EXISTENCE OF MAXIMAL SOLUTIONS FOR THE FINANCIAL STOCHASTIC STEFAN PROBLEM OF A VOLATILE ASSET WITH SPREAD

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Abstract. In this work, we consider the outer Stefan problem for the short-time prediction of the spread of a volatile asset traded in a financial market. The stochastic equation for the evolution of the density of sell and buy orders is the Heat Equation with a non-smooth noise in the sense of Walsh, posed in a moving boundary domain with velocity given by the Stefan condition. This condition determines the dynamics of the spread, and the solid phase $[s^-(t), s^+(t)]$ defines the bid-ask spread area wherein the transactions vanish. We introduce a reflection measure and prove existence and uniqueness of maximal solutions up to stopping times in which the spread $s^+(t)-s^-(t)$ stays a.s. non-negative and bounded. For this, we use a Picard approximation scheme and some of the estimates of [19] for the Green's function and the associated to the reflection measure obstacle problem. Analogous results are obtained for the equation without reflection corresponding to a signed density. Additionally, we apply some formal asymptotics when the noise depends only on time to derive that the spread is given by the integral of the solution of a linear diffusion stochastic equation.

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AMS subject classification: 35K55, 35K40, 60H30, 60H15, 91G80, 91B70.

1. Introduction

1.1. **The Stochastic Stefan problem with spread.** Let $w(x,t)$ be the density of sell and buy orders of a stochastically volatile liquid asset with spread. The moving boundary of the outer Stefan problem for w, $t \geq 0$, is the union of the curves $x = s^+(t)$, $x = s^-(t)$, enclosing the solid phase (or spread area) $\overline{S(t)}$ defined at a given time t by the interval $\overline{S(t)} := [s^-(t), s^+(t)]$. The midpoint $s(t) := (s^-(t) + s^+(t))/2$ is the so-called mid price, and the length $s^+(t) - s^-(t)$ of $\overline{S(t)}$ is the spread at time t. The asset price x has been transformed through a logarithmic scale and in general can take negative and positive values. If x is set in $\overline{S(t)}$, then the asset is not traded, and thus the density $w(x,t)$ of sell and buy orders is zero, otherwise the order is performed.

The Stefan problem for $w = w(x,t)$ satisfying the stochastic Heat equation is written as follows

$$\begin{cases} \partial_t w = \alpha\Delta w + \sigma(\mathrm{dist}(x, \partial S))\dot{W}_s, & x \in \mathbb{R} - \overline{S(t)} \ \ (\text{'liquid' phase}), \ \ t > 0, \\ w = 0, \ \ x \in \overline{S(t)} & (\text{'solid' phase }), \\ V := -\nabla w|_{\partial S} & (\text{Stefan condition}), \\ \partial S(0) = \{s^-(0), s^+(0)\} = \text{given}. \end{cases} \tag{1.1}$$

Here, $\alpha > 0$ stands for the total liquidity index of the market, estimated by the limit order book of the asset, and $\sigma\dot{W}_s$ is the stochastic volatility. The noise diffusion σ is a function of

$$\mathrm{dist}(x, \partial S(t)) = \min\{|x - s^+(t)|, |x - s^-(t)|\} \tag{1.2}$$

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the distance of the price x from the solid phase boundary $\partial S = \partial S(t) = \{s^-(t), s^+(t)\}$, and

$$\dot{W}_s(x,t) := \dot{W}(x - s^+(t), t) \text{ if } x \geq s^+(t), \ \dot{W}_s(x,t) := \dot{W}(-x + s^-(t), t) \text{ if } x \leq s^-(t), \tag{1.3}$$

where $\dot{W}(\pm x \mp s^{\pm}(t), t)$ is the non smooth in space and in time noise defined by Walsh in [34]. The initial condition $w(x,0)$ is considered given for all $x \in \mathbb{R}$.

The limit orders are instructions for trading of a portion of an asset, [25], based on information from the limit order book. The lowest sell order $s^+(t)$ defined as ask price, is the minimum price at which the investor is willing to receive, and $s^-(t)$ is the highest buy order or bid price which is the maximum price at which the investor is willing to pay. An order is executed if the price set (the so-called spot price) lies outside the spread interval $[s^-(t), s^+(t)]$, if not it is sorted in the order book list and not traded, see for example in [16, 24, 31]. We also note that the Gibbs Thomson condition on the moving boundary $\partial S(t)$ which is present in dimensions $d \geq 2$, [28], involving the mean curvature, and the constant value of $w = w_0$ in the solid phase are both replaced by the condition $w = w_0 := 0$ in $\overline{S(t)}$.

The velocity V of $\partial S(t)$ is defined at the boundary points by the Stefan condition

$$\begin{aligned} V(s^+(t), t) &:= \partial_t s^+(t) = -(\nabla w)^+(s^+(t), t), \\ V(s^-(t), t) &:= \partial_t s^-(t) = -(\nabla w)^-(s^-(t), t), \end{aligned} \tag{1.4}$$

for $(\nabla \cdot)^{\pm}$ denoting the derivative from the right ($x > s^+$) and left ($x < s^-$); the Stefan condition describes the change of liquidity. Therefore, the spread dynamics are given by

$$\partial_t s^+(t) - \partial_t s^-(t) = -(\nabla w)^+(s^+(t), t) + (\nabla w)^-(s^-(t), t)). \tag{1.5}$$

The gradients are taken along the 'outer' normal vector, i.e., the direction is towards the liquid phase, so here in $d = 1$ they coincide to the right and left derivatives. Models with a.s. non-negative density w, when for example a reflection measure is introduced to the stochastic heat equation, due to the fact that $w = 0$ at $x = s^{\pm}$ will result in an a.s. decreasing spread. More specifically $(\nabla w)^+(s^+(t), t) \geq 0$ and $(\nabla w)^-(s^-(t), t) \leq 0$ and thus by (1.5) $\partial_t(s^+(t) - s^-(t)) \leq 0$ for all $t \geq 0$ a.s. In contrast, when a signed density is considered the spread is not monotone.

Motivated by the analysis of [28, 29, 6] in higher dimensions, we define the bounded and time independent space domain $\Omega = (a, b)$ by

$$\Omega = \Omega_{\text{Liq}}(t) \cup [s^-(t), s^+(t)], \tag{1.6}$$

for a liquid phase $\Omega_{\text{Liq}} \subset \Omega$ so that for any $x \in \Omega_{\text{Liq}}$

$$0 \leq |x - s^-|, |x - s^+| \leq \lambda, \tag{1.7}$$

for $\lambda = b - a$ a positive constant relatively very larger than the initial spread $s^+(0) - s^-(0)$. The density $w(x,t)$ will be observed for x in Ω. As $\lambda \to \infty$ the liquid phase becomes infinite as in (1.1) and Ω will correspond to $\mathbb{R}$. The problem is one-dimensional and the liquid phase consists of two separate bounded linear segments. This enables the spliting of the Stefan problem equation in two equations posed for $x \in \Omega_{\text{Liq}}$ on $x \geq s^+$ and on $x \leq s^-$ where we shall apply the change of variables

$$y = x - s^+(t) \text{ if } x \geq s^+(t), \ \ y = -x + s^-(t) \text{ if } x \leq s^-(t), \tag{1.8}$$

and thus

$$y_t = -\partial_t s^+(t) \text{ if } x \geq s^+(t), \ \ \partial_t s^-(t) \text{ if } x \leq s^-(t). \tag{1.9}$$

As we shall see the equation is transformed due to the Stefan condition into two independent ones posed each on the fixed space domain $\mathcal{D} := (0, \lambda)$ with Dirichlet b.c. The value $y = 0$ occurs when the price x is $s^{\pm}$, while $y=\lambda$ when the spread is zero and $s^+=s^-$ hits the boundary of Ω. These equations are of the general form

$$v_t(y,t)=\alpha\Delta v(y,t)\mp\nabla v(0^+,t)\nabla v(y,t)\pm\sigma(y)\dot W(y,t)+\dot\eta(y,t),\quad y\in\mathcal{D},\quad t\geq 0, \tag{1.10}$$

for η a reflection measure keeping v a.s. non-negative, while $\eta=0$ will correspond to the unreflected problem and a signed v. We also note that when a system is considered in place of the Stefan problem (1.1) with buy and sell densities observed separately and with different liquidity coefficients α_1, α_2, the same equation of the above general form will appear after the change of variables for $\alpha=\alpha_1,\alpha_2$.

We prove existence of unique maximal solutions (v,η) for the stochastic equation (1.10) for the stopping time $\sup_{M>0}\tau_M$ where

$$\tau_M:=\inf\Big\{T\geq 0:\ \sup_{r\in(0,T)}|\nabla v(0^+,r)|\geq M\Big\}, \tag{1.11}$$

up to which $|\nabla v(0^+,r)|=\nabla v(0^+,r)$ stays a.s. bounded. In the case of the unreflected problem, η is just replaced by zero and the absolute value is kept. In order to return to the initial variables and to the moving boundary problem, the stopping time will be further reduced so that the spread stays a.s. non-negative and the spread area in the domain Ω. These restrictions will be induced by the Stefan condition and the resulting spread dynamics (1.5) on w, the initial spread $s^+(0)-s^-(0)$, and the magnitude of λ.

Deterministic parabolic Stefan problems have been so far extensively studied when describing the phenomenon of phase separation of alloys. In [28], Niethammer introduced the deterministic version of (1.1) in higher dimensions in the physical context of the LSW theory for the Ostwald ripening of alloys; there, a first order approximation was established for the dynamics of the radii of spherical moving boundaries in dimensions $d=3$. In [1, 3, 2], the authors considered the quasi-static problem and obtained second order approximations by taking into account the variable in general geometry of the solid phase. We also refer to [6] for the analysis of the parabolic Stefan problem of [28] in the presence of kinetic undercooling and additive forcing.

Antonopoulou, Bitsaki, and Karali, in [5], derived the rigorous financial interpretation of the parabolic Stefan stochastic model, which applies for a portfolio of assets when $d\geq 2$; a quasi-static version thereof approximates the parabolic one when the diffusion tends to infinity as in the case of very large trading. In contrast to the deterministic Stefan problem where a spherical initial solid phase or the interval $[s^-(0),s^+(0)]$ in dimension $d=1$ are static solutions, in the stochastic case the boundary changes as time evolves due to the random perturbation in the spde; see for example the numerical simulations in [5] when $d=3$. When the sell and buy orders densities are observed separately, then the evolution is described by a system of two stochastic Heat equations with different liquidity coefficients and volatilities depending on the distances $|x-s^+(t)|$, $|x-s^-(t)|$ respectively. Hambly, and Kalsi proved in [19] existence and uniqueness of stochastic solutions for such two phases Stefan systems with reflection, but under the assumption of zero spread for the asset price, i.e., for $s^+(t)=s^-(t)=s(t)$. Considering 2-phases 1-dimensional stochastic Stefan systems for the evolution of sell and buy orders without spread we refer also to [15, 35], and to the more recent results of [27, 20].

1.2. **Limit order books and spread.** An asset is defined as volatile if the corresponding trading price of sell or buy orders deviates from the mid (mean) price. The spread's length which is given as the difference between the actual sell price and the buy price is a measure for the risk of investment, [4]. In particular, highly traded assets tend to have very small spreads, while a relatively large spread indicates a higher risk. An order is a commitment from the traders, a buyer or a seller, to buy or sell respectively at an appropriate price at a given time $t>0$, for which the profit of the trade is maximized for both sides, [16], also called limit price. The spread and the density of transactions reflect asset's liquidity. The total volume of active limit orders in a financial market at a given time is stored in the asset's limit order book. The liquidity coefficient $\alpha > 0$ in the Stochastic Heat equation of the Stefan problem measures the diffusion strength of sell and buy orders and can be approximated, in small time periods, by the total volume of orders divided by an average spread, [5].

The various types of financial contracts require two entities, that is the holder of a financial asset, such as a security, commodity, or currency, who receives the future payments, and the issuer side which has the obligation to deliver the payments according to the initial terms and claims of transaction. An order is a commitment to buy or sell at a given time. Market orders are executed immediately upon submission in contrast to limit orders which remain active until they achieve the expected 'closing' price; this is an automatic execution procedure via online electronic platforms. Electronic trading platforms offer the ability to trade upon information from historical data like past market prices and curves of prices of stocks (old trading view). The market orders are based on the current market prices, while limit orders target to better future prices for maximizing profits, [25]. Limit orders are low risk commitments since the price of execution for sell or buy is predetermined reducing thus the odds of significant failure. However, the process is time consuming and the order may never be executed.

In [7] the German power market liquidity was studied, we also refer to [32] for a statistical analysis of the fluctuations of the average spread where the relation of spread with shares volume and volatility was examined, or to [23] for a stochastic equation model estimating the liquidity risk. In [13], the authors analyzed how transaction costs affect the spreads while in case of zero cost then the market price should act as a Wiener process; see also in [26] for the liquidity risk with respect to the transaction costs and market manipulation under a Brownian motion problem formulation, or in [12, 33, 21, 11], and in [18] for various empirical approaches on spread's forecast. We note that except from the bid-ask spread, there exist several other types of spread like the asset swap spread, the yield spread, the zero volatility spread, the option adjusted spread, the default swap spread, or the bank spreads, see for example in [30, 8, 9, 10, 17, 22].

1.3. **Main Results.** Our analysis covers 3 versions of the Stefan problem.

(1) Let $w_1, w_2 \geq 0$ be the density of sell orders and buy orders respectively. When $x > s^+(t)$ then only sell orders are executed ($w_2 = 0$), while when $x < s^-(t)$ then only buy orders are executed ($w_1 = 0$). Moreover at $x = s^+(t)$ $w_1 = 0$ and at $x = s^-(t)$ $w_2 = 0$. The signed density $w = w_1 - w_2$ is given by

$$w(x,t) = w_1(x,t) \text{ if } x > s^+(t), \;\; w(x,t) = -w_2(x,t) \text{ if } x < s^-(t), \;\; w(x,t) = 0 \text{ otherwise.} \tag{1.12}$$

We introduce in (1.1) the additive term $\dot{\eta}_s$ defined by

$$\dot{\eta}_s(x,t) := \dot{\eta}_1(x - s^+(t), t) \text{ if } x \geq s^+(t), \;\; \dot{\eta}_s(x,t) := -\dot{\eta}_2(-x + s^-(t), t) \text{ if } x \leq s^-(t), \tag{1.13}$$

where η_1, η_2 are reflection measures so that $w_1, w_2 \geq 0$.

(2) We consider the reflected problem where $w \geq 0$. The Stefan condition due to the non-negativity of w which vanishes at $x = s^\pm$ yields an a.s. decreasing spread. The reflection additive term on (1.1) is of the form

$$\dot{\eta}_s(x,t) := \dot{\eta}_1(x - s^+(t), t) \text{ if } x \geq s^+(t), \;\; \dot{\eta}_s(x,t) := \dot{\eta}_2(-x + s^-(t), t) \text{ if } x \leq s^-(t), \tag{1.14}$$

where η_1, η_2 are reflection measures keeping $w \geq 0$ for any $x \in \Omega_{\text{Liq}}$.

(3) The unreflected problem is analyzed with a signed density w where as in (1) the spread is non-monotone.

In all the above cases we derive a system of independent spdes of the form (1.10) for $v = v_1$, $v = v_2$. Then s^+, s^- are specified through integration of the Stefan condition. For stopping times wherein $s^- \leq s^+$ and $(s^-, s^+) \subseteq \Omega$ by applying the change of variables (1.8), $v_1 \to w|_{x \geq s^+}$, $v_2 \to -w|_{x \leq s^-}$ in (1) or $w|_{x \leq s^-}$ in (2) and (3), we return to the initial Stefan problem. The suggested transformation is efficient on representing the stochastic equation of the Stefan problem as a system of independent spdes posed on the fix domain $\mathcal{D} = (0, \lambda)$, of the same general form. Additionally, for the reflected equations, we impose the non-negativity of $v_{1,2}$ by proving existence of the measures $\eta_{1,2}$ on the fix domain which then define the additive reflection term in the initial equation. Our novel approach on transforming first the problem to an spde of reference and then establishing maximal solutions to the initial one by using the Stefan condition for the stopping times is also applicable for various other one-dimensional versions with financial interest being analyzed for example in [19, 20, 27] without spread. Note that our model permits zero spread. The noise diffusion and the noise depend on the distance of x from the spread area boundary and not on the position of x. This yields, since the velocity is given by the standard Stefan condition of (1.1), to spdes in the y variables where $s^{\pm}$, that belong to the initial problem unknowns, are absent. We also mention that variables of the form $y = -x + s(t)$ when $x \leq s$, $y = x - s(t)$ when $x \geq s$ for the zero spread model where s is the sell/buy price, are used in [19, 27]. In [19] the problem is not transformed, a weak solution formulation for proper test functions compactly supported in $[0, 1]$ induces somehow a relevant (not splited) system posed on the fixed domain $(0, 1)$ that seems to facilitate the authors proof of maximal solutions.

In Section 2, we present analytically the change of variables $y = x - s^+$, $y = -x + s^-$ for $x \in \Omega_{\text{Liq}}$, use the Stefan condition and derive per case the Stefan problems as systems of two independent spdes of the form (1.10) for $v = v_1, v_2$, cf. (2.6), (2.11), (2.14). Section 3 is devoted to the existence of unique weak maximal solutions (v, η) of the Dirichlet problem on $\mathcal{D}$ for (1.10) with reflection, and then of maximal solutions to the initial variables with stopping times restricted by the Stefan condition, the non-negativity of spread and the boundedness of the liquid phase. In detail, we write the spde in an integral form using the Green's function of the negative Dirichlet Laplacian and construct an approximate Picard scheme for the truncated problem. In Theorem 3.1 we prove existence and uniqueness a.s. for the Picard approximations, and on the limit existence and uniqueness of the truncated solution. For this, we use some of the Green's estimates of [19] and a proper Banach space introduced therein. The reflection measure η is associated to the obstacle problem estimated in [19]. In Theorem 3.2, using the consistency of the truncated solutions we prove existence of a unique maximal solution (v, η) a.s. in the maximal time interval $[0, \sup\limits_{M>0} \tau_M)$ for τ_M given by (1.11). Given the maximal solution (v, η), for $v = v_{1,2} \geq 0$, $\eta = \eta_{1,2}$, in Theorem 3.3 we prove existence of unique maximal solutions (w_1, η_1), (w_2, η_2) to the reflected Stefan problem (2.3)-(2.15)-(2.16) corresponding to (1), and of $w|_{x \geq s^+} = w_1 \geq 0$, $w|_{x \leq s^-} = -w_2 \leq 0$, in the maximal interval $\mathcal{I}_1 := [0, \hat{\tau})$ for $\hat{\tau} := \min\{\sup\limits_{M>0} \tau_{1M}, \tau_{1s}, \tau_1^*\}$, with $\tau_{1M}, \tau_{1s}, \tau_1^*$ given by (3.38), (3.39), (3.40) for which the spread exists and stays a.s. non-negative for any $t \in \mathcal{I}_1$. An analogous result for the case (2) is proven in Theorem 3.4 but in a different maximal interval $\mathcal{I}_2 := [0, \hat{\tau})$ for $\hat{\tau} := \min\{\sup\limits_{M>0} \tau_{1M}, \tau_{2s}\}$, with τ_{1M}, τ_{2s} given by (3.38), (3.41). There, the decreasing property of the spread is used.

In Section 4 we consider the Stefan problem without reflection, i.e., (3), and the Dirichlet problem on $\mathcal{D}$ for the spde (4.1) that v satisfies. Theorem 4.1 establishes existence and uniqueness a.s. of the truncated equation, and Theorem 4.2 existence of a unique maximal solution v in $[0, \sup\limits_{M>0} \tau_M)$ for τ_M as in (1.11). Then the existence and uniqueness of maximal solution in the initial variables is proven in Theorem 4.3 for the resulting stopping time. We also present some formal asymptotics for very large liquidity coefficient α, when the noise is only time dependent (for example the formal derivative of a Brownian) and for constant noise diffusion. Under the assumption that w approximates a mean-field value $w_\infty(t)$ when the distance from the spread boundary is very large, we derive that the spread is given by the integral of the solution of a stochastic linear diffusion equation, see the dynamics in (4.10), and the linear sde (4.11).

2. The Stefan problems

2.1. **Change of variables.** We consider Ω given by (1.6), $\Omega_{\rm Liq}$ by (1.7), and y defined by (1.8) for any $x \in \Omega_{\rm Liq} \cup \{s^-, s^+\}$. Let $\tilde{w}_1(x,t)$ be defined in $\{x \in \Omega_{\rm Liq} \cup \{s^-, s^+\} : \ x \geq s^+\}$ and $\tilde{w}_2(x,t)$ be defined in $\{x \in \Omega_{\rm Liq} \cup \{s^-, s^+\} : \ x \leq s^-\}$ and set for $y := x - s^+$

$$\tilde{w}_1(x,t) := \tilde{v}_1(y,t) \ \ \forall\, x \in \Omega_{\rm Liq} \cup \{s^-(t), s^+(t)\} : \ x \geq s^+(t),$$

while for $y := -x + s^-$

$$\tilde{w}_2(x,t) := \tilde{v}_2(y,t) \ \ \forall\, x \in \Omega_{\rm Liq} \cup \{s^-(t), s^+(t)\} : \ x \leq s^-(t).$$

If $x \geq s^+(t)$ we get

$$\begin{aligned}
&\tilde{w}_1(x,t) = \tilde{v}_1(x - s^+(t), t) = \tilde{v}_1(y,t), \quad y = x - s^+(t), \ \ y_x = 1, \ \ y_t = -\partial_t s^+(t) \\
&(\tilde{w}_1)_t(x,t) = (\tilde{v}_1)_y(y,t) y_t(y,t) + (\tilde{v}_1)_t(y,t) = -\partial_t s^+(t)(\tilde{v}_1)_y(y,t) + (\tilde{v}_1)_t(y,t), \\
&(\tilde{w}_1)_x(x,t) = (\tilde{v}_1)_y(y,t) y_x = +(\tilde{v}_1)_y(y,t), \\
&(\tilde{w}_1)_{xx}(x,t) = (\tilde{v}_1)_{yy}(y,t)(y_x(y,t))^2 = (\tilde{v}_1)_{yy}(y,t),
\end{aligned} \tag{2.1}$$

and if $x \leq s^-(t)$

$$\begin{aligned}
&\tilde{w}_2(x,t) = \tilde{v}_2(-x + s^-(t), t) = \tilde{v}_2(y,t), \ \ y = -x + s^-(t), \ \ y_x = -1, \ \ y_t = \partial_t s^-(t) \\
&(\tilde{w}_2)_t(x,t) = (\tilde{v}_2)_y(y,t) y_t(y,t) + (\tilde{v}_2)_t(y,t) = \partial_t s^-(t)(\tilde{v}_2)_y(y,t) + (\tilde{v}_2)_t(y,t), \\
&(\tilde{w}_2)_x(x,t) = (\tilde{v}_2)_y(y,t) y_x = -(\tilde{v}_2)_y(y,t), \\
&(\tilde{w}_2)_{xx}(x,t) = (\tilde{v}_2)_{yy}(y,t)(y_x(y,t))^2 = (\tilde{v}_2)_{yy}(y,t).
\end{aligned} \tag{2.2}$$

2.2. **Case 1.** Let for any $x \in \Omega$ the signed density w be given by

$$w(x,t) = w_1(x,t) - w_2(x,t) = \begin{cases} w_1(x,t) & \text{if } \ x > s^+(t), \\ -w_2(x,t) & \text{if } \ x < s^-(t), \\ 0 & \text{otherwise,} \end{cases}$$

for w_1, w_2 the densities of sell orders and buy orders respectively. We then have $w(x,t)|_{x \geq s^+} = w_1(x,t)$, $w(x,t)|_{x \leq s^-} = -w_2(x,t)$.

The equation (1.1) by introducing the additive term $\dot{\eta}_s$ given by (1.13) takes in $\Omega_{\rm Liq}$ the form

$$\partial_t w = \alpha \Delta w + \sigma(\mathrm{dist}(x, \partial S))\dot{W}_s(x,t) + \dot{\eta}_s(x,t), \qquad x \in \Omega_{\rm Liq}, \ \ t > 0,$$

or equivalently for $x \in \Omega_{\rm Liq}$

$$\begin{aligned}
\partial_t w_1 =& \alpha \Delta w_1 + \sigma(x - s^+(t))\dot{W}(x - s^+(t), t) + \dot{\eta}_1(x - s^+(t), t), \qquad\quad x > s^+(t), \ \ t > 0, \\
\partial_t w_2 =& \alpha \Delta w_2 - \sigma(-x + s^-(t))\dot{W}(-x + s^-(t), t) + \dot{\eta}_2(-x + s^-(t), t), \ \ x < s^-(t), \ \ t > 0,
\end{aligned} \tag{2.3}$$

while $w(x,t) = w_1(x,t) = w_2(x,t) = 0, \quad \forall\, x \in [s^-(t), s^+(t)], \ \ \forall\, t > 0$, and $w_1(x,0) = w(x,0)$ for any $x \geq s^+(0)$, $w_2(x,0) = -w(x,0)$ for any $x \leq s^-(0)$. We shall assume that $w_1(x,0), w_2(x,0) \geq 0$.

The reflection measures η_1, η_2 if exist will keep $w_1, w_2 \geq 0$ for all t a.s. Using the Stefan condition (1.4), we obtain

$$\begin{aligned} V(s^+(t),t) =& \partial_t s^+(t) = -(\nabla w)^+(s^+(t),t) = -(\nabla w_1)^+(s^+(t),t), \\ V(s^-(t),t) =& \partial_t s^-(t) = -(\nabla w)^-(s^-(t),t) = (\nabla w_2)^-(s^-(t),t), \end{aligned} \tag{2.4}$$

and so the spread dynamics are given by

$$\partial_t s^+(t) - \partial_t s^-(t) = -(\nabla w_1)^+(s^+(t),t) - (\nabla w_2)^-(s^-(t),t)). \tag{2.5}$$

We apply the change of variables $w_1(x,t) = v_1(y,t)$ for $y = x - s^+$ and so $(\nabla w_1)^+(s^+,t) = \nabla v_1(0^+,t)$, and $w_2(x,t) = v_2(y,t)$ for $y = -x + s^-$ and so $(\nabla w_2)^-(s^-,t) = -\nabla v_2(0^+,t)$, use (2.1), (2.2), and (2.4) which yields that $\partial_t s^+(t) = -(\nabla w_1)^+(s^+(t),t) = -\nabla v_1(0^+,t)$, and that $\partial_t s^-(t) = (\nabla w_2)^-(s^-(t),t) = -\nabla v_2(0^+,t)$, and derive the system of two independent initial and boundary value problems

(2.6)

$$\begin{aligned} \partial_t v_1(y,t) =& \alpha\Delta v_1(y,t) + \partial_t s^+(t)\nabla v_1(y,t) + \sigma(y)\dot{W}(y,t) + \dot{\eta}_1(y,t) \\ =& \alpha\Delta v_1(y,t) - \nabla v_1(0^+,t)\nabla v_1(y,t) + \sigma(y)\dot{W}(y,t) + \dot{\eta}_1(y,t), \;\; y \in (0,\lambda) =: \mathcal{D}, \;\; t > 0, \\ & v_1(0,t) = v_1(\lambda,t) = 0, \;\; t > 0, \;\; v_1(y,0) = w(y + s^+(0),0) \geq 0, \;\; y \in \mathcal{D}, \\ & \text{and} \\ \partial_t v_2(y,t) =& \alpha\Delta v_2(y,t) - \partial_t s^-(t)\nabla v_2(y,t) - \sigma(y)\dot{W}(y,t) + \dot{\eta}_2(y,t) \\ =& \alpha\Delta v_2(y,t) + \nabla v_2(0^+,t)\nabla v_2(y,t) - \sigma(y)\dot{W}(y,t) + \dot{\eta}_2(y,t), \;\; y \in (0,\lambda) =: \mathcal{D}, \;\; t > 0, \\ & v_2(0,t) = v_2(\lambda,t) = 0, \;\; t > 0, \;\; v_2(y,0) = -w(s^-(0) - y,0) \geq 0, \;\; y \in \mathcal{D}, \end{aligned}$$

where we used the Dirichlet b.c. $v_1(\lambda,t) = v_2(\lambda,t) = 0$.

By using (2.5), the spread evolution is given by

$$\partial_t(s^+(t) - s^-(t)) = -\nabla v_1(0^+,t) + \nabla v_2(0^+,t). \tag{2.7}$$

2.3. **Case 2.** Let for any $x \in \Omega$ the density w, and define

$$w_1(x,t) := w(x,t) \;\; x \geq s^+, \;\; w_2(x,t) = w(x,t) \;\; x \leq s^-,$$

and so

$$w(x,t) = \begin{cases} w_1(x,t) & \text{if } x > s^+(t), \\ w_2(x,t) & \text{if } x < s^-(t), \\ 0 & \text{otherwise.} \end{cases}$$

In this case, we have $w(x,t)|_{x \geq s^+} = w_1(x,t)$, $w(x,t)|_{x \leq s^-} = w_2(x,t)$.

We introduce in (1.1) the additive term $\dot{\eta}_s(x,t)$ given by (1.14), for η_1, η_2 there reflection measures keeping $w_1, w_2 \geq 0$ and thus $w \geq 0$. The equation on Ω_{Liq} takes the form

$$\partial_t w = \alpha\Delta w + \sigma(\text{dist}(x,\partial S))\dot{W}_s(x,t) + \dot{\eta}_s(x,t), \qquad x \in \Omega_{\text{Liq}}, \;\; t > 0,$$

or equivalently for $x \in \Omega_{\text{Liq}}$

$$\begin{aligned} \partial_t w_1 =& \alpha\Delta w_1 + \sigma(x - s^+(t))\dot{W}(x - s^+(t),t) + \dot{\eta}_1(x - s^+(t),t), & x > s^+(t), \;\; t > 0, \\ \partial_t w_2 =& \alpha\Delta w_2 + \sigma(-x + s^-(t))\dot{W}(-x + s^-(t),t) + \dot{\eta}_2(-x + s^-(t),t), & x < s^-(t), \;\; t > 0, \end{aligned} \tag{2.8}$$

while $w(x,t) = w_1(x,t) = w_2(x,t) = 0, \quad \forall\, x \in [s^-(t), s^+(t)], \;\; \forall\, t > 0$, and $w_1(x,0) = w(x,0)$ for any $x \geq s^+(0)$, $w_2(x,0) = w(x,0)$ for any $x \leq s^-(0)$. We shall assume that $w_1(x,0), w_2(x,0) \geq 0$.

The reflection measures η_1, η_2 if exist will keep $w_1, w_2, w \geq 0$ for all t a.s. Using the Stefan condition (1.4), we obtain

$$\begin{aligned} V(s^+(t),t) =&\partial_t s^+(t) = -(\nabla w)^+(s^+(t),t) = -(\nabla w_1)^+(s^+(t),t), \\ V(s^-(t),t) =&\partial_t s^-(t) = -(\nabla w)^-(s^-(t),t) = -(\nabla w_2)^-(s^-(t),t), \end{aligned} \tag{2.9}$$

and so the spread dynamics are given by

$$\partial_t s^+(t) - \partial_t s^-(t) = -(\nabla w_1)^+(s^+(t),t) + (\nabla w_2)^-(s^-(t),t)). \tag{2.10}$$

We apply the change of variables $w_1(x,t) = v_1(y,t)$ for $y = x - s^+$ and so $(\nabla w_1)^+(s^+,t) = \nabla v_1(0^+,t)$, and $w_2(x,t) = v_2(y,t)$ for $y = -x + s^-$ and so $(\nabla w_2)^-(s^-,t) = -\nabla v_2(0^+,t)$, use (2.1), (2.2), and (2.4) which yields that $\partial_t s^+(t) = -(\nabla w_1)^+(s^+(t),t) = -\nabla v_1(0^+,t)$, and that $\partial_t s^-(t) = -(\nabla w_2)^-(s^-(t),t) = \nabla v_2(0^+,t)$, to derive the system of two independent initial and boundary value problems

(2.11)
$$\begin{aligned} \partial_t v_1(y,t) =&\alpha\Delta v_1(y,t) + \partial_t s^+(t)\nabla v_1(y,t) + \sigma(y)\dot{W}(y,t) + \dot{\eta}_1(y,t) \\ =&\alpha\Delta v_1(y,t) - \nabla v_1(0^+,t)\nabla v_1(y,t) + \sigma(y)\dot{W}(y,t) + \dot{\eta}_1(y,t), \ \ y \in (0,\lambda) =: \mathcal{D}, \ \ t > 0, \\ & v_1(0,t) = v_1(\lambda,t) = 0, \ \ t > 0, \ \ v_1(y,0) = w(y + s^+(0), 0) \geq 0, \ \ y \in \mathcal{D}, \\ & \text{and} \\ \partial_t v_2(y,t) =&\alpha\Delta v_2(y,t) - \partial_t s^-(t)\nabla v_2(y,t) + \sigma(y)\dot{W}(y,t) + \dot{\eta}_2(y,t) \\ =&\alpha\Delta v_2(y,t) - \nabla v_2(0^+,t)\nabla v_2(y,t) + \sigma(y)\dot{W}(y,t) + \dot{\eta}_2(y,t), \ \ y \in (0,\lambda) =: \mathcal{D}, \ \ t > 0, \\ & v_2(0,t) = v_2(\lambda,t) = 0, \ \ t > 0, \ \ v_2(y,0) = w(s^-(0) - y, 0) \geq 0, \ \ y \in \mathcal{D}. \end{aligned}$$

By using (2.5), the spread evolution is given by

$$\partial_t(s^+(t) - s^-(t)) = -\nabla v_1(0^+,t) - \nabla v_2(0^+,t). \tag{2.12}$$

As we already mentioned, if the reflection measures exist and keep $v_1, v_2 \geq 0$ and since $v_1 = v_2 = 0$ at $y = 0$, then the spread is decreasing.

2.4. **Case 3.** As in Case 2, we consider for any $x \in \Omega$ the signed density w, and define

$$w_1(x,t) := w(x,t) \ \ x \geq s^+, \ \ w_2(x,t) = w(x,t) \ \ x \leq s^-.$$

We do not require here $w \geq 0$ and so we consider the unreflected equation (1.1) posed in $\Omega_{\text{Liq}}, \ \ t > 0$, or equivalently for $x \in \Omega_{\text{Liq}}$

$$\begin{aligned} \partial_t w_1 =&\alpha\Delta w_1 + \sigma(x - s^+(t))\dot{W}(x - s^+(t),t), \qquad\qquad x > s^+(t), \ \ t > 0, \\ \partial_t w_2 =&\alpha\Delta w_2 + \sigma(-x + s^-(t))\dot{W}(-x + s^-(t),t), \ \ x < s^-(t), \ \ t > 0, \end{aligned} \tag{2.13}$$

while $w(x,t) = w_1(x,t) = w_2(x,t) = 0, \quad \forall\, x \in [s^-(t), s^+(t)], \quad \forall\, t > 0$, and $w_1(x,0) = w(x,0)$ for any $x \geq s^+(0)$, $w_2(x,0) = w(x,0)$ for any $x \leq s^-(0)$. Using the Stefan condition (1.4), we obtain (2.9) again for the velocity and the spread dynamics are given by (2.10). We apply the change of variables $w_1(x,t) = v_1(y,t)$ for $y = x - s^+$, and $w_2(x,t) = v_2(y,t)$ for $y = -x + s^-$ to obtain as in

Case 2 the system of two independent initial and boundary value problems

$$\begin{aligned}\partial_t v_1(y,t) =&\alpha\Delta v_1(y,t) - \nabla v_1(0^+,t)\nabla v_1(y,t) + \sigma(y)\dot{W}(y,t), \;\; y\in(0,\lambda)=:\mathcal{D}, \;\; t>0,\\ &v_1(0,t)=v_1(\lambda,t)=0, \;\; t>0, \;\; v_1(y,0)=w(y+s^+(0),0), \;\; y\in\mathcal{D},\\ &\text{and}\\ \partial_t v_2(y,t) =&\alpha\Delta v_2(y,t) - \nabla v_2(0^+,t)\nabla v_2(y,t) + \sigma(y)\dot{W}(y,t), \;\; y\in(0,\lambda)=:\mathcal{D}, \;\; t>0,\\ &v_2(0,t)=v_2(\lambda,t)=0, \;\; t>0, \;\; v_2(y,0)=w(s^-(0)-y,0), \;\; y\in\mathcal{D}.\end{aligned} \tag{2.14}$$

The spread evolution is given as in Case 2 by (2.12), but since v_1, v_2 may change sign even if $v_1=v_2=0$ at $y=0$, the spread is not monotone.

When reflection measures are considered, i.e., for the Cases 1,2, each problem's unknowns for $t\in[0,T]$ is a pair (v,η) where the reflection measure η is defined to satisfy

$$\begin{aligned}&\text{for all measurable functions } \psi:\ \overline{\mathcal{D}}\times(0,T)\to[0,\infty)\\ &\int_0^t\int_{\mathcal{D}}\psi(y,s)\eta(dy,ds) \;\text{ is }\; \mathcal{F}_t-\text{measurable},\end{aligned} \tag{2.15}$$

and the constraint

$$\int_0^T\int_{\mathcal{D}} v(y,s)\eta(dy,ds)=0. \tag{2.16}$$

We shall assume that the noise diffusion σ is a sufficiently smooth function; its minimum regularity will be specified in the sequel. The random measure $W(dy,ds)$ is defined as the 1-dimensional space-time white noise induced by the 2-dimensional Wiener process $W:=\{W(y,t):\ t\in[0,T],\ y\in(0,\lambda)\}$ which generates, for any $t\geq 0$, the filtration $\mathcal{F}_t:=\sigma(W(y,s):\ s\leq t,\ y\in(0,\lambda))$, where the notation σ here denotes the σ-algebra.

Remark 2.1. *In all the above cases, given the solutions v_1,v_2 for $t\in[0,T]$, $s^+(t)$, $s^-(t)$ and the spread $s^+(t)-s^-(t)$ are derived by direct formulae after integration of the Stefan condition in $[0,t]$.*

Remark 2.2. *We observe that the transformed spdes of Cases 1,2,3 are of the general form (2.2), i.e.,*

$$v_t(y,t)=\alpha\Delta v(y,t)\mp\nabla v(0^+,t)\nabla v(y,t)\pm\sigma(y)\dot{W}(y,t)+\dot{\eta}(y,t),$$

posed on $\mathcal{D}:=(0,\lambda)$ for $t\in[0,T]$, with Dirichlet b.c. $v(x,t)=0$ at $\partial\mathcal{D}$, and $v(y,0)$ given, for $v\geq 0$ when η not the zero measure, and signed v when $\eta\equiv 0$.

Remark 2.3. *Given $v_{1,2}$ for any $y\in\mathcal{D}$ and any t in $[0,T]$, then the Stefan condition will determine after integration $s^{\pm}(t)$ in $(0,T]$. Let $x\in\Omega_{\mathrm{Liq}}$ then for any given $t\in[0,T]$ and any $x\geq s^+(t)$ since $y=x-s^+(t)$, $w(x,t)$ will be defined by $v_1(x-s^+(t),t)$, while for any $x\leq s^-(t)$ since $y=-x+s^-(t)$, $w(x,t)$ will be defined by $-v_2(-x+s^-(t),t)$ for case (1) or by $v_2(-x+s^-(t),t)$ for Cases 2,3.*

Remark 2.4. *Evolution for $v_{1,2}$ will be observed as long as $a\leq s^-\leq s^+\leq b$, while $a\leq x\leq b$. In particular, consider $y=\lambda=b-a$. Then if $x\geq s^+$ then $y=b-a=x-s^+\leq b-s^+$ will yield $-a\leq -s^+$ i.e., $s^+\leq a$ and thus $s^+=a$ and $s^-=s^+=a$ and $x=b$ which is the case when the the spread is zero and hits the boundary at b and $v_1(\lambda,t)=0$. If $x\leq s^-$ then $y=b-a=-x+s^-\leq -a+s^-$ will yield $b\leq s^-$ and thus $s^-=b$ and $s^+=s^-=b$ and $x=a$ which is the case when the the spread is zero and hits the boundary at a and $v_2(\lambda,t)=0$. When $y=0$ then either $x=s^-$ and $v_2(0,t)=0$ or $x=s^+$ and $v_1(0,t)=0$. For all $x\in(s^-,s^+)$ the density $w(x,t)$ will be set to 0. The initial values of $v_{1,2}$ are well defined through the initial value*

$w(x,0)$ which is given for all $x\in\mathbb{R}$. We assume that $w(x,0)$ is compactly supported in $\overline{\Omega}$ to obtain a compatibility condition to $v_{1,2}(\lambda,t)=0$ at $t=0$.

We will analyze in detail in the sequel how the restrictions of a non-negative spread and spread area in the domain Ω, i.e., $a<s^-\leq s^+<b$, reduce the stopping time up to which maximal solutions $w_{1,2}$ exist.

3. Existence of maximal solutions with reflection

In what follows we shall present the analytical proof of existence of unique maximal solutions (v,η) for the initial and boundary value problem for

$$v_t(y,t)=\alpha\Delta v(y,t)-\nabla v(0^+,t)\nabla v(y,t)+\sigma(y)\dot{W}(y,t)+\dot{\eta}(y,t), \tag{3.1}$$

posed for any y in $\mathcal{D}=(0,\lambda)$ for $t\in[0,T]$ with Dirichlet b.c., with $v(y,0)$ given, and η a reflection measure satisfying (2.15) and (2.16) keeping v non-negative. As $\alpha>0$ the proof for the 2d i.b.v. problem of (2.6) of Case 1 is completely analogous, while the results for Case 3 (unreflected problem) will be derived at a next section by setting $\eta\equiv 0$. We will keep the absolute values on $\nabla v(0^+,t)$ appearing in the following proofs (even if non-negative in (3.1)) so that the results are applicable for these cases directly.

3.1. Weak formulation.

Let us define an $L^2(\mathcal{D})$ basis of eigenfunctions $w_n:=\sin\Big(\frac{n\pi}{\lambda}x\Big),\ \ n=0,1,2,\cdots$, corresponding to the eigenvalues $\mu_n,\ n=0,1,\cdots$ of $-\Delta u=\mu u,\ \ u(0)=0,\ \ u(\lambda)=0$, where $\mu_n:=\frac{n^2\pi^2}{\lambda^2},\ \ n=0,1,2,\cdots$ The associate Green's function for the negative of the Dirichlet Laplacian can then be given by $\frac{2}{\lambda}\sum\limits_{n=0}^{\infty}e^{-\mu_n t}w_n(x)w_n(y)$, see [14], so that the Green's function corresponding to $-\alpha\Delta$ with Dirichlet b.c. is given by $G(t,x,y)=\frac{2}{\lambda}\sum\limits_{n=0}^{\infty}e^{-\alpha\mu_n t}w_n(x)w_n(y)$.

We say that v is a weak (analytic) solution of (3.1) if it satisfies for all $\phi=\phi(y)$ in $C^2(\overline{\mathcal{D}})$ with $\phi(0)=\phi(\lambda)=0$, the following weak formulation

$$\begin{aligned}\int_{\mathcal{D}}\Big(v(y,t)-v_0(y)\Big)\phi(y)dy&=\int_0^t\int_{\mathcal{D}}\Big(\alpha\Delta\phi(y)v(y,s)+\nabla\phi(y)\nabla v(0^+,s)v(y,s)\Big)dyds\\&+\int_0^t\int_{\mathcal{D}}\phi(y)\sigma(y)W(dy,ds)+\int_0^t\int_{\mathcal{D}}\phi(y)\eta(dy,ds).\end{aligned}\tag{3.2}$$

The solution of (3.1) admits for any $y\in\mathcal{D}$, $t\in[0,T]$, the next integral representation

$$\begin{aligned}v(y,t)=&\int_{\mathcal{D}}v_0(z)G(y,z,t)dz\\&+\int_0^t\int_{\mathcal{D}}\nabla v(0^+,s)\nabla G(y,z,t-s)v(z,s)dzds\\&+\int_0^t\int_{\mathcal{D}}G(y,z,t-s)\sigma(z)W(dz,ds)+\int_0^t\int_{\mathcal{D}}G(y,z,t-s)\eta(dz,ds),\end{aligned}\tag{3.3}$$

and η satisfies (2.15), (2.16).

3.2. Main Theorems.

Let the Banach space $(\mathcal{B},\|\cdot\|_{\mathcal{B}})$

$$\mathcal{B}:=\Big\{f\in C(\overline{\mathcal{D}}):\ \exists f'(0),\ f(0)=f(\lambda)=0\Big\},$$

with the norm $\|\cdot\|_{\mathcal{B}} : \mathcal{B} \to \mathbb{R}^+$, defined by

$$\|f\|_{\mathcal{B}} := \sup_{y\in\mathcal{D}}\Big|\frac{f(y)}{y}\Big|.$$

Let $M>0$ fixed, we define in the Banach space $\mathcal{B}$, as in [19], the operator $\mathcal{T}_M : \mathcal{B} \to \mathcal{B}$ given for any $y \in \overline{\mathcal{D}}$ and $u \in \mathcal{B}$ by

$$\mathcal{T}_M(u)(y,\cdot) = \begin{cases} y\min\left\{\frac{u(y,\cdot)}{y}, M\right\} & y \neq 0, \\ 0 & y = 0. \end{cases} \tag{3.4}$$

We consider a truncated problem through the action of the operator $\mathcal{T}_M$ on the gradient terms of the spde (3.1) for which we prove the next existence-uniqueness theorem.

Theorem 3.1. *Let the noise diffusion σ satisfy*

$$\sigma \in C(\overline{\mathcal{D}}), \;\; \sigma(0) = \sigma(\ell) = 0, \;\; \exists\, \sigma'(0). \tag{3.5}$$

Let also $M>0$ fixed, and some $p \geq p_0 > 8$, and let $v_0(y) \in L^p(\Omega, C[0,T];\mathcal{B})$ be the initial condition of (3.1)*. Then there exists a unique week solution (v^M, η^M) with $v^M \in L^p(\Omega, C[0,T];\mathcal{B})$, depending on M, to the truncated problem*

$$\begin{aligned} v^M_t(y,t) &= \alpha\Delta v^M(y,t) - \nabla(\mathcal{T}_M(v^M))(0^+,t)\nabla(\mathcal{T}_M(v^M))(y,t) \\ &\qquad + \sigma(y)\dot{W}(y,t) + \dot{\eta}^M(y,t), \;\; t\in(0,T], \;\; y \in \mathcal{D}, \\ v^M(y,0) &:= v_0(y), \;\; y\in\mathcal{D}, \\ v^M(0,t) &= v^M(\lambda,t) = 0, \;\; t \in (0,T], \end{aligned} \tag{3.6}$$

where $T := T_M > 0$ such that

$$\sup_{r\in(0,T)} |\nabla(\mathcal{T}_M(v^M))(0^+,r)|^p \leq C_2(M,p) < \infty \;\; a.s. \tag{3.7}$$

More specifically, for any $t \in (0,T)$, v^M satisfies the week formulation

$$\begin{aligned} v^M(y,t) = &\int_{\mathcal{D}} v_0(z)G(y,z,t)dz \\ &+ \int_0^t\int_{\mathcal{D}} \nabla(\mathcal{T}_M(v^M))(0^+,s)\nabla G(y,z,t-s)\mathcal{T}_M(v^M)(z,s)dzds \\ &+ \int_0^t\int_{\mathcal{D}} G(y,z,t-s)\sigma(z)W(dz,ds) \\ &+ \int_0^t\int_{\mathcal{D}} G(y,z,t-s)\eta^M(dz,ds), \end{aligned} \tag{3.8}$$

for $v^M(y,0) := v_0(y)$, and η^M, satisfies (2.15) *and* (2.16)*, i.e.,*

$$\begin{aligned} &\textit{for all measurable functions } \psi: \overline{\mathcal{D}}\times(0,T) \to [0,\infty) \\ &\int_0^t\int_{\mathcal{D}} \psi(y,s)\eta^M(dy,ds) \;\; \textit{is } \mathcal{F}_t - \textit{measurable}, \end{aligned} \tag{3.9}$$

and the constraint

$$\int_0^T\int_{\mathcal{D}} v^M(y,s)\eta^M(dy,ds) = 0. \tag{3.10}$$

Proof. The operator $\mathcal{T}_M : \mathcal{B} \to \mathcal{B}$ is well defined, [19], and thus, for any u in the space $\mathcal{B}$, $\mathcal{T}_M(u)$ returns in $\mathcal{B}$, and so $\mathcal{T}_M(u) \in C(\overline{\mathcal{D}})$ and vanishes at the boundary of $\mathcal{D}$, while the gradient $\nabla(\mathcal{T}_M(u))$ at $x = 0$ exists.

Motivated by the integral representation (3.8), we define through a Picard iteration scheme the approximation v_n^M of v^M as the solution of the approximate problem

$$\begin{aligned} v_n^M(y,t) &= \int_{\mathcal{D}} v_0(z) G(y,z,t) dz \\ &+ \int_0^t \int_{\mathcal{D}} \nabla(\mathcal{T}_M(v^M))(0^+,s) \nabla G(y,z,t-s) \mathcal{T}_M(v_{n-1}^M)(z,s) dz ds \\ &+ \int_0^t \int_{\mathcal{D}} G(y,z,t-s)\sigma(z) W(dz,ds) \\ &+ \int_0^t \int_{\mathcal{D}} G(y,z,t-s) \eta_n^M(dz,ds), \quad n := 1,2\cdots \end{aligned} \tag{3.11}$$

for $v_0^M(y,t) := v_0(y)$, and η_n^M, which approximates η^M, satisfying (2.15) and (2.16), i.e.,

$$\begin{aligned} &\text{for all measurable functions } \psi : \ \overline{\mathcal{D}} \times (0,T) \to [0,\infty) \\ &\int_0^t \int_{\mathcal{D}} \psi(y,s) \eta_n^M(dy,ds) \ \text{ is } \ \mathcal{F}_t - \text{measurable}, \end{aligned} \tag{3.12}$$

and the constraint

$$\int_0^T \int_{\mathcal{D}} v_n^M(y,s) \eta_n^M(dy,ds) = 0. \tag{3.13}$$

In order to keep v_n^M non-negative, and having in mind the integral property (3.13), we will absorb the reflection term η_n^M in the Picard scheme (3.11), by spliting v_n^M as follows

$$v_n^M(y,t) = u_n(y,t) + \mathbb{O}_n(y,t), \tag{3.14}$$

where $\mathbb{O}_n(y,t)$ solves in the weak sense the Heat Equation Obstacle problem for any $y \in \mathcal{D}$, $t \in [0,T]$

$$\begin{aligned} &\partial_t \mathbb{O}_n(y,t) = \alpha \Delta \mathbb{O}_n(y,t) + \tilde{\eta}_n(dy,dt), \quad u_n + \mathbb{O}_n \geq 0 (\Leftrightarrow \mathbb{O}_n \geq -u_n), \\ &\mathbb{O}_n(0,t) = \mathbb{O}_n(\lambda,t) = 0, \quad \mathbb{O}_n(y,0) = 0, \\ &\int_0^T \int_{\mathcal{D}} (u_n(y,s) + \mathbb{O}_n(y,s)) \tilde{\eta}_n(dy,ds) = 0. \end{aligned} \tag{3.15}$$

Note that the above problem has a unique weak solution $(\mathbb{O}_n, \tilde{\eta}_n)$ as long as u_n exists and is smooth. We observe that $\mathbb{O}_n(y,0) = 0$ yields that $u_n(y,0) = v_n^M(y,0) = v_0(y)$.

We define $\eta_n^M := \tilde{\eta}_n$, and as we shall see it satisfies (3.11) when v_n^M satisfies (3.14). Indeed, we replace $v_n^M = u_n(y,t) + \mathbb{O}_n(y,t)$ at the left-hand side of (3.11) and obtain for $\eta_n^M := \tilde{\eta}_n$

$$\begin{aligned}
u_n(y,t) + \mathbb{O}_n(y,t) &= \int_{\mathcal{D}} v_0(z) G(y,z,t) dz \\
&+ \int_0^t \int_{\mathcal{D}} \nabla(\mathcal{T}_M(v^M))(0^+,s) \nabla G(y,z,t-s) \mathcal{T}_M(v_{n-1}^M)(z,s) dz ds \\
&+ \int_0^t \int_{\mathcal{D}} G(y,z,t-s) \sigma(z) W(dz,ds) \\
&+ \int_0^t \int_{\mathcal{D}} G(y,z,t-s) \eta_n^M(dz,ds), \\
&= \int_{\mathcal{D}} v_0(z) G(y,z,t) dz \\
&+ \int_0^t \int_{\mathcal{D}} \nabla(\mathcal{T}_M(v^M))(0^+,s) \nabla G(y,z,t-s) \mathcal{T}_M(v_{n-1}^M)(z,s) dz ds \\
&+ \int_0^t \int_{\mathcal{D}} G(y,z,t-s) \sigma(z) W(dz,ds) \\
&+ \int_0^t \int_{\mathcal{D}} G(y,z,t-s) \tilde{\eta}_n(dz,ds) \;\; n := 1, 2 \cdots
\end{aligned} \tag{3.16}$$

Since $\mathbb{O}_n$ solves in the weak sense (3.15), and $\mathbb{O}_n(y,0) = 0$, then using the same Green's function G for the integral representation of $\mathbb{O}_n$, we see that the last term of (3.16) coincides with $\mathbb{O}_n(y,t)$, so we obtain

$$\begin{aligned}
u_n(y,t) = \int_{\mathcal{D}} & v_0(z) G(y,z,t) dz \\
&+ \int_0^t \int_{\mathcal{D}} \nabla(\mathcal{T}_M(v^M))(0^+,s) \nabla G(y,z,t-s) \mathcal{T}_M(v_{n-1}^M)(z,s) dz ds \\
&+ \int_0^t \int_{\mathcal{D}} G(y,z,t-s) \sigma(z) W(dz,ds), \;\; n := 1, 2 \cdots
\end{aligned} \tag{3.17}$$

We split now v^M by

$$v^M(y,t) = u(y,t) + \mathbb{O}(y,t), \tag{3.18}$$

and set $\eta^M := \tilde{\eta}$, where $(\mathbb{O}(y,t), \tilde{\eta}(y,t))$ solves in the weak sense the Heat Equation Obstacle problem for any $y \in \mathcal{D}$, $t \in [0,T]$

$$\begin{aligned}
&\partial_t \mathbb{O}(y,t) = \alpha \Delta \mathbb{O}(y,t) + \tilde{\eta}(dy,dt), \;\; u + \mathbb{O} \geq 0 (\Leftrightarrow \mathbb{O} \geq -u), \\
&\mathbb{O}(0,t) = \mathbb{O}(\lambda,t) = 0, \;\; \mathbb{O}(y,0) = 0, \\
&\int_0^T \int_{\mathcal{D}} (u(y,s) + \mathbb{O}(y,s)) \tilde{\eta}(dy,ds) = 0.
\end{aligned} \tag{3.19}$$

We observe that $\mathbb{O}(y,0)=0$ yields that $u(y,0)=v^M(y,0)=v_0(y)$, and as we argued for the derivation of (3.17), we obtain that u satisfies

$$\begin{aligned} u(y,t) = & \int_{\mathcal{D}} v_0(z)G(y,z,t)dz \\ & + \int_0^t \int_{\mathcal{D}} \nabla(\mathcal{T}_M(v^M))(0^+,s)\nabla G(y,z,t-s)\mathcal{T}_M(v^M)(z,s)dzds \\ & + \int_0^t \int_{\mathcal{D}} G(y,z,t-s)\sigma(z)W(dz,ds). \end{aligned} \tag{3.20}$$

Using (3.17) for u_n, u_{n-1}, by substraction, we get for $n=2,3,\cdots$

$$\begin{aligned} u_n(y,t)-u_{n-1}(y,t) = & \int_0^t \int_{\mathcal{D}} \Big[\nabla(\mathcal{T}_M(v^M))(0^+,s)\mathcal{T}_M(v^M_{n-1})(z,s) \\ & - \nabla(\mathcal{T}_M(v^M))(0^+,s)\mathcal{T}_M(v^M_{n-2})(z,s)\Big]\nabla G(y,z,t-s)dzds \\ = & \int_0^t \int_{\mathcal{D}} \Big[\nabla(\mathcal{T}_M(v^M))(0^+,s)z\min\Big\{\frac{v^M_{n-1}(z,s)}{z},M\Big\} \\ & - \nabla(\mathcal{T}_M(v^M))(0^+,s)z\min\Big\{\frac{v^M_{n-2}(z,s)}{z},M\Big\}\Big]\nabla G(y,z,t-s)dzds. \end{aligned} \tag{3.21}$$

In the above, we apply $\|\cdot\|_{\mathcal{B}}$-norm at both sides and then take p-powers for some $p>0$, and then $\sup\limits_{t\in(0,T)}$, and then expectation, to obtain for $n=2,3,\cdots$

$$\begin{aligned} & E\Big(\sup_{t\in(0,T)}\|u_n(\cdot,t)-u_{n-1}(\cdot,t)\|^p_{\mathcal{B}}\Big) \\ = & E\Big(\sup_{t\in(0,T)}\Big\|\int_0^t \int_{\mathcal{D}} \Big[\nabla(\mathcal{T}_M(v^M))(0^+,s)z\min\Big\{\frac{v^M_{n-1}(z,s)}{z},M\Big\} \\ & - \nabla(\mathcal{T}_M(v^M))(0^+,s)z\min\Big\{\frac{v^M_{n-2}(z,s)}{z},M\Big\}\Big]\nabla G(y,z,t-s)dzds\Big\|^p_{\mathcal{B}}\Big). \end{aligned} \tag{3.22}$$

In [19], various useful bounds were proven in the norm $\|\cdot\|_{\mathcal{B}}$ for the heat kernel G defined explicitly by a different series representation than the standard trigonometric series (bounds holding obviously true for αt in place of the time variable t, and $\mathcal{D}=(0,\lambda)$ in place of $(0,1)$ there). In particular, we use the estimate of Proposition 4.4. therein, to derive directly for some constant $c=c(T,p)>0$, that

$$E(\sup_{t\in(0,T)}\|J\|^p_{\mathcal{B}}) \le c(T,p)\int_0^T E(\sup_{\tau\in(0,s)}\|f(\cdot,\tau)\|^p_{\mathcal{B}})ds,$$

for

$$J(y,t) := \int_0^t \int_{\mathcal{D}} f(z,s)\nabla G(y,z,t-s)dzds,$$

and

$$\begin{aligned} f(z,s) := & \nabla(\mathcal{T}_M(v^M))(0^+,s)z\min\Big\{\frac{v^M_{n-1}(z,s)}{z},M\Big\} \\ & - \nabla(\mathcal{T}_M(v^M))(0^+,s)z\min\Big\{\frac{v^M_{n-2}(z,s)}{z},M\Big\}. \end{aligned}$$

Using the above in (3.22) yields for $n = 2, 3, \cdots$

$$
\begin{aligned}
&E\Big(\sup_{t\in(0,T)} \|u_n(\cdot,t) - u_{n-1}(\cdot,t)\|^p_{\mathcal{B}}\Big)\\
&\leq cC(T,p)\int_0^T E\Big(\sup_{\tau\in(0,s)}\sup_{z\in\mathcal{D}}\Big|\nabla(\mathcal{T}_M(v^M))(0^+,\tau)\min\Big\{\frac{v^M_{n-1}(z,\tau)}{z}, M\Big\}\\
&\qquad\qquad - \nabla(\mathcal{T}_M(v^M))(0^+,\tau)\min\Big\{\frac{v^M_{n-2}(z,\tau)}{z}, M\Big\}\Big|^p\Big)ds \qquad (3.23)\\
&=cC(T,p)\int_0^T E\Big(\sup_{\tau\in(0,s)}\sup_{z\in\mathcal{D}}\Big|\nabla(\mathcal{T}_M(v^M))(0^+,\tau)\Big|^p\\
&\qquad\qquad \Big|\min\Big\{\frac{v^M_{n-1}(z,\tau)}{z}, M\Big\} - \min\Big\{\frac{v^M_{n-2}(z,\tau)}{z}, M\Big\}\Big|^p\Big)ds.
\end{aligned}
$$

Observing that $a \leq M$ and $b \leq M$ yields $\min\{a, M\} - \min\{b, M\} = a - b$, while $a \geq M$ and $b \geq M$ yields $\min\{a, M\} - \min\{b, M\} = M - M = 0 \leq |a - b|$, while $a \leq M$ and $b \geq M$ yields $\min\{a, M\} - \min\{b, M\} = a - M \leq 0 \leq |a - b|$, we have

$$
|\min\{a, M\} - \min\{b, M\}| \leq |a - b|,
$$

and so, we obtain by (3.23) for $n = 2, 3, \cdots$

$$
\begin{aligned}
&E\Big(\sup_{t\in(0,T)} \|u_n(\cdot,t) - u_{n-1}(\cdot,t)\|^p_{\mathcal{B}}\Big)\\
&\leq cC(T,p)\int_0^T E\Big(\sup_{\tau\in(0,s)}\sup_{z\in\mathcal{D}}\Big|\nabla(\mathcal{T}_M(v^M))(0^+,\tau)\Big|^p\\
&\qquad\qquad \Big|\min\Big\{\frac{v^M_{n-1}(z,\tau)}{z}, M\Big\} - \min\Big\{\frac{v^M_{n-2}(z,\tau)}{z}, M\Big\}\Big|^p\Big)ds \qquad (3.24)\\
&\leq cC(T,p)\int_0^T E\Big(\sup_{\tau\in(0,s)}|\nabla(\mathcal{T}_M(v^M))(0^+,\tau)|^p \sup_{\tau\in(0,s)}\sup_{z\in\mathcal{D}}\Big|\frac{v^M_{n-1}(z,\tau)}{z} - \frac{v^M_{n-2}(z,\tau)}{z}\Big|^p\Big)ds\\
&= cC(T,p)\int_0^T E\Big(\sup_{\tau\in(0,s)}|(\nabla(\mathcal{T}_M(v^M))(0^+,\tau)|^p \sup_{\tau\in(0,s)}\|v^M_{n-1}(\cdot,\tau) - v^M_{n-2}(\cdot,\tau)\|^p_{\mathcal{B}}\Big)ds.
\end{aligned}
$$

But by (3.14), it holds that

$$
\begin{aligned}
\sup_{\tau\in(0,s)}\sup_{z\in\mathcal{D}}\Big|\frac{v^M_{n-1}(z,\tau)}{z} - \frac{v^M_{n-2}(z,\tau)}{z}\Big|^p &\leq c(p)\sup_{\tau\in(0,s)}\sup_{z\in\mathcal{D}}\Big|\frac{u_{n-1}(z,\tau)}{z} - \frac{u_{n-2}(z,\tau)}{z}\Big|^p\\
&\quad + c(p)\sup_{\tau\in(0,s)}\sup_{z\in\mathcal{D}}\Big|\frac{\mathbb{O}_{n-1}(z,\tau)}{z} - \frac{\mathbb{O}_{n-2}(z,\tau)}{z}\Big|^p \qquad (3.25)\\
&\leq c(p)\sup_{\tau\in(0,s)}\sup_{z\in\mathcal{D}}\Big|\frac{u_{n-1}(z,\tau)}{z} - \frac{u_{n-2}(z,\tau)}{z}\Big|^p,
\end{aligned}
$$

where for the last inequality we used the stability bound in $\sup_{\tau\in(0,s)}\sup_{z\in\mathcal{D}}$ of the obstacle problem solutions by the obstacle, cf. [19] in the proof of Theorem 3.2. So, for $s = T$, we obtain by (3.24), (3.25), and for $n = 2, 3, \cdots$

(3.26)

$$\begin{aligned}
&c(p)E\Big(\sup_{t\in(0,T)}\|v_n^M(\cdot,t)-v_{n-1}^M(\cdot,t)\|_{\mathcal{B}}^p\Big)\leq E\Big(\sup_{t\in(0,T)}\|u_n(\cdot,t)-u_{n-1}(\cdot,t)\|_{\mathcal{B}}^p\Big)\\
&\qquad\leq c(p)2^pC(T,p)\int_0^T E\Big(\sup_{\tau\in(0,s)}|\nabla(\mathcal{T}_M(v^M))(0^+,\tau)|^p\sup_{\tau\in(0,s)}\|v_{n-1}^M(\cdot,\tau)-v_{n-2}^M(\cdot,\tau)\|_{\mathcal{B}}^p\Big)ds\\
&\qquad\leq C(T,p)\int_0^T E\Big(\sup_{\tau\in(0,s)}|\nabla(\mathcal{T}_M(v^M))(0^+,\tau)|^p\sup_{\tau\in(0,s)}\|u_{n-1}(\cdot,\tau)-u_{n-2}(\cdot,\tau)\|_{\mathcal{B}}^p\Big)ds.
\end{aligned}$$

We apply the same argumentation as for deriving the above inequality, on (3.17) now. By using that $v_0 \in L^p(\Omega, C[0,T];\mathcal{B})$, and the estimate of Proposition 4.3 from [19], as $p > 8 > 2$, we obtain for $n = 1, 2, \cdots$

$$\begin{aligned}
&c(p)E\Big(\sup_{t\in(0,T)}\|v_n^M(\cdot,t)\|_{\mathcal{B}}^p\Big)\leq E\Big(\sup_{t\in(0,T)}\|u_n(\cdot,t)\|_{\mathcal{B}}^p\Big)\\
&\quad\leq C(T,p)\int_0^T E\Big(\sup_{\tau\in(0,s)}|\nabla(\mathcal{T}_M(v^M))(0^+,\tau)|^p\sup_{\tau\in(0,s)}\|v_{n-1}^M(\cdot,\tau)\|_{\mathcal{B}}^p\Big)ds\\
&\qquad+CT\sup_{\tau\in(0,T)}\|\sigma(\cdot,\tau)\|_{\mathcal{B}}^p+C\\
&\quad\leq C(T,p)\int_0^T E\Big(\sup_{\tau\in(0,s)}|\nabla(\mathcal{T}_M(v^M))(0^+,\tau)|^p\sup_{\tau\in(0,s)}\|u_{n-1}(\cdot,\tau)\|_{\mathcal{B}}^p\Big)ds\\
&\qquad+CT\sup_{\tau\in(0,T)}\|\sigma(\cdot,\tau)\|_{\mathcal{B}}^p+C.
\end{aligned}\tag{3.27}$$

Here, we used (3.14) and the bound of the solution $\mathbb{O}_n$ by the obstacle (comparing with the zero solution) for the first and the third inequality. Moreover, since $p > 8$ we applied the estimate of Proposition 4.5 in [19] to bound the noise term. In the above, note that since σ satisfies (3.5), i.e., $\sigma \in C(\overline{\mathcal{D}})$, $\sigma(0) = \sigma(\ell) = 0$, and that it exists $\sigma'(0)$, then $\|\sigma\|_{\mathcal{B}}$ is well defined. This assumption models a zero volatility at the boundary of $\mathcal{D}$ in accordance to the Dirichlet b.c. for the density of the transactions v, i.e., the solution of (3.1), that vanishes on $\partial\mathcal{D}$.

Thus, by (3.27), since σ satisfies (3.5), v_n^M, u_n stay in $L^p(\Omega, C[0,T];\mathcal{B})$ if $v_0^M(y,t) := v_0(y) \in L^p(\Omega, C[0,T];\mathcal{B})$ and $T = T_M$ such that (4.3) holds true, i.e.,

$$\sup_{r\in(0,T)}|\nabla(\mathcal{T}_M(v^M))(0^+,r)|^p\leq C_2(M,p)<\infty\ \ a.s.$$

Furthermore, by (3.26), we get that v_n^M, u_n are Cauchy in $L^p(\Omega, C[0,T];\mathcal{B})$ while they also stay in $L^p(\Omega, C[0,T];\mathcal{B})$, for $T = T_M$. So, by the completeness of the Banach space $\mathcal{B}$ in this norm, both they converge in $L^p(\Omega, C[0,T];\mathcal{B})$ to some unique $\hat{v}^M, \hat{u} \in L^p(\Omega, C[0,T];\mathcal{B})$ as $n \to \infty$. In details, by using (3.26), we obtain

$$
\begin{aligned}
&E\Big(\sup_{t\in(0,T)}\|v_n^M(\cdot,t)-v_{n-1}^M(\cdot,t)\|_{\mathcal{B}}^p\Big)\leq CE\Big(\sup_{t\in(0,T)}\|u_n(\cdot,t)-u_{n-1}(\cdot,t)\|_{\mathcal{B}}^p\Big)\\
&\quad\leq\tilde{c}\int_0^T E\Big(\sup_{\tau\in(0,s)}|\nabla(\mathcal{T}_M(v^M))(0^+,\tau)|^p\sup_{\tau\in(0,s)}\|v_{n-1}^M(\cdot,\tau)-v_{n-2}^M(\cdot,\tau)\|_{\mathcal{B}}^p\Big)ds\\
&\quad\leq\tilde{c}\int_0^T E\Big(\sup_{\tau\in(0,s)}\|v_{n-1}^M(\cdot,\tau)-v_{n-2}^M(\cdot,\tau)\|_{\mathcal{B}}^p\Big)ds\\
&\quad\leq c^{n-1}\int_0^T\int_0^{s_{n-1}}\int_0^{s_{n-2}}\cdots\int_0^{s_2}1d_{s_1}\cdots d_{s_{n-2}}d_{s_{n-1}}E\Big(\sup_{\tau\in(0,T)}\|v_1^M(\cdot,\tau)-v_0^M(\cdot,\tau)\|_{\mathcal{B}}^p\Big)\\
&\quad\leq C\frac{c^{n-1}}{(n-1)!}\to 0\quad as\ \ n\to\infty,
\end{aligned}
\tag{3.28}
$$

where we used that $v_1^M,\ v_0^M\in L^p(\Omega,C[0,T];\mathcal{B})$ since v_n^M stays in $L^p(\Omega,C[0,T];\mathcal{B})$ for all n if $v_0^M(y,t):=v_0(y)\in L^p(\Omega,C[0,T];\mathcal{B})$. Therefore, v_n^M, u_n are Cauchy in $L^p(\Omega,C[0,T];\mathcal{B})$.

Moreover, we also obtain, as in (3.27)

$$
\begin{aligned}
&E\Big(\sup_{t\in(0,T)}\|v^M(\cdot,t)\|_{\mathcal{B}}^p\Big)\leq CE\Big(\sup_{t\in(0,T)}\|u(\cdot,t)\|_{\mathcal{B}}^p\Big)\\
&\leq C\int_0^T E\Big(\sup_{\tau\in(0,s)}|\nabla(\mathcal{T}_M(v^M))(0^+,\tau)|^p\sup_{\tau\in(0,s)}\|v^M(\cdot,\tau)\|_{\mathcal{B}}^p\Big)ds\\
&\qquad+CT\sup_{\tau\in(0,T)}\|\sigma(\cdot,\tau)\|_{\mathcal{B}}^p+C\\
&\leq C\int_0^T E\Big(\sup_{\tau\in(0,s)}|\nabla(\mathcal{T}_M(v^M))(0^+,\tau)|^p\sup_{\tau\in(0,s)}\|u(\cdot,\tau)\|_{\mathcal{B}}^p\Big)ds\\
&\qquad+CT\sup_{\tau\in(0,T)}\|\sigma(\cdot,\tau)\|_{\mathcal{B}}^p+C.
\end{aligned}
\tag{3.29}
$$

Therefore, by Gronwall's inequality on (3.29), and using (4.3), (3.5), we arrive at

$$
E\Big(\sup_{t\in(0,T)}\|v^M(\cdot,t)\|_{\mathcal{B}}^p\Big)+E\Big(\sup_{t\in(0,T)}\|u(\cdot,t)\|_{\mathcal{B}}^p\Big)\leq C.
\tag{3.30}
$$

Thus, we get that $v^M,u\in L^p(\Omega,C[0,T];\mathcal{B})$. By substraction of the integral representations (3.17), (3.20), we obtain (as previously when deriving the 4th inequality of (3.26)), that u_n,u satisfy for $n=1,2,\cdots$

$$
\begin{aligned}
&E\Big(\sup_{t\in(0,T)}\|u_n(\cdot,t)-u(\cdot,t)\|_{\mathcal{B}}^p\Big)\\
&\quad\leq C\int_0^T E\Big(\sup_{\tau\in(0,s)}|\nabla(\mathcal{T}_M(v^M))(0^+,\tau)|^p\sup_{\tau\in(0,s)}\|u_{n-1}(\cdot,\tau)-u(\cdot,\tau)\|_{\mathcal{B}}^p\Big)ds\\
&\quad\leq\frac{C^{n-1}}{(n-1)!}E\Big(\sup_{t\in(0,T)}\|u_1(\cdot,t)-u(\cdot,t)\|_{\mathcal{B}}^p\Big)\to 0\ \ as\ \ n\to\infty.
\end{aligned}
\tag{3.31}
$$

Here, we used once again (3.5), (4.3), the argument for the last bound of (3.28), together with the fact that, as proven, $u_1,u\in L^p(\Omega,C[0,T];\mathcal{B})$. Therefore, for $T=T_M$, $u_n\to u$ as $n\to\infty$ in $L^p(\Omega,C[0,T];\mathcal{B})$. By uniqueness of the limits, we get that $\hat{u}=u$ a.s.

Since u exists uniquely, then the solution $(\mathbb{O},\tilde{\eta})$ of (3.19) exists uniquely. But it holds that

$$\begin{aligned}\sup_{\tau\in(0,s)}\sup_{z\in\mathcal{D}}\Big|\frac{v_n^M(z,\tau)}{z}-\frac{v^M(z,\tau)}{z}\Big|^p \leq & c\sup_{\tau\in(0,s)}\sup_{z\in\mathcal{D}}\Big|\frac{u_n(z,\tau)}{z}-\frac{u(z,\tau)}{z}\Big|^p\\ &+c\sup_{\tau\in(0,s)}\sup_{z\in\mathcal{D}}\Big|\frac{\mathbb{O}_n(z,\tau)}{z}-\frac{\mathbb{O}(z,\tau)}{z}\Big|^p\\ \leq & c\sup_{\tau\in(0,s)}\sup_{z\in\mathcal{D}}\Big|\frac{u_n(z,\tau)}{z}-\frac{u(z,\tau)}{z}\Big|^p,\end{aligned}$$

where again we used the stability bound in $\sup_{\tau\in(0,s)}\sup_{z\in\mathcal{D}}$ of the obstacle problem solutions by the obstacle, see in [19] in the proof of Theorem 3.2. So, for $T=T_M$, by taking expectation, since, as we have shown, $u_n\to u$ in $L^p(\Omega,C[0,T];\mathcal{B})$, we obtain that $v_n^M\to v^M$ as $n\to\infty$ in $L^p(\Omega,C[0,T];\mathcal{B})$. By uniqueness of the limits, we get that $\hat{v}^M=v^M$ a.s.

So, for all $T=T_M>0$ such that (4.3) holds true, we derive the following:

(1) v^M,u exist and belong in $L^p(\Omega,C[0,T];\mathcal{B})$.
(2) Since u exists, we may define $\eta^M:=\tilde{\eta}$ a.s. for $\tilde{\eta}$ the second term of the solution $(\mathbb{O},\tilde{\eta})$ of (3.19).
(3) The pair (v^M,η^M) exists, $v^M\in L^p(\Omega,C[0,T];\mathcal{B})$, and is the week solution of the truncated problem (3.6) in $(0,T)$, where v^M, η^M satisfy the week formulation (3.8) for $v^M(y,0):=v_0(y)$, where η^M satisfies (3.9) and (3.10).
(4) The pair (v^M,η^M) is unique. Indeed, uniqueness of the limit of v_n^M showed that v^M is unique. Uniqueness of η^M follows by the uniqueness of $\tilde{\eta}$ of the obstacle problem (3.19) since as we have shown u exists uniquely as the limit of u_n in $L^p(\Omega,C[0,T];\mathcal{B})$.

Therefore, there exists a unique solution (v^M,η^N) of the week formulation (3.8) with $v^M\in L^p(\Omega,C[0,T];\mathcal{B})$ and with η^M satisfying (3.9) and (3.10), which completes the proof. ☐

We return to the M-independent problem (3.1), and we shall prove that it admits a unique maximal solution by concatenation of the solution of the M-truncated problem (3.6)-(3.9)-(3.10). This is established by the next Main Theorem.

Theorem 3.2. *Let the noise diffusion σ satisfy* (3.5)*, and $v_0(y)\in L^p(\Omega,C[0,T];\mathcal{B})$ for $p\geq p_0>8$. Then, there exists a unique weak maximal solution (v,η) to the problem* (3.1)-(2.15)-(2.16) *in the maximal interval $[0,\sup_{M>0}\tau_M)$, where*

$$\begin{aligned}\tau_M:=&\inf\Big\{T\geq 0:\ \sup_{r\in(0,T)}|\nabla v(0^+,r)|\geq M\Big\}\\ =&\inf\Big\{T\geq 0:\ \sup_{r\in(0,T)}\nabla v(0^+,r)\geq M\Big\}.\end{aligned}\tag{3.32}$$

Proof. We note that for the reflected problem since $v\geq 0$ a.s. and $v(0,t)=0$, then $\nabla v(0^+,t)\geq 0$ a.s. for any t. As we mentioned, we continue to keep the absolute value on $\nabla v(0^+,t)$ in this proof also in order to present a more general result applicable to the 2d i.b.v. problem of (2.6), and to the problem without reflection of next section.

Let v^M as in Theorem 3.1. We observe first that by the operator definition (3.4), and since $v^M(0,t)=0$, we have

$$\sup_{r\in(0,T)}|\nabla(\mathcal{T}_M(v^M))(0^+,r)|^p\leq\min\Big\{\sup_{r\in(0,T)}|\nabla v^M(0^+,r)|^p,M^p\Big\}\leq M^p,\tag{3.33}$$

and so by (4.3), Theorem 3.1 holds also for any $T = T(M)$ such that

$$\min\Big\{\sup_{r\in(0,T)}|\nabla v^M(0^+,r)|, M\Big\} \leq M < C_2(M,p)^{1/p} < \infty \;\; a.s. \tag{3.34}$$

We fix $M > 0$, and consider arbitrary $\tilde{M} > 0$ such that $\tilde{M} \leq M$. Thus, the weak solution (v^M, η^M) of (3.6)-(3.9)-(3.10) solves weakly the M-independent problem (3.1)-(2.15)-(2.16) (since they share the same initial condition), until the (random) stopping time τ defined as follows

$$\begin{aligned}\tau :=& \inf\Big\{T \geq 0 : \; \min\Big\{\sup_{r\in(0,T)}|\nabla v^M(0^+,r)|, M\Big\} \geq \tilde{M}\Big\}\\ =& \inf\Big\{T \geq 0 : \; \sup_{r\in(0,T)}|\nabla v^M(0^+,r)| \geq \tilde{M}\Big\}.\end{aligned}\tag{3.35}$$

Let now an arbitrary deterministic $t > 0$, then we have as in (3.26)

$$\begin{aligned}&cE\Big(\sup_{s\in(0,\min\{t,\tau\})}\|v^M(\cdot,s) - v^{\tilde{M}}(\cdot,s)\|^p_{\mathcal{B}}\Big) \leq E\Big(\sup_{s\in(0,\min\{t,\tau\})}\|u(\cdot,s) - \tilde{u}(\cdot,s)\|^p_{\mathcal{B}}\Big)\\ &\leq cC(t,p)\int_0^{\min\{t,\tau\}} E\Big(\sup_{r\in(0,s)}|\nabla(\mathcal{T}_{\tilde{M}}(v^{\tilde{M}}))(0^+,r)|^p \sup_{r\in(0,s)}\|v^M(\cdot,r) - v^{\tilde{M}}(\cdot,r)\|^p_{\mathcal{B}}\Big)ds\\ &\leq cC(t,p)\int_0^{\min\{t,\tau\}} E\Big(\sup_{r\in(0,s)}|\nabla(\mathcal{T}_{\tilde{M}}(v^{\tilde{M}}))(0^+,r)|^p \sup_{r\in(0,s)}\|u(\cdot,r) - \tilde{u}(\cdot,r)\|^p_{\mathcal{B}}\Big)ds\\ &\leq C(t,p)\tilde{M}\int_0^{\min\{t,\tau\}} E\Big(\sup_{r\in(0,s)}\sup_{r\in(0,s)}\|u(\cdot,r) - \tilde{u}(\cdot,r)\|^p_{\mathcal{B}}\Big)ds,\end{aligned}\tag{3.36}$$

for $\tilde{u}$ given by

$$\begin{aligned}\tilde{u}(y,t) =& \int_{\mathcal{D}} v_0(z)G(y,z,t)dz\\ &+ \int_0^t\int_{\mathcal{D}} \nabla(\mathcal{T}_{\tilde{M}}(v^{\tilde{M}}))(0^+,s)\nabla G(y,z,t-s)\mathcal{T}_{\tilde{M}}(v^{\tilde{M}})(z,s)dzds\\ &+ \int_0^t\int_{\mathcal{D}} G(y,z,t-s)\sigma(z)W(dz,ds),\end{aligned}\tag{3.37}$$

while

$$\begin{aligned}&cC(t,p)\int_0^{\min\{t,\tau\}} E\Big(\sup_{r\in(0,s)}|\nabla(\mathcal{T}_{\tilde{M}}(v^{\tilde{M}}))(0^+,r)|^p \sup_{r\in(0,s)}\|v^M(\cdot,r) - v^{\tilde{M}}(\cdot,r)\|^p_{\mathcal{B}}\Big)ds\\ &\leq C(t,p)\tilde{M}\int_0^{\min\{t,\tau\}} E\Big(\sup_{r\in(0,s)}\sup_{r\in(0,s)}\|v^M(\cdot,r) - v^{\tilde{M}}(\cdot,r)\|^p_{\mathcal{B}}\Big)ds.\end{aligned}$$

Therefore, by Gronwall's inequality, we get

$$u(\cdot,s) = \tilde{u}(\cdot,s), \quad v^M(\cdot,s) = v^{\tilde{M}}(\cdot,s) \quad \forall\; s \leq \min\{t,\tau\},$$

and thus, by the uniqueness of the obstacle problem solution $\eta^{\tilde{M}}$ when $\tilde{u}$ exists, we arrive at

$$v^M(\cdot,s) = v^{\tilde{M}}(\cdot,s), \quad \eta^M(\cdot,s) = \eta^{\tilde{M}}(\cdot,s) \quad \forall\; s \leq \min\{t,\tau\}.$$

Since t is a deterministic arbitrary constant, the above yields that

$$v^M(\cdot,s) = v^{\tilde{M}}(\cdot,s), \quad \eta^M(\cdot,s) = \eta^{\tilde{M}}(\cdot,s) \;\; \forall\; s \leq \tau \;\; a.s.$$

So, the weak solutions of the M-truncated problem (3.6)-(3.9)-(3.10) are consistent, and we can proceed to concatenation.

Let us define the stochastic process (v, η) such that for all $M > 0$ it coincides with the weak solution (v^M, η^M) of the M-truncated problem (3.6)-(3.9)-(3.10) until the stopping time

$$\begin{aligned}\tau_M &= \inf\Big\{T \geq 0 : \sup_{r\in(0,T)} |\nabla v^M(0^+, r)| \geq M\Big\}\\ &= \inf\Big\{T \geq 0 : \sup_{r\in(0,T)} |\nabla v(0^+, r)| \geq M\Big\}.\end{aligned}$$

By its definition, $(v(\cdot, s), \eta(\cdot, s))$ is a weak solution of the M-independent problem (3.1)-(2.15)-(2.16), for any $s \in [0, \sup_{M>0} \tau_M)$, and τ_M is a localising sequence. Then, $(v(\cdot, s), \eta(\cdot, s))$ is a maximal weak solution of (3.1)-(2.15)-(2.16), since

$$\lim_{t\to\left(\sup_{M>0}\tau_M\right)^-} \sup_{r\in(0,t)} |\nabla v^M(0^+, r)| = \infty \;\; a.s.$$

Uniqueness of the maximal weak solution $(v(\cdot, s), \eta(\cdot, s))$ for $s \in [0, \sup_{M>0} \tau_M)$, follows from the consistency of the solution of the M-truncated problem with which by its definition coincides. □

Let us now consider Case 1. The above analysis is valid for both i.b.v. problems of (2.6), and due to Theorem 3.2, and under its assumptions there exist unique weak maximal solutions (v_1, η_1), (v_2, η_2) satisfying (2.15)-(2.16) in the maximal interval $[0, \sup_{M>0} \tau_{1M})$, where

$$\begin{aligned}\tau_{1M} &:= \inf\Big\{T \geq 0 : \sup_{r\in(0,T)} (|\nabla v_1(0^+, r)| + |\nabla v_2(0^+, r)|) \geq M\Big\}\\ &= \inf\Big\{T \geq 0 : \sup_{r\in(0,T)} (\nabla v_1(0^+, r) + \nabla v_2(0^+, r)) \geq M\Big\}.\end{aligned} \tag{3.38}$$

Recall that $\Omega = (a, b)$, $\lambda = b - a$, and $a \leq s^-(0) \leq s^+(0) \leq b$. We need $a \leq s^-(t) \leq s^+(t) \leq b$ in order to return to the initial variables. This will restrict the stopping time. By using the Stefan condition (2.4) we obtain

$$\partial_t s^-(t) = -\nabla v_2(0^+, t) \leq 0, \;\; \partial_t s^+(t) = -\nabla v_1(0^+, t) \leq 0,$$

and so

$$s^-(t) \leq s^-(0) \leq b, \;\; s^+(t) \leq s^+(0) \leq b,$$

so we need $a \leq s^-(t) \leq s^+(t)$ which yields

$$a \leq s^-(0) - \int_0^t \nabla v_2(0^+, s)ds \leq s^+(0) - \int_0^t \nabla v_1(0^+, s)ds.$$

We define the stopping time

$$\tau_{1s} := \inf\Big\{T > 0 : \sup_{r\in(0,T)} |\nabla v_1(0^+, r) - \nabla v_2(0^+, r)| \geq (T)^{-1}(s^+(0) - s^-(0))\Big\}, \tag{3.39}$$

to keep the spread non-negative and

$$\tau_1^* := \inf\Big\{T > 0 : \sup_{r\in(0,T)} \nabla v_2(0^+, r) \geq (T)^{-1}(s^-(0) - a)\Big\}, \tag{3.40}$$

to keep the spread area in $\mathcal{D}$.

So, the next theorem holds.

Theorem 3.3. *Under the assumptions of Theorem 3.2, and if the initial spread satisfies* $\lambda > s^+(0) - s^-(0) \geq 0$*, then there exist unique weak maximal solutions* (w_1, η_1)*,* (w_2, η_2) *to the reflected Stefan problem* (2.3)-(2.15)-(2.16)*, and* $w|_{x \geq s^+} = w_1$*,* $w|_{x \leq s^-} = -w_2$*, in the maximal interval* $\mathcal{I}_1 := [0, \hat{\tau})$ *for* $\hat{\tau} := \min\{\sup_{M>0} \tau_{1M}, \tau_{1s}, \tau_1^*\}$*, with* $\tau_{1M}, \tau_{1s}, \tau_1^*$ *given by* (3.38)*,* (3.39)*,* (3.40) *for which the spread* $s^+(t) - s^-(t)$ *defined by the Stefan condition* (2.4) *exists and stays a.s. non-negative for any* $t \in \mathcal{I}_1$*.*

We consider now Case 2. Due to Theorem 3.2, and under its assumptions there exist unique weak maximal solutions (v_1, η_1), (v_2, η_2) satisfying (2.15)-(2.16) in the maximal interval $[0, \sup_{M>0} \tau_{1M})$ for τ_{1M} given by (3.38). We need $a \leq s^-(t) \leq s^+(t) \leq b$ in order to return to the initial variables. By using the Stefan condition (2.9) we obtain

$$\partial_t s^-(t) = \nabla v_2(0^+, t) \geq 0, \;\; \partial_t s^+(t) = -\nabla v_1(0^+, t) \leq 0,$$

and so

$$a \leq s^-(0) \leq s^-(t), \;\; s^+(t) \leq s^+(0) \leq b,$$

so we need $s^-(t) \leq s^+(t)$ which yields

$$s^-(0) + \int_0^t \nabla v_2(0^+, s) ds \leq s^+(0) - \int_0^t \nabla v_1(0^+, s) ds.$$

We define the stopping time

$$\tau_{2s} := \inf\Big\{T > 0 : \sup_{r \in (0,T)} (\nabla v_1(0^+, r) + \nabla v_2(0^+, r)) \geq (T)^{-1}(s^+(0) - s^-(0))\Big\}, \tag{3.41}$$

to keep the spread non-negative, while the spread area stays in $\mathcal{D}$ as the spread is decreasing.

So, the next theorem holds.

Theorem 3.4. *Under the assumptions of Theorem 3.2, and if the initial spread satisfies* $\lambda > s^+(0) - s^-(0) \geq 0$*, then there exist unique weak maximal solutions* (w_1, η_1)*,* (w_2, η_2) *to the reflected Stefan problem* (2.8)-(2.15)-(2.16)*, and* $w|_{x \geq s^+} = w_1$*,* $w|_{x \leq s^-} = w_2$*, in the maximal interval* $\mathcal{I}_2 := [0, \hat{\tau})$ *for* $\hat{\tau} := \min\{\sup_{M>0} \tau_{1M}, \tau_{2s}\}$*, with* τ_{1M}, τ_{2s} *given by* (3.38)*,* (3.41)*, for which the spread* $s^+(t) - s^-(t)$ *defined by the Stefan condition* (2.9) *exists and stays a.s. non-negative for any* $t \in \mathcal{I}_2$*.*

4. The problem without reflection

4.1. **Existence of maximal solutions.** We shall consider the unreflected initial and boundary value problem for

$$v_t(y, t) = \alpha \Delta v(y, t) - \nabla v(0^+, t) \nabla v(y, t) + \sigma(y) \dot{W}(y, t), \tag{4.1}$$

posed for any y in $\mathcal{D} = (0, \lambda)$ for $t \in [0, T]$ with Dirichlet b.c., with $v(y, 0)$ given.

In the proofs of the previous section we replace the reflection measure by 0 and keep as presented the absolute value on the changing in general sign $\nabla v(0^+, t)$ (as v may take negative values), and we derive the next results.

Theorem 4.1. *Let the noise diffusion* σ *satisfy the condition* (3.5)*,* $M > 0$ *fixed,* $p \geq p_0 > 8$*, and let* $v_0(y) \in L^p(\Omega, C[0, T]; \mathcal{B})$ *be the initial condition of* (4.1)*. Then there exists a unique week* *solution* $v^M \in L^p(\Omega, C[0,T]; \mathcal{B})$ *to the truncated problem*

$$\begin{aligned} & v_t^M(y,t) = \alpha\Delta v^M(y,t) - \nabla(\mathcal{T}_M(v^M))(0^+,t)\nabla(\mathcal{T}_M(v^M))(y,t) \\ & \qquad\qquad + \sigma(y)\dot{W}(y,t), \;\; t\in(0,T], \;\; y\in\mathcal{D}, \\ & v^M(y,0) := v_0(y), \;\; y\in\mathcal{D}, \\ & v^M(0,t) = v^M(\lambda,t) = 0, \;\; t\in(0,T], \end{aligned} \tag{4.2}$$

where $T := T_M > 0$ *such that*

$$\sup_{r\in(0,T)} |\nabla(\mathcal{T}_M(v^M))(0^+,r)|^p < \infty \;\; a.s., \tag{4.3}$$

where for any $t\in(0,T)$, v^M *satisfies the week formulation*

$$\begin{aligned} v^M(y,t) = & \int_{\mathcal{D}} v_0(z)G(y,z,t)dz \\ & + \int_0^t\int_{\mathcal{D}} \nabla(\mathcal{T}_M(v^M))(0^+,s)\nabla G(y,z,t-s)\mathcal{T}_M(v^M)(z,s)dzds \\ & + \int_0^t\int_{\mathcal{D}} G(y,z,t-s)\sigma(z)W(dz,ds), \end{aligned} \tag{4.4}$$

for $v^M(y,0) := v_0(y)$.

Theorem 4.2. *Let the noise diffusion* σ *satisfy* (3.5), *and* $v_0(y)\in L^p(\Omega, C[0,T];\mathcal{B})$ *for* $p\ge p_0>8$. *Then, there exists a unique weak maximal solution* v *to the problem* (4.1) *in the maximal interval* $[0, \sup_{M>0}\tilde{\tau}_M)$, *where*

$$\tilde{\tau}_M := \inf\Big\{T\ge 0: \sup_{r\in(0,T)}|\nabla v(0^+,r)|\ge M\Big\}. \tag{4.5}$$

We consider now Case 3. Due to Theorem 4.2, and under its assumptions there exist unique weak maximal solutions (v_1,η_1), (v_2,η_2) satisfying (2.15)-(2.16) in the maximal interval $[0, \sup_{M>0}\tau_{3M})$ for τ_{3M} given by

$$\tau_{3M} := \inf\Big\{T\ge 0: \sup_{r\in(0,T)}(|\nabla v_1(0^+,r)|+|\nabla v_2(0^+,r)|)\ge M\Big\}. \tag{4.6}$$

We need $a\le s^-(t)\le s^+(t)\le b$ in order to return to the initial variables. By using the Stefan condition (2.9) we obtain

$$\partial_t s^-(t) = \nabla v_2(0^+,t), \;\; \partial_t s^+(t) = -\nabla v_1(0^+,t),$$

and we need

$$a \le s^-(0) + \int_0^t \nabla v_2(0^+,s)ds \le s^+(0) - \int_0^t \nabla v_1(0^+,s)ds \le b.$$

We define the stopping time

$$\tau_{3s} := \inf\Big\{T>0: \sup_{r\in(0,T)}|\nabla v_1(0^+,r)+\nabla v_2(0^+,r))|\ge (T)^{-1}(s^+(0)-s^-(0))\Big\}, \tag{4.7}$$

to keep the spread non-negative, and

$$\tau_3^* := \inf\Big\{T>0: \sup_{r\in(0,T)}|\nabla v_2(0^+,r)|\ge (T)^{-1}(s^-(0)-a)\Big\}, \tag{4.8}$$

$$\tau_3^{**} := \inf\Big\{T>0: \sup_{r\in(0,T)} |\nabla v_1(0^+,r)| \geq (T)^{-1}(b-s^+(0))\Big\}\Big\}, \tag{4.9}$$

to keep the spread area in $\mathcal{D}$.

So, the next theorem holds.

Theorem 4.3. *Under the assumptions of Theorem 4.2, and if the initial spread satisfies $\lambda > s^+(0)-s^-(0)\geq 0$, then there exist unique weak maximal solutions w_1, w_2 to the unreflected Stefan problem* (2.13)*, and $w|_{x\geq s^+} = w_1$, $w|_{x\leq s^-} = w_2$, in the maximal interval $\mathcal{I}_3 := [0,\hat\tau)$ for $\hat\tau := \min\{\sup_{M>0} \tau_{3M}, \tau_{3s}, \tau_3^*, \tau_3^{**}\}$, with $\tau_{3M}, \tau_{3s}, \tau_3^*, \tau_3^{**}$ given by* (4.6), (4.7), (4.8), (4.9) *for which the spread $s^+(t)-s^-(t)$ defined by the Stefan condition* (2.9) *exists and stays a.s. non-negative for any $t\in\mathcal{I}_3$.*

4.2. **Formal asymptotics.** Let us consider the Stefan problem (1.1) without reflection, and the multiplicative noise defined as $\sigma(y)\dot W(y,t)\equiv \dot W(t)$. We conjecture that the spread s^+-s^- satisfies $\forall\, t>0$

$$\begin{aligned} &\partial_t(s^+-s^-)(t) = -\frac{2}{\lambda}w_\infty,\\ &s^+(0)-s^-(0) = \text{given}, \end{aligned} \tag{4.10}$$

for $w_\infty(t)$ the solution of the initial value stochastic equation problem

$$\partial_t w_\infty(t) = \frac{2\alpha}{\lambda^2}w_\infty(t) + \dot W(t),\ \ t>0, \quad w_\infty(0) = \text{given}. \tag{4.11}$$

We shall present some formal arguments for the derivation of the above system. Let $w_\infty(t)$ be the mean field profile of the diffusion when the domain is infinite. We define $\Omega = \Omega_{\rm Liq}(t)\cup[s^-(t),s^+(t)]$ of diameter $\lambda >> 0$. Moreover, we consider $\alpha >> 0$ so that $\Delta w = 0$ on $\Omega_{\rm Liq}$ approximates the spde of the Stefan problem (1.1). Then w such that

$$w(x,t) = \frac{w_\infty(t)}{\lambda}(x-s^+(t)) \quad x\geq s^+(t), \quad w(x,t) = \frac{w_\infty(t)}{\lambda}(-x+s^-(t)) \quad x\leq s^-(t)$$

satisfies exactly

$$\begin{aligned} &\Delta w(x,t) = 0,\ \ t>0,\\ &w(s^\pm(t),t) = 0,\\ &w(x,t) = w_\infty(t)\ \text{ at }\ x = \lambda + s^+(t)\ \text{ where }\ x - s^+(t) = \lambda,\\ &w(x,t) = w_\infty(t)\ \text{ at }\ x = s^-(t)-\lambda\ \text{ where }\ -x+s^-(t) = \lambda,\\ &\partial_t(s^+-s^-)(t) = -(\nabla w)^+(s^+(t),t) + (\nabla w)^-(s^-(t),t) = -\frac{2}{\lambda}w_\infty(t)\ \ t>0. \end{aligned} \tag{4.12}$$

Here, the value of $w = w_\infty$ at $x = s^\mp \mp \lambda$, for $\lambda >> 0$, approximates the condition $w\to w_\infty$ at infinite distance from the solid phase of [28, 29, 6], i.e., convergence to the mean-field solution.

From the above approximate problem (4.12) we keep the spread evolution through w_∞, i.e., $\partial_t(s^+-s^-)(t) = -\frac{2}{\lambda}w_\infty(t)$, and proceed by matching it to the asymptotics of the solution w of the stochastic parabolic equation (1.1) on $\Omega_{\rm Liq}$. This will yield the sde for $w_\infty(t)$ that is missing. We observe that $\lambda = |\Omega|\simeq|\Omega_{\rm Liq}|$ and so

$$w_\infty(t) \simeq \frac{1}{|\Omega|}\int_{\Omega_{\rm Liq}} w(x,t)dx.$$

By differentiating in time we have

$$\begin{aligned}|\Omega|\partial_t w_\infty \simeq &\int_{\Omega_{\rm Liq}} w_t(x,t)dx - \int_{\partial\Omega_{\rm Liq}} \partial_t(s^+(t)-s^-(t))wdx = \alpha\int_{\Omega_{\rm Liq}} \Delta w(x,t)dx\\ &+\int_{\Omega_{\rm Liq}} \sigma\dot{W}_s(x,t) - \int_{\partial\Omega_{\rm Liq}} \partial_t(s^+(t)-s^-(t))wdx \simeq -\alpha((\nabla w)^+(s^+(t),t)\\ &-(\nabla w)^-(s^-(t),t)) + \int_{\Omega_{\rm Liq}} \sigma\dot{W}_s(x,t) - \int_{\partial\Omega_{\rm Liq}} \partial_t(s^+(t)-s^-(t))wdx\\ =& -\alpha\partial_t(s^+(t)-s^-(t)) + \int_{\Omega_{\rm Liq}} \sigma\dot{W}_s(x,t) - \int_{\partial\Omega_{\rm Liq}} \partial_t(s^+(t)-s^-(t))wdx,\end{aligned}$$

where we assumed that as $x \to \pm\infty$ $w \to w_\infty(t)$ and so $\nabla w \to 0$ (this can be modeled by a Neumann condition for w at $\partial\Omega$ when the diameter λ of Ω is very large). Since $\alpha >> 0$ the last term can be ignored, and therefore

$$|\Omega|\partial_t w_\infty \simeq -\alpha\partial_t(s^+(t)-s^-(t)) + \int_{\Omega_{\rm Liq}} \sigma\dot{W}_s(x,t).$$

Replacing $\lambda = |\Omega|$, and $\partial_t(s^+(t)-s^-(t)) = -\frac{2}{\lambda}w_\infty(t)$, we get for $\sigma = 1$ and $\dot{W}_s(x,t) = \dot{W}(t)$,

$$\partial_t w_\infty \simeq \frac{2\alpha}{\lambda^2}w_\infty + \dot{W}(t), \quad \partial_t(s^+-s^-)(t) = -\frac{2}{\lambda}w_\infty(t),$$

with initial values $w_\infty(0)$ and $s^+(0)-s^-(0)$ (the initial spread).

Remark 4.4. *Setting* $w_\infty := \beta + W$ *we derive the following equivalent problem to* (4.10), (4.11)

$$\begin{aligned}&\partial_t(s^+-s^-)(t) = -\frac{2}{\lambda}(\beta(t)+W(t)),\\ &s^+(0)-s^-(0) = \textit{given},\end{aligned}$$

for $\beta(t)$ *the solution of*

$$\partial_t\beta(t) = \frac{2\alpha}{\lambda^2}(\beta(t)+W(t)), \;\; t>0, \quad \beta(0) = \textit{given}.$$

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