Ordinary Differential Equations

Final exam 11/1/2023. Instructor: Grigoris Fournodavlos

Total points: 13. Maximum grade: 10. Duration: 2,5 hours.

Problem 1 (2 points) *True or False? Justify your response:*

(i) *The IVP* $y' = |y|$, $y(0) = y_0$, *has a unique solution defined near* $t = 0$.

(ii) *The unique solution to the IVP* $y' = y^{\frac{1}{2}}$, $y(0) = 0$, *is the zeroth one.*

(iii) *Let* $f(y)$ *be locally Lipschitz in* $\mathbb{R}$. *Then the IVP* $y' = f(y)$, $y(0) = 1$ *has a global solution.*

(iv) *Let* $f(y)$ *be locally Lipschitz in* $\mathbb{R}$ *and let* y *be the solution to the IVP* $y' = f(y)$, $y(t_0) = y_0$, *with maximal domain of definition* (a, b), *which is also bounded. Then* $(a, b) = \mathbb{R}$.

Problem 2 (2 points) *Let* $f(y)$ *be Lipschitz in all of* $\mathbb{R}$, *ie.* $|f(y_2) - f(y_1)| \leq C|y_2 - y_1|$ *for all* $y_1, y_2 \in \mathbb{R}$, *and let* $y : [0, T) \to \mathbb{R}$, $T > 0$, *be the solution to the IVP* $y' = f(y)$, $y(0) = y_0$. *Show that*

(i) $\sup_{t\in[0,T)} |y(t)|$ *is finite. [Hint: Use Gronwall's inequality.]*

(ii) *The solution* $y(t)$ *extends to* $[0, +\infty)$.

Problem 3 (2 points) (i) *Find the general solution to the system*

$$\vec{y}' = A\vec{y}, \qquad A = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix}.$$

(ii) *Choose three linearly independent solutions and show that their Wronskian is constant.*

Problem 4 (2 points) *The flow* $\vec{\varphi}(t, \vec{\xi})$ *of a vector field* $\vec{v}(\vec{y})$ *solves the IVP*

$$\vec{y}' = \vec{v}(\vec{y}), \qquad \vec{y}(0) = \vec{\xi}$$

and the integral curve that passes through $\vec{\xi}$ *at* $t = 0$ *is given by* $\vec{\varphi}(\cdot, \vec{\xi})$.

(i) *Compute the flow of* $\vec{v}(\vec{y}) = (2y_2, 8y_1)$, *where* $\vec{y} = (y_1, y_2)$.

(ii) *Show that for* $\vec{\xi} = (1, 0)$ *the corresponding integral curve is a hyperbola.*

Problem 5 (2,5 points) *Find the bifurcation values for the family of equations*

$$y' = (y - \mu)(y^2 - 4y + 3), \qquad \mu \in \mathbb{R},$$

the equilibrium points, the stability/instability of the latter, and draw the phase diagrams for the different values of μ.

Problem 6 (2,5 points) (i) *Find the equilibrium points of the system*

$$\begin{cases} y_1' = y_1^3 - y_1 + \frac{1}{2}y_2^2 \\ y_2' = y_2(y_1^2 - \frac{1}{2}y_1) \end{cases}$$

and study their stability via the linearization. (1,5 points)

(ii) *Show that (0,0) is a Lyapunov stable equilibrium point. (1 point)*