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On the Backward Stability of the Schwarzschild Black Hole Singularity

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Abstract: We study the backwards-in-time stability of the Schwarzschild singularity from a dynamical PDE point of view. More precisely, considering a spacelike hypersurface Σ_0 in the interior of the black hole region, tangent to the singular hypersurface $\{r = 0\}$ at a single sphere, we study the problem of perturbing the Schwarzschild data on Σ_0 and solving the Einstein vacuum equations backwards in time. We obtain a local backwards well-posedness result for small perturbations lying in certain weighted Sobolev spaces. No symmetry assumptions are imposed. The perturbed spacetimes all have a singularity at a "collapsed" sphere on Σ_0, where the leading asymptotics of the curvature and the metric match those of their Schwarzschild counterparts to a suitably high order. As in the Schwarzschild backward evolution, the pinched initial hypersurface Σ_0 'opens up' instantly, becoming a regular spacelike (cylindrical) hypersurface. This result thus yields classes of examples of non-symmetric vacuum spacetimes, evolving forward-in-time from regular initial data, which form a Schwarzschild type singularity at a collapsed sphere. We rely on a precise asymptotic analysis of the Schwarzschild geometry near the singularity which turns out to be at the threshold that our energy methods can handle.

Contents

1. Introduction

It is well-known (cf. Birkhoff's theorem [10]) that the only spherically symmetric solution $(\mathcal{M}^{1+3}, g)$ to the Einstein vacuum equations (EVE)

$$\mathrm{Ric}_{ab}(g) = 0, \tag{1.1}$$

is the celebrated Schwarzschild spacetime. It was in fact the first non-trivial solution to the Einstein field equations to be discovered [10]. In Kruskal (null) u, v coordinates the maximally extended metric reads

$$^{S}g = -\Omega^2 du dv + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1.2}$$

where $\Omega^2 = \frac{32M^3}{r}e^{-\frac{r}{2M}}$, $M > 0$, and the radius function r is given implicitly by

$$uv = (1 - \frac{r}{2M})e^{\frac{r}{2M}}. \tag{1.3}$$

Here the underlying manifold $^{S}\mathcal{M}^{1+3}$ is endowed with the differential structure of $\mathcal{U} \times \mathbb{S}^2$, where $\mathcal{U}$ is the open subset $\{uv < 1\}$ in the uv plane; see Fig. 1. The spacetime has an essential curvature singularity at $r = 0$, (the future component of) which is contained in the interior of the black hole region, the quadrant $u > 0, v > 0$. In fact, a short computation shows that the Gauss curvature of the uv-plane equals

$$^{S}\mathrm{K} = \frac{2M}{r^3} \tag{1.4}$$

and hence the manifold is C^2 inextendible past $r = 0$. An interesting feature of this singularity is its spacelike character, that is, it can be viewed as a spacelike hypersurface.

Yet another interesting feature of the Schwarzschild singularity is its unstable nature from the evolutionary dynamical point of view. To illustrate this, consider a global spacelike Cauchy hypersurface Σ^3 in Schwarzschild (Fig. 2). An initial data set for the

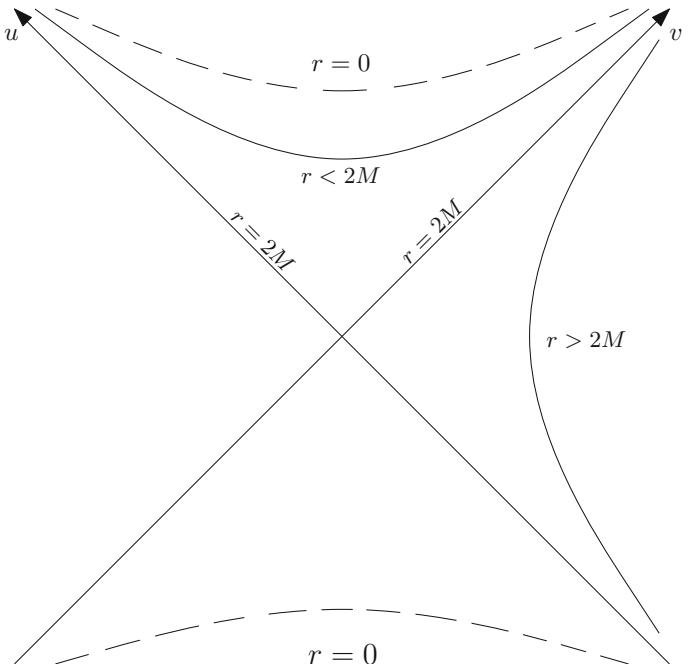

Fig. 1. The Kruskal plane

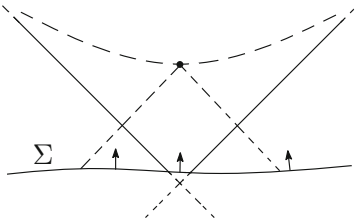

Fig. 2. The future development of Σ in Schwarzschild

EVE consists of a Riemannian metric $\overline{g}$ on Σ and a symmetric two tensor K verifying the constraint equations

$$\begin{cases} \overline{\nabla}^j K_{ij} - \overline{\nabla}_i \mathrm{tr}_{\overline{g}} K = 0 \\ \overline{\mathrm{R}} - |K|^2 + (\mathrm{tr}_{\overline{g}} K)^2 = 0 \end{cases}, \tag{1.5}$$

where $\overline{\nabla}$, $\overline{\mathrm{R}}$ are the covariant derivative and scalar curvature intrinsic to $\overline{g}$.

The instability of the Schwarzschild singularity (w.r.t. the forward Cauchy problem) can already be seen by examining the maximal developments of initial data sets on Σ arising from the celebrated Kerr [9] (explicit) 2-parameter $\mathcal{K}(a, M)$ family of solutions—of which Schwarzschild is a subfamily ($a = 0$). For $a \neq 0$ the singularity completely disappears and the corresponding (maximal) developments extend smoothly up to (and including) the Cauchy horizons. Moreover, taking $|a| \ll 1$, the 'difference' of the

corresponding initial data sets from the Schwarzschild one (with the same $M > 0$), measured in standard Sobolev norms,[1] can be made arbitrarily small.

In fact, the Schwarzschild singularity is conjecturally unstable under *generic* perturbations on Σ. According to a scenario proposed by Belinskiĭ et al. [3] originally formulated for cosmological singularities, in general, one should expect solutions to exhibit oscillatory behaviour towards the singularity. To our knowledge such behaviour has been rigorously studied only in the spatially homogeneous case for the Euler-Einstein system with Bianchi IX symmetry by Ringström [29]. Nonetheless, the heuristic work of [3] has received a lot of attention over the years, see [16,28] and the references therein (and [12] for related numerics). On the other hand, there is a growing expectation that, at least in a neighbourhood of subextremal Kerr, the dominant scenario inside the black hole is the formation of Cauchy horizons and (weak) null singularities. This has been supported by rigorous studies on spherically symmetric charged matter models, see works by Poisson and Israel [26], Ori [25] and recently by Dafermos [9].

However, it is not inferred from the existing literature whether the non-oscillatory type of singularity observed in Scwarzschild is an isolated phenomenon for the EVE in some neighbourhood of the Schwarzschild initial data on Σ or part of a larger family. A priori it is not clear what to expect, since one might argue that such a special singularity is a mathematical artefact due to spherical symmetry. Therefore, we pose the following question:

Is there a class of non-spherically symmetric Einstein vacuum spacetimes which develop a first singularity of Schwarzschild type?

The goal of the present paper is to answer the preceding question in the affirmative. A Schwarzschild type singularity here has the meaning of a first singularity in the vacuum development which has the same geometric blow up profile with Schwarzschild and which can be seen by a foliation of uniformly spacelike hypersurfaces; hence, not contained in a Cauchy horizon. We confine the question to the formation of one singular sphere in the vacuum development in the same manner as in Schwarzschild, where each point on the sphere can be understood as a distinct ideal singular point of the spacetime in the language of TIPs [13]. Ideally, one would like to study the forward problem and identify initial data for the EVE on Σ (Fig. 2) that lead to such singularities. Although this is a very interesting problem, we find it far beyond reach at the moment. Instead, we study the existence problem backwards-in-time.

More precisely, we adopt the following plan: Let Σ_0^3 be a spacelike hypersurface in Schwarzschild, tangent[2] at a single sphere of the singular hyperbola $r = 0$ inside the black hole; Fig. 3. We assume, without loss of generality,[3] that the tangent sphere is $(u = 1, v = 1)$ in Kruskal coordinates (1.2). Consider now initial data sets $(\overline{g}, K)$ on Σ_0 for the EVE (1.1), which have the same singular behaviour to leading order at $(u = 1, v = 1)$ with the induced Schwarzschild initial data set $({}^S\overline{g}, {}^SK)$ on Σ_0 and solve the EVE backwards, as depicted in the 2-dim Fig. 3, without symmetries.

Realizing the above plan we thus prove the existence of a class of non-spherically symmetric vacuum spacetimes for which (1) the leading asymptotics of the blow up of

[1] The difference can be defined, for example, component wise for the two pairs of 2-tensors with respect to a common coordinate system and measured in $W^{s,p}$ Sobolev spaces used in the literature [5].

[2] The tangency here should be understood with respect to the differential structure of the Kruskal maximal extension induced by the standard u, v, θ, ϕ coordinates (1.2).

[3] Recall that the vector field tangent to the $r = \text{const.}$ hypersurfaces (Fig. 1) is Killing and we may hence utilize it to shift Σ_0 and $(u = 1, v = 1)$ to whichever point on $\{uv = 1\}$ we wish; Fig. 3.

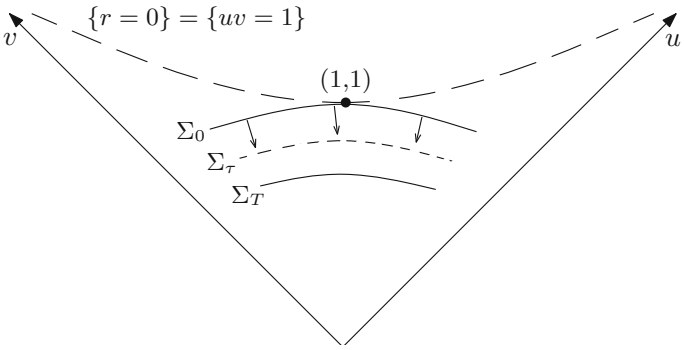

Fig. 3. The black hole region in Kruskal's extension

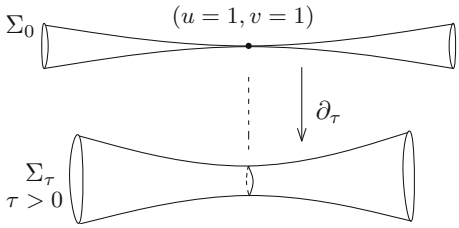

Fig. 4. The foliation $\{\Sigma_\tau\}$ in 3d

curvature and in general of all the geometric quantities (metric, second fundamental form, etc.) coincide with their Schwarzschild counterparts, as one approaches the singularity, and (2) the singularity is realized as the limit of uniformly spacelike hypersurfaces, which in the forward direction "pinch off" in finite time at one sphere. Conversely, we visualise the backward evolution of $(\Sigma_0, \overline{g}, K)$ in the following manner: At 'time' $\tau = 0$ the initial slice Σ_0 is a two ended spacelike (3-dim) hypersurface with a sphere singularity at $(u = 1, v = 1)$. Once Σ_0 evolves through (1.1), it becomes instantaneously a regular spacelike hypersurface Σ_τ, $\tau > 0$ and the singular *pinch* opens up; Fig. 4.

The main difficulty to overcome in the backward local existence problem is the singularity on Σ_0, which of course renders it beyond the scope of the classical local existence theorem for the Einstein equations [4], even its latest state of the art improvement by Klainerman et al. [18], which requires at the very least the curvature of the initial hypersurface to be in L^2. For the Schwarzschild initial data set $({}^S\overline{g}, {}^S K)$ on Σ_0, and hence for perturbed initial data sets $(\overline{g}, K)$ with the same leading order geometry at $(u = 1, v = 1)$, it is not hard to check (Sect. 3) that the initial curvature is at the singular level

$$\overline{\mathrm{R}} \notin L^p(\Sigma_0), \quad \nabla K \notin L^p(\Sigma_0) \qquad p \geq \frac{5}{4}. \tag{1.6}$$

Thus, we must rely heavily on the background Schwarzschild geometry to control the putative backward evolution. A very useful fact for analysis is the opening up (smoothing out) of the singularity (Fig. 4) in the backward direction.

To our knowledge, general local existence results, without symmetry assumptions, for the EVE (1.1) with singular initial curvature not in L^2 have been achieved only fairly recently by Luk-Rodnianski [20,21] and Luk [19] for the characteristic initial value

problem, where they consider delta curvature singularities and weak null singularities respectively. However, their context is much different from ours and the results do not seem applicable to singularities of Schwarzschild type.

We proceed now to formulate a first version of our main results; for more precise statements, in terms of weighted Sobolev spaces, see Theorems 4.6, 4.8, 6.7.

Theorem 1.1. *There exists $\alpha > 0$ sufficiently large, such that for every triplet $(\Sigma_0, \overline{g}, K)$ verifying:*

- (i) *the constraints* (1.5)*,*
- (ii) $\overline{g} = {}^S\overline{g} + r^{\alpha}\mathcal{O}$, $K = {}^SK + r^{\alpha-\frac{3}{2}}u$, *where $\mathcal{O}, u$ are 2-tensors on Σ_0 bounded in H^4, H^3 respectively,*
- (iii) $\|\overline{g} - {}^S\overline{g}\|_{L^\infty(\Sigma_0)} \ll 1$, *there exists a H^4 local solution g to the Einstein vacuum equations* (1.1) *with initial data $(\overline{g}, K)$, unique up to isometry, in the backward region to Σ_0, foliated by $\{\Sigma_\tau\}_{\tau\in[0,T]}$ (Fig. 3); the time of existence $T > 0$ depends continuously on the norms of $\mathcal{O}$, u and the exponent $\alpha > 0$.*

The fact that non-trivial initial data sets in compliance with Theorem 1.1 exist is not at all obvious nor standard. We need to show essentially that for any large parameter $\alpha > 0$, there exist non-spherically symmetric solutions to the constraint equations (1.5), having the asymptotics (ii). We construct such solutions using the *conformal method*, which we set up in Sect. 6.

Theorem 1.2. *Let $\alpha > 0$ be sufficiently large, consistent with Theorem 1.1. Then for every choice of the transverse, traceless part of the second fundamental form on Σ_0, compatible with the assumptions in Theorem 1.1, there exists a solution to the constraints* (1.5) *localized near the singular sphere and verifying the asymptotics* (ii) *above.*

Let us emphasize the fact that the above spacetimes are very special in that they agree with Schwarzschild at the singularity to a high (but finite) order—this is captured by the large exponent $\alpha > 0$ in Theorem 1.1—and therefore are *non-generic*. The need to choose α large may be seen however natural to some extent in view of the instability of the Schwarzschild singularity, from the point of view of the forwards-in-time problem. Indeed, the stable perturbations of the Schwarzschild singularity must form a strict subclass of all perturbed vacuum developments.

1.1. Method of proof and outline. The largest part of the paper is concerned with the evolutionary part of the problem, i.e., proving Theorem 1.1. Due to the singular nature of the backward existence problem described above, Fig. 3, the choice of framework must be carefully considered. The standard wave coordinates approach [4] does not seem to be feasible in our situation; one expects that coordinates would be highly degenerate at the singularity. Also, the widely used CMC gauge condition is not applicable, since the mean curvature of the initial hypersurface Σ_0 blows up (Sect. 3). Instead, we find it more suitable to use orthonormal frames and rewrite the EVE one order higher as a quasilinear Yang-Mills hyperbolic system of equations [18,24], under a Lorenz gauge condition,[4] for the corresponding connection 1-forms. We recall briefly this framework in Sect. 2.

[4] The analogue of a wave gauge for orthonormal frames.

However, even after expressing the EVE in the above framework, the singular level of initial configurations does not permit a direct energy estimate approach. In addition to (1.6), one can see (Sect. 3) that neither is the second fundamental form in L^2

$$K \notin L^2(\Sigma_0). \tag{1.7}$$

Note that the latter is at the level of one derivative in the metric. Hence, near the singularity the perturbed spacetimes we wish to construct do not even make sense as weak solutions of the EVE (1.1). Therefore, it is crucial that we use the background Schwarzschild spacetime to recast the evolution equations in a new form having more regular initial data. We do this in Sect. 4 by considering a new system of equations for the 'difference' between the putative perturbed spacetime and Schwarzschild. The resulting equations now have regular initial data and they are eligible for an energy method, but there is a price to pay. The coefficients of the new system will depend on the Schwarzschild geometry and will necessarily be highly singular at $r = 0$. We compute in Sect. 3 the precise blow up orders of the Schwarzschild connection coefficients, curvature, etc. Nevertheless, the issue of evolving singular initial data has become the more tractable problem of finding appropriate weighted solution spaces for the final singular equations.

In Sect. 4.2 we introduce the weighted Sobolev spaces which yield the desired flexibility in proving energy estimates. The right weights are given naturally by the singularities in the coefficients of the resulting equations, namely, powers of the Schwarzschild radius function r with a certain analogy corresponding to the order of each term. After stating the general local existence theorems in Sect. 4.3 and a more precise version of Theorem 1.1, we proceed to its proof via a contraction mapping argument which occupies Sect. 5. Therein, we derive the main weighted energy estimates by exploiting the asymptotic analysis at $r = 0$ of the Schwarzschild components (Sect. 3). It is necessary in our result that the power of r, $\alpha > 0$, in the weighted norms is sufficiently large; cf. assumption (ii) in Theorem 1.1. In the estimating process certain *critical* terms are inevitably generated, because of the singularities in the coefficients of the system we are working with; these terms are critical in that they appear with larger weights than the ones in the energy we are trying to control and thus prevent the estimates from closing. The exponent $\alpha > 0$ is then picked sufficiently large such that these critical terms have an overall favourable sign; this allows us to drop the critical terms and close the estimates.

The largeness of α forces the perturbed spacetime to agree asymptotically with Schwarzschild to a high order at the singularity. Although the latter may seem restrictive, it is quite surprising to us that there even exists a suitable choice of α which makes the argument work in the first place. A closer inspection of our method reveals that it is very sensitive with respect to certain asymptotics of the coefficients in the equations that happen to be just borderline to allow an energy-based argument to close. The most important of these are the blow up order of the sectional curvature (1.4) and the rate of growth of the Schwarzschild radius function r backwards in time. The latter corresponds to the 'opening up' rate of the *neck pinch* of the singular initial hypersurface Σ_0, Fig. 4. In this sense the Schwarzschild singularity is exactly at the threshold that our energy-based method can tolerate.

In the last section, Sect. 6, we study the constraint equations (1.5) in a perturbative manner about the Schwarzschild singular initial data set $({}^S\overline{g}, {}^SK)$ on Σ_0, following the conformal approach. We prove Theorem 1.2 by employing the inverse function theorem. More precisely, we prove that the linearized conformal constraint map (about Schwarz-

schild) is Fredholm in suitable weighted Sobolev spaces,[5] capturing the asymptotics needed for Theorem 1.1 to be applied, see Proposition 6.6, which we prove in Sect. 6.2. In the case where Σ_0 is localized in a small neighborhood of its singularity, the elliptic estimates we derive can be improved to yield that the linearized conformal map is actually an isomorphism. It is worth noting that the solutions to the constraints that we produce have unbounded mean curvature. Given the highly singular nature of the problem, we must exploit the features of the Schwarzschild background in order to obtain the special initial data sets needed in Theorem 1.1. These include good signs of certain dangerous terms appearing in the resulting elliptic system and a delicate decoupling of specific parts of the unknown variables.

1.2. Final comments; possible applications. To our surprise the present evolution bears some resemblance at an analytical level with a prior work on the stability of singular Ricci solitons [1]. Although of different nature, hyperbolic/parabolic (respectively), they share a couple of key features such as the opening up rate of the singularity and the "borderline" singularities in the coefficients involved.

The understanding of the question of stability of singularities in Einstein's equations and the behaviour of solutions near them is of great significance in the field. However, in general very little is known. In terms of rigorous results, substantial progress has been made in spherical symmetry in the presence of matter [6,7,9,29]. Moreover, certain matter models enjoy the presence of a monotonic quantity, which has been employed to study the stability of singularity formation in the general non-symmetric regime, cf. recent work of Rodnianski-Speck [31] on the FLRW big bang singularity. This is in contrast with the vacuum case of black hole interior and the unstable nature of the Schwarzschild singularity. We emphasize again the fact that the method developed herein does not impose any symmetry assumptions nor does it rely on any monotonicity. It should be noted as well that it does not depend on whether the particular singularity type is generic or not. On the contrary, we hope that the method developed herein can be employed to produce classes of examples of other singular solutions to the Einstein field equations, which until now are only known to exist under special symmetry assumptions and for which the general stability question may be out of reach.

The idea of constructing singular spacetimes by prescribing a specific singular behaviour and solving for a spacetime 'starting from the singularity' is not new. There exists an extensive literature regarding the construction of cosmological spacetimes exhibiting Kasner type singularities at each point of their 'big bang' hypersurface[6] using Fuchsian techniques [16,17,28]. However, the results in this category rely on the undesirable assumption of analyticity [2] and or on various symmetry assumptions, see relevant work on Gowdy spacetimes [27,30]. Yet, we believe that the usual Fuchsian algorithm cannot be applied to Schwarzschild type singularities due to their more singular nature.[7]

After our treatment of singular initial data containing a single sphere of $\{uv = 1\}$, a reasonable next step would be to study whether the construction of non-spherically

[5] We note that the spaces we use for the constraint equations differ from those we use for the evolutionary part of the problem.

[6] At each point of the usual singular spacelike hypersurface the spacetime metric approaches asymptotically the metric of a Kasner spacetime, with the Kasner parameters generally varying from point to point, what is called AVTD behaviour [16].

[7] The reason should be understood in an effort to reduce the Einstein equations to Fuchsian type equations for a Schwarzschild type singularity. In this case the singularities in the coefficients of the reduced evolution equations would be stronger than the ones encountered in the literature.

symmetric vacuum spacetimes containing an arc of the singular hyperbola (Fig. 3) is possible or even the whole singularity $r = 0$. Certainly this is a more restrictive question and at first glance not so obvious how to formulate it as a backward initial value problem for the EVE. However, we hope that the method developed herein could help approach this direction.

Lastly, one could try to perform a global instead of a local construction by considering a Cauchy hypersurface Σ_0 extending to spacelike infinity. We expect this follows readily from the work here, but we do not pursue it further. Perhaps a gluing construction could also be achieved.

2. The Einstein Equations as a Quasilinear Yang–Mills System

The Einstein vacuum equations (1.1), by virtue of the second Bianchi identity, imply the vanishing of the divergence of the Riemann curvature tensor. Decomposing the latter with respect to an orthonormal frame, which satisfies a suitable gauge condition, it results to a quasilinear second order hyperbolic system of equations for the connection 1-forms corresponding to that frame, which bears resemblance to the semilinear Yang–Mills [24]. Recently this formulation of the EVE played a key role in the resolution of the bounded L^2 curvature conjecture [18]. In this section we express the EVE (1.1) in the above setting, which we are going to use to directly solve the Cauchy problem. This necessitates some technical details which are carried out in Appendix A. Also, to avoid additional computations we write all equations directly in scalar non-tensorial form.[8]

All indices below range from 0 to 3 unless otherwise stated.

2.1. Cartan formalism. Let $(\mathcal{M}^{1+3}, g)$ be a Lorenzian manifold and let $\{e_0, e_1, e_2, e_3\}$ be an orthonormal frame; $g_{ab} := m_{ab} = \text{diag}(-1, 1, 1, 1)$. Assume also that $\mathcal{M}^{1+3}$ has the differential structure of $\Sigma \times [0, T]$, where each leaf $\Sigma \times \{\tau\} =: \Sigma_\tau$ is a 3-dim spacelike hypersurface. We denote the connection 1-forms associated to the preceding frame by

$$(A_X)_{ij} := g(\nabla_X e_i, e_j) = -(A_X)_{ji}, \tag{2.1}$$

where ∇ is the g-compatible connection of $\mathcal{M}^{1+3}$. Recall the definition of the Riemann curvature tensor

$$R_{\mu\nu ij} := g(\nabla_{e_\mu}\nabla_{e_\nu} e_i - \nabla_{e_\nu}\nabla_{e_\mu} e_i, e_j). \tag{2.2}$$

By the former definition of connection 1-forms, using m_{ab} to raise and lower indices, we write

$$\nabla_{e_a} e_b = (A_a)_b{}^k e_k.$$

Hence, we have

$$\begin{aligned}\nabla_{e_\mu}\nabla_{e_\nu} e_i &= \nabla_{e_\mu}(\nabla_{e_\nu} e_i) - \nabla_{\nabla_{e_\mu} e_\nu} e_i = \nabla_{e_\mu}\big((A_\nu)_i{}^k e_k\big) - (A_\mu)_\nu{}^k (A_k)_i{}^c e_c \\ &= e_\mu(A_\nu)_i{}^k e_k + (A_\nu)_i{}^k (A_\mu)_k{}^d e_d - (A_\mu)_\nu{}^k (A_k)_i{}^c e_c\end{aligned}$$

[8] It will be clear though which are the covariant expressions; see also [18].

Therefore, we get the following expression for the components of the Riemann curvature

$$\begin{aligned} R_{\mu\nu ij} = {} & e_\mu(A_\nu)_{ij} - e_\nu(A_\mu)_{ij} + (A_\nu)_i{}^k(A_\mu)_{kj} - (A_\mu)_i{}^k(A_\nu)_{kj} \\ & - (A_\mu)_\nu{}^k(A_k)_{ij} + (A_\nu)_\mu{}^k(A_k)_{ij} \end{aligned} \tag{2.3}$$

or setting

$$([A_\mu, A_\nu])_{ij} = (A_\mu)_i{}^k(A_\nu)_{kj} - (A_\nu)_i{}^k(A_\mu)_{kj} \tag{2.4}$$

we rewrite

$$(F_{\mu\nu})_{ij} := R_{\mu\nu ij} = e_\mu(A_\nu)_{ij} - e_\nu(A_\mu)_{ij} - ([A_\mu, A_\nu])_{ij} - (A_{[\mu})_{\nu]}{}^k(A_k)_{ij}, \tag{2.5}$$

where by standard convention

$$(A_{[\mu})_{\nu]}{}^k(A_k)_{ij} := (A_\mu)_\nu{}^k(A_k)_{ij} - (A_\nu)_\mu{}^k(A_k)_{ij}.$$

In the same manner we compute the covariant derivative of the Riemann tensor:

$$\begin{aligned} \nabla_\sigma R_{\mu\nu ij} = {} & e_\sigma(F_{\mu\nu})_{ij} - (A_\sigma)_\mu{}^k(F_{k\nu})_{ij} - (A_\sigma)_\nu{}^k(F_{\mu k})_{ij} \\ & -(A_\sigma)_i{}^k(F_{\mu\nu})_{kj} - (A_\sigma)_j{}^k(F_{\mu\nu})_{ik} \\ = {} & e_\sigma(F_{\mu\nu})_{ij} - (A_\sigma)^k{}_{[\mu}(F_{\nu]k})_{ij} - ([A_\sigma, F_{\mu\nu}])_{ij} \end{aligned} \tag{2.6}$$

Recall the transformation law of the above quantities under change of frames: Let $\{\tilde{e}_i\}_0^3$ be an orthonormal frame on $\mathcal{M}^{1+3}$ such that

$$\tilde{e}_a = O_a^k e_k \tag{2.7}$$

and let $(\tilde{A}_X)_{ij} := g(\nabla_X \tilde{e}_i, \tilde{e}_j)$ be the corresponding connection 1-forms. Then

$$(\tilde{A}_X)_{ij} = O_i^b O_j^c (A_X)_{bc} + X(O_i^b) O_j^c m_{bc}. \tag{2.8}$$

In addition, from (2.7) we have

$$\begin{aligned} \nabla_X \tilde{e}_a &= X(O_a^k) e_k + O_a^k \nabla_X e_k \\ (\tilde{A}_X)_a{}^d \tilde{e}_d &= X(O_a^k) e_k + O_a^k (A_X)_k{}^d e_d \end{aligned}$$

or

$$X(O_a^l) = (\tilde{A}_X)_a{}^d O_d^l - O_a^k (A_X)_k{}^l. \tag{2.9}$$

2.2. $\nabla \times Ric = 0$. Now we proceed by assuming that the *curl* of the Ricci tensor of the metric g vanishes:

$$\nabla_i R_{\nu j} - \nabla_j R_{\nu i} = 0, \tag{2.10}$$

where $R_{ab} := R_{\mu ab}{}^{\mu}$. A direct implication of the (contracted) second Bianchi identity is that the divergence of the Riemann curvature tensor satisfies

$$\nabla^{\mu} R_{ij\nu\mu} = \nabla_i R_{\nu j} - \nabla_j R_{\nu i} = 0. \tag{2.11}$$

Thus, it follows from (2.6) that

$$e^{\mu}(F_{\mu\nu})_{ij} - (A^{\mu})^{k}{}_{[\mu}(F_{\nu]k})_{ij} - ([A^{\mu}, F_{\mu\nu}])_{ij} = 0 \tag{2.12}$$

or by (2.5)

$$\begin{aligned} \boxdot\,(A_\nu)_{ij} - e^{\mu} e_\nu (A_\mu)_{ij} - e^{\mu}([A_\mu, A_\nu])_{ij} - e^{\mu}\big((A_{[\mu})_{\nu]}{}^{k}(A_k)_{ij}\big) \\ = \ (A^{\mu})^{k}{}_{[\mu}(F_{\nu]k})_{ij} + ([A^{\mu}, F_{\mu\nu}])_{ij}, \end{aligned} \tag{2.13}$$

where $\boxdot := -e_0^2 + e_1^2 + e_2^2 + e_3^2$ is the non-covariant box with respect to the frame e_i. Since

$$[e_\mu, e_\nu] = \nabla_{e_\mu} e_\nu - \nabla_{e_\nu} e_\mu = (A_{[\mu})_{\nu]}{}^{k} e_k,$$

(2.13) takes the equivalent form

$$\begin{aligned} \boxdot (A_\nu)_{ij} - e_\nu e^{\mu}(A_\mu)_{ij} = (A^{[\mu})_{\nu]}{}^{k} e_k (A_\mu)_{ij} + e^{\mu}([A_\mu, A_\nu])_{ij} + e^{\mu}\big((A_{[\mu})_{\nu]}{}^{k}(A_k)_{ij}\big) \\ + (A^{\mu})^{k}{}_{[\mu}(F_{\nu]k})_{ij} + ([A^{\mu}, F_{\mu\nu}])_{ij}, \end{aligned} \tag{2.14}$$

$\nu, i, j = 0, 1, 2, 3$. We remark that (2.14) is an equation of scalar functions.

2.3. Choice of gauge. Note that the preceding equation is not of hyperbolic type. We convert (2.14) into a quasilinear hyperbolic system of equations by imposing a Lorenz gauge condition on the orthonormal frame $\{e_i\}_0^3$:[9]

$$A^2 = (\mathrm{div} A)_{ij} := \nabla^{\mu}(A_\mu)_{ij} - (A_{\nabla_{e^\mu} e_\mu})_{ij} = e^{\mu}(A_\mu)_{ij} - (A^{\mu})_{\mu}{}^{k}(A_k)_{ij}, \tag{2.15}$$

where by A^2 we denote some quadratic expression in the connection coefficients $(A_\nu)_{ij}$ varying in ij. This a freedom one has in choosing the frame e_i; see Lemma A.1. Under (2.15), the Eq. (2.14) becomes the quasilinear second order

$$\begin{aligned} \boxdot\,(A_\nu)_{ij} = (A^{[\mu})_{\nu]}{}^{k} e_k (A_\mu)_{ij} + e^{\mu}([A_\mu, A_\nu])_{ij} + e^{\mu}\big((A_{[\mu})_{\nu]}{}^{k}(A_k)_{ij}\big) \\ + (A^{\mu})^{k}{}_{[\mu}(F_{\nu]k})_{ij} + ([A^{\mu}, F_{\mu\nu}])_{ij} + e_\nu(A^2) + e_\nu\big((A^{\mu})_{\mu}{}^{k}(A_k)_{ij}\big) \end{aligned} \tag{2.16}$$

[9] A wave type gauge essentially for e_i. The Coulomb gauge is another alternative which is used in [18]. We do not employ it here.

2.4. The reduced equations; initial data for EVE. Following (2.11)–(2.16) we actually see that the equation

$$\nabla_i \mathrm{R}_{\nu j} - \nabla_j \mathrm{R}_{\nu i} + e_\nu \big(\mathrm{div} A - A^2\big)_{ij} \\ =: H_{\nu ij} = (\text{LHS of (2.16)}) - (\text{RHS of (2.16)}), \tag{2.17}$$

holds true for every Lorentzian metric g and orthonormal frame $\{e_i\}_0^3$, without any additional assumptions or gauge condition. We call $H_{\nu ij} = 0$, i.e., the system (2.16), the reduced equations. We note that even after the gauge fixing, the reduced equations are not equivalent to the EVE (1.1), but only imply the vanishing of the *curl* of the Ricci tensor (2.10). However, one may suitably prescribe initial data for (2.16) such that they lead to solutions of the EVE and which are consistent with the Lorenz gauge condition (2.15).

Now we address the initial value problem for the reduced equations $H_{\nu ij} = 0$ aiming to the EVE. To solve the Eq. (2.16) one needs an equation relating the evolution of the orthonormal frame $\{e_i\}_0^3$ to that of the connection 1-forms. Let $\partial_0, \partial_1, \partial_2, \partial_3$ be a reference frame[10] in $\Sigma \times [0, T]$ (∂_0 transversal direction). We express e_i in terms of ∂_a:

$$e_i = O_i^a \partial_a \tag{2.18}$$

By virtue of the diffeomorphism invariance of the EVE, we may assume that the timelike unit vector of the orthonormal frame $\{e_i\}_0^3$ of the spacetime we solve for is $e_0 = \partial_0$. Doing so we deduce

$$\partial_0(O_i^a) = \mathcal{L}_{e_0}\big(\hat{\partial}_a(e_i)\big) = \mathcal{L}_{\partial_0}(\hat{\partial}_a) e_i + \hat{\partial}_a([\partial_0, e_i]) = O_i^b \mathcal{L}_{\partial_0}(\hat{\partial}_a)\partial_b + \hat{\partial}_a([e_0, e_i]),$$

where $\mathcal{L}$ denotes the Lie derivative and $\hat{\partial}_a$ is the 1-form dual to ∂_a. Setting $[\partial_0, \partial_b] =: \Gamma^c_{[0b]}\partial_c$ we rewrite

$$\partial_0(O_i^a) = -O_i^b \Gamma^a_{[0b]} + (A_{[0)i]}{}^k O_k^a. \tag{2.19}$$

Now we proceed to formulate the necessary and sufficient conditions on the initial data set of the reduced equations (2.16), coupled to (2.19), such that the corresponding solution yields a solution to the EVE. The following proposition is proved in Appendix A.1.

Proposition 2.1. *Let $(A_\nu)_{ij}$, O_i^a be a solution of (2.16),(2.19), arising from initial configurations subject to*

$$(A_\nu)_{ij}(\tau = 0) = -(A_\nu)_{ji}(\tau = 0) \qquad \partial_0(A_\nu)_{ij}(\tau = 0) = -\partial_0(A_\nu)_{ji}(\tau = 0) \\ O_0^a(\tau = 0) = I_0{}^a \tag{2.20}$$

and

$$(divA)_{ij} - A^2 = 0 \Longleftrightarrow e^\mu (A_\mu)_{ij} - (A^\mu)_\mu{}^k (A_k)_{ij} - A^2 = 0 \\ Ric_{ab}(g) = 0 \Longleftrightarrow e^\mu (A_\nu)_{i\mu} - e_\nu (A^\mu)_{i\mu} - ([A^\mu, A_\nu])_{i\mu} - (A^{[\mu})_{\nu]}{}^k (A_k)_{i\mu} = 0 \tag{2.21}$$

on Σ_0. Then the latter solution corresponds to an Einstein vacuum spacetime $(\mathcal{M}^{1+3}, g)$ and furthermore the frame $\{e_i\}_0^3$ (2.18) is g-orthonormal, $e_0 = \partial_0$, and satisfies the Lorenz gauge condition (2.15).

[10] Not orthonormal or coordinates, simply a basis frame.

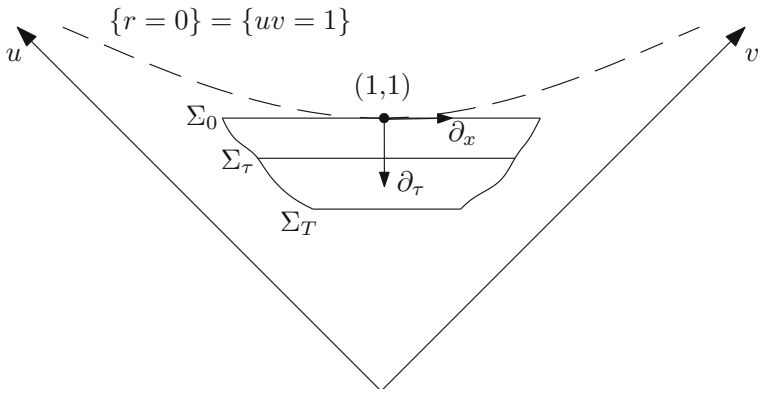

Fig. 5. The foliation (3.1) in the interior of the black hole

Remark 2.2. Note that the second part of (2.21) includes the constraints (1.5); $\mathrm{R}_{0b} = \mathrm{R}_{00} - \frac{1}{2}\mathrm{R} = 0$, $b = 1, 2, 3$, on Σ_0. The condition (2.21) is necessary and sufficient (as we show in A.1) to yield the propagation of the gauge and the EVE themselves. Once we have chosen the orthonormal frame initially and the initial data components $(A_0)_{ij}(\tau = 0)$, which correspond to the ∂_0 derivative of $\{e_i\}_0^3$, then the rest of the initial data set of (2.16) is fixed by the condition (2.21), i.e., the Lorenz gauge and the EVE on the initial hypersurface Σ_0, see Remark A.2.

3. The Schwarzschild Components

We fix an explicit Schwarzschild orthonormal reference frame and compute the corresponding connection coefficients, which we then use to find the leading asymptotics of the second fundamental form and curvature of the initial singular hypersurface Σ_0 in Schwarzschild. Knowing the precise leading blow up behaviour of these quantities is crucial for the study of the backwards well-posedness in the next section. For distinction, we denote Schwarzschild components with an upper left script ^{S}.

Let us consider a specific foliation of spacelike hypersurfaces Σ_τ, $\tau \in [0, T]$, for the backward problem in a neighbourhood of $(u = 1, v = 1)$; Fig. 3. For convenience[11] let

$$\Sigma_\tau : -\frac{1}{2}(u+v)+1 = \tau \qquad (u,v) \in (1-\epsilon, 1+\epsilon)^2, \ \tau \in [0,T]. \tag{3.1}$$

In temporal and spatial coordinates τ, x

$$\partial_\tau := -\partial_u - \partial_v, \quad \partial_x := \partial_u - \partial_v$$
$$x = \frac{1}{2}(u-v), \tag{3.2}$$

the metric (1.2) takes the form

$${}^S g = -\Omega^2 d\tau^2 + \Omega^2 dx^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \qquad \Omega^2 = \frac{32M^3}{r} e^{-\frac{r}{2M}}. \tag{3.3}$$

[11] It is easy to see that the following leading asymptotics we derive are independent of the particular choice of foliation.

By (1.3), (3.2) r is related to τ, x via

$$(1-\tau)^2 - x^2 = (1 - \frac{r}{2M})e^{\frac{r}{2M}}, \tag{3.4}$$

from which one can derive the following formulas:

$$\begin{aligned} \partial_\tau r &= \frac{\Omega^2}{4M}(1-\tau), \qquad \partial_x r = \frac{\Omega^2}{4M}x \\ \partial_\tau \Omega^2 &= -\frac{\Omega^4}{4M}(\frac{1}{r}+\frac{1}{2M})(1-\tau), \quad \partial_x \Omega^2 = -\frac{\Omega^4}{4M}(\frac{1}{r}+\frac{1}{2M})x \end{aligned} \tag{3.5}$$

Remark 3.1. The above first two identities yield the leading asymptotics:

$$r^2 \sim 16M^2(\frac{x^2}{2}+\tau), \qquad \text{as } \tau, x \to 0. \tag{3.6}$$

Directly from the form of the induced metric on Σ_τ,

$$^S\overline{g} = \Omega^2 dx^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.7}$$

we compute the corresponding induced volume form

$$d\mu_{S_{\overline{g}}} = \Omega r^2 \sin\theta dx d\theta d\phi = \left[4\sqrt{2}M^{\frac{3}{2}}r^{\frac{3}{2}} + O(r^2)\right]\sin\theta dx d\theta d\phi \tag{3.8}$$

and its rate of change along ∂_τ using (3.5):

$$\partial_\tau d\mu_{S_{\overline{g}}} = \left[\frac{12M^2}{r^2}(1-\tau) + O(\frac{1}{r})\right]d\mu_{S_{\overline{g}}}. \tag{3.9}$$

Normalizing, we define the Schwarzschild orthonormal frame

$$\partial_0 = \frac{1}{\Omega}\frac{\partial}{\partial\tau} \qquad \partial_1 = \frac{1}{\Omega}\frac{\partial}{\partial x} \qquad \partial_2 = \frac{1}{r}\frac{\partial}{\partial\theta} \qquad \partial_3 = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \tag{3.10}$$

and the relative connection coefficients $^S(A_\mu)_{ij} = {}^S g(^S\nabla_{\partial_\mu}\partial_i, \partial_j)$ associated to it. A tedious computation[12] shows that the non-zero components read

$$\begin{aligned} {}^S(A_0)_{01} &= -\frac{\Omega}{8M}(\frac{1}{r}+\frac{1}{2M})x \\ {}^S(A_1)_{01} &= -\frac{\Omega}{8M}(\frac{1}{r}+\frac{1}{2M})(1-\tau) \\ {}^S(A_2)_{02} &= {}^S(A_3)_{03} = \frac{\Omega}{4M}\frac{1-\tau}{r} \\ {}^S(A_2)_{12} &= {}^S(A_3)_{13} = \frac{\Omega}{4M}\frac{x}{r} \\ {}^S(A_3)_{23} &= \frac{\cot\theta}{r} \end{aligned} \tag{3.11}$$

[12] One may calculate the connection coefficients using the Koszul formula

$$^S(A_\mu)_{ij} = \frac{1}{2}\Big[{}^S g([\partial_\mu, \partial_i], \partial_j) - {}^S g([\partial_i, \partial_j], \partial_\mu) + {}^S g([\partial_j, \partial_\mu], \partial_i)\Big].$$

Recall the (spacetime) divergence formula of the connection 1-forms $X \to {}^S(A_X)_{ij}$

$$ {}^S(\mathrm{div}A)_{ij} := \partial^{\mu S}(A_\mu)_{ij} - {}^S(A_{\nabla_{\partial^\mu}\partial_\mu})_{ij} = \partial^{\mu S}(A_\mu)_{ij} - {}^S(A^\mu)_\mu{}^{bS}(A_b)_{ij} \tag{3.12} $$

Utilizing (3.5) and (3.11), we check that the first order term in the RHS of (3.12) vanishes

$$ \partial^{\mu S}(A_\mu)_{ij} = 0, \tag{3.13} $$

leaving

$$ {}^S(\mathrm{div}A)_{ij} = {}^S(A_3)_{23}{}^S(A_2)_{ij}. \tag{3.14} $$

Remark 3.2. Thus, the orthonormal frame (3.10) satisfies a Lorenz gauge type condition (2.15).

Remark 3.3. Summarizing the above identities and formulas we obtain the following leading asymptotics at $r = 0$:

$$ \partial_0 \sim \frac{1}{4\sqrt{2}M^{\frac{3}{2}}} r^{\frac{1}{2}} \partial_\tau \quad \partial_1 \sim \frac{1}{4\sqrt{2}M^{\frac{3}{2}}} r^{\frac{1}{2}} \partial_x \tag{3.15} $$

$$ |{}^S A| \leq \frac{C}{r^{\frac{3}{2}}} \quad |\partial^{(k)\,S} A| \leq \frac{C}{r^{(k+1)\frac{3}{2}}}, \tag{3.16} $$

where C depends on $M > 0$ and k. Notice that the latter asymptotics are sharp for $k = 0$ and when $\partial^{(k)} = \partial_0^{(k)}$. In fact, the components of the second fundamental form of the slices ${}^S K_{ii} = {}^S(A_i)_{0i}$, $i = 1, 2, 3$, are exactly at this level. In more geometric terms we have (up to constants)

$$ |{}^S K| \sim \frac{1}{r^{\frac{3}{2}}} \quad |\mathrm{tr}_{{}^S\overline{g}}{}^S K| \sim \frac{1}{r^{\frac{3}{2}}} \quad |{}^S\overline{\mathrm{R}}| \sim \frac{1}{r^2}. \tag{3.17} $$

Thus, employing (3.8), (3.6) for $\tau = 0$, we see that both the scalar curvature and the second fundamental of the initial singular hypersurface Σ_0 are far from being square integrable

$$ \begin{aligned} \int_{\Sigma_0} |{}^S K|^2 d\mu_{{}^S\overline{g}} \sim \int_0^\epsilon \frac{1}{x^3} x^{\frac{3}{2}} dx = \int_0^\varepsilon \frac{1}{x^{\frac{3}{2}}} dx = +\infty \\ \int_{\Sigma_0} |{}^S\overline{\mathrm{R}}|^2 d\mu_{{}^S\overline{g}} \sim \int_0^\epsilon \frac{1}{x^4} x^{\frac{3}{2}} dx = \int_0^\epsilon \frac{1}{x^{\frac{5}{2}}} dx = +\infty \end{aligned} \tag{3.18} $$

The same holds for the mean curvature of Σ_0. In fact, a similar calculation shows $\mathrm{tr}_{{}^S\overline{g}}{}^S K \notin L^p$, $p \geq \frac{5}{3}$.

Remark 3.4. The precise leading asymptotics of all computed quantities in this section play a crucial role in the analysis of the backward existence problem and the proofs of the main theorems in the next section. However, exact formulas, like (3.11), are not really needed. We could have chosen as well a foliation of the form $\Sigma_\tau : \tau = f(x)$, instead of (3.1), i.e., $\tau = \mathrm{const.}$, for some smooth function $f(x)$, $f'(0) = 0$. It is easy to see by computing the induced metric and second fundamental form that the leading asymptotics of all relevant quantities of interest remain the same.

4. The Local-in Time Backwards Well-posedness

4.1. Perturbed spacetime; a transformed system. Let $(\overline{g}, K)$ be a perturbation of the Schwarzschild initial data set $({}^S\overline{g}, {}^SK)$ on Σ_0, verifying the constraints (1.5), and let $\{e_i\}_1^3$ be an orthonormal frame of $(\Sigma_0, \overline{g})$. We fix a reference frame $\{\partial_i\}_0^3$ in $\mathcal{M}^{1+3} = \{\Sigma_\tau\}_{\tau\in[0,T]}$, namely, the Schwarzschild orthonormal frame (3.10); Fig. 5. Let $\{e_i\}_0^3$, $e_0 = \partial_0$, be a frame extension in $\mathcal{M}^{1+3}$ expressed in terms of ∂_d via

$$e_c = O_c^d \partial_d. \tag{4.1}$$

Consider now the (unique) metric g for which e_i is orthonormal, $g_{ab} := m_{ab} = \mathrm{diag}(-1, 1, 1, 1)$, and the corresponding connection coefficients $(A_\nu)_{ij} = g(\nabla_{e_\nu} e_i, e_j)$. Then Proposition 2.1 asserts that the EVE (1.1) for g, under the Lorenz gauge condition[13]

$$(\mathrm{div} A)_{ij} = (A_3)_{23}(A_2)_{ij}, \tag{4.2}$$

reduce to the system of scalar equations

$$\begin{aligned}
\boxdot(A_\nu)_{ij} &= (A^{[\mu})_{\nu]}{}^k e_k(A_\mu)_{ij} + e^\mu([A_\mu, A_\nu])_{ij} + e^\mu\big((A_{[\mu})_{\nu]}{}^k (A_k)_{ij}\big) \\
&\quad + (A^\mu)^k{}_{[\mu}(F_{\nu]k})_{ij} + ([A^\mu, F_{\mu\nu}])_{ij} + e_\nu\big((A_3)_{23}(A_2)_{ij}\big) + e_\nu\big((A^\mu)_\mu{}^k (A_k)_{ij}\big) \\
\partial_0(O_c^d) &= -O_c^{bS}(A_{[0})_{b]}{}^d + (A_{[0})_{c]}{}^k O_k^d, \quad \nu, i, j, c, d \in \{0, 1, 2, 3\}
\end{aligned} \tag{4.3}$$

where $\boxdot := -e_0^2 + e_1^2 + e_2^2 + e_3^2$ and ${}^S(A_{[0})_{b]}{}^d = [\partial_0, \partial_b]^d$.

However, the system (4.3) has singular initial data in the Schwarzschild background which do not permit an energy approach directly. For this reason we recast the equations in a way that captures the closeness to the Schwarzschild spacetime. Let

$$(u_\nu)_{ij} := (A_\nu)_{ij} - {}^S(A_\nu)_{ij} : \{\Sigma_\tau\}_{\tau\in[0,T]} \to \mathbb{R} \qquad \nu, i, j \in \{0, 1, 2, 3\}, \tag{4.4}$$

where the components ${}^S(A_\nu)_{ij}$ are the Schwarzschild connection coefficients corresponding to the frame $\{\partial_i\}_0^3$ (3.10) and they are given by (3.11). We are going to use these new functions to control the evolution of the perturbed spacetime.

Consider now the analogous system to (4.3) satisfied by the Schwarzschild components ${}^S(A_\nu)_{ij}, \partial_c$. In view of the asymptotics (3.15), we define Γ_q to be a smooth function satisfying the bound

$$|\Gamma_q| \leq \frac{C_q}{r^q} \qquad\qquad |\partial^{(k)}\Gamma_q| \leq \frac{C_{q,k}}{r^{q+\frac{3}{2}k}}, \tag{4.5}$$

for constants $C_q, C_{q,k}$ depending on $M > 0$. Taking the difference of the two analogous systems we obtain a new system for the functions $(u_\nu)_{ij}$, $O_c^d - I_c{}^d$ written schematically in the form:

$$\begin{aligned}
h^{ab}\partial_a\partial_b(u_\nu)_{ij} &= O\Gamma_{\frac{3}{2}}\partial u + O\Gamma_3 u + O\Gamma_{\frac{9}{2}}(O - I) + O\Gamma_3\partial(O - I) \\
&\quad + \Gamma_3 u^2 + Ou\partial u + u^3 + O\partial(O - I)\partial u \\
\partial_0(O_c^d - I_c{}^d) &= \Gamma_{\frac{3}{2}}(O - I) + (O - I)u + u,
\end{aligned} \tag{4.6}$$

[13] We choose now a specific type based on the one satisfied by the Schwarzschild reference frame (3.14).

where

$$h^{ab} := m^{cd} O_c^a O_d^b = g^{ab} \tag{4.7}$$

and each term in the RHS denotes some algebraic combination of finite number of terms of the depicted type (varying in ν, i, j) where the particular indices do not matter.

Remark 4.1. Evidently, the systems (4.3) and (4.6) are equivalent. The benefit is that the assumption on the perturbed spacetime, being close to Schwarzschild, implies that the functions $(u_\nu)_{ij}$, $O_c^d - I_c{}^d$ are now small and regular. Thus, we have reduced the evolutionary problem to solving the PDE-ODE system of equations (4.6). However, the issue of singular initial data in (4.3) has become an issue of singularities in the coefficients of the resulting equations (4.6), at $\tau = x = 0$, which do not make it possible to apply the energy procedure in standard spaces; see also (3.18). These singularities, in large part, are due to the intrinsic curvature blow up and cannot be gauged away; in particular the coefficients Γ_3 of the potential terms in (4.6) correspond to the Schwarzschild curvature (1.4). Some of the functions Γ_q that appear in (4.6), expressed in terms of Schwarzschild connection coefficients (3.11) and their derivatives, are less singular than (4.5), but representatives of the exact bound do appear in all the terms.

Remark 4.2. Another crucial asymptotic behaviour that our method heavily depends on is that of the radius function r. According to (3.6), we observe that the best $L^\infty_{\Sigma_\tau}$ bound one could hope for the ratio $1/r^2$ is of the form

$$\| \frac{1}{r^2} \|_{L^\infty(\Sigma_\tau)} \leq \frac{C}{\tau}, \tag{4.8}$$

which obviously fails to be integrable in time $\tau \in [0, T]$, for any $T > 0$. This fact lies at the heart of the difficulty of closing a Gronwall type estimate.

4.2. The weighted H^s spaces. In order to study the well-posedness of (4.6) we introduce certain weighted norms. It turns out that the weights which yield the desired flexibility in obtaining energy estimates are the following.

Definition 4.3. Given $\alpha > 0$ and $\tau \in [0, T]$, we define the (time dependent) weighted Sobolev space $H^{s,\alpha}[\tau]$, as a subspace of the standard H^s space on Σ_τ with the Schwarzschild induced volume form satisfying:

$$H^{s,\alpha}[\tau]: \quad u \in H^s(\Sigma_\tau), \quad \|u\|^2_{H^{s,\alpha}[\tau]} := \sum_{k \leq s} \int_{\Sigma_\tau} \frac{[\partial^{(k)} u]^2}{r^{2\alpha - 3(k-1)}} d\mu_{s_{\overline{g}}} < +\infty, \tag{4.9}$$

where by $\partial^{(k)}$ we denote any order k combination of directional derivatives with respect to the components $\partial_1, \partial_2, \partial_3$ of the Schwarzschild frame (3.10). For convenience, we drop τ from the notation whenever the context is clear.

Remark 4.4. Observe that the weights in the norm $\| \cdot \|_{H^{s,\alpha}}$ in (4.9) blow up only at $\tau = 0, x = 0$. For $\tau > 0$ fixed, the weights are uniformly bounded above by some positive constant C_τ, which becomes infinite as $\tau \to 0^+$. The dependence of the power $2\alpha - 3(k-1)$ on the number k of derivatives corresponds to the singularities in the coefficients of the equation (4.6).

Lemma 4.5. *The weighted $H^{s,\alpha}$ spaces satisfy the properties:*

$$\begin{aligned} H^{s_1,\alpha} &\subset H^{s_2,\alpha} \quad s_1 < s_2 \\ r^{-\frac{3}{2}l} u \in H^{s,\alpha-\frac{3}{2}l}, &\quad \text{whenever } u \in H^{s,\alpha} \\ \partial^{(k)} u \in H^{s-k,\alpha-\frac{3}{2}k} &\quad k \leq s,\ u \in H^{s,\alpha} \end{aligned} \tag{4.10}$$

Proof. They are immediate consequences of Definition 4.3 and the fact that

$$|\partial_1(r^{-\frac{3}{2}l})| \leq C l r^{-\frac{3}{2}l-\frac{1}{2}} \quad |\partial_2(r^{-\frac{3}{2}l})| = |\partial_3(r^{-\frac{3}{2}l})| = 0,$$

cf. (3.5), (3.10). □

4.3. Local existence theorems. Let

$$\begin{aligned} \mathcal{E}(u, O; \alpha, T) := \sum_{\nu,i,j=0}^{3} \Bigg[&\sup_{\tau\in[0,T]} \left(\|(u_\nu)_{ij}\|^2_{H^{3,\alpha}} + \|\partial_0(u_\nu)_{ij}\|^2_{H^{2,\alpha-\frac{3}{2}}} \right) \\ &+ \int_0^T \left(\|(u_\nu)_{ij}\|^2_{H^{3,\alpha+1}} + \|\partial_0(u_\nu)_{ij}\|^2_{H^{2,\alpha-\frac{1}{2}}} \right) d\tau \Bigg] \\ + \sum_{c,d=0}^{3} \Bigg[&\sup_{\tau\in[0,T]} \|O_c^d - {I_c}^d\|^2_{H^{3,\alpha+\frac{3}{2}}} + \int_0^T \|O_c^d - {I_c}^d\|^2_{H^{3,\alpha+\frac{5}{2}}} d\tau \Bigg] \end{aligned} \tag{4.11}$$

be the total weighted energy of the functions $(u_\nu)_{ij}$, $O_c^d - {I_c}^d$ defined in $\{\Sigma_\tau\}_{\tau\in[0,T]}$ (3.1), Fig. 3.1, the backward domain of dependence of Σ_0 with respect to the metric g we are solving for. Since the actual domain depends on the unknown solution, it will be fully determined in the end; see Sect. 5. For brevity we denote by

$$\begin{aligned} \mathcal{E}_0 := \sum_{\nu,i,j\in\{0,1,2,3\}} &\left[\|(u_\nu)_{ij}(\tau=0)\|^2_{H^{3,\alpha}} + \|\partial_0(u_\nu)_{ij}(\tau=0)\|^2_{H^{2,\alpha-\frac{3}{2}}} \right] \\ + \sum_{c,d\in\{0,1,2,3\}} &\|O_c^d - {I_c}^d\|^2_{H^{3,\alpha+\frac{3}{2}}(\Sigma_0)} \end{aligned} \tag{4.12}$$

the energy at the initial singular slice Σ_0.

The following theorem is our first main local well-posedness result for the system (4.6), whose proof occupies Sect. 5.

Theorem 4.6. *There exist $\alpha > 0$ sufficiently large and $\varepsilon > 0$ small such that if*

$$\mathcal{E}_0 < +\infty \qquad \|O_c^d - {I_c}^d\|_{L^\infty(\Sigma_0)} < \varepsilon, \qquad c, d = 0, 1, 2, 3, \tag{4.13}$$

then the system (4.6) admits a unique solution, up to some small time $T = T(\mathcal{E}_0, \alpha) > 0$, in the spaces

$$\begin{aligned} (u_\nu)_{ij} &\in C([0,T]; H^{3,\alpha}) \cap L^2([0,T]; H^{3,\alpha+1}) \qquad \nu, i, j \in \{0,1,2,3\} \\ \partial_0(u_\nu)_{ij} &\in C([0,T]; H^{2,\alpha-\frac{3}{2}}) \cap L^2([0,T]; H^{2,\alpha-\frac{1}{2}}) \\ O_c^d - {I_c}^d &\in C([0,T]; H^{3,\alpha+\frac{3}{2}}) \cap L^2([0,T]; H^{3,\alpha+\frac{5}{2}}) \qquad c, d \in \{0,1,2,3\} \end{aligned} \tag{4.14}$$

Remark 4.7. (i) The second part of condition (4.13), $\varepsilon > 0$ small, is necessary for the equation (4.6) to be hyperbolic, yielding sufficient pointwise control on the h^{ab}'s (4.7)

$$|h^{bb} - m^{bb}| < \frac{1}{2} \qquad |h^{bc}| \leq C\varepsilon^2, \qquad b, c = 0, 1, 2, 3, b \neq c. \tag{4.15}$$

It could be obviously replaced by the stronger assumption that $\mathcal{E}_0 < \varepsilon$, since the energy $\mathcal{E}(u, O; \alpha, T)$ controls the L^∞ norm of u, O by standard Sobolev embedding.

(ii) How large the exponent α has to be depends on the coefficients of the system (4.6). In the final inequalities in Sect. 5 $\alpha > 0$ is picked large enough so that certain 'critical' terms can be absorbed in the LHS and the estimates can close.

The above theorem is a local existence result for the system (4.6). Imposing now the proper conditions on the initial data set of (4.6), the solution (4.14) yields a solution of (4.3) which in turn corresponds to an Einstein vacuum spacetime (1.1).

Theorem 4.8. *Let α, ε be such as in Theorem 4.6 and let $(\Sigma_0, \overline{g}, K)$ be an initial data set for the Einstein vacuum equations (1.1) satisfying the constraints (1.5), such that the components*

$$(u_\nu)_{ij} \in H^{3,\alpha}(\Sigma_0) \qquad \nu, i, j = 1, 2, 3, \tag{4.16}$$

$$O_c^d - I_c{}^d \in H^{3,\alpha+\frac{3}{2}}(\Sigma_0) \qquad \|O_c^d - I_c{}^d\|_{L^\infty(\Sigma_0)} < \varepsilon \qquad c, d = 1, 2, 3, \tag{4.17}$$

computed with respect to an orthonormal frame $\{e_i\}_1^3$ on $(\Sigma_0, \overline{g})$, and

$$(u_i)_{0j}(\tau = 0) := K_{ij} - {}^S K_{ij} \in H^{3,\alpha}(\Sigma_0) \qquad i, j = 1, 2, 3. \tag{4.18}$$

Then, there exists a solution g to the EVE (1.1) in the backward region to Σ_0, foliated by $\{\Sigma_\tau\}_{t\in[0,T]}$, with induced initial data set $(\overline{g}, K)$ on Σ_0 and an orthonormal frame extension $\{e_i\}_0^3$ for which the corresponding (spacetime) functions $(u_\nu)_{ij}$, $O_c^d - I_c^d$ (4.4), (4.1) lie in the spaces (4.14).

If in addition $O_c^d - I_c{}^d \in C([0, T]; H^{4,\alpha+\frac{3}{2}})$, $c, d = 1, 2, 3$, then the Einsteinian vacuum development is unique up to isometry.

The fact that such (non-spherically symmetric) initial data sets $(\Sigma_0, \overline{g}, K)$ exist, in compliance with Theorem 4.8, is shown in Sect. 6.

Proof of Theorem 4.8. We want to invoke Theorem 4.6. For this purpose, we prescribe initial data for the system (4.6):

(i) The components (4.16), (4.17), (4.18) are given.

(ii) Since in the beginning of Sect. 4.1 we assumed $e_0 = \partial_0$ and since $\{e_i\}_1^3$ is initially tangent to Σ_0, we set

$$O_0^b(\tau = 0) = I_0{}^b \qquad O_a^0(\tau = 0) = I_a{}^0 \qquad a, b = 0, 1, 2, 3. \tag{4.19}$$

(iii) We (freely) assign[14]

$$(u_0)_{ab}(\tau = 0) := (A_0)_{ab} - {}^S(A_0)_{ab} \in H^{3,\alpha}(\Sigma_0), \quad a, b = 0, 1, 2, 3. \tag{4.20}$$

[14] The functions $(u_0)_{ab}(\tau = 0)$ or equivalently $(A_0)_{ab}(\tau = 0)$ fix the ∂_0 derivative of the frame $\{e_i\}_0^3$ on Σ_0; see Lemma A.1 and Remark A.2.

Once we have prescribed the above, the components $\partial_0(u_\nu)_{ij}(\tau = 0)$ are fixed by the assumption (2.21) on the initial data of the original system (4.3); see Remark A.2. Indeed, subtracting the corresponding Schwarzschild components from (A.15), (A.16), which obvisouly satisfy the same initial relations, cf. (3.14), we obtain schematically:

$$\partial_0(u_\nu)_{ij} = O\partial_a u + \Gamma_{\frac{3}{2}} u + \Gamma_3(O - I) + u^2 \quad \text{on } \Sigma_0,\ a = 1, 2, 3. \tag{4.21}$$

By (4.10) and standard Sobolev embedding we conclude that

$$\partial_0(u_\nu)_{ij}(\tau = 0) \in H^{2,\alpha-\frac{3}{2}} \quad \nu, i, j = 0, 1, 2, 3. \tag{4.22}$$

Thus, the assumption (4.13) is verified and Theorem 4.6 can be invoked. From Proposition 2.1 it follows that the solution (4.14) of (4.6) and hence of (4.3) yields indeed an Einstein vacuum spacetime $(\{\Sigma_\tau\}_{\tau\in[0,T]}, g)$.

To prove uniqueness (up to isometry) we rely on the uniqueness statement in Theorem 4.6. Suppose there is another Einsteinian vacuum development $(\tilde{\mathcal{M}}^{1+3}, \tilde{g})$ of the initial data set $(\Sigma_0, \overline{g}, K)$, diffeomorphic to $\{\Sigma_\tau\}_{\tau\in[0,T]}$, satisfying the hypothesis (4.16), (4.17), (4.18); defined by pulling back the relevant quantities through the preceding diffeomorphism, taking differences etc. In order to use the uniqueness statement in Theorem 4.6, we need the two spacetimes to have the same initial data for the system (4.6). The part of the initial data set given by the assumptions in the statement of Theorem 4.8 is of course identical for both spacetimes. The remaining components that we want to agree, other than the $(\tilde{u}_0)_{ab}(\tau = 0)$'s, as noted in the previous paragraph, can be fixed by condition (2.21). Therefore, we get identical initial data components for the system (4.6) by constructing a Lorenz gauge frame (4.2) $\{\tilde{e}_i\}_0^3$ for $\tilde{g}$, which is initially equal to $\{e_i\}_0^3$ on Σ_0 and such that $(\tilde{u}_0)_{ab}(\tau = 0) = (u_0)_{ab}(\tau = 0)$ as well; see Lemma A.1. The only assumption to be verified is the well-posedness of the system (A.1) for functions in the solution spaces (4.14), after taking differences with the equation for the frame $\{e_i\}_0^3$. However, this falls in the category of the system (4.6) [in fact simpler, being semilinear] to which Theorem 4.6 can be applied. The extra derivative that we have to assume in order to close, $\tilde{O}_c^d - I_c{}^d \in H^{4,\alpha+\frac{3}{2}}$, is due to the div$A$ term in the RHS of (A.1). □

5. Proof of Theorem 4.6

Throughout this section we will use the notation $X \lesssim Y$ to denote an inequality between the quantities X, Y of the form $X \leq CY$, where C is an absolute positive constant depending only on the Schwarzschild mass $M > 0$. The same for the standard notation $O(X)$, for a quantity bounded by $|O(X)| \leq CX$, $X > 0$. Furthermore, all the estimates regard only the Schwarzschild region foliated by $\{\Sigma_\tau\}_{\tau\in[0,T]}$; Fig. 3.

5.1. Proof outline. We prove Theorem 4.6 via a contraction mapping argument. First we establish an energy estimate in the relevant weighted H^3 spaces in Sect. 5.3. Then we obtain a contraction, in Sect. 5.4, in the corresponding spaces of one derivative less, see (5.48), which together with the energy estimate yield the desired solution (4.14).

To derive these estimates we have to eliminate some *critical* terms which are generated due to the singularities in the coefficients of the equations, having larger weights than the ones in the norm (4.9), and which prevent us from closing (see Propositions 5.3, 5.6). This is where the role of the weights (4.9) comes in. The parameter $\alpha > 0$ helps

generate critical terms with a favourable sign. Being large enough, but finite, α provides an overall negative sign for the critical terms, hence, rendering them removable from the RHS of the final inequalities. This enables us to close the estimates and complete the proof. The precise asymptotics of the singularities in the coefficients of the equations (4.6), at $\tau = x = 0$, and the opening up rate of the radius function r in $\tau > 0$ play a crucial role here.[15]

5.2. Basic estimates. Let v be a scalar function defined on Σ_τ, represented by

$$v \circ \psi_\tau : U_\tau \to \mathbb{R}, \tag{5.1}$$

where $\psi_\tau : U_\tau \to \Sigma_\tau$ is the (x, θ, ϕ) coordinate chart. We recall some standard inequalities: the classical Sobolev embedding of $H^2(U)$ in $L^\infty(U)$

$$\|v\|_{L^\infty(U)} \lesssim \|v\|_{H^2(U)} \tag{5.2}$$

and the interpolation inequality

$$\|v\|_{L^4(U)} \leq C\|v\|^{\frac{1}{4}}_{L^2(U)}\|\nabla v\|^{\frac{3}{4}}_{L^2(U)} \quad v \in C^\infty_c(U), \tag{5.3}$$

for a bounded domain $U \subset \mathbb{R}^3$ with (piecewise) C^2 boundary. In the following proposition v is assumed to be regular enough such that the RHSs make sense.

Proposition 5.1. *For a general function $v : \Sigma_\tau \to \mathbb{R}$, $\tau \in [0, T]$, with the appropriate regularity, the following inequalities hold:*

The L^∞ bound

$$\|\frac{v}{r^k}\|_{L^\infty(\Sigma_\tau)} \lesssim (k+1)^2\|v\|_{H^{2,k+3+\frac{1}{4}}(\Sigma_\tau)} \tag{5.4}$$

and the L^4 estimate

$$\|\frac{v}{r^k}\|_{L^4(\Sigma_\tau)} \lesssim (k+1)^{\frac{3}{4}}\|v\|_{H^{1,k+1+\frac{1}{4}}(\Sigma_\tau)}. \tag{5.5}$$

Proof. From (5.2) we have

$$\|\frac{v}{r^k}\|_{L^\infty(\Sigma_\tau)} \overset{(5.1)}{=} \|\frac{v}{r^k} \circ \psi_\tau\|_{L^\infty(U_\tau)} \lesssim \|\frac{v}{r^k} \circ \psi_\tau\|_{H^2(U_\tau)}$$

substituting (3.8) and the frame (3.10)

$$\lesssim (k+1)^2\|v\|_{H^{2,k+3+\frac{1}{4}}(\Sigma_\tau)}.$$

We argue similarly in the case of (5.5). □

[15] If we were to tweak the leading orders just by $\epsilon > 0$, the previous procedure would fail no matter how large $\alpha > 0$ is to begin with.

5.3. Energy estimate in $H^{3,\alpha}$. We set up now the iteration scheme we are going to follow. Let $\{\overline{u}, \overline{O}\} := \{(\overline{u}_\nu)_{ij}, \overline{O}_c^d : \nu, i, j, c, d = 0, 1, 2, 3\}$ be a set of spacetime functions in the solution spaces (4.14), verifying $|\overline{O}_c^d - {I_c}^d| < \varepsilon$ initially on Σ_0. We assume without loss of generality[16]

$$\mathcal{E}(\overline{u}, \overline{O}; \alpha, T) \leq 2\mathcal{E}_0. \tag{5.6}$$

We also assume

$$\|\partial_0(\overline{O}_c^d)\|^2_{H^{2,\alpha}[\tau]} \lesssim \mathcal{E}_0^2 + \mathcal{E}_0 \qquad \forall \tau \in [0, T], \quad c, d = 0, 1, 2, 3. \tag{5.7}$$

Iteration step: Consider the following linear version of the system (4.6), where we replace the functions u, O in the following specific terms by the corresponding ones from the set $\{\overline{u}, \overline{O}\}$:

$$\begin{aligned} \overline{h}^{ab}\partial_a\partial_b(u_\nu)_{ij} &= \overline{O}\Gamma_{\frac{3}{2}}\partial u + \overline{O}\Gamma_3 u + \overline{O}\Gamma_{\frac{9}{2}}(O - I) + \overline{O}\Gamma_3\partial(O - I) \\ &\quad + \Gamma_3\overline{u}^2 + \overline{O}\overline{u}\partial\overline{u} + \overline{u}^3 + \overline{O}\partial(\overline{O} - I)\partial\overline{u} \\ \partial_0(O_c^d - {I_c}^d) &= \Gamma_{\frac{3}{2}}(O - I) + (\overline{O} - I)\overline{u} + u, \end{aligned} \tag{5.8}$$

where $\overline{h}^{ab} = m^{cd}\overline{O}_c^a\overline{O}_d^b$. Observe that we kept in the RHS of (5.8) the functions u, O attached to the most singular coefficients of the system. This is actually very important to our strategy in order to avoid further complications.

We assume now there exists a solution $(u_\nu)_{ij}$, $O_c^d - {I_c}^d$ of (5.8) lying in the solution space (4.14). The existence of such a solution is based mainly on the energy estimate we will derive below and a standard duality argument which we omit.

Claim For a chosen large enough $\alpha > 0$ and $T > 0$ sufficiently small (depending on $\mathcal{E}_0, \alpha$) the following estimate holds

$$\mathcal{E}(u, O; \alpha, T) \leq 2\mathcal{E}_0. \tag{5.9}$$

The preceding H^3-weighted energy estimate, cf. (4.11), will be used in the next subsection to close the contraction argument that yields the existence and uniqueness of the solution (4.14) to (4.6). Now we begin the proof of (5.9):

First note that by the fundamental theorem of calculus, following a ∂_0 integral curve and employing (5.4), we readily obtain from our initial assumptions and (5.7) the pointwise bound

$$\sup_{\tau\in[0,T]} \|\overline{O} - I\|_{L^\infty(\Sigma_\tau)} \leq \varepsilon + CT\mathcal{E}_0 < 2\varepsilon, \tag{5.10}$$

provided $\alpha \geq \frac{1}{2} + 3 + \frac{1}{4}$ and $T < \frac{\varepsilon}{C\mathcal{E}_0}$.

All the more, directly from the ODE in (5.8) we deduce the estimate: [applying the bounds (5.4), (5.6) to $(\overline{O} - I)\overline{u}$ and employing the asymptotics (4.5)]

$$\|\partial_0(O_c^d)\|^2_{H^{2,\alpha}[\tau]} \lesssim \mathcal{E}_0^2 + \|O - I\|^2_{H^{2,\alpha+\frac{3}{2}}[\tau]} + \|u\|^2_{H^{2,\alpha}[\tau]}, \tag{5.11}$$

for all $\tau \in [0, T]$, $c, d = 0, 1, 2, 3$.[17]

[16] Any assumptions that we make on the functions $\overline{u}$, $\overline{O}$, we must derive for the next set of functions u, f below.

[17] This estimate, together with (5.9) in the end, imply the analogue of (5.7) for the functions $\partial_0(O_c^d)$.

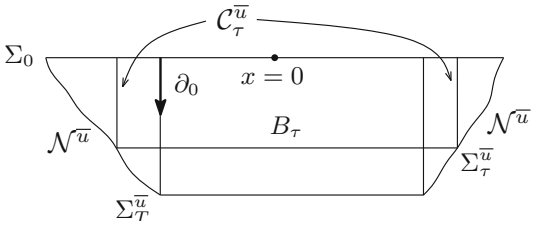

Fig. 6. The backward domain of dependence of Σ_0

We derive (5.9) in the backward domain of dependence of Σ_0 w.r.t. the metric $(g_{ab})_{\overline{u}} := g_{\overline{u}}(\partial_a, \partial_b)$, $a, b = 0, 1, 2, 3$, whose inverse is given by $g_{\overline{u}}^{ab} := \overline{h}^{ab}$; compare to (4.7). The boundary of the domain is the backward incoming $g_{\overline{u}}$-null hypersurface $\mathcal{N}^{\overline{u}}$ emanating from $\partial\Sigma_0$ (Fig. 6). We foliate the domain by the τ = const. hypersurfaces $\Sigma_\tau^{\overline{u}}$ inside $\mathcal{N}^{\overline{u}}$. Let ρ be the scalar function defined near $\mathcal{N}^{\overline{u}}$ via

$$\rho(\mathcal{C}_\tau^{\overline{u}}) := T - \tau, \tag{5.12}$$

where $\mathcal{C}_\tau^{\overline{u}}$ is the cylinder obtained from the flow of $\partial\Sigma_\tau^{\overline{u}}$ backwards along the integral curves of ∂_0. Using ρ we may write each leaf of the foliation as

$$\Sigma_\tau^{\overline{u}} = \bigcup_{t^* \in [\tau, T]} \{\rho_\tau = T - t^*\} \bigcup B_\tau \qquad \tau \in [0, T], \tag{5.13}$$

where $\rho_\tau := \rho\big|_{\Sigma_\tau^{\overline{u}}}$ and B_τ is simply the projection of $\Sigma_T^{\overline{u}}$ onto $\Sigma_\tau^{\overline{u}}$ through the integral curves of ∂_0.

Since by definition $\rho + \tau - T$ is zero on $\mathcal{N}^{\overline{u}}$, it follows that the $g_{\overline{u}}$-gradient of $\rho + \tau - T$, on $\mathcal{N}^{\overline{u}}$, lies on the hypersurface itself and furthermore it is $g_{\overline{u}}$-null, i.e., ρ satisfies the eikonal equation

$$\begin{aligned} \left|\nabla_{g_{\overline{u}}}(\rho + \tau - T)\right|^2_{g_{\overline{u}}} &= \overline{h}^{AB}\partial_A(\rho)\partial_B(\rho) + \Omega^{-2}\overline{h}^{00} + 2\Omega^{-1}\overline{h}^{A0}\partial_A(\rho) \\ &= 0 \quad \text{on } \mathcal{N}^{\overline{u}}, \end{aligned} \tag{5.14}$$

where $A, B = 1, 2, 3$.

Remark 5.2. The backward domain of definition of the variables u, $O - I$ depends on $\overline{u}$, $\overline{O} - I$. For the iteration scheme and the contraction mapping argument in Sect . 5.4 to be well-defined, all functions involved in the process must have a common domain of definition. To solve this issue is to begin with a slightly 'larger' initial hypersurface $\widetilde{\Sigma}_0 \supset \Sigma_0$ extending Σ_0 at both ends and to solve at each iteration step for the new variables in a 'smaller' domain contained in the interior of the domain of the previous iterate by shrinking the initial hypersurface $\widetilde{\Sigma}_0$. Since we are also proving a contraction mapping at the same time (see Sect. 5.4) we can make sure that the shrinking of $\widetilde{\Sigma}_0$ stops at Σ_0 giving the final backward domain in the limit.

We define the following adapted energy, which controls the part of the total energy (4.11) that refers to u:

$$E_{s+1,\alpha}[u](\tau) := \frac{1}{2}\sum_{\nu,i,j}\sum_{|J|\leq s}\int_{\Sigma_\tau^{\overline{u}}}\Big[-\overline{h}^{00}\frac{\big[\partial_0(u_\nu)_{ij,J}\big]^2}{r^{2\alpha-3|J|}} + \overline{h}^{AB}\frac{\partial_A(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}\frac{\partial_B(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}} + \frac{(u_\nu)^2_{ij,J}}{r^{2\alpha-3(|J|-1)}}\Big]d\mu_{s_{\overline{g}}}, \tag{5.15}$$

where $(u_\nu)_{ij,J} := \partial^{(J)}(u_\nu)_{ij}$ and J is a spatial multi-index (containing only directions $\partial_1, \partial_2, \partial_3$). It is evident from (5.10), $\overline{h}^{ab} = m^{cd}\overline{O}^a_c\overline{O}^b_d$, that $E_{3,\alpha}$ is equivalent to the weighted $H^{3,\alpha}\times H^{2,\alpha-\frac{3}{2}}$ norm of u on $\Sigma_\tau^{\overline{u}}$.

We summarize in the following proposition the main energy estimates derived below.

Proposition 5.3. *The following two energy estimates hold:*

$$\begin{aligned}&\partial_\tau E_{3,\alpha}[u] + 8M^2e^{-1}(1-\tau)\alpha E_{3,\alpha+1}[u]\\ &\lesssim (\mathcal{E}_0^{\frac{1}{2}} + \mathcal{E}_0 + \alpha^2 + \alpha^3\mathcal{E}_0)E_{3,\alpha}[u] + E_{3,\alpha+1}[u] + \mathcal{E}_0\|O-I\|^2_{H^{3,\alpha+\frac{3}{2}}}\\ &\quad + \|O-I\|^2_{H^{3,\alpha+\frac{5}{2}}} + \alpha^3\mathcal{E}_0^2 + \mathcal{E}_0^3 \qquad (5.16)\\ &\frac{1}{2}\partial_\tau\sum_{c,d}\|O^d_c - {I_c}^d\|^2_{H^{3,\alpha+\frac{3}{2}}} + 4M^2e^{-1}(1-\tau)\alpha\sum_{c,d}\|O^d_c - {I_c}^d\|^2_{H^{3,\alpha+\frac{5}{2}}}\\ &\lesssim \|O-I\|^2_{H^{3,\alpha+\frac{5}{2}}} + E_{3,\alpha+1}[u] + \mathcal{E}_0^2, \qquad (5.17)\end{aligned}$$

for all $\tau\in(0,T)$.

The overall energy estimate (5.9) follows from Proposition 5.3: adding (5.16), (5.17) we wish to close the estimate by employing the standard Gronwall lemma. However, this is not possible in general, because of the critical energies in the RHS, having larger weights than the ones differentiated in the LHS, namely, $E_{3,\alpha+1}[u]$, $\|O-I\|^2_{H^{3,\alpha+\frac{5}{2}}}$ instead of $E_{3,\alpha}[u]$, $\|O-I\|^2_{H^{3,\alpha+\frac{3}{2}}}$. It is precisely at this point that the role of the weights we introduced is revealed. Choosing $\alpha > 0$ large enough to begin with, how large depending on the constants in the above inequalities, we absorb the critical terms

$$E_{3,\alpha+1}[u],\ \|O-I\|^2_{H^{3,\alpha+\frac{5}{2}}}$$

in the LHS and then the standard Gronwall lemma applies to give (5.9).

Proof of (5.16). Let

$$P_{J,\alpha} := \frac{1}{2}\Big[-\overline{h}^{00}\frac{\big[\partial_0(u_\nu)_{ij,J}\big]^2}{r^{2\alpha-3|J|}} + \overline{h}^{AB}\frac{\partial_A(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}\frac{\partial_B(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}} + \frac{(u_\nu)^2_{ij,J}}{r^{2\alpha+3-3|J|}}\Big], \tag{5.18}$$

for any spatial multi-index J with $|J|\leq 2$; recall $(u_\nu)_{ij,J} := \partial^{(J)}(u_\nu)_{ij}$.

It follows from (5.13) and the coarea formula that

$$\partial_\tau \int_{\Sigma_\tau^{\overline{u}}} P_{J,\alpha} d\mu_{s_{\overline{g}}} = -\int_{\partial\Sigma_\tau^{\overline{u}}} \frac{P_{J,\alpha}}{|{}^S\overline{\nabla}\rho|} dS + \int_{\Sigma_\tau^{\overline{u}}} \partial_\tau P_{J,\alpha} d\mu_{s_{\overline{g}}} + \int_{\Sigma_\tau^{\overline{u}}} P_{J,\alpha} \partial_\tau d\mu_{s_{\overline{g}}}, \tag{5.19}$$

where ${}^S\overline{\nabla}\rho$ stands for the gradient of ρ with respect to the intrinsic connection on $(\Sigma_\tau, {}^S\overline{g})$ and dS is the Schwarzschild induced volume form on $\partial\Sigma_\tau^{\overline{u}}$. Note that the boundary term in (5.19) has a favourable sign. Since $\mathcal{N}^{\overline{u}}$ is $g_{\overline{u}}$-incoming null, the sum of all arising boundary terms should have a good sign and therefore can be dropped in the end. Indeed, this is the case and it can be easily seen by keeping track of the few boundary terms that appear below. To analyse the last two terms in (5.19), we recall the ∂_τ differentiation formulas of the radius function r (3.5), the estimate on volume form $d\mu_{s_{\overline{g}}}$ (3.9) and the commutator relation $[\partial_0, \partial_B] = {}^S(A_{[0]})_B{}^c\partial_c \overset{(3.15)}{=} \Gamma_{\frac{3}{2}}\partial$:

$$\begin{aligned}
&\int_{\Sigma_\tau^{\overline{u}}} \partial_\tau P_{J,\alpha} d\mu_{s_{\overline{g}}} + \int_{\Sigma_\tau^{\overline{u}}} P_{J,\alpha} \partial_\tau d\mu_{s_{\overline{g}}} \\
&= -8M^2(1-\tau)\alpha \int_{\Sigma_\tau^{\overline{u}}} e^{-\frac{r}{2M}} P_{J,\alpha+1} d\mu_{s_{\overline{g}}} + \int_{\Sigma_\tau^{\overline{u}}} P_{J,\alpha} O(\frac{1}{r^2}) d\mu_{s_{\overline{g}}} \\
&\quad + \frac{1}{2}\int_{\Sigma_\tau^{\overline{u}}} \Omega\Big[-\partial_0(\overline{h}^{00}) \frac{\big[\partial_0(u_\nu)_{ij,J}\big]^2}{r^{2\alpha-3|J|}} + \partial_0(\overline{h}^{AB}) \frac{\partial_A(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}} \frac{\partial_B(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}} \Big] d\mu_{s_{\overline{g}}} \\
&\quad + \int_{\Sigma_\tau^{\overline{u}}} \Omega\Big[-\overline{h}^{00} \frac{\partial_0(u_\nu)_{ij,J} \partial_0^2(u_\nu)_{ij,J}}{r^{2\alpha-3|J|}} + \overline{h}^{AB} \frac{\partial_A(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}} \frac{\partial_B\partial_0(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}} \Big] d\mu_{s_{\overline{g}}} \\
&\quad + \int_{\Sigma_\tau^{\overline{u}}} \Omega \overline{h}^{AB} \frac{\partial_A(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}} \frac{\Gamma_{\frac{3}{2}}\partial(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}} d\mu_{s_{\overline{g}}} + \int_{\Sigma_\tau^{\overline{u}}} \Omega \frac{(u_\nu)_{ij,J}\partial_0(u_\nu)_{ij,J}}{r^{2\alpha+3-|J|}} d\mu_{s_{\overline{g}}}
\end{aligned} \tag{5.20}$$

The first term on the LHS of (5.20) is critical having a favourable sign of magnitude α. We use this term alone to absorb all arising critical terms in the process. Recall $|\overline{h}| = |\overline{O}^2| \leq 1$, cf. (5.10), and the asymptotics (4.5). Also, applying (5.4) to $\partial_0\overline{h}$ and (5.7) we derive

$$|\Omega\partial_0(h)| \lesssim \mathcal{E}(\overline{u}, \overline{O}; \alpha, T)^{\frac{1}{2}}, \qquad \Omega \lesssim \frac{1}{r^{\frac{1}{2}}}, \qquad |\Gamma_{\frac{3}{2}}| \lesssim \frac{1}{r^{\frac{3}{2}}}.$$

Hence, by Cauchy's inequality and (5.6) we have

$$\begin{aligned}
&\frac{1}{2}\int_{\Sigma_\tau^{\overline{u}}} \Omega\Big[-\partial_0(\overline{h}^{00}) \frac{\big[\partial_0(u_\nu)_{ij,J}\big]^2}{r^{2\alpha-3|J|}} + \partial_0(\overline{h}^{AB}) \frac{\partial_A(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}} \frac{\partial_B(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}} \Big] d\mu_{s_{\overline{g}}} \\
&\quad + \int_{\Sigma_\tau^{\overline{u}}} \Omega \overline{h}^{AB} \frac{\partial_A(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}} \frac{\Gamma_{\frac{3}{2}}\partial(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}} d\mu_{s_{\overline{g}}} + \int_{\Sigma_\tau^{\overline{u}}} \Omega \frac{(u_\nu)_{ij,J}\partial_0(u_\nu)_{ij,J}}{r^{2\alpha+3-|J|}} d\mu_{s_{\overline{g}}} \\
&\quad \lesssim \mathcal{E}(\overline{u}, \overline{O}; \alpha, T)^{\frac{1}{2}} E_{3,\alpha}[u] + E_{3,\alpha+1}[u] \\
&\quad \lesssim \mathcal{E}_0^{\frac{1}{2}} E_{3,\alpha}[u] + E_{3,\alpha+1}[u]
\end{aligned} \tag{5.21}$$

For the next term we proceed by integrating by parts[18] (IBP), denoting by $N := \overline{g}^{BB} N_B \partial_B$ the outward unit normal on $\partial \Sigma_\tau^{\overline{u}}$ w.r.t. Schwarzschild metric $\overline{g}$ on $\Sigma_\tau^{\overline{u}}$:

$$\begin{aligned}
&\int_{\Sigma_\tau^{\overline{u}}} \Omega \Bigg[-\overline{h}^{00} \frac{\partial_0 (u_\nu)_{ij,J} \partial_0^2 (u_\nu)_{ij,J}}{r^{2\alpha-3|J|}} + \overline{h}^{AB} \frac{\partial_A (u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}} \frac{\partial_B \partial_0 (u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}} \Bigg] d\mu_{s_{\overline{g}}} \\
&= -\int_{\Sigma_\tau^{\overline{u}}} \Omega \frac{\partial_0 (u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}} \Bigg[\overline{h}^{00} \frac{\partial_0^2 (u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}} + \overline{h}^{AB} \frac{\partial_B \partial_A (u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}} \Bigg] d\mu_{s_{\overline{g}}} \\
&\quad + \int_{\partial \Sigma_\tau^{\overline{u}}} \Omega \overline{h}^{AB} \frac{\partial_A (u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}} \frac{\partial_0 (u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}} N_B dS \\
&\quad - \int_{\Sigma_\tau^{\overline{u}}} \Bigg[\partial_B \Big(\frac{\Omega \overline{h}^{AB}}{r^{2\alpha-3|J|}} \Big) \partial_A (u_\nu)_{ij,J} + \Omega \overline{h} \frac{\Gamma_{\frac{3}{2}} \partial (u_\nu)_{ij,J}}{r^{2\alpha-3|J|}} \Bigg] \partial_0 (u_\nu)_{ij,J} d\mu_{s_{\overline{g}}} \qquad (5.22)
\end{aligned}$$

It is immediate from the definition of the frame (3.10) and (3.5) that

$$\Big| \partial_1 \Big(\frac{\Omega}{r^{2\alpha-3}} \Big) \Big| \lesssim \frac{\alpha}{r^{2\alpha-2}} \qquad \partial_2 \Big(\frac{\Omega}{r^{\alpha-\frac{3}{2}}} \Big) = \partial_3 \Big(\frac{\Omega}{r^{\alpha-\frac{3}{2}}} \Big) = 0.$$

Hence, similarly to (5.21)

$$\begin{aligned}
&-\int_{\Sigma_\tau^{\overline{u}}} \Bigg[\partial_B \Big(\frac{\Omega \overline{h}^{AB}}{r^{2\alpha-3|J|}} \Big) \partial_A (u_\nu)_{ij,J} + \Omega \overline{h} \frac{\Gamma_{\frac{3}{2}} \partial (u_\nu)_{ij,J}}{r^{2\alpha-3|J|}} \Bigg] \partial_0 (u_\nu)_{ij,J} d\mu_{s_{\overline{g}}} \qquad (5.23) \\
&\lesssim (\mathcal{E}_0^{\frac{1}{2}} + \alpha^2) E_{3,\alpha}[u] + E_{3,\alpha+1}[u]. \qquad (|J| \leq 2)
\end{aligned}$$

□

Remark. The term in the RHS of the preceding estimate with coefficient α^2 is not critical. This is very important otherwise the overall estimates would not close, since the critical term with favourable sign in (5.20) is only of magnitude α.

We proceed to the boundary term in the RHS of (5.22). Recall that ρ is constant on $\partial \Sigma_\tau^{\overline{u}}$ (5.12), and decreasing in the interior direction of $\Sigma_\tau^{\overline{u}}$. Hence, the outward unit normal N is the Schwarzschild normalized gradient of ρ on $\Sigma_\tau^{\overline{u}}$, $N = \frac{{}^S\overline{\nabla}\rho}{|{}^S\overline{\nabla}\rho|}$. Since $(\overline{h}^{AB})_{A,B=1,2,3}$ is a symmetric positive definite matrix, the following standard inequality holds:

$$\begin{aligned}
\Bigg| \overline{h}^{AB} \frac{\partial_A (u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}} \Omega N_B \Bigg|^2 &\leq \Bigg(\overline{h}^{AB} \frac{\partial_A (u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}} \frac{\partial_B (u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}} \Bigg) \big(\Omega^2 \overline{h}^{AB} N_A N_B \big) \\
&= \Bigg(\overline{h}^{AB} \frac{\partial_A (u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}} \frac{\partial_B (u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}} \Bigg) \frac{\Omega^2 \overline{h}^{AB} \partial_A(\rho) \partial_B(\rho)}{|{}^S\overline{\nabla}\rho|^2} \\
&= \Bigg(\overline{h}^{AB} \frac{\partial_A (u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}} \frac{\partial_B (u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}} \Bigg) \frac{-\overline{h}^{00} - 2\Omega \overline{h}^{A0} \partial_A(\rho)}{|{}^S\overline{\nabla}\rho|^2} \\
&\qquad \text{(by (5.14))}
\end{aligned}$$

[18] We integrate by parts using the spatial part of the Schwarzschild frame $\partial_1, \partial_2, \partial_3$. Doing so we pick up connection coefficients, since it is not covariant IBP.

Therefore, we have the bound

$$\begin{aligned}
&\int_{\partial\Sigma_\tau^{\overline{u}}} \Omega\overline{h}^{AB}\frac{\partial_A(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}\frac{\partial_0(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}N_B dS\\
&\quad\leq \int_{\partial\Sigma_\tau^{\overline{u}}} \Big|\frac{\partial_0(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}\Big|\sqrt{\frac{-\overline{h}^{00}-2\Omega\overline{h}^{A0}\partial_A(\rho)}{|{}^S\overline{\nabla}\rho|}}\sqrt{\frac{\overline{h}^{AB}}{|{}^S\overline{\nabla}\rho|}\frac{\partial_A(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}\frac{\partial_B(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}}dS\\
&\quad\leq \frac{1}{2}\int_{\partial\Sigma_\tau^{\overline{u}}} -\frac{\overline{h}^{00}}{|{}^S\overline{\nabla}\rho_\tau|}\frac{\big[\partial_0(u_\nu)_{ij,J}\big]^2}{r^{2\alpha-3|J|}}-\frac{2\Omega\overline{h}^{A0}\partial_A(\rho)}{|{}^S\overline{\nabla}\rho_\tau|}\frac{\big[\partial_0(u_\nu)_{ij,J}\big]^2}{r^{2\alpha-3|J|}}dS\\
&\qquad+\frac{1}{2}\int_{\partial\Sigma_\tau^{\overline{u}}}\frac{\overline{h}^{AB}}{|{}^S\overline{\nabla}\rho|}\frac{\partial_A(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}\frac{\partial_B(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}dS
\end{aligned}\tag{5.24}$$

The remaining term to be estimated is the one on first line in the RHS of (5.22), which we rewrite

$$\begin{aligned}
&-\int_{\Sigma_\tau^{\overline{u}}}\Omega\Big[\overline{h}^{00}\frac{\partial_0^2(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}+\overline{h}^{AB}\frac{\partial_B\partial_A(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}\Big]\frac{\partial_0(u_\nu)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|}}d\mu_{S_{\overline{g}}}\\
&\quad=-\int_{\Sigma_\tau^{\overline{u}}}\big(\overline{h}^{ab}\partial_a\partial_b(u_\nu)_{ij,J}\big)\Omega\frac{\partial_0(u_\nu)_{ij,J}}{r^{2\alpha-3|J|}}d\mu_{S_{\overline{g}}}\\
&\qquad+\int_{\Sigma_\tau^{\overline{u}}}2\Omega\overline{h}^{A0}\frac{\partial_A\partial_0(u_\nu)_{ij,J}\partial_0(u_\nu)_{ij,J}}{r^{2\alpha-3|J|}}+\Omega h\frac{\Gamma_{\frac{3}{2}}\partial(u_\nu)_{ij,J}\partial_0(u_\nu)_{ij,J}}{r^{2\alpha-3|J|}}d\mu_{S_{\overline{g}}}
\end{aligned}\tag{5.25}$$

By taking the $\partial^{(J)}$ derivative (J spatial multi-index $|J|\leq 2$) of the first equation in (5.8) and commuting the differentiation in the LHS we obtain the equation

$$\begin{aligned}
&\overline{h}^{ab}\partial_a\partial_b(u_\nu)_{ij,J}\\
&\quad=\partial^{(J)}\Big[\overline{O}\Gamma_{\frac{3}{2}}\partial u+\overline{O}\Gamma_3 u+\overline{O}\Gamma_{\frac{9}{2}}(O-I)+\overline{O}\Gamma_3\partial(O-I)\\
&\qquad+\Gamma_3\overline{u}^2+\overline{O}\overline{u}\partial\overline{u}+\overline{u}^3+\overline{O}\partial(\overline{O}-I)\partial\overline{u}\Big]+[\overline{h}^{ab}\partial_a\partial_b,\partial^{(J)}](u_\nu)_{ij},
\end{aligned}\tag{5.26}$$

where the commutator can in turn be written schematically as: [recall (3.15),(4.5)]

$$\begin{aligned}
[\overline{h}^{ab}\partial_a\partial_b,\partial^{(J)}](u_\nu)_{ij}&=\partial^2(\overline{h})\partial^2(u_\nu)_{ij}+\Big[\Gamma_{\frac{3}{2}}\partial(\overline{h})+\Gamma_3\overline{h}\Big]\partial^2(u_\nu)_{ij}\\
&\quad+\partial(\overline{h})\partial^3(u_\nu)_{ij}+\overline{h}\Gamma_{\frac{3}{2}}\partial^3(u_\nu)_{ij}\\
&\quad+\Big[\partial(\overline{h})\Gamma_3+\overline{h}\Gamma_{\frac{9}{2}}\Big]\partial(u_\nu)_{ij}\qquad\text{if } |J|=2\\
[\overline{h}^{ab}\partial_a\partial_b,\partial^{(J)}](u_\nu)_{ij}&=\partial(\overline{h})\partial^2(u_\nu)_{ij}+\overline{h}\Gamma_{\frac{3}{2}}\partial^2(u_\nu)_{ij}\qquad\text{if } |J|=1\\
&\quad+\overline{h}\Gamma_3\partial(u_\nu)_{ij}
\end{aligned}\tag{5.27}$$

We integrate by parts in the second term on the RHS of (5.25) and argue similarly to (5.23) to get

$$\begin{aligned}
&\int_{\Sigma_\tau^{\overline{u}}} 2\Omega\overline{h}^{A0}\frac{\partial_A\partial_0(u_v)_{ij,J}\partial_0(u_v)_{ij,J}}{r^{2\alpha-3|J|}}+\Omega\overline{h}\frac{\Gamma_{\frac{3}{2}}\partial(u_v)_{ij,J}\partial_0(u_v)_{ij,J}}{r^{2\alpha-3|J|}}d\mu_{s_{\overline{g}}}\\
&=\int_{\partial\Sigma_\tau^{\overline{u}}}\Omega\overline{h}^{A0}\frac{\left[\partial_0(u_v)_{ij,J}\right]^2}{r^{2\alpha-3|J|}}N_a dS-\int_{\Sigma_\tau^{\overline{u}}}\Omega\partial_A(\overline{h}^{A0})\frac{\left[\partial_0(u_v)_{ij,J}\right]^2}{r^{2\alpha-3|J|}}d\mu_{s_{\overline{g}}}\\
&\quad-\int_{\Sigma_\tau^{\overline{u}}}\overline{h}^{A0}\partial_A\Big(\frac{\Omega}{r^{2\alpha-3|J|}}\Big)\left[\partial_0(u_v)_{ij,J}\right]^2 d\mu_{s_{\overline{g}}}+\int_{\Sigma_\tau^{\overline{u}}}\Omega\overline{h}\Gamma_{\frac{3}{2}}\frac{\left[\partial_0(u_v)_{ij,J}\right]^2}{r^{2\alpha-3|J|}}d\mu_{s_{\overline{g}}}\\
&\quad+\int_{\Sigma_\tau^{\overline{u}}}\Omega\overline{h}\frac{\Gamma_{\frac{3}{2}}\partial(u_v)_{ij,J}\partial_0(u_v)_{ij,J}}{r^{2\alpha-3|J|}}d\mu_{s_{\overline{g}}}\\
&\leq\int_{\partial\Sigma_\tau^{\overline{u}}}\Omega\overline{h}^{A0}\frac{\left[\partial_0(u_v)_{ij,J}\right]^2}{r^{2\alpha-3|J|}}N_A dS+C(\mathcal{E}_0^{\frac{1}{2}}+\alpha^2)E_{3,\alpha}[u]+CE_{3,\alpha+1}[u] \qquad (5.28)
\end{aligned}$$

Finally, for the last and main term in the first line of the RHS of (5.25) we recall that $|\Omega|\lesssim\frac{1}{r^{\frac{1}{2}}}$ to obtain directly from Cauchy's inequality

$$\begin{aligned}
&-\int_{\Sigma_\tau^{\overline{u}}}\big(\overline{h}^{ab}\partial_a\partial_b(u_v)_{ij,J}\big)\Omega\frac{\partial_0(u_v)_{ij,J}}{r^{2\alpha-3|J|}}d\mu_{s_{\overline{g}}}\\
&\lesssim\Big\|\frac{\overline{h}^{ab}\partial_a\partial_b(u_v)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|-\frac{1}{2}}}\Big\|_{L^2}^2+\Big\|\frac{\partial_0(u_v)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|+1}}\Big\|_{L^2}^2\\
&\lesssim\Big\|\frac{\overline{h}^{ab}\partial_a\partial_b(u_v)_{ij,J}}{r^{\alpha-\frac{3}{2}|J|-\frac{1}{2}}}\Big\|_{L^2}^2+\|\partial_0(u_v)_{ij}\|_{H^{2,\alpha-\frac{1}{2}}}^2 \qquad (5.29)
\end{aligned}$$

We proceed by plugging the RHS of (5.26) into the first term in the last inequality (5.29) above and treat each arising group of terms separately. Employing the basic inequalities in Proposition 5.1 along with the bounds of $\overline{O}$, $\partial(\overline{O})$, $\overline{u}$ (5.6), (5.7) and (5.10) we derive:

$$\begin{aligned}
&\Big\|\frac{\partial^{(J)}\left[\overline{O}\Gamma_{\frac{3}{2}}\partial u+\overline{O}\Gamma_3 u\right]}{r^{\alpha-\frac{3}{2}|J|-\frac{1}{2}}}\Big\|_{L^2}^2\\
&\lesssim\|\overline{O}\|_{L^\infty}^2E_{3,\alpha+1}[u]+\|\partial(\overline{O})\|_{L^\infty}^2E_{2,\alpha}[u]+\|u\|_{H^{2,\alpha+1}}^2\\
&\quad+\Big\|\frac{\partial^{(J)}(\overline{O})\Gamma_{\frac{3}{2}}\partial u}{r^{\alpha-\frac{3}{2}|J|-\frac{1}{2}}}\Big\|_{L^2}^2+\Big\|\frac{\partial^{(J)}(\overline{O})\Gamma_3 u}{r^{\alpha-\frac{3}{2}|J|-\frac{1}{2}}}\Big\|_{L^2}^2 \qquad (5.30)
\end{aligned}$$

(the last two terms appear only in the case $|J|=2$)

$$\begin{aligned}
&\lesssim\mathcal{E}(\overline{u},\overline{O};\alpha,T)E_{3,\alpha}[u]+E_{3,\alpha+1}[u]+\big(\|u\|_{L^\infty}^2+\|\partial u\|_{L^\infty}^2\big)\Big\|\frac{\partial^{(J)}(\overline{O})}{r^{\alpha-\frac{1}{2}}}\Big\|_{L^2}^2\\
&\lesssim\mathcal{E}_0E_{3,\alpha}[u]+E_{3,\alpha+1}[u]\\
&\Big\|\frac{\partial^{(J)}\left[\overline{O}\Gamma_{\frac{9}{2}}(O-I)+\overline{O}\Gamma_3\partial(O-I)\right]}{r^{\alpha-\frac{3}{2}|J|-\frac{1}{2}}}\Big\|_{L^2}^2
\end{aligned}$$

$$\begin{aligned}
&\lesssim \|O-I\|^2_{H^{3,\alpha+\frac{5}{2}}} + \|\partial\overline{O}\|^2_{L^\infty}\|O-I\|^2_{H^{2,\alpha+\frac{3}{2}}} \\
&\quad + \Big(\|\frac{O-I}{r^{\frac{3}{2}}}\|^2_{L^\infty} + \|\partial(O-I)\|^2_{L^\infty}\Big)\|\frac{\partial^{(J)}(\overline{O})}{r^{\alpha-\frac{1}{2}}}\|^2_{L^2}
\end{aligned}$$

(we include the last term only when $|J|=2$ and utilize (5.7),(5.11))

$$\begin{aligned}
&\lesssim \|O-I\|^2_{H^{3,\alpha+\frac{5}{2}}} + (\mathcal{E}_0^2+\mathcal{E}_0)\|O-I\|^2_{H^{2,\alpha+\frac{3}{2}}} + \mathcal{E}_0\|O-I\|^2_{H^{3,\alpha+\frac{3}{2}}} \\
&\quad + \mathcal{E}_0^3 + \mathcal{E}_0\|u\|^2_{H^{2,\alpha}} \qquad (5.31)
\end{aligned}$$

$$\begin{aligned}
&\Big\|\frac{\partial^{(J)}\big[\Gamma_3\overline{u}^2 + \overline{O}\overline{u}\partial\overline{u} + \overline{u}^3 + \overline{O}\partial(\overline{O}-I)\partial\overline{u}\big]}{r^{\alpha-\frac{3}{2}|J|-\frac{1}{2}}}\Big\|^2_{L^2} \\
&\lesssim \Big[\|\frac{\overline{u}}{r^{\frac{3}{2}}}\|^2_{L^\infty} + \|\frac{\partial\overline{u}}{r^{\frac{3}{2}}}\|^2_{L^\infty} + \|\partial\overline{O}\|^2_{L^\infty}\big(\|\overline{u}\|^2_{L^\infty} + \|\partial\overline{u}\|^2_{L^\infty}\big)\Big]E_{3,\alpha}[\overline{u}] \\
&\quad + \|\overline{u}\|^2_{L^\infty}\|\partial\overline{u}\|^2_{L^\infty}\|\overline{O}-I\|^2_{H^{2,\alpha+\frac{3}{2}}} + \big(\|\overline{u}\|^4_{L^\infty} + \|\overline{u}\|^2_{L^\infty}\|\partial\overline{u}\|^2_{L^\infty}\big)\|\overline{u}\|^2_{H^{2,\alpha}} \\
&\quad + \|\partial\overline{O}\|^2_{L^\infty}E_{3,\alpha}[\overline{u}] + \|\frac{\partial^2(\overline{O}-I)\partial^2\overline{u}}{r^{\alpha-3-\frac{1}{2}}}\|^2_{L^2} + \|\partial\overline{u}\|^2_{L^\infty}\|\overline{O}-I\|^2_{H^{3,\alpha+\frac{3}{2}}} \\
&\quad + \big(\|\partial\overline{O}\|^4_{L^\infty} + \|\partial\overline{O}\|^2_{L^\infty}\|\partial\overline{u}\|^2_{L^\infty}\big)\big(\|\overline{O}-I\|^2_{H^{2,\alpha+\frac{3}{2}}} + E_{2,\alpha}[\overline{u}]\big) \\
&\lesssim \mathcal{E}(\overline{u},\overline{O};\alpha,T)^2 + \mathcal{E}(\overline{u},\overline{O};\alpha,T)^3 + \|\frac{\partial^2(\overline{O})}{r^{\frac{\alpha}{2}-\frac{3}{2}-\frac{1}{4}}}\|^2_{L^4}\|\frac{\partial^2(\overline{u})}{r^{\frac{\alpha}{2}-\frac{3}{2}-\frac{1}{4}}}\|^2_{L^4} \\
&\text{(employing the } L^4 \text{ estimate (5.5))} \lesssim \mathcal{E}_0^2 + \mathcal{E}_0^3 + \alpha^3\mathcal{E}_0^2 \qquad (5.32)
\end{aligned}$$

By (5.30)–(5.32) we have the following lemma.

Lemma 5.4. $\partial_0^2(u_v)_{ij} \in C([0,T];H^{1,\alpha-3}) \cap L^2([0,T];H^{1,\alpha-2})$ *and moreover the following estimate holds:*

$$\begin{aligned}
&\Big\|\frac{\partial^{(J)}\partial_0^2(u_v)_{ij}}{r^{\alpha-\frac{3}{2}|J|-\frac{1}{2}}}\Big\|^2_{L^2[\tau]} \\
&\lesssim \mathcal{E}_0\big(E_{3,\alpha}[u] + \|O-I\|^2_{H^{3,\alpha+\frac{3}{2}}}\big) + E_{3,\alpha+1}[u] + \|O-I\|^2_{H^{3,\alpha+\frac{5}{2}}} \\
&\quad + \alpha^3\mathcal{E}_0^2 + \mathcal{E}_0^3, \qquad (5.33)
\end{aligned}$$

for $|J|\le 1$, $J\subset\{1,2,3\}$, $\tau\in(0,T)$.

Proof. The proof follows by solving for $\partial_0^2(u_v)_{ij}$ in the Eq. (5.8) and summing up the above estimates (5.30)–(5.32). □

To bound the commutator (5.27) we treat the cases $|J|=2$, $|J|=1$ separately. For $|J|=1$:

$$\begin{aligned}
&\Big\| \frac{[\overline{h}^{ab}\partial_a\partial_b, \partial^{(J)}](u_\nu)_{ij}}{r^{\alpha-\frac{3}{2}-\frac{1}{2}}} \Big\|_{L^2}^2 \\
&= \Big\| \frac{\partial(\overline{h})\partial^2(u_\nu)_{ij} + \overline{h}\Gamma_{\frac{3}{2}}\partial^2(u_\nu)_{ij} + \overline{h}\Gamma_3\partial(u_\nu)_{ij}}{r^{\alpha-\frac{3}{2}-\frac{1}{2}}} \Big\|_{L^2}^2 \\
&\lesssim \|\partial\overline{h}\|_{L^\infty}^2 \Big\| \frac{\partial^2(u_\nu)_{ij}}{r^{\alpha-\frac{3}{2}-\frac{1}{2}}} \Big\|_{L^2}^2 + \|\overline{h}\|_{L^\infty}^2 \Big\| \frac{\partial^2(u_\nu)_{ij}}{r^{\alpha-\frac{1}{2}}} \Big\|_{L^2}^2 + \|\overline{h}\|_{L^\infty}^2 \Big\| \frac{\partial(u_\nu)_{ij}}{r^{\alpha+1}} \Big\|_{L^2}^2
\end{aligned}$$

(employing (5.33) in the case $\partial^2(u_\nu)_{ij} = \partial_0^2(u_\nu)_{ij}$)

$$\begin{aligned}
&\lesssim \mathcal{E}_0\big(E_{3,\alpha}[u] + \|O - I\|^2_{H^{3,\alpha+\frac{3}{2}}}\big) + E_{3,\alpha+1}[u] + \|O - I\|^2_{H^{3,\alpha+\frac{5}{2}}} \\
&\quad + \alpha^3\mathcal{E}_0^2 + \mathcal{E}_0^3
\end{aligned} \tag{5.34}$$

When $|J| = 2$ we have

$$\begin{aligned}
&\Big\| \frac{[\overline{h}^{ab}\partial_a\partial_b, \partial^{(J)}](u_\nu)_{ij}}{r^{\alpha-3-\frac{1}{2}}} \Big\|_{L^2}^2 \\
&\lesssim \Big\| \frac{\partial^2(\overline{h})\partial^2(u_\nu)_{ij} + \big[\Gamma_{\frac{3}{2}}\partial(\overline{h}) + \Gamma_3\overline{h}\big]\partial^2(u_\nu)_{ij}}{r^{\alpha-3-\frac{1}{2}}} \Big\|_{L^2}^2 \\
&\quad + \Big\| \frac{\partial(\overline{h})\partial^3(u_\nu)_{ij} + \overline{h}\Gamma_{\frac{3}{2}}\partial^3(u_\nu)_{ij} + \big[\partial(\overline{h})\Gamma_3 + \overline{h}\Gamma_{\frac{9}{2}}\big]\partial(u_\nu)_{ij}}{r^{\alpha-3-\frac{1}{2}}} \Big\|_{L^2}^2
\end{aligned}$$

(note that term $\partial^3(u_\nu)_{ij}$ contains at most two ∂_0 derivatives)

$$\begin{aligned}
&\lesssim \Big\| \frac{\partial^2\overline{h}}{r^{\frac{\alpha}{2}-\frac{3}{2}-\frac{1}{4}}} \Big\|_{L^4}^2 \Big\| \frac{\partial^2(u_\nu)_{ij}}{r^{\frac{\alpha}{2}-\frac{3}{2}-\frac{1}{4}}} \Big\|_{L^4}^2 + \|\partial\overline{h}\|_{L^2}^2\big(E_{3,\alpha}[u] + \|\partial_0^2(u_\nu)_{ij}\|^2_{H^{1,\alpha-3}}\big) \\
&\quad + \|\overline{h}\|_{L^\infty}^2\big(E_{3,\alpha+1}[u] + \|\partial_0^2(u_\nu)_{ij}\|^2_{H^{1,\alpha-2}}\big)
\end{aligned}$$

(employing the L^4 estimate (5.5) and (5.33))

$$\begin{aligned}
&\lesssim \alpha^3\mathcal{E}_0E_{3,\alpha}[u] + \mathcal{E}_0E_{3,\alpha}[u] + E_{3,\alpha+1}[u] + \mathcal{E}_0\|O - I\|^2_{H^{3,\alpha+\frac{3}{2}}} + \|O - I\|^2_{H^{3,\alpha+\frac{5}{2}}} \\
&\quad + \alpha^3\mathcal{E}_0^2 + \mathcal{E}_0^3
\end{aligned} \tag{5.35}$$

Summary Incorporating (5.20)–(5.35) in (5.19) we conclude that

$$\begin{aligned}
&\partial_\tau \int_{\Sigma_\tau^{\overline{u}}} P_{J,\alpha} d\mu_{s_{\overline{g}}} + 8M^2e^{-1}(1-\tau)\alpha \int_{\Sigma_\tau^{\overline{u}}} P_{J,\alpha+1} d\mu_{s_{\overline{g}}} \\
&\lesssim (\mathcal{E}_0^{\frac{1}{2}} + \mathcal{E}_0 + \alpha^2 + \alpha^3\mathcal{E}_0)E_{3,\alpha}[u] + E_{3,\alpha+1}[u] + \|O - I\|^2_{H^{3,\alpha+\frac{5}{2}}} \\
&\quad + \mathcal{E}_0\|O - I\|^2_{H^{3,\alpha+\frac{3}{2}}} + \alpha^3\mathcal{E}_0^2 + \mathcal{E}_0^3
\end{aligned} \tag{5.36}$$

Summing over the indices ν, i, j and $J, |J| \leq 2$, we arrive at the desired estimate (5.16).

Proof of (5.17). Let $J, |J| \leq 3$, be a spatial multi-index. Like in the case of (5.19), it follows from the coarea formula and the asymptotics (3.5), (3.9) that

$$
\begin{aligned}
&\frac{1}{2}\partial_\tau \| \frac{O^d_{c,J} - {I_c}^d{}_{,J}}{r^{\alpha+3-\frac{3}{2}|J|}} \|^2_{L^2(\Sigma^{\overline{u}}_\tau)} \\
&= -\frac{1}{2}\int_{\partial\Sigma^{\overline{u}}_\tau} \frac{(O^d_{c,J} - {I_c}^d{}_{,J})^2}{|{}^S\overline{\nabla}\rho| r^{2\alpha+6-3|J|}} dS \\
&\quad -(\alpha+3-\frac{3|J|}{2})\int_{\Sigma^{\overline{u}}_\tau} \frac{(O^d_{c,J} - {I_c}^d{}_{,J})^2}{r^{2\alpha+7-3|J|}} \partial_\tau r d\mu_{s_{\overline{g}}} \\
&\quad + \int_{\Sigma^{\overline{u}}_\tau} \Omega \frac{(O^d_{c,J} - {I_c}^d{}_{,J})\partial_0(O^d_{c,J})}{r^{2\alpha+6-3|J|}} d\mu_{s_{\overline{g}}} + \frac{1}{2}\int_{\Sigma^{\overline{u}}_\tau} \frac{(O^d_{c,J} - {I_c}^d{}_{,J})^2}{r^{2\alpha+6-3|J|}} \partial_\tau d\mu_{s_{\overline{g}}} \\
&\leq -4M^2 e^{-1}\alpha(1-\tau) \| \frac{O^d_{c,J} - {I_c}^d{}_{,J}}{r^{\alpha+4-\frac{3}{2}|J|}} \|^2_{L^2(\Sigma^{\overline{u}}_\tau)} \\
&\quad + \int_{\Sigma^{\overline{u}}_\tau} \Omega \frac{(O^d_{c,J} - {I_c}^d{}_{,J})\partial_0(O^d_{c,J})}{r^{2\alpha+6-3|J|}} d\mu_{s_{\overline{g}}} + C \| \frac{O^d_{c,J} - {I_c}^d{}_{,J}}{r^{\alpha+4-\frac{3}{2}|J|}} \|^2_{L^2(\Sigma^{\overline{u}}_\tau)}, \qquad (5.37)
\end{aligned}
$$

where $O^d_{c,J} - {I_c}^d{}_{,J} := \partial^{(J)}(O^d_c - {I_c}^d)$. By Cauchy's inequality we have ($\Omega \lesssim \frac{1}{r^{\frac{1}{2}}}$)

$$
\int_{\Sigma^{\overline{u}}_\tau} \Omega \frac{(O^d_{c,J} - {I_c}^d{}_{,J})\partial_0(O^d_{c,J})}{r^{2\alpha+6-3|J|}} d\mu_{s_{\overline{g}}} \lesssim \| \frac{O^d_{c,J} - {I_c}^d{}_{,J}}{r^{\alpha+4-\frac{3}{2}|J|}} \|^2_{L^2} + \| \frac{\partial_0(O^d_{c,J})}{r^{\alpha+\frac{5}{2}-\frac{3}{2}|J|}} \|^2_{L^2} \qquad (5.38)
$$

Taking the $\partial^{(J)}$ derivative of the ODE in (5.8) we obtain

$$
\partial_0(O^d_{c,J} - {I_c}^d{}_{,J}) = \partial^{(J)}\left[\Gamma_{\frac{3}{2}}(O-I) + (\overline{O}-I)\overline{u} + u\right] + [\partial_0, \partial^{(J)}](O^d_c - {I_c}^d) \qquad (5.39)
$$

The commutator in the RHS of (5.39) schematically reads

$$
\begin{aligned}
[\partial_0, \partial^{(J)}](O^d_c - {I_c}^d) &= \Gamma_{\frac{3}{2}}\partial(O^d_c - {I_c}^d) && \text{if } |J| = 1 \\
&= \Gamma_3 \partial(O^d_c - {I_c}^d) + \Gamma_{\frac{3}{2}}\partial^2(O^d_c - {I_c}^d) && \text{if } |J| = 2 \\
&= \Gamma_{\frac{9}{2}}\partial(O^d_c - {I_c}^d) + \Gamma_3\partial^2(O^d_c - {I_c}^d) && \text{if } |J| = 3 \\
&\quad + \Gamma_{\frac{3}{2}}\partial^3(O^d_c - {I_c}^d), && \qquad (5.40)
\end{aligned}
$$

where we note that at most one ∂_0 derivative of $O^d_c - {I_c}^d$ appears in the preceding expressions. Hence, we deduce directly from (5.39):

$$
\begin{aligned}
&\| \frac{\partial_0(O^d_{c,J} - {I_c}^d{}_{,J})}{r^{\alpha+\frac{5}{2}-\frac{3}{2}|J|}} \|^2_{L^2} \\
&\lesssim \|\Gamma_{\frac{3}{2}}(O-I)\|^2_{H^{3,\alpha+1}} + \|(\overline{O}-I)\overline{u}\|^2_{H^{3,\alpha+1}} + +\|u\|^2_{H^{3,\alpha+1}} \\
&\quad + \| \frac{[\partial_0, \partial^{(J)}](O^d_c - {I_c}^d)}{r^{\alpha+\frac{5}{2}-\frac{3}{2}|J|}} \|^2_{L^2}
\end{aligned}
$$

(exmploying Lemma 4.5 and applying the L^∞ bound on $(\overline{O}-I)\overline{u}$)

$$\begin{aligned}
&\lesssim \|O-I\|^2_{H^{3,\alpha+\frac{5}{2}}} + E_{3,\alpha+1}[u] + \|\overline{O}-I\|^2_{H^{3,\alpha+\frac{3}{2}}}\|\overline{u}\|^2_{H^{3,\alpha}} \\
&\quad + \|\frac{\Gamma_{\frac{3}{2}}\partial(O_c^d - I_c{}^d)}{r^{\alpha+\frac{5}{2}-\frac{3}{2}}}\|^2_{L^2} + \|\frac{\Gamma_3\partial(O_c^d - I_c{}^d) + \Gamma_{\frac{3}{2}}\partial^2(O_c^d - I_c{}^d)}{r^{\alpha+\frac{5}{2}-3}}\|^2_{L^2} \\
&\quad + \|\frac{\Gamma_{\frac{9}{2}}\partial(O_c^d - I_c{}^d) + \Gamma_3\partial^2(O_c^d - I_c{}^d) + \Gamma_{\frac{3}{2}}\partial^3(O_c^d - I_c{}^d)}{r^{\alpha+\frac{5}{2}-\frac{9}{2}}}\|^2_{L^2} \\
&\lesssim \|O-I\|^2_{H^{3,\alpha+\frac{5}{2}}} + \mathcal{E}(\overline{u},\overline{O};\alpha,T)^2 + E_{3,\alpha+1}[u] \qquad (5.41)
\end{aligned}$$

Combining (5.37)–(5.41) we derive

$$\begin{aligned}
&\frac{1}{2}\partial_\tau\|\frac{O^d_{c,J} - I_c{}^d{}_{,J}}{r^{\alpha+3-\frac{3}{2}|J|}}\|^2_{L^2(\Sigma^{\overline{u}}_\tau)} + 4M^2e^{-1}\alpha(1-\tau)\|\frac{O^d_{c,J} - I_c{}^d{}_{,J}}{r^{\alpha+4-\frac{3}{2}|J|}}\|^2_{L^2(\Sigma^{\overline{u}}_\tau)} \\
&\quad \lesssim \|O-I\|^2_{H^{3,\alpha+\frac{5}{2}}} + E_{3,\alpha+1}[u] + \mathcal{E}_0^2 \qquad (5.42)
\end{aligned}$$

Taking into account the set of indices c, d and J, $|J| \leq 3$, we complete the proof of (5.17) and hence of Proposition 5.3. □

5.4. Contraction mapping in $H^{2,\alpha}$. We proceed to show that the mapping defined via (5.8) in the beginning of Sect. 5.3 is a contraction. Let us consider another set of spacetime functions $(\tilde{u}_\nu)_{ij}$, $\tilde{O}^d_c, \tilde{\overline{u}}, \tilde{\overline{O}}$ solving the coupled system analogous to (5.8). Setting

$$(du_\nu)_{ij} = (u_\nu)_{ij} - (\tilde{u}_\nu)_{ij}, \quad d\overline{u} = \overline{u} - \tilde{\overline{u}}, \quad dO^d_c = O^d_c - \tilde{O}^d_c, \quad d\overline{O} = \overline{O} - \tilde{\overline{O}} \qquad (5.43)$$

we obtain schematically the new system of equations (depicting only the types of terms in the RHS suppressing the particular indices)

$$\begin{aligned}
&\overline{h}^{ab}\partial_a\partial_b(du_\nu)_{ij} \\
&\quad = \overline{O}\Gamma_{\frac{3}{2}}\partial(du) + \overline{O}\Gamma_3 du + \overline{O}\Gamma_{\frac{9}{2}}dO \\
&\qquad + d\overline{O}\big[\Gamma_{\frac{3}{2}}\partial\tilde{u} + \Gamma_3\tilde{u} + \Gamma_{\frac{9}{2}}(\tilde{O}-I) + \Gamma_3\partial\tilde{O}\big] \\
&\qquad + \overline{O}\Gamma_3\partial(df) + (\overline{O}+\tilde{\overline{O}})d\overline{O}\partial^2(\tilde{u}_\nu)_{ij} + G(d\overline{u}, d\overline{O}), \qquad (5.44)
\end{aligned}$$

where

$$\begin{aligned}
G(d\overline{u}, d\overline{O}) &= \Gamma_{\frac{3}{2}}d\overline{u}(\overline{u} + \tilde{\overline{u}} + \overline{u}^2 + \tilde{\overline{u}}^2 + \overline{u}\tilde{\overline{u}}) + \overline{O}\overline{u}\partial(d\overline{u}) \\
&\quad + \overline{O}d\overline{u}\partial\tilde{\overline{u}} + d\overline{O}\tilde{\overline{u}}\partial\tilde{\overline{u}} + \overline{O}\partial(\overline{O})\partial(d\overline{u}) \\
&\quad + \overline{O}\partial(d\overline{O})\partial\tilde{\overline{u}} + d\overline{O}\partial(\tilde{\overline{O}})\partial\tilde{\overline{u}} \qquad (5.45)
\end{aligned}$$

and

$$\partial_0(dO^d_c) = \Gamma_{\frac{3}{2}}dO + (\overline{O}-I)d\overline{u} + \tilde{\overline{u}}d\overline{O} + du \qquad (5.46)$$

Further, we assume that both sets of variables we have introduced are consistent with the energy estimate (5.9) we have established in the previous subsection:

$$\mathcal{E}(u, O;\alpha,T),\ \mathcal{E}(\overline{u},\overline{O};\alpha,T),\ \mathcal{E}(\tilde{u},\tilde{O};\alpha,T),\ \mathcal{E}(\tilde{\overline{u}},\tilde{\overline{O}};\alpha,T) \leq 2\mathcal{E}_0. \qquad (5.47)$$

Claim For large enough $\alpha > 0$ and $T > 0$ is sufficiently small the following contraction holds:

$$E_{2,\alpha}[du] + \sum_{c,d} \|dO_c^d\|^2_{H^{2,\alpha+\frac{3}{2}}} \leq \kappa\big(E_{2,\alpha}[d\overline{u}] + \sum \|d\overline{O}\|^2_{H^{2,\alpha+\frac{3}{2}}}\big), \tag{5.48}$$

for some $0 < \kappa < 1$.

Remark 5.5. We are forced to close the contraction mapping argument in $H^{2,\alpha}$, having one derivative less than the space of the energy estimate (5.47), see Sect. 5.3, as it is common in 2^{nd}-order quasilinear hyperbolic PDE [5], because of the problematic term $(\overline{O} + \tilde{\overline{O}})d\overline{O}\partial^2(\tilde{u}_v)_{ij}$ in (5.44), which is generated from the difference of the top order terms in the LHS.

Proposition 5.6. *Under the above considerations, the following estimates hold:*

$$\begin{aligned}
&\partial_\tau E_{2,\alpha}[du] + 8M^2e^{-1}(1-\tau)\alpha E_{2,\alpha+1}[du] \\
&\quad\lesssim (\mathcal{E}_0^{\frac{1}{2}} + \mathcal{E}_0 + \alpha^2)E_{2,\alpha}[du] + E_{2,\alpha+1}[du] + \|dO\|^2_{H^{2,\alpha+\frac{5}{2}}} \\
&\qquad + (\mathcal{E}_0 + \mathcal{E}_0^2 + \alpha^3\mathcal{E}_0)\big(E_{2,\alpha}[d\overline{u}] + \|d\overline{O}\|^2_{H^{2,\alpha+\frac{3}{2}}}\big) \qquad (5.49)\\
&\frac{1}{2}\partial_\tau \sum_{c,d} \|dO_c^d\|^2_{H^{2,\alpha+\frac{3}{2}}} + 4M^2e^{-1}(1-\tau)\alpha \sum_{c,d} \|dO_c^d\|^2_{H^{2,\alpha+\frac{5}{2}}} \\
&\quad\lesssim \|dO\|^2_{H^{2,\alpha+\frac{5}{2}}} + E_{2,\alpha+1}[du] + \mathcal{E}_0\big(E_{2,\alpha}[d\overline{u}] + \|d\overline{O}\|^2_{H^{2,\alpha+\frac{3}{2}}}\big), \qquad (5.50)
\end{aligned}$$

for all $\tau \in (0, T)$.

Assuming Proposition 5.6 we prove the above claim (5.48). After summing (5.49),(5.50), we absorb into the LHS the critical terms

$$E_{2,\alpha+1}[du],\ \|dO\|^2_{H^{2,\alpha+\frac{5}{2}}},$$

which appear in the RHS of the above inequalities. This is done by picking the parameter α sufficiently large (but finite). The contraction estimate (5.48) then follows from Gronwall's inequality for $T > 0$ suitably small.

Proof of Proposition 5.6. The proof follows exactly the lines of the proof of Proposition 5.3. The only notable difference lies in the estimation of the analogous term to (5.29), derived in (5.30)–(5.35). We sketch the argument in the present situation:

Let J denote at most one spatial index, $|J| \leq 1$, either 1, 2 or 3. The main term to be estimated is

$$\begin{aligned}
&-\int_{\Sigma_\tau^{\overline{u}}} \partial^{(J)}\Big[\text{RHS of (5.44)}\Big]\Omega\frac{\partial_0(du_v)_{ij,J}}{r^{2\alpha-3}}d\mu_{S_{\overline{g}}} \\
(\text{recall } \Omega \lesssim \tfrac{1}{r^{\frac{1}{2}}}) \quad &\lesssim \Big\|\frac{\partial_0(du_v)_{ij,J}}{r^{\alpha-\frac{1}{2}}}\Big\|^2_{L^2} + \Big\|\frac{\partial^{(J)}\big[\text{RHS of (5.44)}\big]}{r^{\alpha-2}}\Big\|^2_{L^2}, \qquad (5.51)
\end{aligned}$$

where $(du_\nu)_{ij,J} := \partial^{(J)}(du_\nu)_{ij}$. Plugging in (5.44) and using the basic estimates in Proposition 5.1, along with the assumption (5.47) we obtain

$$
\begin{aligned}
&\Big\| \frac{\partial^{(J)}\big[\text{RHS of (5.44)}\big]}{r^{\alpha-2}} \Big\|_{L^2}^2 \\
&\lesssim \Big\| \frac{\partial^{(J)}\big(\overline{O}\Gamma_{\frac{3}{2}}\partial(du) + \overline{O}\Gamma_3 du + \overline{O}\Gamma_{\frac{9}{2}} dO\big)}{r^{\alpha-2}} \Big\|_{L^2}^2 \\
&\quad + \Big\| \frac{\partial^{(J)}\big(d\overline{O}\big[\Gamma_{\frac{3}{2}}\partial\tilde{u} + \Gamma_3\tilde{u} + \Gamma_{\frac{9}{2}}(\tilde{O}-I) + \Gamma_3\partial\tilde{O}\big]\big)}{r^{\alpha-2}} \Big\|_{L^2}^2 \\
&\quad + \Big\| \frac{\partial^{(J)}\big(\overline{O}\Gamma_3\partial(dO)\big)}{r^{\alpha-2}} \Big\|_{L^2}^2 + \Big\| \frac{\partial^{(J)}\big[(\overline{O}+\tilde{\overline{O}})d\overline{O}\partial^2(\tilde{u}_\nu)_{ij}\big]}{r^{\alpha-2}} \Big\|_{L^2}^2 \\
&\quad + \Big\| \frac{\partial^{(J)} G(d\overline{u}, d\overline{O})}{r^{\alpha-2}} \Big\|_{L^2}^2
\end{aligned}
$$

(recall the asymptotics (4.5))

$$
\begin{aligned}
&\lesssim \Big\| \frac{\partial^{(J)}\partial(du)}{r^{\alpha-\frac{1}{2}}} \Big\|_{L^2}^2 + \Big\| \frac{\partial(du)}{r^{\alpha+1}} \Big\|_{L^2}^2 + \Big\| \frac{du}{r^{\alpha+\frac{5}{2}}} \Big\|_{L^2}^2 + \mathcal{E}_0\Big(\Big\| \frac{\partial(du)}{r^{\alpha-\frac{1}{2}}} \Big\|_{L^2}^2 + \Big\| \frac{du}{r^{\alpha+1}} \Big\|_{L^2}^2 \Big) \\
&\quad + \Big\| \frac{\partial^{(J)}(dO)}{r^{\alpha+\frac{5}{2}}} \Big\|_{L^2}^2 + \Big\| \frac{dO}{r^{\alpha+4}} \Big\|_{L^2}^2 + \mathcal{E}_0 \Big\| \frac{dO}{r^{\alpha+\frac{5}{2}}} \Big\|_{L^2}^2 \\
&\quad + \mathcal{E}_0 \Big\| \frac{\partial^{(J)}(d\overline{O})}{r^{\alpha+\frac{3}{2}}} \Big\|_{L^2}^2 + \Big\| \frac{d\overline{O}}{r^{\frac{3}{2}}} \Big\|_{L^\infty}^2 \big(\|\tilde{u}\|_{H^{2,\alpha}}^2 + \|\tilde{O} - I\|_{H^{2,\alpha+\frac{3}{2}}}^2 \big) \\
&\quad + \Big\| \frac{\partial^{(J)}\partial(dO)}{r^{\alpha+1}} \Big\|_{L^2}^2 + \Big\| \frac{\partial(dO)}{r^{\alpha+\frac{5}{2}}} \Big\|_{L^2}^2 + \mathcal{E}_0 \Big\| \frac{\partial(dO)}{r^{\alpha+1}} \Big\|_{L^2}^2 \\
&\quad + \Big\| \frac{\partial^{(J)}\big[(\overline{O}+\tilde{\overline{O}})d\overline{O}\partial^2(\tilde{u}_\nu)_{ij}\big]}{r^{\alpha-2}} \Big\|_{L^2}^2 + \Big\| \frac{\partial^{(J)} G(d\overline{u}, d\overline{O})}{r^{\alpha-2}} \Big\|_{L^2}^2 \\
&\lesssim E_{2,\alpha+1}[du] + \mathcal{E}_0 E_{2,\alpha}[du] + \|dO\|_{H^{2,\alpha+\frac{5}{2}}}^2 + \mathcal{E}_0 \|dO\|_{H^{2,\alpha+\frac{3}{2}}}^2 \\
&\quad + \mathcal{E}_0 \|d\overline{O}\|_{H^{2,\alpha+\frac{3}{2}}}^2 + \Big\| \frac{\partial^{(J)}\big[(\overline{O}+\tilde{\overline{O}})d\overline{O}\partial^2(\tilde{u}_\nu)_{ij}\big]}{r^{\alpha-2}} \Big\|_{L^2}^2 + \Big\| \frac{\partial^{(J)} G(d\overline{u}, d\overline{O})}{r^{\alpha-2}} \Big\|_{L^2}^2
\end{aligned}
\tag{5.52}
$$

We proceed to the problematic term $(\overline{O}+\tilde{\overline{O}})d\overline{O}\partial^2(\tilde{u}_\nu)_{ij}$ which can be controlled only in H^1:

$$
\begin{aligned}
&\Big\| \frac{\partial^{(J)}\big[(\overline{O}+\tilde{\overline{O}})d\overline{O}\partial^2(\tilde{u}_\nu)_{ij}\big]}{r^{\alpha-2}} \Big\|_{L^2}^2 \\
&\lesssim \|\partial^{(J)}(\overline{O}+\tilde{\overline{O}})\|_{L^\infty}^2 \|d\overline{O}\|_{L^\infty}^2 \Big\| \frac{\partial^2(\tilde{u}_\nu)_{ij}}{r^{\alpha-2}} \Big\|_{L^2}^2 + \|\overline{O}+\tilde{\overline{O}}\|_{L^\infty}^2 \Big\| \frac{\partial^{(J)}(d\overline{O})}{r^{\frac{\alpha}{2}-1}} \Big\|_{L^4}^2 \\
&\quad \cdot \Big\| \frac{\partial^2(\tilde{u}_\nu)_{ij}}{r^{\frac{\alpha}{2}-1}} \Big\|_{L^4}^2 + \|\overline{O}+\tilde{\overline{O}}\|_{L^\infty}^2 \Big\| \frac{d\overline{O}}{r} \Big\|_{L^\infty}^2 \Big\| \frac{\partial^{(J)}\partial^2(\tilde{u}_\nu)_{ij}}{r^{\alpha-2}} \Big\|_{L^2}^2
\end{aligned}
$$

(employing the L^4 estimate (5.5))

$$
\lesssim (\mathcal{E}_0^2 + \alpha^3\mathcal{E}_0^2)\|d\overline{O}\|_{H^{2,\alpha+\frac{3}{2}}}^2 \tag{5.53}
$$

Finally, plugging in the nonlinearity (5.45), we have the bound

$$
\begin{aligned}
&\Big\| \frac{\partial^{(J)} G(d\overline{u}, d\overline{O})}{r^{\alpha-2}} \Big\|_{L^2}^2 \\
&\lesssim \Big\| \frac{\partial^{(J)} \big(\Gamma_{\frac{3}{2}} d\overline{u} (\overline{u} + \tilde{\overline{u}} + \overline{u}^2 + \tilde{\overline{u}}^2 + \overline{u}\tilde{\overline{u}}) \big)}{r^{\alpha-2}} \Big\|_{L^2}^2 + \Big\| \frac{\partial^{(J)} \big(\overline{O}\overline{u} \partial(d\overline{u}) \big)}{r^{\alpha-2}} \Big\|_{L^2}^2 \\
&\quad + \Big\| \frac{\partial^{(J)} \big(\overline{O} d\overline{u} \partial\tilde{\overline{u}} + d\overline{O}\tilde{\overline{u}} \partial\tilde{\overline{u}} + \overline{O}\partial(\overline{O})\partial(d\overline{u}) \big)}{r^{\alpha-2}} \Big\|_{L^2}^2 \\
&\quad + \Big\| \frac{\partial^{(J)} \big(\overline{O}\partial(d\overline{O})\partial\tilde{\overline{u}} + d\overline{O}\partial(\tilde{\overline{O}})\partial\tilde{\overline{u}} \big)}{r^{\alpha-2}} \Big\|_{L^2}^2 \\
&\lesssim (\mathcal{E}_0 + \mathcal{E}_0^2) E_{2,\alpha}[d\overline{u}] + (\mathcal{E}_0 + 1) \|d\overline{u}\|_{L^\infty}^2 \Big\| \frac{\partial^{(J)} \partial\tilde{\overline{u}}}{r^{\alpha-2}} \Big\|_{L^2}^2 + \Big\| \frac{\partial^2(\overline{O})\partial(d\overline{u})}{r^{\alpha-2}} \Big\|_{L^2}^2 \\
&\quad + (\mathcal{E}_0 + \mathcal{E}_0^2) \|d\overline{O}\|_{H^{2,\alpha+\frac{3}{2}}}^2 + \Big\| \frac{\partial(d\overline{O})\partial^2\tilde{\overline{u}}}{r^{\alpha-2}} \Big\|_{L^2}^2 + \|d\overline{O}\|_{L^\infty}^2 \mathcal{E}_0^2 \\
&\lesssim (\mathcal{E}_0 + \mathcal{E}_0^2) \big(E_{2,\alpha}[d\overline{u}] + \|d\overline{O}\|_{H^{2,\alpha+\frac{3}{2}}}^2 \big) + \Big\| \frac{\partial^2 \overline{O}}{r^{\frac{\alpha}{2}-1}} \Big\|_{L^4}^2 \Big\| \frac{\partial(d\overline{u})}{r^{\frac{\alpha}{2}-1}} \Big\|_{L^4}^2 \\
&\quad + \Big\| \frac{\partial(d\overline{O})}{r^{\frac{\alpha}{2}-1}} \Big\|_{L^4}^2 \Big\| \frac{\partial^2\tilde{\overline{u}}}{r^{\frac{\alpha}{2}-1}} \Big\|_{L^4}^2
\end{aligned}
$$

(by the L^4 estimate (5.5))

$$
\lesssim (\mathcal{E}_0 + \mathcal{E}_0^2 + \alpha^3 \mathcal{E}_0) \big(E_{2,\alpha}[d\overline{u}] + \|d\overline{O}\|_{H^{2,\alpha+\frac{3}{2}}}^2 \big) \tag{5.54}
$$

□

6. The Constraint Equations in a Singular Background of Unbounded Mean Curvature

In this section we prove Theorem 1.2, our main stability result for the constraint equations (1.5) about the Schwarzschild singular initial data. We use the conformal method to construct the desired initial data sets for the EVE. The proof is an application of the inverse function theorem. Although similar results have been achieved in the smooth case and some rough backgrounds (see [8] for a general exposition), to our knowledge, the singular Schwarzschild background (Sect. 3) eludes any past references in the literature.

In order to employ the inverse function theorem we derive suitable weighted, elliptic estimates for the linearized conformal constraint map. We show that it is in general Fredholm between the weighted H^s spaces that we work with and an isomorphism in the case where the initial hypersurface Σ_0 is contained in sufficiently small neighbourhood of its singularity at $x = 0$. To define the appropriate spaces of our maps and derive coercive estimates we exploit heavily the features of the background Schwarzschild spacetime, including a delicate decoupling of the momentum constraint for the divergence free part of the projected unknown vector field onto the rotationally symmetric spheres of the Schwarzschild initial data set.

The weighted norms that we use to derive our estimates differ slightly from the ones we use for the hyperbolic part of the problem Sects. 4.2, 5. This is due to the fact that different singular terms in the resulting system have different leading orders. We are forced to take this into account to obtain useful elliptic estimates.

One of the main aspects of the problem that make it non-standard is the unboundedness of the mean curvature $\mathrm{tr}_{\overline{g}}K$ of the perturbation. In fact, one can check (Sect. 3) that

$$\mathrm{tr}_{\overline{g}}K \notin L^p(\Sigma_0), \qquad p \geq \frac{5}{3}.$$

The blow up orders of the second fundamental form of Σ_0 and the mean curvature in particular happen to be the most singular of the curvature terms in the equations. A very useful fact that we exploit is that in certain crucial terms they appear with a favourable sign.

The results in the literature of the constraints using the conformal method are mostly restricted to the constant mean curvature (CMC) or 'near CMC' regime [5]. Recently, there have been a number of advances to the case of large mean curvature, 'far from CMC', [11,15,22]. However, due to a smallness assumption on one of the variables, these results can be thought of in a sense as 'near CMC' [14,23]. All the more, they contain certain regularity assumptions which in particular imply that the mean curvature is in L^∞ and therefore do not directly apply to our case. Although our theorem generates initial data sets for the EVE which have unbounded mean curvature, they are also perturbative in the sense that they are close to the corresponding Schwarzschild induced initial data, measured in suitable norms.

6.1. The conformal approach; linearization and stability. We wish to construct initial data sets $(\overline{g}, K)$ on $\Sigma_0 = (-\epsilon, \epsilon)_x \times r^2\mathbb{S}^2$ for the EVE, i.e., solutions to the constraints (1.5), which are close to the Schwarzschild induced initial data and asymptote to Schwarzschild at a high order towards the singularity $r = 0$, see Theorem 1.1. Recall that the Schwarzschild induced metric on Σ_0 and its second fundamental form are given by

$$\begin{aligned} {}^S\overline{g} = \Omega^2 dx^2 + r^2 g_{\mathbb{S}^2}, \qquad \Omega^2 = \frac{32M^3}{r} e^{-\frac{r}{2M}}, \quad r^2 \sim 8M^2x^2 \\ {}^SK_{11} = -\frac{1}{2}\frac{\Omega}{4Mr} + l.o.t., \qquad {}^SK_{22} = {}^SK_{33} = \frac{\Omega}{4Mr}, \end{aligned} \tag{6.1}$$

where

$$\partial_1 = \frac{1}{\Omega}\partial_x \quad \partial_2 = \frac{1}{r}\partial_\theta, \quad \partial_3 = \frac{1}{r\sin\theta}\partial_\phi. \tag{6.2}$$

All derivations below involving spatial indices are carried out using the Schwarzschild frame (6.2). We will look for solutions of (1.5) of the form

$$\overline{g} = \varphi^4 {}^S\overline{g} \quad K_{ij} = \varphi^{-2}(\sigma_{ij} + LW_{ij}) + \frac{1}{3}\varphi^4 {}^S\overline{g}_{ij}\mathrm{tr}_{{}^S\overline{g}}{}^SK, \tag{6.3}$$

where

$$LW_{ij} := {}^S\overline{\nabla}_i W_j + {}^S\overline{\nabla}_j W_i - \frac{2}{3}{}^S\overline{g}_{ij}\,{}^S\overline{\nabla}^k W_k \tag{6.4}$$

In this set-up we have the freedom to choose the symmetric, traceless and transverse (TT) 2-tensor σ. Then the constraint equations reduce to an elliptic system of equations ([5]) for the conformal factor φ and the vector field W:

$$\begin{aligned} {}^S\overline{\nabla}^j LW_{ij} - \frac{2}{3}\phi^6 {}^S\overline{\nabla}_i(\mathrm{tr}_{s_{\overline{g}}} {}^S K) = 0 \\ -{}^S\overline{\Delta}\phi + \frac{1}{8}{}^S\overline{\mathrm{R}}\phi - \frac{1}{8}|\sigma + LW|^2\phi^{-7} + \frac{1}{12}(\mathrm{tr}_{s_{\overline{g}}} {}^S K)^2\phi^5 = 0 \end{aligned} \tag{6.5}$$

We prefer to analyse the top order term in the first equation of (6.5) after commuting derivatives

$${}^S\overline{\nabla}^j LW_{ij} = ({}^S\overline{\Delta}W)_i + \frac{1}{3}{}^S\overline{\nabla}_i({}^S\overline{\nabla}^k W_k) + {}^S\overline{\mathrm{Ric}}_i{}^j W_j \tag{6.6}$$

It is easy to see that the Schwarzschild induced initial data on Σ_0 can be parametrized in this fashion by choosing

$${}^S\varphi = 1, \quad \sigma = 0, \quad {}^S W = \mathrm{grad}_{s_{\overline{g}}} f(x), \quad f(x) \sim \frac{a}{\sqrt{|x|}}, \tag{6.7}$$

where f is a spherically symmetric function on Σ_0 solving the ODE[19]

$$\frac{4}{3}\partial_1^3 f + \frac{8}{3}\frac{\Omega}{4Mr}x \cdot \partial_1^2 f + \frac{3}{2}\frac{\Omega^2}{16M^2r^2}x + \frac{1}{2}\frac{\Omega^2}{32M^3r}x = 0, \qquad \partial_1 \sim \sqrt{|x|}\partial_x. \tag{6.8}$$

Setting $\sigma = 0$, $Y = W - {}^S W$, $\eta = \phi - 1$, the linearization of the system (6.5) about $Y = 0$, $\eta = 0$ with inhomogeneous terms Z, h reads

$$\begin{aligned} ({}^S\overline{\Delta}Y)_1 + \frac{1}{3}{}^S\overline{\nabla}_1({}^S\overline{\nabla}^k Y_k) + {}^S\overline{\mathrm{R}}_{11}Y_1 - 4({}^S\overline{\nabla}_1 \mathrm{tr}_{s_{\overline{g}}} {}^S K)\eta = Z_1 \\ ({}^S\overline{\Delta}Y)_i + \frac{1}{3}{}^S\overline{\nabla}_i({}^S\overline{\nabla}^k Y_k) + {}^S\overline{\mathrm{R}}_{ii}Y_i = Z_i, \quad i = 2, 3 \\ -{}^S\overline{\Delta}\eta + \frac{1}{8}{}^S\overline{\mathrm{R}}\eta + \frac{7}{8}|{}^S LW|^2\eta + \frac{5}{12}(\mathrm{tr}_{s_{\overline{g}}} {}^S K)^2\eta - \frac{1}{2}{}^S LW^{ij}\,{}^S\overline{\nabla}_j Y_i = h \end{aligned} \tag{6.9}$$

Recall (3.11) to compute the leading asymptotics, as $x \to 0$, of the (singular Schwarzschild) coefficients of (6.9):

$$\begin{aligned} {}^S\overline{\mathrm{R}}_{11} = -\frac{1}{4Mr} + O(1), \qquad {}^S\overline{\mathrm{R}}_{22} = {}^S\overline{\mathrm{R}}_{33} = \frac{1}{r^2} + O(\frac{1}{r}), \qquad ({}^S\overline{\nabla}_1 \mathrm{tr}_{s_{\overline{g}}} {}^S K) \sim \frac{c}{r^2} \\ {}^S\overline{\mathrm{R}} = \frac{2}{r^2} + O(\frac{1}{r}), \qquad (\mathrm{tr}_{s_{\overline{g}}} {}^S K)^2 = \frac{9}{4}\frac{\Omega^2}{16M^2r^2}, \qquad \Omega^2 = \frac{32M^3}{r}e^{-\frac{r}{2M}} \\ r^2 \sim 8M^2x^2, \qquad {}^S LW_{11} = -\frac{\Omega}{4Mr}, \qquad {}^S LW_{22} = {}^S LW_{33} = \frac{1}{2}\frac{\Omega}{4Mr}. \end{aligned} \tag{6.10}$$

Remark 6.1. Observe that the most singular coefficients in (6.9) are of order r^{-3} and they correspond to the zeroth order terms of the third equation. Fortunately they come with a good sign. This fact plays a crucial role in the analysis below.

[19] The first equation in (6.5) for $i = 1$ reduces to (6.8) in spherical symmetry, whereas the $i = 2, 3$ parts of the vector equation for W are automatically satisfied.

We exploit the fact that the Schwarzschild background is spherically symmetric and split the variables $\phi, \eta, W_1, Y_1, Z_1$ into

$$\begin{aligned} \phi &= \phi_0 + \phi_1 \quad W_1 = W_{10} + W_{11} \\ \eta &= \eta_0 + \eta_1 \quad Y_1 = Y_{10} + Y_{11} \\ h &= h_0 + h_1 \quad Z_1 = Z_{10} + Z_{11}, \end{aligned} \tag{6.11}$$

where $\phi_0, \eta_0, W_{10}, Y_{10}, Z_{10}$ are the spherically symmetric parts of the corresponding functions.

Let $W^\top, Y^\top, Z^\top$ denote the projections of the vector fields W, Y, Z onto the rotationally symmetric spheres $r^2\mathbb{S}^2$. There is a difficulty in obtaining coercive estimates for the part of these vector fields lying in the first eigenspace of the vector Laplacian on the spheres. The former space corresponds to the spherical conformal Killing vector fields. It turns out that the "genuinely" conformal Killing fields (which are not Killing) do not obstruct our analysis, by taking advantage of the divergence term in the second equation of (6.9). However, we must exclude the projections of $W^\top, Y^\top, Z^\top$ on the Killing vector fields of the round spheres $r^2\mathbb{S}^2$, $r > 0$. We achieve that by imposing first that $W^\top, Y^\top$ are orthogonal to the space of the spherical Killing vector fields:

$$W^\top, Y^\top \in \mathcal{K}^\perp, \qquad \mathcal{K} := \{X \in T(r^2\mathbb{S}^2), r > 0 : \nabla\!\!\!/_i X_j + \nabla\!\!\!/_j X_i = 0\}, \tag{6.12}$$

where $\nabla\!\!\!/$ denotes the covariant differentiation on $r^2\mathbb{S}^2$. The next lemma is the useful observation that the orthogonality assumption (6.12) holds automatically for $Z^\top$ as well.

Lemma 6.2. *If $W^\top, Y^\top \in \mathcal{K}^\perp$, then $Z^\top \in \mathcal{K}^\perp$. In other words the orthogonality condition* (6.12) *is preserved by the linearized operator corresponding to* (6.9) *and in fact by the actual non-linear conformal map* (6.5) *as well.*

Proof. Notice that the $i = 2, 3$ part of the vector equation in (6.5),

$$0 = {}^S\overline{\nabla}^j L W_{ij} = ({}^S\overline{\Delta} W)_i + \frac{1}{3} {}^S\overline{\nabla}_i ({}^S\overline{\nabla}^k W_k) + {}^S\overline{\mathrm{R}}_{ii} W_i$$

is equivalent to the second equation in (6.9) by just setting $Y^\top = W^\top$. Hence, the last assertion follows once we show $Z^\top \in \mathcal{K}^\perp$.

Let X be a Killing vector field of $\mathbb{S}^2$ and $A(x)$ a test function, compactly supported in $(-\epsilon, 0) \cup (0, \epsilon)$. We then have

$$\begin{aligned} \int_{\Sigma_0} {}^S\overline{g}\big(A(x)X, Z^\top\big) d\mu_{s_{\overline{g}}} &= \int_{\Sigma_0} A(x) {}^S\overline{g}\Big(X, {}^S\overline{\Delta} Y + \frac{1}{3}\nabla\!\!\!/ ({}^S\overline{\nabla}^k Y_k) + \frac{1}{r^2} Y^\top\Big) d\mu_{s_{\overline{g}}} \\ \text{(IBP and using (6.12))} &= \int_{\Sigma_0} A(x) {}^S\overline{g}(X, {}^S\overline{\Delta} Y) - \frac{1}{3} ({}^S\overline{\nabla}^k Y_k) \nabla\!\!\!/^i X_i d\mu_{s_{\overline{g}}} \\ (X \text{ is divergence free}) &= \int_{\Sigma_0} A(x) {}^S\overline{g}(X, {}^S\overline{\Delta} Y) d\mu_{s_{\overline{g}}} \end{aligned}$$

$$\begin{aligned} ({}^S\overline{\Delta} = {}^S\overline{\nabla}_{11} + {}^S(A^i)_{1i} {}^S\overline{\nabla}_1 + \not{\Delta}) \\ &= \int_{\Sigma_0} A(x) {}^S\overline{g}(X, {}^S\overline{\nabla}_{11} Y + {}^S(A^i)_{1i} {}^S\overline{\nabla}_1 Y) d\mu_{s_{\overline{g}}} \\ &\quad + \int_{\Sigma_0} A(x) {}^S\overline{g}(\not{\Delta} X, Y) d\mu_{s_{\overline{g}}} \end{aligned}$$

$$
\begin{aligned}
(\not{\Delta} X + \tfrac{1}{r^2} X = 0) \qquad & = \int_{\Sigma_0} A(x)\, {}^S\overline{g}(X, {}^S\overline{\nabla}_{11} Y + {}^S(A^i)_{1i}\, {}^S\overline{\nabla}_1 Y) d\mu_{s_{\overline{g}}} \\
\text{(IBP)} \qquad & = \int_{-\epsilon}^{\epsilon} (\text{function of } x) \int_{\mathbb{S}^2} {}^S\overline{g}(X, Y) d\mu_{\mathbb{S}^2} dx \\
& = 0
\end{aligned}
$$

Since $A(x)$ is arbitrary, it follows that $Z^\top \perp X$ for all $X \in T(r^2\mathbb{S}^2)$, $X \in \mathcal{K}$, for *any* sphere $r^2\mathbb{S}^2$. Thus $Z^\top \in K^\perp$. □

We make use of (6.12) below by employing the following lemma.

Lemma 6.3. *Let X be a vector field on $r^2\mathbb{S}^2$ satisfying the orthogonality assumption* (6.12). *Then*

$$\int_{r^2\mathbb{S}^2} |\nabla\!\!\!/ X|^2 + \frac{1}{3} |\nabla\!\!\!/^i X_i|^2 d\mu_{r^2\mathbb{S}^2} \geq \frac{4}{3} \int_{r^2\mathbb{S}^2} \frac{|X|^2}{r^2} d\mu_{r^2\mathbb{S}^2}. \tag{6.13}$$

Proof. The inequality

$$\int_{r^2\mathbb{S}^2} |\nabla\!\!\!/ X|^2 d\mu_{r^2\mathbb{S}^2} \geq \int_{r^2\mathbb{S}^2} \frac{|X|^2}{r^2} d\mu_{r^2\mathbb{S}^2} \tag{6.14}$$

is standard and valid for all spherical vector fields. Moreover, equality in (6.14) is achieved if and only if X lies in the first eigenspace of the vector sphere Laplacian, i.e., if X is a conformal Killing vector field. For the part of X orthogonal to the space of conformal Killing vector fields, denoted by $X_{>1}$, we deduce using vector spherical harmonics that

$$\int_{r^2\mathbb{S}^2} |\nabla\!\!\!/ X_{>1}|^2 d\mu_{r^2\mathbb{S}^2} \geq (\lambda_2 - 1) \int_{r^2\mathbb{S}^2} \frac{|X_{>1}|^2}{r^2} d\mu_{r^2\mathbb{S}^2} \tag{6.15}$$

where $\lambda_2 = 6$ is the second eigenvalue of $-\Delta_{\mathbb{S}^2}$. If now X is conformal Killing, but not Killing, $X \in \mathcal{K}^\perp$, then we have

$$\int_{r^2\mathbb{S}^2} |\nabla\!\!\!/^i X_i|^2 d\mu_{r^2\mathbb{S}^2} = \int_{r^2\mathbb{S}^2} \frac{|X|^2}{r^2} d\mu_{r^2\mathbb{S}^2} \tag{6.16}$$

Combining thus all above cases we derive (6.13). □

We proceed now to define the weighted H^s spaces we are going to work with:

$$
\begin{aligned}
H^{s,\alpha}_{\text{vf-0}} : \quad & v \in H^s \quad \& \quad \sum_{|j|\leq s} \int_{\Sigma_0} \frac{({}^S\overline{\nabla}_1^{(j)} v)^2}{|x|^{2\alpha - |j| + 1}} d\mu_{s_{\overline{g}}} < +\infty \\
H^{s,\alpha}_{\text{vf-1}} : \quad & v \in H^s \quad \& \quad \sum_{|j|+|k|\leq s} \int_{\Sigma_0} \frac{({}^S\overline{\nabla}_1^{(j)} \nabla\!\!\!/_l^{(k)} v)^2}{|x|^{2\alpha - 2(|j|+|k|-1)}} d\mu_{s_{\overline{g}}} < +\infty \\
H^{s,\alpha}_{\text{sc}} : \quad & v \in H^s \quad \& \quad \sum_{|j|+|k|\leq s} \int_{\Sigma_0} \frac{({}^S\overline{\nabla}_1^{(j)} \nabla\!\!\!/^{(k)} v)^2}{|x|^{2\alpha - 3(|j|+|k|-1)}} d\mu_{s_{\overline{g}}} < +\infty,
\end{aligned} \tag{6.17}
$$

where v is either a scalar or a vector field.

Remark 6.4. The precise ordering of the above derivatives does not matter since the Schwarzschild connection coefficients ${}^{S}(A_\mu)_{jk} = O(|x|^{-\frac{1}{2}})$, $1 \in \{\mu jk\}$, see (3.11). Also, note that we can use either covariant or non-covariant differentiation since ${}^{S}(A_\mu)_{jk} = O(|x|^{-1})$ for all indices, ${}^{S}(A_1)_{jk} = 0$, and thus the extra terms arising from the various ${}^{S}(A_\mu)_{jk}$'s can be incorporated in the norms.

Define the operator

$$\begin{aligned}\Psi(W_{10} - {}^{S}W_{10}, W_{11}, W^{\top}, \phi - 1, \sigma) : H^{4,\alpha}_{\text{vf-0}} \times H^{4,\alpha}_{\text{vf-1}} \times \left(H^{4,\alpha}_{\text{vf-1}} \cap \mathcal{K}^{\perp}\right) \times H^{4,\alpha}_{\text{sc}} \times \mathcal{B}_\sigma \\ \to H^{2,\alpha-1}_{\text{vf-0}} \times H^{2,\alpha-2}_{\text{vf-1}} \times \left(H^{2,\alpha-2}_{\text{vf-1}} \cap \mathcal{K}^{\perp}\right) \times H^{2,\alpha-3}_{\text{sc}}, \\ \Psi = (\text{LHS of the system (6.5)}),\end{aligned} \tag{6.18}$$

where $\mathcal{B}_\sigma$ can be any of the above spaces of sufficiently high regularity with similar weights.[20]

Lemma 6.5. *Ψ is well-defined, bounded and C^1 (Fréchet).*

Proof. We realize Ψ is well-defined by matching the singular orders of the coefficients (6.5) with the analogies in the weights of the norms (6.17) and by recalling Lemma 6.2.

Express Ψ as differences of the variables $\phi - 1$, $W - {}^{S}W$. The boundedness of Ψ then follows by applying Sobolev embedding to the arising non-linear terms, see (5.4), and by controlling the linear terms, which can be read from the linearized system (6.9), in the weighted H^s norms (6.17) that where carefully defined to for this exact purpose. The same argument actually implies that Ψ is C^1. □

By definition we have

$$\begin{aligned}D\Psi_{(W_{10}-{}^{S}W_{10}, W_{11}, W^{\top}, \phi-1)}(\mathbf{0})(Y_{10}, Y_{11}, Y^{\top}, \eta) =: D\Psi(Y_{10}, Y_{11}, Y^{\top}, \eta) : \\ H^{4,\alpha}_{\text{vf-0}} \times H^{4,\alpha}_{\text{vf-1}} \times \left(H^{4,\alpha}_{\text{vf-1}} \cap \mathcal{K}^{\perp}\right) \times H^{4,\alpha}_{\text{sc}} \\ \to H^{2,\alpha-1}_{\text{vf-0}} \times H^{2,\alpha-2}_{\text{vf-1}} \times \left(H^{2,\alpha-2} \cap \mathcal{K}^{\perp}\right) \times H^{2,\alpha-3}_{\text{sc}}, \\ D\Psi = (\text{LHS of (6.9)})\end{aligned} \tag{6.19}$$

Proposition 6.6. *The bounded operator*

$$\begin{aligned}D\Psi : \left[H^{4,\alpha}_{vf\text{-}0} \times H^{4,\alpha}_{vf\text{-}1} \times \left(H^{4,\alpha}_{vf\text{-}1} \cap \mathcal{K}^{\perp}\right) \times H^{4,\alpha}_{sc}\right] \cap H^{1}_{0} \\ \to H^{2,\alpha-1}_{vf\text{-}0} \times H^{2,\alpha-2}_{vf\text{-}1} \times \left(H^{2,\alpha-2}_{vf\text{-}1} \cap \mathcal{K}^{\perp}\right) \times H^{2,\alpha-3}_{sc}\end{aligned} \tag{6.20}$$

is Fredholm, i.e., it has finite dim kernel and cokernel, for any α sufficiently large, consistent with Theorem 4.8. In the case that Σ_0 is contained in a sufficiently small neighbourhood of $x = 0$, $D\Psi$ is in fact an isomorphism.

We postpone the proof Proposition 6.6 for Sect. 6.2 and proceed to formulate our stability result for the constraints.

[20] Owing to Sect. 4, $\sigma \in H^{3,\alpha}$ would be fine.

Theorem 6.7. *Let α be sufficiently large, given by Theorem 4.8. Also, let $\Sigma_0 = (-\epsilon, \epsilon)_x \times r^2\mathbb{S}^2$ be an initial singular hypersurface for ϵ sufficiently small such that the second part of Proposition 6.6 is valid. Then for any $\sigma \in H^{3,\alpha}$ with sufficiently small norm, there exists a solution to the conformal constraint equations (6.5) in the spaces*

$$\begin{aligned}(W_{10} - {}^S W_{10}, W_{11}, W^\top, \phi - 1) &\in H^{4,\alpha}_{vf\text{-}0} \times H^{4,\alpha}_{vf\text{-}1} \times \left(H^{4,\alpha}_{vf\text{-}1} \cap \mathcal{K}^\perp\right) \times H^{4,\alpha}_{sc} \\ W - {}^S W, \phi - 1 &\in H^1_0 \end{aligned} \tag{6.21}$$

In particular, the pairs $(\overline{g}, K)$ given by (6.3) verify the constraints (1.5) and the assumptions of Theorem 4.8.

Proof. The main assertion regarding the solution to the conformal constraint equations follows from the inverse function theorem, since $D\Psi$ (6.20) is an isomorphism, the level set $\Psi^{-1}(\{0\})$ is the set of solutions to (6.5) and $\Psi(0) = 0$. Although the domain of Ψ is slightly different from the space of initial data sets in Theorem 4.8, picking α larger than required, we can ensure that the pairs $(\overline{g}, K)$ we construct in this section, given by (6.3), satisfy the initial conditions in Theorem 4.8. □

6.2. Proof of Proposition 6.6. We derive elliptic estimates for $D\Psi$ in the spaces (6.17) defined earlier. The system

$$D\Psi(\eta, Y) = (h, Z). \tag{6.22}$$

is by definition (6.9). Recall briefly the notation (6.11), $Y^\top$, $Z^\top$, and let

$$\tilde{Y} = Y_{11}\partial_1 + Y^\top \qquad \tilde{Z} = Z_{11}\partial_1 + Z^\top \tag{6.23}$$

Then it is easy to see that (6.22) reduces to two systems, one for $\tilde{Y}, \eta_1, \tilde{Z}, h_1$: [which we write by replacing the singular coefficients (6.10) with their leading orders, recall $r^2 \sim 8M^2x^2$]

$$\begin{aligned} ({}^S\overline{\Delta}\tilde{Y})_1 + \frac{1}{3}{}^S\overline{\nabla}_1({}^S\overline{\nabla}^k\tilde{Y}_k) - \frac{1}{4Mr}\tilde{Y}_1 + O(\frac{1}{|x|^2})\eta_1 &= \tilde{Z}_1 \\ ({}^S\overline{\Delta}\tilde{Y})_i + \frac{1}{3}\not{\nabla}({}^S\overline{\nabla}^k\tilde{Y}_k) + \frac{1}{r^2}\tilde{Y}_i &= \tilde{Z}_i, \quad i = 2, 3 \\ -{}^S\overline{\Delta}\eta_1 + \frac{b}{|x|^3}\eta_1 + O(\frac{1}{|x|^{\frac{3}{2}}}){}^S\overline{\nabla}_k\tilde{Y}_j &= h_1, \quad b > 0 \end{aligned} \tag{6.24}$$

and one for the spherically symmetric parts of the variables, $\eta_0, Y_{10}, h_0, Z_{10}$:

$$\begin{aligned} \partial_1^2 Y_{10} + O(\frac{1}{|x|^{\frac{1}{2}}})\partial_1 Y_{10} - \frac{1}{4Mr}Y_{10} + O(\frac{1}{|x|^2})\eta_0 &= Z_{10} \\ -\partial_1^2\eta_0 + O(\frac{1}{|x|^{\frac{1}{2}}})\partial_1\eta_0 + \frac{b}{|x|^3}\eta_0 + O(\frac{1}{|x|^{\frac{3}{2}}})\partial_1 Y_{10} + O(\frac{1}{|x|^2})Y_{10} &= h_0, \quad b > 0, \end{aligned} \tag{6.25}$$

where $O(|x|^a)$ denotes a smooth function, $x \neq 0$, satisfying $\partial_1^{(j)} O(|x|^a) = O(|x|^{a-\frac{|j|}{2}})$. Recall that $\partial_1 \sim \sqrt{|x|}\partial_x$.

The potential terms $\frac{1}{r^2}\tilde{Y}_i$, $i = 2, 3$ in (6.24) are troublesome, because they come with a bad sign and it is precisely the reason we need (6.12) and Lemma (6.3) to handle them.

Note that the zeroth order term $-\frac{1}{4Mr}Y_{10}$ in the first equation of (6.25) has a favourable sign, but it is one order weaker than the favourable term coming from the sphere Laplacian in the equations, see also (6.29) below. This fact forces us to treat the spherically symmetric part of Y separately.

Proposition 6.8 (A priori elliptic estimate I). *Assume* $h_1 \in H^{2,\alpha-3}_{sc}$, $\tilde{Z} \in H^{2,\alpha-2}_{vf\text{-}I}$, $Z^\top \in \mathcal{K}^\perp$, *and* $\eta_1 \in H^{4,\alpha}_{sc} \cap H^1_0$, $\tilde{Y} \in H^{4,\alpha}_{vf\text{-}I} \cap H^1_0$, $Y^\top \in \mathcal{K}^\perp$ *solving* (6.24). *Then the following estimate holds:*

$$\|\eta_1\|^2_{H^{4,\alpha}_{sc}} + \|\tilde{Y}\|^2_{H^{4,\alpha}_{vf\text{-}I}} \lesssim \|h_1\|^2_{H^{2,\alpha-3}_{sc}} + \|\tilde{Z}\|^2_{H^{2,\alpha-2}_{vf\text{-}I}} + \|\eta_1\|^2_{L^2} + \|\tilde{Y}\|^2_{L^2} \tag{6.26}$$

If in addition $x \in (-\epsilon, \epsilon)$, $\epsilon > 0$ *sufficiently small* (*how small depending on the coefficients of the system* (6.24) *and* α), *then* (6.26) *can be improved to*

$$\|\eta_1\|^2_{H^{4,\alpha}_{sc}} + \|\tilde{Y}\|^2_{H^{4,\alpha}_{vf\text{-}I}} \lesssim \|h_1\|^2_{H^{2,\alpha-3}_{sc}} + \|\tilde{Z}\|^2_{H^{2,\alpha-2}_{vf\text{-}I}} \tag{6.27}$$

Proof. We will employ the inequality for $Y^\top$ given by Lemma 6.3:

$$\int_{r^2\mathbb{S}^2} \frac{|\not\nabla Y^\top|^2}{|x|^{2\alpha}} + \frac{1}{3}\frac{|\not\nabla^i Y_i^\top|^2}{|x|^{2\alpha}} d\mu_{r^2\mathbb{S}^2} \geq \frac{4}{3}\int_{r^2\mathbb{S}^2} \frac{|Y^\top|^2}{r^2|x|^{2\alpha}} d\mu_{r^2\mathbb{S}^2}. \tag{6.28}$$

and the standard one for $\tilde{Y}_1$

$$\int_{\Sigma_0} \frac{|\not\nabla \tilde{Y}_1|^2}{|x|^{2\alpha}} d\mu_{s_{\overline{g}}} \geq 2\int_{\Sigma_0} \frac{\tilde{Y}_1^2}{r^2|x|^{2\alpha}} d\mu_{s_{\overline{g}}} \geq c\int_{\Sigma_0} \frac{\tilde{Y}_1^2}{|x|^{2\alpha+2}} d\mu_{s_{\overline{g}}}, \tag{6.29}$$

which is immediate by definition (6.11), (6.23). Multiplying (6.24) with $\frac{1}{\varepsilon}\frac{\tilde{Y}_i}{x^{2\alpha}}$, $\frac{\eta_1}{x^{2\alpha}}$ respectively, subtracting the first two equations from the last, integrating on Σ_0 and integrating by parts we arrive at the inequality

$$\begin{aligned}
&\int_{\Sigma_0} \frac{|{}^S\overline{\nabla}\eta_1|^2}{|x|^{2\alpha}} + \frac{1}{\varepsilon}\frac{|{}^S\overline{\nabla}\tilde{Y}|^2}{|x|^{2\alpha}} + \frac{1}{3\varepsilon}\frac{|{}^S\overline{\nabla}^i\tilde{Y}_i|^2}{|x|^{2\alpha}} + \frac{1}{\varepsilon}\frac{\tilde{Y}_1^2}{4Mr|x|^{2\alpha}} + \frac{\eta_1^2}{|x|^{2\alpha+3}} d\mu_{s_{\overline{g}}} \\
&(\partial_1 \sim \sqrt{|x|}\partial_x) \\
&\leq \int_{\Sigma_0} C\alpha\frac{|\eta_1\partial_1\eta_1|}{|x|^{2\alpha+\frac{1}{2}}} + \frac{1}{\varepsilon}C\alpha\frac{|\tilde{Y}^S\overline{\nabla}_1\tilde{Y}|}{|x|^{2\alpha+\frac{1}{2}}} + \frac{1}{\varepsilon}C\frac{|\eta_1\tilde{Y}_1|}{|x|^{2\alpha+2}} + \frac{1}{\varepsilon}\frac{|Y^\top|^2}{r^2|x|^{2\alpha}} \\
&\quad + C\frac{|\eta_1{}^S\overline{\nabla}_j\tilde{Y}_i|}{|x|^{2\alpha+\frac{3}{2}}} + \frac{h_1\eta_1}{|x|^{2\alpha}} - \frac{1}{\varepsilon}\frac{\tilde{Z}_i\tilde{Y}^i}{|x|^{2\alpha}} d\mu_{s_{\overline{g}}} \\
&\leq \int_{\Sigma_0} \frac{1}{2}\frac{(\partial_1\eta_1)^2}{|x|^{2\alpha}} + C\alpha^2\frac{\eta_1^2}{|x|^{2\alpha+1}} + \frac{|{}^S\overline{\nabla}_1\tilde{Y}|^2}{|x|^{2\alpha}} + \frac{C}{\varepsilon^2}\alpha^2\frac{|\tilde{Y}|^2}{|x|^{2\alpha+1}} + \frac{C}{\varepsilon^2}\frac{\eta_1^2}{|x|^{2\alpha+2}} \\
&\quad + \frac{\tilde{Y}_1^2}{|x|^{2\alpha+2}} + \frac{1}{\varepsilon}\frac{|Y^\top|^2}{r^2|x|^{2\alpha}} + \frac{1}{2}\frac{\eta_1^2}{|x|^{2\alpha+3}} + C\frac{|{}^S\overline{\nabla}_j\tilde{Y}_i|^2}{|x|^{2\alpha}} \\
&\quad + \frac{1}{4}\frac{\eta_1^2}{|x|^{2\alpha+3}} + C\frac{h_1^2}{|x|^{2\alpha-3}} + \frac{|\tilde{Y}|^2}{|x|^{2\alpha+2}} + \frac{C}{\varepsilon^2}\frac{|\tilde{Z}|^2}{|x|^{2\alpha-2}} d\mu_{s_{\overline{g}}}
\end{aligned} \tag{6.30}$$

Now the desired estimate at the level of H^1 (i.e., the parts of the relevant norms that depend on up to one derivative of $\eta_1, \tilde{Y}$) follows by comparing the powers in the denominators on both sides, taking $\varepsilon > 0$ sufficiently small utilizing the inequalities (6.28), (6.29). If $|x| \ll 1$, then it is easy to see that all weighted $\eta_1, \tilde{Y}$ terms can be absorbed in the LHS. The full $H^{4,\alpha}$ estimate is obtained by using (6.24), differentiating the system in the spatial directions and applying a similar procedure. We only derive the second order estimate: Multiply the system (6.24) with $\frac{1}{\varepsilon}\frac{{}^S\overline{\nabla}_{22}\tilde{Y}}{|x|^{2\alpha-2}}$, $\frac{{}^S\overline{\nabla}_{22}\eta_1}{|x|^{2\alpha-3}}$, integrate over Σ_0, subtract the first two equations from the third one and integrate by parts twice to deduce

$$\begin{aligned}
&\int_{\Sigma_0} \frac{|{}^S\overline{\nabla}{}^S\overline{\nabla}_2\eta_1|^2}{|x|^{2\alpha-3}} + \frac{1}{\varepsilon}\sum_{i=1}^{3}\frac{|{}^S\overline{\nabla}{}^S\overline{\nabla}_2\tilde{Y}|^2}{|x|^{2\alpha-2}} + \frac{1}{\varepsilon}\frac{|{}^S\overline{\nabla}_2\tilde{Y}_1|^2}{4Mr|x|^{2\alpha-2}} + \frac{|{}^S\overline{\nabla}_2\eta_1|^2}{|x|^{2\alpha}} d\mu_{s_{\overline{g}}} \\
&\leq \int_{\Sigma_0} C\alpha\frac{|{}^S\overline{\nabla}_{21}\eta_1\,{}^S\overline{\nabla}_2\eta_1|}{|x|^{2\alpha-3+\frac{1}{2}}} + \frac{|{}^S\overline{\mathrm{Ric}}\,{}^S\overline{\nabla}\eta_1\,{}^S\overline{\nabla}_2\eta_1|}{|x|^{2\alpha-3}} + \frac{C}{\varepsilon}\alpha\frac{|{}^S\overline{\nabla}_{21}\tilde{Y}\,{}^S\overline{\nabla}_2\tilde{Y}|}{|x|^{2\alpha-2+\frac{1}{2}}} \\
&\quad + \frac{1}{\varepsilon}\frac{|{}^S\overline{\mathrm{Ric}}\,{}^S\overline{\nabla}\tilde{Y}\,{}^S\overline{\nabla}_2\tilde{Y}|}{|x|^{2\alpha-2}} + \frac{1}{\varepsilon}\frac{1}{3}\frac{\not{\nabla}({}^S\overline{\nabla}^k\tilde{Y}_k)\,{}^S\overline{\nabla}_{22}\tilde{Y}}{|x|^{2\alpha-2}} \\
&\quad + \frac{1}{\varepsilon}\frac{1}{3}\frac{{}^S\overline{\nabla}_1(\not{\nabla}^k\tilde{Y}_k)\,{}^S\overline{\nabla}_{22}\tilde{Y}}{|x|^{2\alpha-2}} + \frac{C}{\varepsilon}\frac{|\eta_1\,{}^S\overline{\nabla}_{22}\tilde{Y}_1|}{|x|^{2\alpha}} + C\frac{|{}^S\overline{\nabla}_j\tilde{Y}_i\,{}^S\overline{\nabla}_{22}\eta_1|}{|x|^{2\alpha-\frac{3}{2}}} \\
&\quad - \frac{h_1\,{}^S\overline{\nabla}_{22}\eta_1}{|x|^{2\alpha-3}} + \frac{1}{\varepsilon}\frac{\tilde{Z}_i\,{}^S\overline{\nabla}_{22}\tilde{Y}^i}{|x|^{2\alpha-2}} d\mu_{s_{\overline{g}}} \\
&(|{}^S\overline{\mathrm{Ric}}| \sim r^{-2}) \\
&\leq \int_{\Sigma_0} \frac{1}{2}\frac{|{}^S\overline{\nabla}_{21}\eta_1|^2}{|x|^{2\alpha-3}} + C\alpha^2\frac{|{}^S\overline{\nabla}_2\eta_1|^2}{|x|^{2\alpha-2}} + C\frac{|{}^S\overline{\nabla}\eta_1|^2}{|x|^{2\alpha-1}} + \frac{1}{2\varepsilon}\frac{|{}^S\overline{\nabla}_{21}\tilde{Y}|^2}{|x|^{2\alpha-2}} \\
&\quad + \frac{C}{\varepsilon}\alpha^2\frac{|{}^S\overline{\nabla}_2\tilde{Y}|^2}{|x|^{2\alpha-1}} + \frac{C}{\varepsilon}\frac{|{}^S\overline{\nabla}\tilde{Y}|^2}{|x|^{2\alpha}} + \frac{1}{4\varepsilon}\frac{|{}^S\overline{\nabla}_{22}\tilde{Y}|^2}{|x|^{2\alpha-2}} + \frac{1}{9\varepsilon}\frac{|\not{\nabla}{}^S\overline{\nabla}\tilde{Y}|^2}{|x|^{2\alpha-2}} \\
&\quad + \frac{1}{4\varepsilon}\frac{|{}^S\overline{\nabla}_{22}\tilde{Y}|^2}{|x|^{2\alpha-2}} + \frac{1}{9\varepsilon}\frac{|{}^S\overline{\nabla}_1\not{\nabla}\tilde{Y}|^2}{|x|^{2\alpha-2}} + C\frac{\eta_1^2}{|x|^{2\alpha+2}} + \frac{1}{20\varepsilon}\frac{|{}^S\overline{\nabla}_{22}\tilde{Y}|^2}{|x|^{2\alpha-2}} \\
&\quad + C\frac{|{}^S\overline{\nabla}_j\tilde{Y}_i|^2}{|x|^{2\alpha}} + \frac{1}{2}\frac{|{}^S\overline{\nabla}_{22}\eta_1|^2}{|x|^{2\alpha-3}} + C\frac{h_1^2}{|x|^{2\alpha-3}} + \frac{1}{20\varepsilon}\frac{|{}^S\overline{\nabla}_{2j}\tilde{Y}_i|^2}{|x|^{2\alpha-2}} \\
&\quad + \frac{C}{\varepsilon}\frac{|\tilde{Z}|^2}{|x|^{2\alpha-2}} d\mu_{s_{\overline{g}}}
\end{aligned} \tag{6.31}$$

We obtain a bound for the weighted norms of the second derivatives of the variables, including only one ∂_1 derivative, by adding to (6.31) its analogue for the terms ${}^S\overline{\nabla}{}^S\overline{\nabla}_3\eta_1$, ${}^S\overline{\nabla}{}^S\overline{\nabla}_3\tilde{Y}_i$, absorbing the second order terms of the RHS in the LHS and by applying the H^1 estimate we derived above to the lower order terms. Finally, in order to bound the corresponding norms of ${}^S\overline{\nabla}_{11}\tilde{Y}_i$, ${}^S\overline{\nabla}_{11}\eta_1$ as well, we use directly the system (6.24) to move the derivatives we have already controlled to the RHS. Then we first take the $\|\frac{\cdot}{|x|^{\alpha-1}}\|_{L^2}$ norm of the first two equations in (6.24):

$$\sum_{i=1}^{3}\|\frac{{}^S\overline{\nabla}_{11}\tilde{Y}}{|x|^{\alpha-1}}\|_{L^2}\lesssim\|\frac{\not\nabla^2\tilde{Y}}{|x|^{\alpha-1}}\|_{L^2}+\|\frac{{}^S\overline{\nabla}_1\not\nabla\tilde{Y}}{|x|^{\alpha-1}}\|_{L^2}+\|\frac{{}^S\overline{\nabla}\tilde{Y}}{|x|^{\alpha}}\|_{L^2}$$
$$+\|\frac{\tilde{Y}}{|x|^{\alpha+1}}\|_{L^2}+\|\frac{\eta_1}{|x|^{\alpha+1}}\|_{L^2}+\|\frac{\tilde{Z}}{|x|^{\alpha-1}}\|_{L^2}\tag{6.32}$$

and the $\|\frac{\cdot}{|x|^{\alpha-\frac{3}{2}}}\|_{L^2}$ norm of the third equation of (6.24) to infer that

$$\|\frac{{}^S\overline{\nabla}_{11}\eta_1}{|x|^{\alpha-\frac{3}{2}}}\|_{L^2}\leq\|\frac{\not\nabla^2\eta_1}{|x|^{\alpha-\frac{3}{2}}}\|_{L^2}+\|\frac{{}^S\overline{\nabla}_1\not\nabla\eta_1}{|x|^{\alpha-\frac{3}{2}}}\|_{L^2}+\|\frac{{}^S\overline{\nabla}\eta_1}{|x|^{\alpha}}\|_{L^2}$$
$$+\|\frac{\eta_1}{|x|^{\alpha+\frac{3}{2}}}\|_{L^2}+\|\frac{{}^S\overline{\nabla}_j\tilde{Y}_i}{|x|^{\alpha}}\|_{L^2}+\|\frac{h_1}{|x|^{\alpha-\frac{3}{2}}}\|_{L^2}\tag{6.33}$$

□

For the spherically symmetric parts of η, Y (6.11) we prove the following:

Proposition 6.9 (A priori elliptic estimate II). *Let $h_0\in H^{2,\alpha-3}_{sc}$, $Z_{10}\in H^{2,\alpha-1}_{vf\text{-}0}$ and $\eta_0\in H^{4,\alpha}_{sc}\cap H^1_0$, $Y_{10}\in H^{4,\alpha}_{vf\text{-}0}\cap H^1_0$, all x variable functions solving (6.25). Then for α sufficiently large the following estimate holds:*

$$\|\eta_0\|^2_{H^{4,\alpha}_{sc}}+\|Y_{10}\|^2_{H^{4,\alpha}_{vf\text{-}0}}\lesssim\|h_0\|^2_{H^{2,\alpha-3}_{sc}}+\|Z_{10}\|^2_{H^{2,\alpha-1}_{vf\text{-}0}}+\|\eta_0\|^2_{L^2}+\|Y_{10}\|^2_{L^2}\tag{6.34}$$

If in addition $x\in(-\epsilon,\epsilon)$, $\epsilon>0$ sufficiently small, how small depending on the coefficients of the system (6.9) and α, then in fact

$$\|\eta_0\|^2_{H^{4,\alpha}_{sc}}+\|Y_{10}\|^2_{H^{4,\alpha}_{vf\text{-}0}}\lesssim\|h_0\|^2_{H^{2,\alpha-3}_{sc}}+\|Z_{10}\|^2_{H^{2,\alpha-1}_{vf\text{-}0}}\tag{6.35}$$

Proof. We multiply the first equation above with $\frac{\partial_1 Y_{10}}{|x|^{2\alpha-\frac{1}{2}}}$, integrate over Σ_0 and integrate by parts: [note that the boundary terms are either zero or have a good sign]

$$\int_{\Sigma_0}\alpha\frac{(\partial_1 Y_{10})^2}{|x|^{2\alpha}}d\mu_{s_{\overline{g}}}\leq\int_{\Sigma_0}C\frac{(\partial_1 Y_{10})^2}{|x|^{2\alpha}}+C\frac{|Y_{10}\partial_1 Y_{10}|}{|x|^{2\alpha+\frac{1}{2}}}+C\frac{|\eta_0\partial_1 Y_{10}|}{|x|^{2\alpha+\frac{3}{2}}}$$
$$+\frac{Z_{10}\partial_1 Y_{10}}{|x|^{2\alpha-\frac{1}{2}}}d\mu_{s_{\overline{g}}}$$
$$\leq\int_{\Sigma_0}C\frac{Y_{10}^2}{|x|^{2\alpha+1}}+C\frac{(\partial_1 Y_{10})^2}{|x|^{2\alpha}}+\frac{1}{4}\frac{\eta_0^2}{|x|^{2\alpha+3}}$$
$$+\frac{Z_{10}^2}{|x|^{2\alpha-1}}d\mu_{s_{\overline{g}}}\tag{6.36}$$

On the other hand, multiplying the second equation in (6.25) with $\frac{\eta_0}{|x|^{2\alpha}}$, integrating over Σ_0 and integrating by parts we have

$$\begin{aligned}(b>0)\quad & \int_{\Sigma_0}\frac{(\partial_1\eta)^2}{|x|^{2\alpha}}+b\frac{\eta_0^2}{|x|^{2\alpha+3}}d\mu_{s_{\overline{g}}}\\ &\leq\int_{\Sigma_0}C\alpha\frac{|\eta_0\partial_1\eta_0|}{|x|^{2\alpha+\frac{1}{2}}}+C\frac{|\eta_0\partial_1 Y_{10}|}{|x|^{2\alpha+\frac{3}{2}}}+C\frac{\eta_0 Y_{i0}}{|x|^{2\alpha+2}}+\frac{\eta_0 h_0}{|x|^{2\alpha}}d\mu_{s_{\overline{g}}}\\ &\leq\int_{\Sigma_0}\frac{1}{2}\frac{(\partial_1\eta_0)^2}{|x|^{2\alpha}}+C\alpha^2\frac{\eta_0^2}{|x|^{2\alpha+1}}+\frac{1}{2}\frac{\eta_0^2}{|x|^{2\alpha+3}}+C\frac{(\partial_1 Y_{10})^2}{|x|^{2\alpha}}\\ &\quad+C\frac{Y_{10}^2}{|x|^{2\alpha+1}}+C\frac{h_0^2}{|x|^{2\alpha-3}}d\mu_{s_{\overline{g}}}\end{aligned}\tag{6.37}$$

Adding (6.32), (6.33) we employ Hardy's inequality

$$\int_{-\epsilon}^{\epsilon}\frac{Y_{10}^2}{|x|^{2\alpha+1}}dx\leq\frac{1}{\alpha^2}\int_{-\epsilon}^{\epsilon}\frac{(\partial_x Y_{10})^2}{|x|^{2\alpha-1}}dx,\qquad \partial_x\sim\frac{1}{\sqrt{|x|}}\partial_1.\tag{6.38}$$

and take advantage of the largeness of α to absorb most terms in the LHS and obtain a weighted H^1 estimate for Y_{10}, η_0. The higher order norms are controlled in turn using the system (6.25) and differentiating in ∂_1. If in addition $|x|\ll 1$, we deduce the improved estimate (6.35) by absorbing in the LHS all the η_0, Y_{10} terms appearing in the final inequalities. □

The Propositions 6.8, 6.9 combined imply that $D\Psi$ (6.20) is semi-Fredholm, i.e., it has finite dimensional kernel and closed range. Since similar type of estimates can also be derived for the adjoint operator, it follows that the linearized map is Fredholm. In the case where $|x|\ll 1$, we proved that the estimates can be improved to yield that $D\Psi$ is an isomorphism. This completes the proof of Proposition 6.6.

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Appendix A. Changing Frames Freedom; Propagating Identities; Retrieving the EVE from the Reduced Equations

Given a spacetime $(\mathcal{M}^{1+3}, g)$ and an orthonormal frame $\{e_i\}_0^3$, one may change to a Lorenz gauge frame $\{\tilde{e}_i\}_0^3$ by solving the following semilinear system of equations, which is derived by taking the divergence of (2.9):

$$\begin{aligned}\Box_g(O_a^l)=&\ (\mathrm{div}\tilde{A})_a{}^d O_d^l+\tilde{A}\partial O+A\partial O+O_a^k(\mathrm{div}A)_k{}^l\\ \text{(by (2.15) for } \tilde{A})\quad=&\ \tilde{A}^2O+\tilde{A}\partial O+A\partial O+O\mathrm{div}A\\ \text{(from (2.8))}\quad=&\ O^5A^2+O^3(\partial O)^2+AO^4\partial O+AO^2\partial O\\ &+O(\partial O)^2+A\partial O+O\mathrm{div}A,\end{aligned}$$

where the terms without indices in the RHS stand for an algebraic expression of a finite number terms of the depicted type.

Lemma A.1. *If the above system* (*which we write schematically as*)

$$\Box_g(O_i^b) = O^5A^2 + O^3(\partial O)^2 + AO^4\partial O + AO^2\partial O + O(\partial O)^2 + A\partial O + OdivA. \tag{A.1}$$

is well-posed in a certain solution space, then there exists a unique orthonormal frame

$$\tilde{e}_i = O_i^b e_b \tag{A.2}$$

with O_i^b lying in that particular space, which is identical to $\{e_i\}_0^3$ on the initial hypersurface Σ_0, verifies the Lorenz gauge condition (2.15) *and such that the connection coefficients $(\tilde{A}_0)_{ij} := g(\nabla_{\tilde{e}_0}\tilde{e}_i, \tilde{e}_j)$, $i < j$, are equal to a priori assigned functions on Σ_0; within the corresponding space of one order of regularity less than O_i^b.*

Proof. It suffices to show that the initial data for (A.1) is uniquely determined by the assertions. We set

$$O_i^b(\tau = 0) := I_i{}^b \qquad \text{(i.e., } \tilde{e}_i = e_i \text{ on } \Sigma_0\text{)}. \tag{A.3}$$

Let

$$\tilde{e}_0(O_i^b)(\tau = 0) = e_0(O_i^b)(\tau = 0) =: h_i^b, \qquad h_i^b m_{bj} = -h_j^b m_{bi}. \tag{A.4}$$

Then the transition formula (2.8) for $X = \tilde{e}_0$ reads

$$(\tilde{A}_0)_{ij}(\tau = 0) = (A_0)_{ij}(\tau = 0) + h_i^b m_{bj}. \tag{A.5}$$

Thus, the components $(\tilde{A}_0)_{ij}$ can be freely prescribed initially by choosing h_i^b in (A.4) accordingly. □

A.1. Proof of proposition 2.1. We will leave the reader to fill in the details for the fact that the solution $(A_\nu)_{ij}$, O_i^a of (2.16), (2.19) corresponds to a spacetime $(\mathcal{M}^{1+3}, g)$. This is a consequence of the necessary initial assumption (2.20). One such immediate consequence follows from (2.19) for $i = 0$:

$$\partial_0(O_0^a) = -O_0^b\Gamma^a_{[0b]}, \qquad O_0^a(\tau = 0) - I_0{}^a = 0, \tag{A.6}$$

which implies $O_0^a = I_0{}^a$ and hence $e_0 = \partial_0$ everywhere, since $\Gamma^a_{[00]} = 0$. The set of functions O_i^a defines the orthonormal frame $\{e_i\}_0^3$ in $\mathcal{M}^{1+3}$ through (2.18) and hence completely determines the metric g. What remains to be verified is that the connection coefficients of $\{e_i\}_0^3$ are indeed the $(A_\nu)_{ij}$'s of the given solution. In other words, we have to show that the connection D induced by the solution set $(A_\nu)_{ij}$,

$$D_{e_\nu}e_i := (A_\nu)_i{}^k e_k, \tag{A.7}$$

is the Levi–Civita connection ∇ of the metric g. Formally, one cannot take this for granted. It has to be retrieved from the equations (2.16), (2.19) and the initial assumption (2.20). For example, the compatibility of D with respect to g is encoded in the skew-symmetry of the $(A_\nu)_{ij}$'s

$$D(g) = 0, \qquad \text{iff} \qquad (A_\nu)_{ij} + (A_\nu)_{ji} = 0, \tag{A.8}$$

which also has to be verified, since it is a priori valid only initially (2.20). The way to do this is by deriving the following new system of equations from (2.16) for the symmetric sums:

$$\begin{aligned}\boxdot\big((A_\nu)_{ij}+(A_\nu)_{ji}\big) &= (A^{[\mu})_{\nu]}{}^{k}e_k\big((A_\mu)_{ij}+(A_\mu)_{ji}\big)+e^{\mu}\big((A_{[\mu})_{\nu]}{}^{k}\big[(A_k)_{ij}+(A_k)_{ji}\big]\big)\\ &\quad+e_\nu\big(A\big[(A)_{ij}+(A)_{ji}\big]\big)+e_\nu\big((A^{\mu})_{\mu}{}^{k}\big[(A_k)_{ij}+(A_k)_{ij}\big]\big),\end{aligned}\tag{A.9}$$

where we have assumed that the sum $(A^2)_{ij}+(A^2)_{ji}$ corresponding to the term A^2 in the gauge condition (2.15) can be expressed as $A\big[(A)_{ij}+(A)_{ji}\big]$. Since (A.9) has zero initial data (2.20), the symmetric sums are zero everywhere and hence the skew-symmetry (A.8) propagates.[21]

Proof of proposition 2.1; EVE and Lorenz gauge. Recall (2.17) and the reduced equations $H_{\nu ij}=0$. By assumption $(A_\nu)_{ij}$ is a solution of (2.16), i.e., the RHS of (2.17) vanishes. Taking the divergence of (2.17) with respect to the index ν, the first part of the LHS of (2.17), corresponding to the curl of the Ricci tensor, vanishes and we are left with

$$\begin{aligned}&\Box_g\big(\mathrm{div}A-A^2\big)_{ij}\\ &\qquad=(A^{\nu})_i{}^{c}e_\nu\big(\mathrm{div}A-A^2\big)_{cj}+(A^{\nu})_j{}^{c}e_\nu\big(\mathrm{div}A-A^2\big)_{ic}.\end{aligned}\tag{A.10}$$

The Lorenz gauge condition is valid initially (2.21). If the e_0 derivative of $\big(\mathrm{div}A-A^2\big)_{ij}$ is zero as well on Σ_0, then the Lorenz gauge is valid in all of $\mathcal{M}^{1+3}=\Sigma\times[0,T]$.[22] This is in fact implied by (2.17), putting $\nu=0$ we have

$$e_0\big(\mathrm{div}A-A^2\big)_{ij}=\nabla_j\mathrm{R}_{0i}-\nabla_i\mathrm{R}_{0j}=0\qquad\text{on }\Sigma_0\tag{A.11}$$

by virtue of the vanishing of $\mathrm{R}_{ab}(\tau=0)$ (2.21) and the (twice contracted) second Bianchi identity, $\nabla^a\mathrm{R}_{ab}=\frac{1}{2}\mathrm{R}$, to replace if necessary a transversal derivative with tangential ones to Σ_0.

On the other hand, taking the ∇^i divergence of (2.17) and commuting derivatives we obtain

$$\begin{aligned}\Box_g\mathrm{R}_{\nu j}=\nabla^i\nabla_j\mathrm{R}_{\nu i}&=\frac{1}{2}\nabla_j\nabla_\nu\mathrm{R}+\mathrm{R}^{i}{}_{j}{}^{c}{}_{\nu}\mathrm{R}_{ci}+\mathrm{R}^{i}{}_{j}{}^{c}{}_{i}\mathrm{R}_{\nu c}\\ &=\mathrm{R}^{i}{}_{j}{}^{c}{}_{\nu}\mathrm{R}_{ci}+\mathrm{R}_{j}{}^{c}\mathrm{R}_{\nu c},\end{aligned}\tag{A.12}$$

where we employed again the twice contracted second Bianchi identity and the fact that the scalar curvature R vanishes everywhere: [contracting $\{\nu j\}$ in (2.17)]

$$0=\nabla_i\mathrm{R}-\frac{1}{2}\nabla_i\mathrm{R}=\frac{1}{2}\nabla_i\mathrm{R},\qquad\mathrm{R}\big|_{\Sigma_0}=0.\tag{A.13}$$

Now that we know the Lorenz gauge is valid, the identities (2.21) and (2.17) $i=0$ imply

$$\mathrm{R}_{\nu j}=0,\qquad\nabla_0\mathrm{R}_{\nu j}=\nabla_j\mathrm{R}_{\nu 0},\qquad\text{on }\Sigma_0.\tag{A.14}$$

[21] This follows by a basic a priori energy estimate for linear systems like (A.9), which in the singular Schwarzschild background is derived in Sect. 5.3 for the more involved quasilinear system (4.3).

[22] Note however that the term $e_0\big(\mathrm{div}A-A^2\big)_{ij}$ is of second order in A and hence not at the level of initial data for (2.16), which we are allowed to prescribe. If zero initially, this should be a consequence of the geometric nature of the equations.

Utilizing the second Binachi identity $\nabla^a \mathrm{R}_{ab} = \frac{1}{2}\nabla \mathrm{R} = 0$ once more, we conclude that $\nabla_0 \mathrm{R}_{\nu j}$ vanishes and hence the initial data set of (A.12) is the trivial one. Thus, the initial condition $\mathrm{R}_{\nu j}(\tau = 0) = 0$ (2.21) propagates and the spacetime $(\mathcal{M}^{1+3}, g)$ obtained from the solution of (2.16) verifies the EVE (1.1). □

Remark A.2. Given the frame $\{e_i\}_0^3$ initially on Σ_0, and once the components $(A_0)_{ij}(\tau = 0)$ have been chosen,[23] then the initial data set of (2.13) is fixed by condition (2.21), i.e., the EVE and Lorenz gauge on Σ_0. Indeed, the components $(A_\nu)_{ij}(\tau = 0)$, $\nu, i, j = 1, 2, 3$, are determined uniquely by the orthonormal frame $\{e_i\}_1^3$ on $(\Sigma_0, \overline{g})$. The $(A_i)_{0j}(\tau = 0)$'s correspond to the components of second fundamental form K_{ij} of Σ_0, which is given by the solution to the constraints (1.5), included in (2.21). Moreover, the expression of (2.21) in terms of A, for $\nu, i = 1, 2, 3$, reads (schematically)

$$\begin{aligned} e_0(A_\nu)_{0i} &= e_\nu(A) + A^2 \\ e_0(A_0)_{ij} &= \sum_{\mu=1}^{3} e_\mu(A_\mu)_{ij} + A^2 \quad \text{on } \Sigma_0. \end{aligned} \tag{A.15}$$

Hence, the LHS functions are expressed in terms of already determined components. Finally, the rest components $e_0(A_\nu)_{ij}(\tau = 0)$, $\nu, i, j = 1, 2, 3$, are fixed by the algebraic property of the Riemann tensor

$$\begin{aligned} \mathrm{R}_{0\nu ij} &= \mathrm{R}_{ij0\nu} \\ e_0(A_\nu)_{ij} - e_\nu(A_0)_{ij} - ([A_\mu, A_\nu])_{ij} - (A_{[\mu})_{\nu]}{}^k (A_k)_{ij} &= \\ e_i(A_j)_{0\nu} - e_j(A_i)_{0\nu} - ([A_i, A_j])_{0\nu} - (A_{[i})_{j]}{}^k (A_k)_{0\nu} & \end{aligned}$$

or

$$e_0(A_\nu)_{ij} = e_\nu(A_0)_{ij} + e_i(A_j)_{0j} - e_j(A_i)_{0\nu} + A^2, \quad \text{on } \Sigma_0, \tag{A.16}$$

since all the terms in the RHS have been accounted for. Notice that the definition of Riemann curvature was implicitly used in deriving (A.12) upon commuting covariant derivatives.

References

1. Alexakis, S., Chen, D., Fournodavlos, G.: Singular Ricci solitons and their stability under the Ricci flow. Commun. Partial Differ. Equ. **40**(12), 2123–2172 (2015)
2. Andersson, L., Rendall, A.D.: Quiescent cosmological singularities. Commun. Math. Phys. **218**(3), 479–511 (2001)
3. Belinskii, V.A., Khalatnikov, I.M., Lifshitz, E.M.: Oscillatory approach to a singular point in the relativistic cosmology. Adv. Phys. **19**, 525 (1970)
4. Choquet-Bruhat, Y.: Théoreme d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires. Acta Math. **88**, 141–225 (1952)
5. Choquet-Bruhat, Y.: General relativity and the Einstein equations. Oxford University Press, Oxford (2009)
6. Christodoulou, D.: Violation of cosmic censorship in the gravitational collapse of a dust cloud. Commun. Math. Phys. **93**(2), 171–195 (1984)
7. Christodoulou, D.: The formation of black holes and singularities in spherically symmetric gravitational collapse. Commun. Pure Appl. Math. **44**, 339–373 (1991)

[23] The $(A_0)_{ij}$'s are not fixed by the Lorenz gauge condition; cf. Lemma A.1. They correspond to the ∂_0 derivative of the frame components e_i, which we can freely assign initially.

8. Chruściel, P., Delay, E.: On mapping properties of the general relativistic constraints operator in weighted function spaces, with applications. Mém. Soc. Math. Fr. (N.S.) No. 94 (2003), vi+103 pp.
9. Dafermos, M.: Black holes without spacelike singularities. Commun. Math. Phys. **332**, 729–757 (2014)
10. Dafermos, M. and Rodnianski I.: Lectures on black holes and linear waves. In: Clay Mathematics Proceedings, vol. 17, pp. 97–205. Amer. Math. Soc., Providence, RI (2013). arXiv:0811.0354
11. Dilts, J., Isenberg, J., Mazzeo, R., Meier, C.: Non-CMC solutions of the Einstein constraint equations on asymptotically Euclidean manifolds. Class. Quantum Gravity **31**, 1–10 (2014)
12. Garfinkle, D.: Numerical simulations of singular spacetimes. Class. Quantum Gravity **29**, 7 (2012)
13. Geroch, R., Kronheimer, E., Penrose, R.: Ideal points in space-time. Proc. R. Soc. Lond. Ser. A **327**, 545–567 (1972)
14. Gicquaud, R., Ngô, Q.A.: On the far from constant mean curvature solutions to the Einstein constraint equations. Class. Quantum Gravity **31**, 19504 (2014)
15. Holst, M., Nagy, G., Tsogtgerel, G.: Rough solutions to the Einstein constraint equations on closed manifolds without near-CMC conditions. Commun. Math. Phys. **288**, 547–613 (2009)
16. Isenberg, J., Moncrief, V.: Asymptotic behaviour in polarized and half-polarized $U(1)$ symmetric vacuum spacetimes. Class. Quantum Gravity **19**(21), 5361–5386 (2002)
17. Kichenassamy, S., Rendall, A.D.: Analytic description of singularities in Gowdy spacetimes. Class. Quantum Gravity **15**(5), 1339–1355 (1998)
18. Klainerman, S., Rodnianski, I., Szeftel, J.: The bounded L^2 curvature conjecture. Invent. Math. **202**(1), 91–216 (2015)
19. Luk, J.: Weak null singularities in general relativity. arXiv:1311.4970
20. Luk, J., Rodnianski, I.: Local propagation of impulsive gravitational waves. Commun. Pure Appl. Math. **68**(4), 511–624 (2015)
21. Luk, J., Rodnianski, I.: Nonlinear interaction of impulsive gravitational waves for the vacuum Einstein equations. arXiv:1301.1072
22. Maxwell, D.: A class of solutions of the vacuum Einstein constraint equations with freely specified mean curvature. Math. Res. Lett. **16**, 627–645 (2009)
23. Maxwell, D.: A model problem for conformal parameterizations of the Eins tein constraint equations. Commun. Math. Phys. **302**(3), 697–736 (2011)
24. Moncrief, V.: An integral equation for spacetime curvature in general relativity. Isaac Newton Institute Preprints (2005)
25. Ori, A.: Perturbative approach to the inner structure of a rotating black hole. Gen. Relativ. Gravit. **29**(7), 881–929 (1997)
26. Poisson, E., Israel, W.: Internal structure of black holes. Phys. Rev. D (3) **41**(6), 1796–1809 (1990)
27. Rendall, A.D.: Fuchsian analysis of singularities in Gowdy spacetimes beyond analyticity. Class. Quantum Gravity **17**(16), 3305–3316 (2000)
28. Rendall, A.D.: Fuchsian methods and spacetime singularities. Class. Quantum Gravity **21**, 295–304 (2004)
29. Ringström, H.: The Bianchi IX attractor. Ann. Henri Poincar. **2**(3), 405–500 (2001)
30. Ringström, H.: On Gowdy vacuum spacetimes. Math. Proc. Camb. Philos. Soc. **136**(2), 485–512 (2004)
31. Rodnianski, I., Speck, J.: Stable Big Bang Formation in Near-FLRW Solutions to the Einstein-Scalar Field and Einstein-Stiff Fluid Systems. arXiv:1407.6298

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