=0$, the last term is not present. Expanding the operator in the LHS gives

$$\begin{aligned}N^l(\partial_t^2-\Phi^2\Delta_g)\underline{\partial}^ru=&\,N^l\partial_t^2\underline{\partial}^ru-N^l(\Phi^2g^{AB}\partial_A\partial_B\underline{\partial}^ru)+N^l(\Phi^2\partial g\underline{\partial}^r\partial u)-\Phi^2N^{l+2}\underline{\partial}^ru\\&-\sum_{l_1+l_2=l,\,l_2\le l-1}N^{l_1}(\Phi^2)N^{l_2+2}\underline{\partial}^ru,\end{aligned} \tag{3.14}$$

where the last term appears only for $l \geq 1$. Solving for $N^{l+2}\underline{\partial}^r u$ using (3.14) and employing (3.13), (3.7), (3.6) yields the inequality

$$
\begin{aligned}
&\|N^{l+2}\underline{\partial}^r u\|_{L^2(\tilde{\Sigma}_t)} \\
&\lesssim \sum_{l'\leq l,\, r'\leq r} \{\|N^{l'}\underline{\partial}^{r'} k\|_{L^2(\tilde{\Sigma}_t)} + \|N^{l'}\underline{\partial}^{r'}\partial k\|_{L^2(\tilde{\Sigma}_t)}\} + \sum_{l_1+l_2\leq l,\, r_1+r_2\leq r} \|N^{l_1}\underline{\partial}^{r_1} k\|_{H^1(\tilde{\Sigma}_t)}\|N^{l_2}\underline{\partial}^{r_2}\partial_t k\|_{H^1(\tilde{\Sigma}_t)} \\
&+ \sqrt{E_{total}(t)} + \sum_{r'=0}^{s+1}\|\partial_t^{r'} f\|_{H^{s+1-r'}(\tilde{\Sigma}_t)} + \|N^l\underline{\partial}^{r+2}u\|_{L^2(\tilde{\Sigma}_t)} + \|N^l\underline{\partial}^{r+2}u\|_{L^2(\tilde{\Sigma}_t)} \\
&+ \sum_{l'\leq l-1}\|N^{l'+2}\underline{\partial}^r u\|_{L^2(\tilde{\Sigma}_t)} + \delta_{l+r=s-1}\|k\|_{H^2(\tilde{\Sigma}_t)} + \sum_{l'\leq l,\, r'\leq r-1}\|N^{l'}\underline{\partial}^{r'}\partial^2 k\|_{L^2(\tilde{\Sigma}_t)}
\end{aligned}
\tag{3.15}
$$

Notice that for $l = r = 0$, the terms in the last line do not appear, hence, every term in the previous RHS is controlled by the total energy and the boundary data. This gives a bound for the L^2 norm of $N^2 u$. Arguing by finite induction in $l + r = 0, 1, \dots s-1$, and using the bootstrap assumption (3.6), we conclude the estimate:

$$
\|N^{l+2}\underline{\partial}^r u\|_{L^2(\tilde{\Sigma}_t)} \lesssim \sqrt{E_{total}(t)} + \sum_{r'=0}^{s+1}\|\partial_t^{r'} f\|_{H^{s+1-r'}(\tilde{\Sigma}_t)}, \tag{3.16}
$$

for every $l + r = 0, 1, \dots, s-1$, $u = k_i{}^j, \widetilde{k}_A{}^B$. Note that commuting the derivatives within the norm in the previous LHS generates terms that are controllable from the total energy, using the L^∞ estimate in (3.7) and the bootstrap assumption (3.6). Thus, squaring (3.16) gives the desired bound (3.9). □

Lemma 3.2. *Assume the bootstrap assumption* (3.6) *holds, for a fixed* $s \geq 3$. *Then the part of the total energy* (3.3) *corresponding to* g_{ij}, g^{ij}, Φ, *as well as the less than top order energies of* $k_i{}^j$, *satisfy:*

$$
\sum_{r=0}^{s+1}\sum_{i,j=1}^{3}\left[\|\partial_t^r g_{ij}\|^2_{H^{s+1-r}(\tilde{\Sigma}_t)} + \|\partial_t^r g^{ij}\|^2_{H^{s+1-r}(\tilde{\Sigma}_t)}\right] + \sum_{r=0}^{s+1}\|\partial_t^r\Phi\|^2_{H^{s+2-r}(\tilde{\Sigma}_t)} \leq C_{init}e^{tC_*} + tC_*, \tag{3.17}
$$

$$
\sum_{r=0}^{s-1}\sum_{i,j=1,2,N}\left[\|\partial_t\underline{\partial}^r k_i{}^j\|^2_{H^{s+1-r}(\tilde{\Sigma}_t)} + \|\nabla\underline{\partial}^r k_i{}^j\|^2_{H^{s+1-r}(\tilde{\Sigma}_t)}\right] \leq C_{init}e^{tC_*} + tC_*, \tag{3.18}
$$

for all $t \in [0, T_0]$, *where* $C_{init} > 0$ *is a constant depending on the initial energy* $E_{total}(0)$, *but is independent of* C_0, *while* $C_* > 0$ *is allowed to also depend on* C_0 *and the* $H^{s+1-r}(\tilde{\Sigma}_t)$ *norms of* $f_A{}^B$, $A, B = 1, 2$, $r = 0, \dots, s+1$.

Moreover, $\partial_t g_{ij}, \partial_t g^{ij}, \partial_t\Phi$ *gain a derivative in* L^2 *compared to* (3.17), *but with a worse control on the corresponding norms:*

$$
\sum_{r=0}^{s}\sum_{i,j=1}^{3}\left[\|\partial_t^{r+1} g_{ij}\|^2_{H^{s+1-r}(\tilde{\Sigma}_t)} + \|\partial_t^{r+1} g^{ij}\|^2_{H^{s+1-r}(\tilde{\Sigma}_t)}\right] + \sum_{r=0}^{s}\|\partial_t^{r+1}\Phi\|^2_{H^{s+2-r}(\tilde{\Sigma}_t)} \leq C_*, \tag{3.19}
$$

for all $t \in [0, T_0]$, *and a different* C_* *than that in* (3.17).

Proof. Step 1: Control of the lower order norms from the top ones. Assume that the top order estimate (3.19) holds. Then the estimate (3.17) follows straightforwardly. We illustrate the argument in the case of the $H^{s+1}(\tilde{\Sigma}_t)$ norm of g_{ij}, using as well Cauchy-Schwartz, the Sobolev estimates (3.7), and the bootstrap assumption (3.6):

$$
\begin{aligned}
\frac{1}{2}\partial_t\|g_{ij}\|^2_{H^{s+1}(\tilde{\Sigma}_t)} &= \sum_{r\leq s+1}\int_{\tilde{\Sigma}_t}\partial^r g_{ij}\partial_t\partial^r g_{ij}\mathrm{vol}_{\tilde{\Sigma}_t} + \sum_{r\leq s+1}\int_{\tilde{\Sigma}_t}\frac{1}{2}(\partial^r g_{ij})^2\partial_t\mathrm{vol}_{\tilde{\Sigma}_t} \\
&\leq \|g_{ij}\|_{H^{s+1}(\tilde{\Sigma}_t)}\|\partial_t g_{ij}\|_{H^{s+1}(\tilde{\Sigma}_t)} + CC_0\|g_{ij}\|^2_{H^{s+1}(\tilde{\Sigma}_t)} \qquad (\partial_t\mathrm{vol}_{\tilde{\Sigma}_t} = \Phi\mathrm{tr}k\mathrm{vol}_{\tilde{\Sigma}_t}) \\
\Rightarrow \quad \partial_t\|g_{ij}\|_{H^{s+1}(\tilde{\Sigma}_t)} &\leq \|\partial_t g_{ij}\|_{H^{s+1}(\tilde{\Sigma}_t)} + CC_0\|g_{ij}\|_{H^{s+1}(\tilde{\Sigma}_t)} \\
\Rightarrow \quad \|g_{ij}\|_{H^{s+1}(\tilde{\Sigma}_t)} &\leq e^{CC_0t}\|g_{ij}\|_{H^{s+1}(\tilde{\Sigma}_0)} + t\sqrt{C_*}e^{CC_0+CC_0t},
\end{aligned}
\tag{3.20}
$$

for all $t \in [0, T_0]$. The derivations for the rest of the norms in the LHS of (3.17) are the same. (3.18) is proven similarly, since the top order part of the energy $E_k(t)$ contains one more time derivative $\underline{\partial} = \partial_t$.

Step 2: Control of the top order norms of g_{ij}, g^{ij}. Hence, we need only estimate the top order norms in the LHS of (3.19). We proceed to estimate the top order norms of g_{ij} using the differentiated equation (2.38), the L^∞ estimate in (3.7), the bootstrap assumption (3.6):

$$
\sum_{r=0}^{s}\|\partial_t^{r+1} g_{ij}\|^2_{H^{s+1-r}(\tilde{\Sigma}_t)} = \sum_{r=0}^{s}\sum_{l\leq s+1-r}\int_{\tilde{\Sigma}_t}[\partial^l\partial_t^r(2\Phi g_{aj}k_i{}^a)]^2\mathrm{vol}_{\tilde{\Sigma}_t} \leq CC_0^4\sum_{a=1}^{3}\sum_{r=0}^{s}\|\partial_t^r k_i{}^a\|^2_{H^{s+1-r}(\tilde{\Sigma}_t)} \tag{3.21}
$$

Employing Lemma 3.1 gives (3.19) for g_{ij}. The derivation of (3.19) for g^{ij} is the same.

Step 3: Control of the top order norms of Φ. We begin with controlling the $H^2(\tilde{\Sigma}_t)$ norm of tangential derivatives of Φ, using the commuted equation (2.40). In particular, using the L^∞ estimate in (3.7) and the bootstrap assumption (3.6) we have:

$$\|\Delta_g \underline{\partial}^r\Phi\|_{L^2(\tilde{\Sigma}_t)} \leq \|\underline{\partial}^r(|k|^2\Phi)\|_{L^2(\tilde{\Sigma}_t)} + \Big\| \sum_{r_1+r_2\leq r,\, r_2<r} (\underline{\partial}^{r_1} g \underline{\partial}^{r_2}\partial^2\Phi + \underline{\partial}^{r_1}\partial g \underline{\partial}^{r_2}\partial\Phi) \Big\|_{L^2(\tilde{\Sigma}_t)} \leq CC_0^3, \tag{3.22}$$

for all $r = 0, \dots, s$. All terms in the previous RHS have $s+1$ derivatives in L^2, apart from the factor ∂g in the last term, unless one of the tangential derivatives $\underline{\partial} = \partial_t$, since we then plug in the equation (2.38). In this case it holds as well:

$$\sum_{r=0}^{s} \|\Delta_g \underline{\partial}^r \partial_t\Phi\|_{L^2(\tilde{\Sigma}_t)} \leq CC_0^3. \tag{3.23}$$

To obtain elliptic estimates from (3.23), we integrate by parts twice, to derive the standard identity:

$$\begin{aligned}
\|\Delta_g v\|^2_{L^2(\tilde{\Sigma}_t)} &= \int_{\tilde{\Sigma}_t} \nabla^i\nabla_i v \nabla^j\nabla_j \mathrm{vol}_{\tilde{\Sigma}_t} = -\int_{\tilde{\Sigma}_t} \nabla_i v \nabla^i\nabla^j\nabla_j v \mathrm{vol}_{\tilde{\Sigma}_t} + \int_{\partial\tilde{\Sigma}_t} \nabla_N v \nabla^j\nabla_j v \mathrm{vol}_{\partial\tilde{\Sigma}_t} \\
&= \int_{\tilde{\Sigma}_t} \nabla^j\nabla_i v \nabla^i\nabla_j v \mathrm{vol}_{\tilde{\Sigma}_t} - \int_{\tilde{\Sigma}_t} R^{ia}\nabla_i v\nabla_a v \mathrm{vol}_{\tilde{\Sigma}_t} + \int_{\partial\tilde{\Sigma}_t} \nabla_N v \nabla^j\nabla_j v \mathrm{vol}_{\partial\tilde{\Sigma}_t} \\
&\quad - \int_{\partial\tilde{\Sigma}_t} \nabla_i v \nabla^i \nabla_N v \mathrm{vol}_{\partial\tilde{\Sigma}_t}
\end{aligned} \tag{3.24}$$

for $v = \underline{\partial}^r\partial_t\Phi$, $r = 0, \dots, s$. We estimate the boundary terms in (3.24) using the boundary condition $\Phi = 1$ on $\mathcal{T}, \mathcal{T}_{ind}$, the bootstrap assumption (3.6), and Young's inequality:

$$\begin{aligned}
&\Big| \int_{\partial\tilde{\Sigma}_t} \nabla_N v \nabla^j\nabla_j v \mathrm{vol}_{\partial\tilde{\Sigma}_t} - \int_{\partial\tilde{\Sigma}_t} \nabla_i v \nabla^i\nabla_N v \mathrm{vol}_{\partial\tilde{\Sigma}_t} \Big| \\
=& \Big| \int_{\partial\tilde{\Sigma}_t} g^{AB}(\nabla_N v \nabla_A\nabla_B v - \nabla_A v \nabla^B \nabla_N v) \mathrm{vol}_{\partial\tilde{\Sigma}_t} \Big| \\
=& \Big| \int_{\partial\tilde{\Sigma}_t} \partial g \partial v \partial v \mathrm{vol}_{\partial\tilde{\Sigma}_t} \Big| = \Big| \int_{\tilde{\Sigma}_t} N(\partial g \partial v \partial v) + (\mathrm{div}N)\partial g \partial v \partial v \mathrm{vol}_{\tilde{\Sigma}_t} \Big| \\
\leq& \varepsilon \|\partial^2 v\|^2_{L^2(\tilde{\Sigma}_t)} + \frac{C}{\varepsilon} C_0^2 \|\partial v\|^2_{L^2(\tilde{\Sigma}_t)}
\end{aligned} \tag{3.25}$$

Expressing schematically the rest of the terms in (3.24), using the bootstrap assumption (3.6) to control the Ricci term R^{ia}, the bounds (3.22)-(3.23) on the Laplacian of $v = \underline{\partial}^r\partial_t\Phi$, and taking $\varepsilon > 0$ sufficiently small, we deduce the estimate:

$$\sum_{r=0}^{s} \|\partial^2 \underline{\partial}^r \partial_t \Phi\|^2_{L^2(\tilde{\Sigma}_t)} \leq CC_0^3 + CC_0^2 \sum_{r=0}^{s} \|\partial \underline{\partial}^r \partial_t\Phi\|^2_{L^2(\tilde{\Sigma}_t)} \overset{(3.6)}{\leq} CC_0^4 \tag{3.26}$$

Since the bootstrap assumption (3.6) controls the lower order terms in the $H^2(\tilde{\Sigma}_t)$ norm of $\underline{\partial}^r\partial_t\Phi$, we in fact have

$$\sum_{r=0}^{s} \|\underline{\partial}^r\partial_t\Phi\|^2_{H^2(\tilde{\Sigma}_t)} \leq CC_0^4. \tag{3.27}$$

The previous bound gives (3.19) for the $H^2(\tilde{\Sigma}_t)$ norm of $\partial_t^{s+1}\Phi$.

We proceed to control the N derivatives of Φ at top order as well. Differentiating the equation (2.40), for $\underline{\partial}^r = \underline{\partial}^{r'}\partial_t$, with N^l and expanding the Laplacian gives:

$$\begin{aligned}
N^{l+2}\underline{\partial}^{r'}\partial_t\Phi = N^l \Big[&\underline{\partial}^{r'}\partial_t(|k|^2\Phi) + \sum_{\underline{\partial}^{r_1}\underline{\partial}^{r_2} = \underline{\partial}^{r'}\partial_t,\, r_2<r'+1} (\underline{\partial}^{r_1} g \underline{\partial}^{r_2}\partial^2\Phi + \underline{\partial}^{r_1}\partial g \underline{\partial}^{r_2}\partial\Phi) \\
&- g^{AB}\partial_A\partial_B\underline{\partial}^{r'}\partial_t\Phi + \partial g \partial \underline{\partial}^{r'}\partial_t\Phi \Big]
\end{aligned} \tag{3.28}$$

For $l + r' = s$, all terms in the preceding RHS, apart from $|k|^2 N^l \underline{\partial}^{r'}\partial_t\Phi$, $g^{AB}N^l\partial_A\partial_B\underline{\partial}^{r'}\partial_t\Phi$, can be controlled using the Sobolev inequalities (3.7), the bootstrap assumption (3.6), Lemma 3.1, and the top order estimate (3.19) for g_{ij}. The remaining two terms can be controlled via a finite induction argument in l. Indeed, for $l \leq 2$, the aforementioned terms are already bounded by virtue of (3.27). Increasing $l \leq s - r'$ step by step and using (3.28) gives the desired inequality:

$$\sum_{l+r'=s} \|N^{l+2}\underline{\partial}^{r'}\partial_t\Phi\|_{L^2(\tilde{\Sigma}_t)} \leq C_* \tag{3.29}$$

The desired top order estimate (3.19) for Φ is a combination of (3.23), (3.29) and the bootstrap assumption (3.6). □

With the previous two lemmas at our disposal, we proceed to the local well-posedness of the initial boundary value problem for the reduced system of equations.

Proposition 3.3. *Let $s \geq 3$. Then the reduced system of equations* (2.2), (2.7), (2.18), *subject to the boundary conditions* (2.21), (2.26), (2.27), (2.28), (2.29), *and the compatibility conditions (see Section 2.2), on both $\mathcal{T}, \mathcal{T}_{ind}$ (see Figure 2), has a locally well-posed initial boundary value problem in the spaces $\partial_t^r g_{ij}, \partial_t^r k_{ij} \in H^{s+1-r}(\tilde{\Sigma}_t)$, $\partial_t^r \Phi \in H^{s+2-r}(\tilde{\Sigma}_t)$, $r = 0, \dots, s+1$.*

Proof. It is realized in the following three steps: 1. We derive a priori estimates for k, which, together with the estimates in Lemma 3.2, yield the quantified version (3.42) of the bootstrap assumption (3.6). This is the main part of the proof. 2. The latter estimate can be upgraded to a Picard iteration in a, for the most part, standard way. 3. We show that the linear step in such an iteration is well-defined, by deriving the adjoint boundary value problem, to which the usual duality argument applies.

Step 1: A priori energy estimates for k. Let g_{ij}, k_{ij}, Φ be a solution to the reduced system (2.2),(2.7),(2.18), such that (3.6) holds. Taking the time derivative of $E_k(t)$, integrating by parts using (3.2), and plugging in the wave equations (2.35), (2.37), yields in the following energy inequality:

$$\begin{aligned}
\frac{1}{2}\partial_t E_k(t) \leq & \sum_{r=0}^{s} \int_{\partial\tilde{\Sigma}_t} \Phi^2 \big\{ N\underline{\partial}^r k_C{}^C \partial_t \underline{\partial}^r k_C{}^C + 4g^{AB} N\underline{\partial}^r k_{NA} \partial_t \underline{\partial}^r k_{NB} + N\underline{\partial}^r k_{NN} \partial_t \underline{\partial}^r k_{NN} \big\} \mathrm{vol}_{\partial\tilde{\Sigma}_t} \\
& + CC_0 E_k(t) + \sum_{r=0}^{s} \int_{\tilde{\Sigma}_t} \partial_t \underline{\partial}^r k \Big[\underline{\partial}^r \{ \Phi^{-1}\partial_t \Phi \partial^2 \Phi + \Phi^{-1} \partial g \partial_t \Phi \partial \Phi + \partial^2 \partial_t \Phi + \partial g \partial \partial_t \Phi + k \partial \Phi \partial \Phi \\
& + \Phi \partial k \partial \Phi + \Phi k \partial_t k + \Phi k^3 + \Phi^{-1} \partial_t \Phi \partial k + \Phi k \partial^2 \Phi + \Phi k \partial g \partial \Phi + \Phi^2 k \partial^2 g + \Phi^2 k \partial g \partial g + \Phi^2 \partial g \partial k \} \\
& + \sum_{r_1+r_2+r_3 \leq r,\, r_3 < r} \{ \underline{\partial}^{r_1}(\Phi^2) \underline{\partial}^{r_2} g \underline{\partial}^{r_3} \partial^2 k + \underline{\partial}^{r_1}(\Phi^2) \underline{\partial}^{r_2} \partial g \underline{\partial}^{r_3} \partial k \} \\
& + \Phi^2 \{ \partial^2 \underline{\partial}^r f + \partial g \partial \underline{\partial}^r f - \Phi^{-2} \partial_t^2 \underline{\partial}^r f \} \Big] \mathrm{vol}_{\tilde{\Sigma}_t},
\end{aligned} \tag{3.30}$$

where the first term in the second line comes from when ∂_t hits $g, \Phi, \mathrm{vol}_{\tilde{\Sigma}_t}$ in (3.4), after using the L^∞ estimate in (3.7) and the bootstrap assumption (3.6). Also, for brevity, we have written the bulk term coming from plugging in the wave equations (2.35), (2.37) in a schematic form.

Notice that apart from the term $\partial_t \underline{\partial}^r k \underline{\partial}^r (\Phi^2 k \partial^2 g)$, for $r = s$, when all derivatives hit $\partial^2 g$ and they are all spatial, the rest of the terms in the bulk can be controlled by employing the Sobolev inequalities (3.7), the bootstrap assumption (3.6), the estimates in Lemmas 3.1, 3.2, and Young's inequality. Hence, we deduce from (3.30) the energy estimate:

$$\begin{aligned}
\frac{1}{2}\partial_t E_k(t) \leq & \sum_{r=0}^{s} \int_{\partial\tilde{\Sigma}_t} \Phi^2 \big\{ N\underline{\partial}^r k_C{}^C \partial_t \underline{\partial}^r k_C{}^C + 4g^{AB} N\underline{\partial}^r k_{NA} \partial_t \underline{\partial}^r k_{NB} + N\underline{\partial}^r k_{NN} \partial_t \underline{\partial}^r k_{NN} \big\} \mathrm{vol}_{\partial\tilde{\Sigma}_t} \\
& + C_* + \int_{\tilde{\Sigma}_t} \Phi^2 \partial_t \underline{\partial}^s k \underline{\partial}^s \partial^2 g \mathrm{vol}_{\tilde{\Sigma}_t},
\end{aligned} \tag{3.31}$$

where C_* depends on C_0 and $\sup_{\tau \in [0,t]} \|\partial_t^r f_A{}^B\|_{H^{s+2-r}(\tilde{\Sigma}_\tau)}$, $A, B = 1, 2$, $r = 0, \dots, s+2$.

To handle the last term in (3.31) we integrate by parts one spatial tangential derivative $\underline{\partial} = \partial_1, \partial_2$ and then pull out ∂_t. Since the integral is relative to the intrinsic volume, integrating by parts generates an extra term containing the divergence of $\underline{\partial}$ and pulling out ∂_t can hit the volume form, however, these are at the level of $\partial g, k$ which are controlled in L^∞:

$$\begin{aligned}
\int_{\tilde{\Sigma}_t} \Phi^2 \partial_t \underline{\partial}^s k \underline{\partial}^s \partial^2 g \mathrm{vol}_{\tilde{\Sigma}_t} =& - \int_{\tilde{\Sigma}_t} \Phi^2 \partial_t \underline{\partial}^{s+1} k \underline{\partial}^{s-1} \partial^2 g \mathrm{vol}_{\tilde{\Sigma}_t} - \int_{\tilde{\Sigma}_t} \partial_t \underline{\partial}^s k \underline{\partial}^{s-1} \partial^2 g [\underline{\partial}(\Phi^2) + \partial g] \mathrm{vol}_{\tilde{\Sigma}_t} \\
\leq & - \partial_t \int_{\tilde{\Sigma}_t} \Phi^2 \underline{\partial}^{s+1} k \underline{\partial}^{s-1} \partial^2 g \mathrm{vol}_{\tilde{\Sigma}_t} + \int_{\tilde{\Sigma}_t} \Phi^2 \underline{\partial}^{s+1} k \partial_t \underline{\partial}^{s-1} \partial^2 g \mathrm{vol}_{\tilde{\Sigma}_t} + CC_0^3 \\
\leq & - \partial_t \int_{\tilde{\Sigma}_t} \Phi^2 \underline{\partial}^{s+1} k \underline{\partial}^{s-1} \partial^2 g \mathrm{vol}_{\tilde{\Sigma}_t} + C_*
\end{aligned} \tag{3.32}$$

where in the last inequality we employed the top order estimate (3.19) for g, together with the bootstrap assumption.

Plugging (3.32) into (3.31), integrating in $[0,t]$, $t \leq T_0$, using Young's inequality and the lower order estimate (3.17), we deduce

$$\begin{aligned}
\frac{1}{2}E_k(t) \leq & \sum_{r=0}^{s} \int_0^t \int_{\partial\tilde{\Sigma}_\tau} \Phi^2 \big\{ N\underline{\partial}^r k_C{}^C \partial_\tau \underline{\partial}^r k_C{}^C + 4g^{AB} N\underline{\partial}^r k_{NA} \partial_\tau \underline{\partial}^r k_{NB} + N\underline{\partial}^r k_{NN} \partial_\tau \underline{\partial}^r k_{NN} \big\} \mathrm{vol}_{\partial\tilde{\Sigma}_\tau} \\
& + \frac{1}{2}E_k(0) + tC_* + CE_{total}(0) + \varepsilon E_k(t) + \frac{C}{\varepsilon}(C_{init} e^{tC_*} + tC_*),
\end{aligned} \tag{3.33}$$

for all $t\in[0,T_0]$. Here, it is important for the overall estimate below that C, C_{init} are independent of C_0.

Therefore, it remains to treat the boundary terms in (3.33). First, we simplify the terms in the brackets using conditions (2.27), (2.41), (2.42) (which hold on $\partial\tilde{\Sigma}_t$ by assumption):

$$\begin{aligned}
&N\underline{\partial}^r k_C{}^C\partial_t\underline{\partial}^r k_C{}^C+4g^{AB}N\underline{\partial}^r k_{NA}\partial_t\underline{\partial}^r k_{NB}+N\underline{\partial}^r k_{NN}\partial_t\underline{\partial}^r k_{NN}\\
=&\,\partial_t\underline{\partial}^r k_C{}^C(N\underline{\partial}^r k_C{}^C-N\underline{\partial}^r k_{NN})+4g^{AB}\partial_t\underline{\partial}^r k_{NB}\Big[-\frac{1}{2}\partial_A\underline{\partial}^r k_C{}^C+\underline{\partial}^r(k\partial g+\partial f)_A\\
&+\sum_{r_1+r_2\le r,\,r_2<r}(\underline{\partial}^{r_1}g\underline{\partial}^{r_2}\partial k)_A\Big]\\
=&\,2\partial_t\underline{\partial}^r k_C{}^C\partial_A\underline{\partial}^r k_N{}^A-2g^{AB}\partial_t\underline{\partial}^r k_{NB}\partial_A\underline{\partial}^r k_C{}^C+\partial_t\underline{\partial}^r k\Big[\underline{\partial}^r(k\partial g+\partial f)+\sum_{r_1+r_2\le r,\,r_2<r}(\underline{\partial}^{r_1}g\underline{\partial}^{r_2}\partial k)\Big]\\
=&-2\partial_A(\partial_t\underline{\partial}^r k_N{}^A\underline{\partial}^r k_C{}^C)+2\partial_t(\partial_A\underline{\partial}^r k_N{}^A\underline{\partial}^r k_C{}^C)+\sum_{\underline{\partial}^{r_1}\underline{\partial}^{r_2}=\partial_t\underline{\partial}^r,\,r_2<r+1}\underline{\partial}^{r_1}g\underline{\partial}^{r_2}k\partial_A\underline{\partial}^r k\\
&+\partial_t\underline{\partial}^r k\Big[\underline{\partial}^r(k\partial g+\partial f)+\sum_{r_1+r_2\le r,\,r_2<r}(\underline{\partial}^{r_1}g\underline{\partial}^{r_2}\partial k)\Big]
\end{aligned}\tag{3.34}$$

Plugging (3.34) into the boundary terms in (3.33) and integrating by parts in ∂_A,∂_t gives:

$$\begin{aligned}
&\sum_{r=0}^s\int_0^t\int_{\partial\tilde{\Sigma}_\tau}\Phi^2\big\{N\underline{\partial}^r k_C{}^C\partial_\tau\underline{\partial}^r k_C{}^C+4g^{AB}N\underline{\partial}^r k_{NA}\partial_\tau\underline{\partial}^r k_{NB}+N\underline{\partial}^r k_{NN}\partial_\tau\underline{\partial}^r k_{NN}\big\}\mathrm{vol}_{\partial\tilde{\Sigma}_\tau}\\
=&\sum_{r=0}^s\int_{\partial\tilde{\Sigma}_\tau}2\Phi^2\partial_A\underline{\partial}^r k_N{}^A\underline{\partial}^r k_C{}^C\mathrm{vol}_{\partial\tilde{\Sigma}_\tau}\Big|_0^t\\
&+\sum_{r=0}^s\int_0^t\int_{\partial\tilde{\Sigma}_\tau}\Big[(\Phi\underline{\partial}\Phi+\Phi^3k+\Phi^2\partial g)\underline{\partial}^{r+1}k\underline{\partial}^r k+\sum_{\underline{\partial}^{r_1}\underline{\partial}^{r_2}=\partial_\tau\underline{\partial}^r,\,r_2<r+1}\Phi^2\underline{\partial}^{r_1}g\underline{\partial}^{r_2}k\partial_A\underline{\partial}^r k\\
&+\Phi^2\partial_\tau\underline{\partial}^r k\big\{\underline{\partial}^r(k\partial g+\partial f)+\sum_{r_1+r_2\le r,\,r_2<r}(\underline{\partial}^{r_1}g\underline{\partial}^{r_2}\partial k)\big\}\Big]\mathrm{vol}_{\partial\tilde{\Sigma}_\tau}
\end{aligned}\tag{3.35}$$

The first boundary term in the RHS of (3.35) needs careful handling to prove that it can be absorbed in the LHS of the final energy estimate. We are more flexible with the bulk of boundary terms in the last two lines. Indeed, using the identity

$$\int_{\partial\tilde{\Sigma}_t}u\mathrm{vol}_{\partial\tilde{\Sigma}_t}=\int_{\tilde{\Sigma}_t}Nu+(\mathrm{div}N)u\mathrm{vol}_{\tilde{\Sigma}_t},\tag{3.36}$$

the L^∞ estimate in (3.7), the bootstrap assumption (3.6), Lemma 3.1, the top order estimate (3.19) for g, we bound all terms apart from those with k factors containing $s+2$ derivatives. In these we integrate by parts one tangential derivative and reapply the previous estimates:

$$\begin{aligned}
&\sum_{r=0}^s\int_0^t\int_{\partial\tilde{\Sigma}_\tau}\Big[(\Phi\underline{\partial}\Phi+\Phi^3k+\Phi^2\partial g)\underline{\partial}^{r+1}k\underline{\partial}^r k+\sum_{\underline{\partial}^{r_1}\underline{\partial}^{r_2}=\partial_\tau\underline{\partial}^r,\,r_2<r+1}\Phi^2\underline{\partial}^{r_1}g\underline{\partial}^{r_2}k\partial_A\underline{\partial}^r k\\
&+\Phi^2\partial_\tau\underline{\partial}^r k\big\{\underline{\partial}^r(k\partial g+\partial f)+\sum_{r_1+r_2\le r,\,r_2<r}(\underline{\partial}^{r_1}g\underline{\partial}^{r_2}\partial k)\big\}\Big]\mathrm{vol}_{\partial\tilde{\Sigma}_\tau}\\
\le&\, tC_*+\int_0^t\int_{\tilde{\Sigma}_\tau}\Big[(\Phi\underline{\partial}\Phi+\Phi^3k+\Phi^2\partial g)(N\underline{\partial}^{s+1}k)\underline{\partial}^s k+\sum_{\underline{\partial}^{r_1}\underline{\partial}^{r_2}=\partial_\tau\underline{\partial}^s,\,r_2<s+1}\Phi^2\underline{\partial}^{r_1}g\underline{\partial}^{r_2}k(N\partial_A\underline{\partial}^s k)\\
&+\Phi^2(N\partial_\tau\underline{\partial}^s k)\big\{\underline{\partial}^s(k\partial g+\partial f)+\sum_{r_1+r_2\le s,\,r_2<s}(\underline{\partial}^{r_1}g\underline{\partial}^{r_2}\partial k)\big\}\Big]\mathrm{vol}_{\tilde{\Sigma}_\tau}\\
\le&\, CE_{total}(0)+\varepsilon E_k(t)+\frac{C}{\varepsilon}(C_{init}e^{tC_*}+tC_*)\qquad\text{(after commuting } N,\underline{\partial}\text{ and IBP in }\underline{\partial})
\end{aligned}\tag{3.37}$$

where C_* depends on C_0 and $\sup_{\tau\in[0,t]}\|\partial_\tau^r f_A{}^B\|_{H^{s+2-r}(\tilde{\Sigma}_\tau)}$, $A,B=1,2$, $r=0,\dots,s+2$. Note that the initial energy comes up only in the case where $\underline{\partial}^{s+1}=\partial_t^{s+1}$ after integrating by parts.

For the more delicate first boundary term in the RHS of (3.35), before employing (3.36), we split the term using the boundary condition (2.27), otherwise it will be non-absorbable in the LHS below (see Remark 3.4). In addition, we make use of the estimate (3.18) to control any less than top energies of $k_i{}^j$:

$$\sum_{r=0}^s\int_{\partial\tilde{\Sigma}_\tau}2\Phi^2\partial_A\underline{\partial}^r k_N{}^A\underline{\partial}^r k_C{}^C\mathrm{vol}_{\partial\tilde{\Sigma}_\tau}\Big|_0^t$$

$$=\sum_{r=0}^{s}\int_{\partial\tilde{\Sigma}_\tau}\Phi^2\{\partial_A\underline{\partial}^r k_N{}^A\underline{\partial}^r k_C{}^C-\partial_A\underline{\partial}^r k_N{}^A\underline{\partial}^r k_{NN}\}\mathrm{vol}_{\partial\tilde{\Sigma}_\tau}\Big|_0^t \tag{3.38}$$

$$\leq\sum_{r=0}^{s}\int_{\tilde{\Sigma}_\tau}\Phi^2\{(N\partial_A\underline{\partial}^r k_N{}^A)\underline{\partial}^r k_C{}^C+\partial_A\underline{\partial}^r k_N{}^A(N\underline{\partial}^r k_C{}^C)-(N\partial_A\underline{\partial}^r k_N{}^A)\underline{\partial}^r k_{NN} \qquad \text{(by (3.36))}$$

$$-\partial_A\underline{\partial}^r k_N{}^A(N\underline{\partial}^r k_{NN})\}\mathrm{vol}_{\tilde{\Sigma}_\tau}\Big|_0^t+\varepsilon E_k(t)+\frac{C}{\varepsilon}(C_{init}e^{tC_*}+tC_*) \tag{3.39}$$

$$\leq\int_{\tilde{\Sigma}_t}\Phi^2\{-(N\underline{\partial}^s k_N{}^A)\partial_A\underline{\partial}^s k_C{}^C+\partial_A\underline{\partial}^s k_N{}^A(N\underline{\partial}^s k_C{}^C)+(N\underline{\partial}^s k_N{}^A)\partial_A\underline{\partial}^s k_{NN} \qquad \text{(IBP in } \partial_A)$$

$$-\partial_A\underline{\partial}^s k_N{}^A(N\underline{\partial}^s k_{NN})\}\mathrm{vol}_{\tilde{\Sigma}_t}+CE_{total}(0)+\varepsilon E_k(t)+\frac{C}{\varepsilon}(C_{init}e^{tC_*}+tC_*)$$

$$\leq\int_{\tilde{\Sigma}_t}\Phi^2\{2\eta(g^{AB}\partial^i\underline{\partial}^s k_{NA}\partial_i\underline{\partial}^s k_{NB})+\frac{1}{4\eta}|\nabla\underline{\partial}^s k_C{}^C|^2 \qquad \text{(by Young's inequality } ab\leq\eta a^2+\tfrac{1}{4\eta}b^2)$$

$$+\frac{1}{4\eta}|\nabla\underline{\partial}^s k_{NN}|^2\}\mathrm{vol}_{\tilde{\Sigma}_t}+CE_{total}(0)+\varepsilon E_k(t)+\frac{C}{\varepsilon}(C_{init}e^{tC_*}+tC_*)$$

where C, C_{init} are independent of C_0 and $\eta>0$ is chosen below.

Combining (3.33), (3.35)-(3.38), we obtain the inequality:

$$\frac{1}{2}E_k(t)\leq\frac{1}{2}E_k(0)+tC_*+CE_{total}(0)+\varepsilon E_k(t)+\frac{C}{\varepsilon}(C_{init}e^{tC_*}+tC_*)$$
$$+\int_{\tilde{\Sigma}_t}\Phi^2\{2\eta(g^{AB}\partial^i\underline{\partial}^s k_{NA}\partial^i\underline{\partial}^s k_{NB})+\frac{1}{4\eta}|\nabla\underline{\partial}^s k_C{}^C|^2+\frac{1}{4\eta}|\nabla\underline{\partial}^s k_{NN}|^2\}\mathrm{vol}_{\tilde{\Sigma}_t}, \tag{3.40}$$

for all $t\in[0,T_0]$. Recall the definition (3.4) of $E_k(t)$. Setting $\eta=\frac{3}{4}$ and taking ε sufficiently small, we observe that $\varepsilon E_k(t)$ and the terms in the last line can be absorbed in the LHS, giving

$$E_k(t)\leq C(C_{init}e^{tC_*}+tC_*), \tag{3.41}$$

where C, C_{init} are independent of C_0, while C_* depends on C_0 and $\sup_{\tau\in[0,t]}\|\partial_t^r f_A{}^B\|_{H^{s+2-r}(\tilde{\Sigma}_\tau)}$, $A,B=1,2$, $r=0,\dots,s+2$. In view of the estimate (3.17) and the definition (3.3), the same type of bounds holds for the total energy

$$E_{total}(t)\leq C(C_{init}e^{tC_*}+tC_*), \tag{3.42}$$

for all $t\in[0,T_0]$.

Remark 3.4. We would like to emphasize the somehow surprising appearance of the boundary terms (3.38) and their seemingly delicate nature. If we had not split up the terms using (2.27), then the terms in the second line of (3.40) would be bounded by $\max\{\eta,\frac{1}{\eta}\}E_k(t)\geq E_k(t)$, which would render them barely non-absorbable in the LHS of (3.40).

Step 2: A picard iteration scheme. Consider a sequence of iterates g^n, k^n, Φ^n, $n\in\mathbb{N}$, defined over $\{\tilde{\Sigma}_t\}_{t\in[0,T_0]}$, with k^0, g^0, Φ^0 equal to their initial values everywhere, satisfying the following linear system of equations:

$$\begin{aligned}\partial_t g_{ij}^{n+1}&=-2\Phi^n k_{ij}^n,\\ \partial_t^2-(\Phi^n)^2\Delta_{g^n}(k^{n+1})_i{}^j&=(F^n)_i{}^j,\\ \Delta_{g^n}\Phi^{n+1}&=|k^n|^2\Phi^n,\end{aligned} \tag{3.43}$$

where $(F^n)_i{}^j$ is equal to the RHS of (2.34) with all n-th iterates, with the same initial data as the original variables (see Section 2.2), and subject to the boundary conditions

$$\begin{aligned}\Phi^{n+1}=1,\qquad (\widetilde{k}^{n+1})_A{}^B&=\mathrm{tr}_{g^n}k^{n+1}=0,\\ \nabla_N k_{NA}^{n+1}=-\nabla_B(k^{n+1})_A{}^B&=-\nabla_B(\widehat{k}^{n+1})_A{}^B-\frac{1}{2}\nabla_A(k^{n+1})_C{}^C, \qquad \text{on } \{\partial\tilde{\Sigma}_t\}_{t\in[0,T_0]}\\ \frac{1}{2}[\nabla_N(k^{n+1})_{NN}-\nabla_N(k^{n+1})_A{}^A]&=-\nabla_A(k^{n+1})_N{}^A,\end{aligned} \tag{3.44}$$

where ∇, N here are the Levi-Civita connection of g^n, the outward g^n-normal to the boundary $\partial\tilde{\Sigma}_t$, and $(\widetilde{k}^{n+1})_A{}^B$ is defined analogously to (2.36).

Assume that the variables $\partial_t^r g^n, \partial_t^r k^n\in H^{s+1-r}(\tilde{\Sigma}_t)$, $\partial_t^r\Phi^n\in H^{s+2-r}(\tilde{\Sigma}_t)$, $r=0,\dots,s+1$, satisfy (3.6) for some fixed $s\geq3$, with C_0, T_0 to be chosen appropriately, where $\mathrm{vol}_{\tilde{\Sigma}_t}, \nabla, \Phi, g^{AB}, g^{ij}$ that appear in the energies (3.3)-(3.5) are defined relative to the $n-1$ iterates, $n\geq1$. Then by re-deriving the estimates in Lemmas 3.1, 3.2, and those for k derived above, we infer that $g^{n+1}, k^{n+1}, \Phi^{n+1}$ satisfy (3.42) for their corresponding total energy. Now we choose C_0 sufficiently large and T_0 sufficiently small to begin with, such that

$$C(C_{init}e^{tC_*}+tC*)\leq C_0. \tag{3.45}$$

This is possible, since C, C_{init} are independent of C_0. Note that they are also independent of n, therefore, C_0, T_0 are uniform in n. We conclude by induction that the iterates have uniformly bounded total energy. A contraction mapping can then be proved in the same energy spaces, by subtracting (3.43) from the analogous system satisfied by g^n, k^n, Φ^n, and re-deriving essentially the same estimates as above for $g^{n+1}-g^n, k^{n+1}-k^n, \Phi^{n+1}-\Phi^n$, using the uniform boundedness of the total energy of the iterates to control the coefficients of the resulting system for the differences. Thus, the sequence g^n, k^n, Φ^n converges in a strong sense, sufficient to yield a classical solution to the non-linear system (2.2), (2.18), (2.34), subject to the boundary conditions (2.21), (2.26), (2.27), (2.28), (2.29). We omit the details, since the argument is standard once the a priori energy estimates have been derived.

Step 3: The linear step in the iteration. Finally, we have to show that the above sequence of iterates is well-defined, that is, a solution to the linear system (3.43), subjected to the conditions (3.44), actually exists in the relevant spaces. The first and third equations in (3.43) are trivially solved, since one is an ODE in t and the solution to the other is given by the standard Dirichlet problem. Existence of a solution to the system of wave equations for the components of k^{n+1} is established by a duality argument. This is based on the a priori estimates that we have already derived, and a study of the adjoint problem. The argument would be standard, if it were not for the rather involved boundary conditions (3.44). We will show that the adjoint system has a similar boundary value problem, for which a priori estimates can be derived along the same lines as the ones in Step 1. Then a solution is furnished by the Riesz representation theorem, see [13, §5.2.2].

Since the equations for $(\widetilde{k}^{n+1})_A{}^B, \mathrm{tr}_{g^n}k^{n+1}$ decouple from those for $(k^{n+1})_C{}^C - k^{n+1}_{NN}, k^{n+1}_{NA}$, and they are subject to homogeneous Dirichlet boundary conditions, existence for the former is standard. We proceed to derive the dual boundary conditions for the adjoint system satisfied by $(k^{n+1})_C{}^C - k^{n+1}_{NN}, k^{n+1}_{NA}$, which we write, for convenience, in covariant form:

$$\begin{aligned}(e_0^2-\Delta_{g^n})[(k^{n+1})_C{}^C - k^{n+1}_{NN}] &= \mathcal{R}^n,\\ (e_0^2-\Delta_{g^n})(k^{n+1})_{NA} &= \mathcal{R}^n_A,\end{aligned} \tag{3.46}$$

where $e_0 = (\Phi^n)^{-1}\partial_t$. Consider the test functions

$$v_i{}^j \in C^\infty_c\big([0,T_0)\times\tilde{\Sigma}_t\big),\ i,j=1,2,N, \qquad v_C{}^C = -v_{NN} \text{ on } \{\partial\tilde{\Sigma}_t\}_{t\in[0,T_0]}. \tag{3.47}$$

Note that $\tilde{\Sigma}_t$ is compact, hence, the compact support of the test functions is equivalent to the vanishing of $v_i{}^j$ at $t=T_0$. Multiplying (3.46) with $v_C{}^C - v_{NN}, 8v_N{}^A$, integrating in $\{\tilde{\Sigma}_t\}_{t\in[0,T_0]}$, and integrating by parts in ∂_t, ∇ we have:

$$\begin{aligned}&\int_0^{T_0}\int_{\tilde{\Sigma}_t}(v_C{}^C - v_{NN})\mathcal{R}^n + 8v_N{}^A\mathcal{R}^n_A \mathrm{vol}_{\tilde{\Sigma}_t}dt\\ &=\int_0^{T_0}\int_{\tilde{\Sigma}_t}(v_C{}^C - v_{NN})(e_0^2-\Delta_{g^n})[(k^{n+1})_C{}^C - k^{n+1}_{NN}] + 8v_{NA}(e_0^2-\Delta_{g^n})(k^{n+1})_N{}^A \mathrm{vol}_{\tilde{\Sigma}_t}dt\\ &=\int_0^{T_0}\int_{\tilde{\Sigma}_t}\Big[[(k^{n+1})_C{}^C - k^{n+1}_{NN}](e_0^2-\Delta_{g^n})(v_C{}^C - v_{NN}) + 8k^{n+1}_{NA}(e_0^2-\Delta_{g^n})v_N{}^A\\ &\quad+\mathcal{N}(v,k,\partial_t k,\Phi^n,\partial_t\Phi^n)\Big]\mathrm{vol}_{\tilde{\Sigma}_t}dt + \int_0^{T_0}\int_{\partial\tilde{\Sigma}_t}\Big[[(k^{n+1})_C{}^C - k^{n+1}_{NN}]\nabla_N(v_C{}^C - v_{NN})\\ &\quad-(v_C{}^C - v_{NN})\nabla_N[(k^{n+1})_C{}^C - k^{n+1}_{NN}] + 8k^{n+1}_{NA}\nabla_N v_N{}^A - 8v_N{}^A\nabla_N k^{n+1}_{NA}\Big]\mathrm{vol}_{\partial\tilde{\Sigma}_t}dt,\end{aligned} \tag{3.48}$$

where $\mathcal{N}(v,k,\partial_t k,\Phi^n,\partial_t\Phi^n)$ contains the terms generated when ∂_t hits $\mathrm{vol}_{\tilde{\Sigma}_t}, \Phi^n$. The adjoint system of equations for $v_C{}^C - v_{NN}, v_{NA}$ can be read from (3.48), by setting the spacetime integral equal to zero. Evidently, it is of the form (3.46) and the dual boundary conditions are derived by setting the boundary integral in (3.48) equal to zero:

$$\begin{aligned}0 &= \int_0^{T_0}\int_{\partial\tilde{\Sigma}_t}\Big[[(k^{n+1})_C{}^C - k^{n+1}_{NN}]\nabla_N(v_C{}^C - v_{NN}) - (v_C{}^C - v_{NN})\nabla_N[(k^{n+1})_C{}^C - k^{n+1}_{NN}]\\ &\quad+8k^{n+1}_{NA}\nabla_N v_N{}^A - 8v_N{}^A\nabla_N k^{n+1}_{NA}\Big]\mathrm{vol}_{\partial\tilde{\Sigma}_t}dt\\ &=\int_0^{T_0}\int_{\partial\tilde{\Sigma}_t}\Big[2(k^{n+1})_C{}^C\nabla_N(v_C{}^C - v_{NN}) - 2v_C{}^C 2\nabla^A k^{n+1}_{NA} \qquad (k^{n+1}_{NN} = -(k^{n+1})_C{}^C,\ v_{NN} = -v_C{}^C)\\ &\quad+8k^{n+1}_{NA}\nabla_N v_N{}^A + 8v_N{}^A[\nabla_B(\widehat{k}^{n+1})_A{}^B + \tfrac{1}{2}\nabla_A(k^{n+1})_C{}^C]\Big]\mathrm{vol}_{\partial\tilde{\Sigma}_t}dt \qquad (\text{by } (3.44))\\ &=\int_0^{T_0}\int_{\partial\tilde{\Sigma}_t}\Big[2(k^{n+1})_C{}^C(\nabla_N v_C{}^C - \nabla_N v_{NN} - 2\nabla_A v_N{}^A) \qquad (\text{IBP in } \nabla_A)\\ &\quad+4k^{n+1}_{NA}(2\nabla_N v_N{}^A + \nabla^A v_C{}^C) + 8v_N{}^A\partial_B f_A{}^B + v_N{}^A\partial g^n k^{n+1} + v_C{}^C\partial g^n k^{n+1}\Big]\mathrm{vol}_{\partial\tilde{\Sigma}_t}dt\end{aligned} \tag{3.49}$$

The last two terms are lower order and can be incorporated in the spacetime integral in (3.48) using the trace identity (3.36), modifying the adjoint system for $v_C{}^C - v_{NN}, v_{NA}$ accordingly. Thus, we conclude that for the resulting system, the boundary conditions are up to lower order terms the same as (3.44):

$$\nabla_N v_C{}^C - \nabla_N v_{NN} = 2\nabla^A v_{NA}, \qquad \nabla_N v_{NA} = -\frac{1}{2}\nabla_A v_C{}^C. \tag{3.50}$$

The energy estimates for the adjoint problem can therefore be derived as in Step 1. This proves existence for $(k^{n+1})_C{}^C - k^{n+1}_{NN}, k^{n+1}_{NA}$ and hence, for the overall linear system (3.43). □

4 Vanishing of the Ricci tensor: Solution to the EVE

In this section we show that a solution to the reduced system of equations (2.2), (2.7), (2.18), subject to the boundary conditions (2.21), (2.26), (2.27), (2.28), (2.29), on $\{\partial\tilde{\Sigma}_t\}_{t\in[0,T_0]}$ (see Figure 2), is an actual solution to (1.3), epxressed in the maximal gauge (1.4). By the localization procedure outlined in Section 3.1, this completes the proof Theorem 1.1.

4.1 The equations satisfied by the vanishing quantities

Let $\mathbf{g}$ be the metric of the form (2.1), where g, k, Φ is the solution furnished by Proposition 3.3. Then, according to Proposition 2.1, the spacetime Ricci tensor satisfies the propagation equation (2.6). Considering the Einstein tensor $G_{ij} = \mathbf{R}_{ij} - \frac{1}{2}g_{ij}\mathbf{R}$, (2.6) implies the equation:

$$e_0 G_{ij} = \nabla_i\mathcal{G}_j + \nabla_j\mathcal{G}_i - g_{ij}\nabla^a\mathcal{G}_a - \nabla_i\nabla_j\mathrm{tr}k + \frac{1}{2}g_{ij}\Delta_g\mathrm{tr}k + k_{ij}\mathbf{R} + \frac{1}{2}g_{ij}e_0\mathbf{R}_{00} - g_{ij}k^{ab}\mathbf{R}_{ab} \tag{4.1}$$

The above equation is coupled to the wave equation (2.19) for tr$k$:

$$e_0^2\mathrm{tr}k - \Delta_g\mathrm{tr}k = e_0[(\mathrm{tr}k)^2] + 4\Phi^{-1}(\nabla^a\Phi)\mathcal{G}_a \tag{4.2}$$

On the other hand, from (2.17) and (2.18) we also have

$$\mathbf{R}_{00} = e_0\mathrm{tr}k. \tag{4.3}$$

Combining (4.2)-(4.3), the equation (4.1) becomes

$$\begin{aligned} e_0 G_{ij} =& \nabla_i\mathcal{G}_j + \nabla_j\mathcal{G}_i - g_{ij}\nabla^a\mathcal{G}_a - \nabla_i\nabla_j\mathrm{tr}k + g_{ij}\Delta_g\mathrm{tr}k + k_{ij}\mathbf{R} - g_{ij}k^{ab}\mathbf{R}_{ab} \\ &+ \frac{1}{2}g_{ij}e_0[(\mathrm{tr}k)^2] + 2g_{ij}\Phi^{-1}(\nabla^a\Phi)\mathcal{G}_a, \end{aligned} \tag{4.4}$$

where we can also plug in

$$\begin{aligned} &g^{ab}G_{ab} = -\frac{1}{2}g^{ab}\mathbf{R}_{ab} + \frac{3}{2}\mathbf{R}_{00} \quad\Rightarrow\quad g^{ab}\mathbf{R}_{ab} = -2g^{ab}G_{ab} + 3e_0\mathrm{tr}k \\ \Rightarrow\quad &\mathbf{R} = -2g^{ab}G_{ab} + 2e_0\mathrm{tr}k, \qquad \mathbf{R}_{ij} = G_{ij} - g_{ij}g^{ab}G_{ab} + g_{ij}e_0\mathrm{tr}k \end{aligned} \tag{4.5}$$

to rewrite

$$\begin{aligned} e_0 G_{ij} =& \nabla_i\mathcal{G}_j + \nabla_j\mathcal{G}_i - g_{ij}\nabla^a\mathcal{G}_a - \nabla_i\nabla_j\mathrm{tr}k + g_{ij}\Delta_g\mathrm{tr}k + k_{ij}(2e_0\mathrm{tr}k - 2g^{ab}G_{ab}) \\ &- g_{ij}(k^{ab}G_{ab} - \mathrm{tr}k g^{ab}G_{ab}) + 2g_{ij}\Phi^{-1}(\nabla^a\Phi)\mathcal{G}_a, \end{aligned} \tag{4.6}$$

Next, we use the contracted second Bianchi identity to derive a propagation equation for $\mathcal{G}_i$:

$$\begin{aligned} e_0\mathcal{G}_i = e_0\mathbf{R}_{0i} &= D_0\mathbf{R}_{0i} + \Phi^{-1}\nabla^j\Phi\mathbf{R}_{ji} - k_i{}^j\mathbf{R}_{0j} + \Phi^{-1}\nabla_i\Phi\mathbf{R}_{00} \\ &= D_j\mathbf{R}_i{}^j - \frac{1}{2}\partial_i\mathbf{R} + \Phi^{-1}\nabla^j\Phi\mathbf{R}_{ji} - k_i{}^j\mathbf{R}_{0j} + \Phi^{-1}\nabla_i\Phi\mathbf{R}_{00} \\ &= \nabla_j\mathbf{R}_i{}^j - \frac{1}{2}\partial_i\mathbf{R} + \Phi^{-1}\nabla^j\Phi\mathbf{R}_{ji} - k_i{}^j\mathbf{R}_{0j} + \Phi^{-1}\nabla_i\Phi\mathbf{R}_{00} \\ &\quad + \mathrm{tr}k\mathbf{R}_{0i} + k_i{}^j\mathbf{R}_{0j} \\ &= \nabla_j G_i{}^j + \Phi^{-1}\nabla^j\Phi(G_{ij} - g_{ij}g^{ab}G_{ab} + g_{ij}e_0\mathrm{tr}k) + \mathrm{tr}k\mathcal{G}_i + \Phi^{-1}\nabla_i\Phi e_0\mathrm{tr}k \end{aligned} \tag{4.7}$$

Taking the e_0 derivative of (4.7) and plugging in (4.4), (2.6), (4.2), we derive the following wave equation for $\mathcal{G}_i$:

$$\begin{aligned} e_0^2\mathcal{G}_i - \Delta_g\mathcal{G}_i =& \nabla_j\big\{k_i{}^j(2e_0\mathrm{tr}k - 2g^{ab}G_{ab}) - \delta_i^j(k^{ab}G_{ab} - \mathrm{tr}k g^{ab}G_{ab}) + 2\delta_i^j\Phi^{-1}\nabla^a\Phi\,\mathcal{G}_a\big\} \\ &+ R_i{}^a(\mathcal{G}_a - \nabla_a\mathrm{tr}k) + (e_0\Gamma^j_{ja})G_i{}^a - (e_0\Gamma^a_{ji})G_a{}^j + \Phi^{-1}\nabla_j\Phi\, e_0 G_i{}^j + \nabla_j(2k^{ja}G_{ia}) \\ &+ e_0\big\{\Phi^{-1}\nabla^j\Phi(G_{ij} - g_{ij}g^{ab}G_{ab} + g_{ij}e_0\mathrm{tr}k) + \mathrm{tr}k\mathcal{G}_i + \Phi^{-1}\nabla_i\Phi e_0\mathrm{tr}k\big\} \end{aligned} \tag{4.8}$$

The equations (4.2), (4.6), (4.8) form a closed system of equations for the variables $G_{ij}, \mathcal{G}_i, \mathrm{tr}k$ that should vanish, for a solution to the EVE in the maximal gauge.

4.2 Boundary conditions induced by the solution to the reduced system

The conditions (2.27)-(2.29), combined with the Coddazi identity (2.15), imply the following boundary conditions on $\{\partial\tilde{\Sigma}\}_{t\in[0,T_0]}$:

$$\begin{aligned} \mathrm{tr}k=0, \qquad \mathcal{G}_A=\mathbf{R}_{0A}=\partial_A\mathrm{tr}k-\nabla^i k_{Ai}=\partial_A\mathrm{tr}k=0,\\ \mathcal{G}_N=\mathbf{R}_{0N}=\nabla_N\mathrm{tr}k-\nabla^i k_{Ni}=\nabla_N {k_A}^A-\nabla^A k_{NA}=\frac{1}{2}\nabla_N\mathrm{tr}k \end{aligned} \tag{4.9}$$

These conditions, together with (4.2), (4.6), (4.8), define a well-determined boundary value problem for $\mathrm{tr}k,\mathbf{G}_{ij},\mathcal{G}_i$.

4.3 Modified vanishing quantities and equations with homogeneous boundary conditions

In order to avoid losing derivatives in the energy estimates for the vanishing quantities below, we need to modify the $\mathcal{G}_i$'s, such that they all have homogeneous Dirichlet data on the boundary. For this purpose we set:

$$\tilde{\mathcal{G}}_i:=\mathcal{G}_i-\frac{1}{2}\nabla_i\mathrm{tr}k \tag{4.10}$$

Then (4.6) becomes

$$\begin{aligned} e_0G_{ij}=&\nabla_i\tilde{\mathcal{G}}_j+\nabla_j\tilde{\mathcal{G}}_i-g_{ij}\nabla^a\tilde{\mathcal{G}}_a+\frac{1}{2}g_{ij}\Delta_g\mathrm{tr}k+k_{ij}(2e_0\mathrm{tr}k-2g^{ab}G_{ab})\\ &-g_{ij}(k^{ab}G_{ab}-\mathrm{tr}k g^{ab}G_{ab})+2g_{ij}\Phi^{-1}(\nabla^a\Phi)(\tilde{\mathcal{G}}_a+\frac{1}{2}\nabla_a\mathrm{tr}k) \end{aligned} \tag{4.11}$$

On the other hand, $\tilde{\mathcal{G}}_i$ satisfies a wave equation of the same form as (4.8), since the additional terms generated in the LHS, after plugging in (4.10), equal

$$\begin{aligned} \frac{1}{2}e_0^2\nabla_i\mathrm{tr}k-\frac{1}{2}\Delta_g\nabla_i\mathrm{tr}k&=\frac{1}{2}\Phi^{-1}\partial_t(\Phi^{-1}\partial_t\partial_i\mathrm{tr}k)-\frac{1}{2}\nabla_j\nabla^j\nabla_i\mathrm{tr}k\\ &=\frac{1}{2}\nabla_i(e_0^2\mathrm{tr}k-\Delta_g\mathrm{tr}k)+\frac{1}{2}e_0(\Phi^{-1}\nabla_i\Phi e_0\mathrm{tr}k)+\frac{1}{2}\Phi^{-1}\nabla_i\Phi e_0^2\mathrm{tr}k+\frac{1}{2}{R_i}^a\nabla_a\mathrm{tr}k\\ &=\frac{1}{2}\nabla_i\big\{e_0[(\mathrm{tr}k)^2]+4\Phi^{-1}(\nabla^a\Phi)(\tilde{\mathcal{G}}_a+\frac{1}{2}\nabla_a\mathrm{tr}k)\big\}+\frac{1}{2}e_0(\Phi^{-1}\nabla_i\Phi e_0\mathrm{tr}k)\\ &\quad+\frac{1}{2}\Phi^{-1}\nabla_i\Phi e_0^2\mathrm{tr}k+\frac{1}{2}{R_i}^a\nabla_a\mathrm{tr}k \end{aligned} \tag{4.12}$$

Hence, we obtain an equation of the form

$$e_0^2\tilde{\mathcal{G}}_i-\Delta_g\tilde{\mathcal{G}}_i=L(\tilde{\mathcal{G}}_a,\nabla\tilde{\mathcal{G}}_a,G_{ab},\nabla G_{ab},\mathrm{tr}k,\nabla\mathrm{tr}k,e_0\mathrm{tr}k,e_0e_0\mathrm{tr}k,\nabla e_0\mathrm{tr}k,\nabla\nabla\mathrm{tr}k) \tag{4.13}$$

where $L(\tilde{\mathcal{G}}_a,\nabla\tilde{\mathcal{G}}_a,G_{ab},\nabla G_{ab},\mathrm{tr}k,\nabla\mathrm{tr}k,e_0\mathrm{tr}k,e_0e_0\mathrm{tr}k,\nabla e_0\mathrm{tr}k,\nabla\nabla\mathrm{tr}k)$ is treated as a linear expression in the depicted variables with C^1 coefficients. The advantage of working with $\tilde{\mathcal{G}}_i$ is that they all satisfy homogeneous Dirichlet boundary conditions:

$$\tilde{\mathcal{G}}_i:=\mathcal{G}_i-\frac{1}{2}\nabla_i\mathrm{tr}k=\mathbf{R}_{0i}-\frac{1}{2}\nabla_i\mathrm{tr}k\overset{(4.9)}{=}0, \qquad \text{on } \{\partial\tilde{\Sigma}_t\}_{t\in[0,T_0]}. \tag{4.14}$$

4.4 Energy estimates for the vanishing quantities

Recall that the initial data for g,k are such that the Einstein tensor, and $\mathrm{tr}k,\partial_t\mathrm{tr}k$ vanish initially on $\tilde{\Sigma}_0$, see (2.30)-(2.32) in Section 2.2. Moreover, from the Bianchi equation (4.7), we also infer the vanishing of

$$e_0\mathcal{G}_i=0, \quad \text{on } \tilde{\Sigma}_0. \tag{4.15}$$

In order to prove the vanishing of the Einstein tensor everywhere, together with the validity of the maximal gauge $\mathrm{tr}k=0$, we need to establish an energy estimate for the system (4.2), (4.11), (4.13), subject to the boundary conditions (4.14), $\mathrm{tr}k=0$ on $\{\tilde{\Sigma}_t\}_{t\in[0,T_0]}$.

Proposition 4.1. *Consider a solution to the initial boundary value problem for the reduced system of equations furnished by Proposition 3.3. Then the spacetime metric* $\mathbf{g}$ *given by* (2.1) *satisfies* (1.3)*, and the second fundamental form* k *of the* $\tilde{\Sigma}_t$ *hypersurfaces is traceless.*

Proof. The variables $\mathrm{tr}k,G_{ij},\tilde{\mathcal{G}}_i$ satisfy a homogeneous system of evolutions equations, with homogeneous Dirichlet boundary conditions and trivial initial data. We will infer their vanishing everywhere by propagating standard energy estimates. Note that the equations (4.2), (4.11), (4.13) can be viewed as a linear system in $\mathrm{tr}k,G_{ij},\tilde{\mathcal{G}}_i$ with bounded coefficients, since in Lemma 3.1 and Proposition 3.3 $s\geq3$, hence, controlling up to two mixed derivatives of g,k,Φ in L^∞. The energy estimates would be immediate, if it were not for the ∇G terms in (4.13), which could lead to a derivative loss, since (4.11) does not gain a spatial derivative. However, this issue can be resolved by integrating by parts in the energy estimates, both in ∇, ∂_t, similarly to how we treated the term (3.32) in the proof of Proposition 3.3.

The energy for $\tilde{\mathcal{G}}_i$ reads

$$E_{\tilde{\mathcal{G}}}(t) := \int_{\tilde{\Sigma}_t} g^{ij} e_0\tilde{\mathcal{G}}_i e_0\tilde{\mathcal{G}}_j + \nabla^j\tilde{\mathcal{G}}^i\nabla_j\tilde{\mathcal{G}}_i + \tilde{\mathcal{G}}^i\mathcal{G}_i \mathrm{vol}_{\tilde{\Sigma}_t} \tag{4.16}$$

Going back to (4.2), we notice that there is room to commute once with any tangential derivative $\underline{\partial} = \partial_t, \partial_1, \partial_2$. Thus, by a standard Gronwall argument, we can control the following energy of $\mathrm{tr}k$:

$$E_{\mathrm{tr}k}(t) := \int_{\tilde{\Sigma}_t} (e_0\underline{\partial}\mathrm{tr}k)^2 + (\nabla\underline{\partial}\mathrm{tr}k)^2 + (e_0\mathrm{tr}k)^2 + (\nabla\mathrm{tr}k)^2 + (\mathrm{tr}k)^2\mathrm{vol}_{\tilde{\Sigma}_t} \tag{4.17}$$

by

$$\sup_{t\in[0,T_0]} E_{\mathrm{tr}k}(t) \lesssim T_0 \sup_{t\in[0,T_0]} E_{\tilde{\mathcal{G}}}(t) \tag{4.18}$$

Then, using the wave equation (4.2), we can also control $\nabla_N\nabla_N\mathrm{tr}k$ in L^2, hence, controlling all second derivatives of $\mathrm{tr}k$:

$$\sup_{t\in[0,T_0]} \|\nabla_N\nabla_N\mathrm{tr}k\|^2_{L^2(\tilde{\Sigma}_t)} \lesssim T_0 \sup_{t\in[0,T_0]} E_{\tilde{\mathcal{G}}}(t) + \sum_i \sup_{t\in[0,T_0]} \|\tilde{\mathcal{G}}_i\|^2_{L^2(\tilde{\Sigma}_t)} \tag{4.19}$$

Note that although the last term in the preceding inequality has no smallness in T, we can directly bound it from its time derivative:

$$\partial_t\|\tilde{\mathcal{G}}_i\|^2_{L^2(\tilde{\Sigma}_t)} \lesssim \|\tilde{\mathcal{G}}_i\|^2_{L^2(\tilde{\Sigma}_t)} + \|\partial_t\tilde{\mathcal{G}}_i\|^2_{L^2(\tilde{\Sigma}_t)} \quad \overset{\mathcal{G}_i|_{\tilde{\Sigma}_0}=0}{\Longrightarrow} \quad \|\tilde{\mathcal{G}}_i\|^2_{L^2(\tilde{\Sigma}_t)} \lesssim T_0\|e_0\tilde{\mathcal{G}}_i\|^2_{L^2(\tilde{\Sigma}_t)} \tag{4.20}$$

On the other hand, we are forced to estimate G_{ij} only in L^2, due to the first order terms $\nabla_j\tilde{\mathcal{G}}_i$ in the RHS of (4.11). The standard energy estimate for (4.11), using (4.18)-(4.20), gives

$$\sum_{i,j=1}^{3} \|G_{ij}\|^2_{L^2(\tilde{\Sigma}_t)} \lesssim T_0 \sup_{t\in[0,T_0]} E_{\tilde{\mathcal{G}}}(t) \tag{4.21}$$

Finally, the standard energy estimate for (4.13) with boundary conditions (4.14), using (4.18)-(4.21), gives an inequality of the form:

$$E_{\tilde{\mathcal{G}}}(t) \leq CT_0 \sup_{t\in[0,T_0]} E_{\mathcal{G}}(t) + \left|\int_0^t\int_{\tilde{\Sigma}_\tau} k\nabla G\partial_\tau\tilde{\mathcal{G}}\mathrm{vol}_{\tilde{\Sigma}_\tau}d\tau\right|, \tag{4.22}$$

Integrating by parts in ∇, ∂_t we have:

$$\begin{aligned}\int_0^t\int_{\tilde{\Sigma}_\tau} k\nabla G\partial_\tau\tilde{\mathcal{G}}\mathrm{vol}_{\tilde{\Sigma}_\tau}d\tau &\overset{(4.14)}{=} -\int_0^t\int_{\tilde{\Sigma}_\tau}(kG\partial_\tau\nabla\tilde{\mathcal{G}} + k\partial_\tau\Gamma G\tilde{\mathcal{G}} + G\partial_\tau\tilde{\mathcal{G}}\nabla k)\mathrm{vol}_{\tilde{\Sigma}_\tau}d\tau \\ &= \int_0^t\int_{\tilde{\Sigma}_\tau}[k\partial_\tau G\nabla\tilde{\mathcal{G}} + (k^2+\partial_\tau k)G\nabla\tilde{\mathcal{G}} - k\partial_\tau\Gamma G\tilde{\mathcal{G}} - G\partial_\tau\tilde{\mathcal{G}}\nabla k]\mathrm{vol}_{\tilde{\Sigma}_\tau}d\tau \\ &\quad - \int_{\tilde{\Sigma}_t} kG\nabla\tilde{\mathcal{G}}\mathrm{vol}_{\tilde{\Sigma}_t}\end{aligned} \tag{4.23}$$

Plugging in (4.11), using Cauchy-Schwartz, the above estimates (4.18)-(4.20), and Young's inequality for the last term in the preceding RHS, we deduce the estimate

$$\left|\int_0^t\int_{\tilde{\Sigma}_\tau} k\nabla G\partial_\tau\tilde{\mathcal{G}}\mathrm{vol}_{\tilde{\Sigma}_\tau}d\tau\right| \leq \varepsilon\|\nabla\tilde{\mathcal{G}}\|^2_{L^2(\tilde{\Sigma}_t)} + \frac{C}{\varepsilon}T_0 \sup_{t\in[0,T_0]} E_{\tilde{\mathcal{G}}}(t) \leq (\varepsilon + \frac{C}{\varepsilon}T_0) \sup_{t\in[0,T_0]} E_{\tilde{\mathcal{G}}}(t) \tag{4.24}$$

Combining (4.22), (4.24), it follows that

$$\sup_{t\in[0,T_0]} E_{\tilde{\mathcal{G}}}(t) \leq (\varepsilon + \frac{C}{\varepsilon}T_0) \sup_{t\in[0,T_0]} E_{\tilde{\mathcal{G}}}(t), \tag{4.25}$$

for a different constant C. Thus, choosing ε, T_0 sufficiently small such that $\varepsilon + \frac{C}{\varepsilon}T_0 < 1$, we conclude that $\mathcal{G}_i$ vanishes everywhere, which implies by virtue of (4.18), (4.20), that $G_{ij}, \mathrm{tr}k$ are everywhere vanishing as well. Going back to the definition (4.10) and (4.3), we have $\mathbf{R}_{0i} = \mathbf{R}_{00} = 0$. Combined with the vanishing of $G_{ij} = \mathbf{R}_{ij} - \frac{1}{2}\mathbf{R}$, this shows that the solution to the reduced equations (2.2),(2.7),(2.18) is indeed a solution to the EVE, satisfying the maximal gauge. □

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