THE ROLE OF VISUALIZATION
In the Teaching and Learning of Mathematical Analysis

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ABSTRACT

In this paper a brief introduction is presented to the nature and different types of mathematical visualization. Then we shall examine some of the influences visualization has had on the development of mathematics and its teaching, exploring in particular its current status. We then inspect the particular role it may have in what concerns mathematical analysis and the difficulties that surround the correct use of it, with or without the computer. Finally a sample of exercises in visualization in basic real analysis is presented in order to show with examples its possible role in the teaching and learning of this subject.

1 What is visualization in Mathematics?

The following story may convey the savor of visualization much better than many analyses. The protagonist here is the great Norbert Wiener, but I am sure that most mathematicians have been able to observe something similar happening to more than one of his or her teachers or colleagues. Wiener was giving one of his lectures at the MIT before a numerous audience.He was immersed in the intricate details of a complicated proof. The blackboard was almost full of formulas and he was marching on unblinkingly towards his goal. Suddenly he got stuck. One minute, two minutes,... To the students it seemed the end of the world... The great Wiener stuck...incredible! He was looking at the formulas, he was messing his hair, he was humming..., until he seemed to know what to do. He went with decision to one of the still empty corners of the blackboard and there he stayed for a little while drawing some mysterious pictures. He did not say a word, his great shoulders almost concealing from everybody what he was doing. Finally he sighted with relief, erased with care what he had drawn and went back to the point he had interrupted his proof and concluded it without any hesitation.

Mathematical concepts, ideas, methods, have a great richness of visual relationships that are intuitively representable in a variety of ways. The use of them is clearly very beneficial from the point of view of their presentation to others, their manipulation when solving problems and doing research.

The experts in a particular field own a variety of visual images, of intuitive ways to perceive and manipulate the most usual concepts and methods in the subject on which they work. By means of them they are capable of relating, in a versatile manner the constellations of facts and results of the theory that are frequently too complex to be handled in a more analytic and logic manner. In a direct way, similar to the one in which we recognize a familiar face, they are able to select, through what to others seems to be an intricate mess of facts, the most appropriate ways of attacking the most difficult problems of the subject.

The basic ideas of mathematical analysis, for instance order, distance, operations with numbers,... are born from very concrete and visualizable situations. Every expert is conscious of the usefulness to relate to such concrete aspects when he is handling the corresponding abstract objects. The same thing happens with other more abstract parts of mathematics. This way of acting with explicit attention to the possible concrete representations of the objects one is manipulating in order to have a more efficient approach to the abstract relationships one is handling is what we call mathematical visualization.

The fact that visualization is a very important aspect of mathematics is something quite natural if we have into account the meaning of the mathematical activity and the structure of the human mind. Through the mathematical activity man tries to explore many different structures of reality that are apt to be handled by the process we call mathematization in the following way. Initially we have the perception of certain similarities in the real objects that guide us to the abstraction from these perceptions of what is common and to submit it to a peculiar rational and symbolic elaboration that allow us to efficiently handle the structures which lie behind such perceptions.

Arithmetic, for example, arises with the intention to rationally dominate the multiplicity what is present in reality. Geometry tries to rationalize the properties of the form and extension in space. Algebra, in a second order abstraction process, explores the structures lying behind numbers and operations related to them. It deals with a sort of symbol of symbol. Mathematical analysis arose in order to deal with the structures of change of real things in time and in space,...

The mathematization process has proved to be extraordinarily useful in order to better understand and manipulate the common structures of many real things. Our human perception is very strongly visual and so it is not surprising at all that the continuous support on its visual aspect is so entrained in many of the tasks related to mathematization, not only in those that, like geometry, deal more directly and specifically with spatial aspects, but also in some others, like mathematical analysis, that arose in order to explore different kinds of changes occurring in material things.

Even in those mathematical activities in which abstraction seems to take us much beyond what is perceptible to our material vision, mathematicians very often use symbolic processes, visual diagrams, and many other forms of mental processes involving the imagination that accompany them in their work. They help them to acquire what we could call a certain intuition of the abstract, a set of mental reflexes, a special familiarity with the object at hand that affords them something like a holistic, unitary and relaxed vision of the relationships between the different objects of their contemplation. In this way they seem to know in advance how these different objects are going to react when they introduce some convenient changes in some part of the structure.

Visualization appears in this way like something absolutely natural not only in the birth of the mathematical thought but also in the discovery of new relations between mathematical objects and also, of course, in the transmission and communication processes which are proper to the mathematical activity.

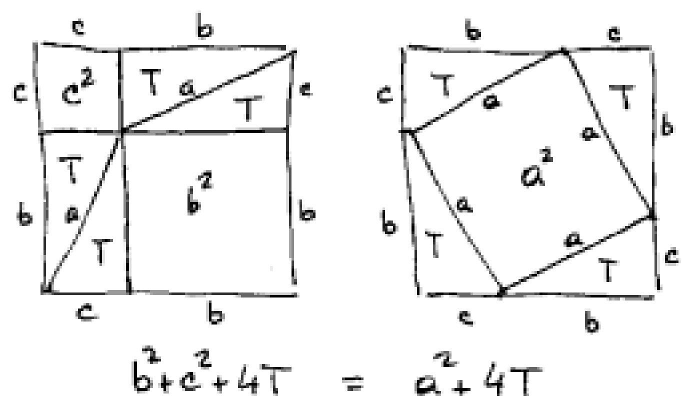
2 Different types of visualization

Our human visualization, even the apparently superficial phenomenon that we call "vision" in its more physiological sense, is not a process that merely involves the optical processes of our eyes. It is much more complex, since it entails in a quite important form, the activity of our brain. Perhaps in the newly born child the phenomenon that takes place is much more similar to the one occurring in a photographic camera, but the cerebral processes that immediately start taking place in his brain cause that, after a rather short time, after experimenting with the objects of the world outside the child transforms his vision into a true mental interpretation of what before was a simple physical optical phenomenon.

The visualization experiences with which we are going to deal here have a much more interpretation weight. In many of the forms of visualization we are going to experiment we have to follow a true process of codification and decodification in which intervene very crucially a whole world of personal and social interchanges, a good part of them firmly rooted in the history of the mathematical activity.

This makes the process of visualization largely based in the interaction with many person around us and in the immersion and enculturation in the historical and social context of mathematics. Visualization is therefore not an immediate vision of the relationships, but rather an interpretation of what is presented to our contemplation that we can only do when we have learned to appropriately read the type of communication it offers us. Here we have an example.

The following figure uses to be presented as a paradigm of a visualization in mathematics, a proof of Pythagoras' theorem. Probably the novice who looks with attention to this drawing arrives to see, with some luck, two equal squares that have been dissected in two different ways and perhaps will be able to understand, through the written indications, that the square over the hypotenuse of the rectangular triangle that arises, that seems to be copy of the other two that appear in different positions in the figure, have an area that is equal to the sum of the areas of the other two squares over the other two sides of the triangle.



But in order to arrive to the Pythagoras' theorem it will be necessary that he may prove that those triangles marked with T are of the same area, and that this same situation appears in any possible rectangular triangle, i.e. he needs to perceive that he is having before his eyes a generic situation.

The purported absolute immediacy of this dissection in order to show the general truth of Pythagoras' theorem is to a certain point deceiving, since it requires for such a purpose an involved work of decodification that is obvious to the expert, but far to be open to the novice. This consideration is one of the reasons why the introduction to visualization, for example in the teaching and learning of mathematics, is not an easy task that requires the clear conscience that the transparency of the process, perhaps real for the teacher because of the familiarity, acquired by the continued practice along many years, may be absent at all for the one who starts with this type of process.

But the presence of this type of decodification process in any visualization makes clear that mathematical visualization is not going to be a univocal term at all. According to the degree of correspondence between the mathematical situation and the concrete way of representation, that can be more or less close, natural, symbolic, even more or less personal and perhaps incommunicable... there are going to be many different types of visualization. In what follows I am going to try to distinguish several of them. At the light of some examples we can try to perceive the deep differences among them and some of the difficulties inherent to their practice.

Isomorphic visualization

The objects may have an "exact" correspondence with the representations we make of them. This means that, in principle, it would be possible to establish a set of rules to translate the elements of our visual representation and the mathematical relations of the objects they represent they represent. In this way the visual manipulations of the objects could be transformed, if we so desire, into abstract mathematical relationships. This kind of representation might be called an isomorphic visualization.

The modelization of a mathematical problem, which in many cases is possible, may be in many cases an isomorphic visualization. Its usefulness is rather manifest. The manipulation of the objects that we perceive with our senses or with our imagination is normally easier and more direct than the handling of abstract objects, that frequently may be rather complicated in its structure.

An example: Josephus problem.

In his book *De bello judaico*, Heggesipus tells about the siege by the Romans of the city of Jotapat. Josephus and other 40 Jewish men took refuge in a cave near the city and decided to kill themselves rather than surrendering. To Josephus and to a friend the idea was not making them very happy. They decided to take their measures. They suggested to do it in a certain order. All men should set themselves in a circle and, starting by an enthusiast who by all means wanted to be the first in killing himself, they would commit suicide by turn counting three. Josephus's idea, of course, was to place himself and his friend in such a way that they would be the two last ones in this order and so, being in absolute majority after the massacre of all the others, to decide to stop it. What places should Josephus and his friend take in order to accomplish their purpose?

The solution is rather easy. One takes 41 little stones, marks each one of them with a number 1, 2, 3,..., 41. One simulates the suicides and looks which two stones are left at the end.

The handling of the problem is clearly isomorphic and shows one of the shortcomings that can accompany visualization. We have been able to solve this particular problem, but the solution is going to vary when, for instance, there are 47 instead of 41 stones or when one counts five instead of three. Our visualization solves our particular problem, but the mathematician is interested in knowing what to do when there are m stones, one puts them in a circle and takes out the n-th one starting by a particular stone in a definite orientation.

We are confronted with a situation similar to the previous one concerning Pythagoras theorem. Will it happen in general what I observe in this particular triangle? There, after a rather simple conceptual elaboration one can arrive to the fact that the situation is in fact generic, independent of the rectangular triangle considered. Here, however, our manipulation only has solved our particular problem. Not a little achievement and besides, from such concrete manipulations very often arise very illuminating ideas which lead us to the general solution of our abstract problem.

A great part of our visualizations in mathematical analysis is of this isomorphic kind. They are probably the ones that mathematicians accept and use more profusely without objections. The visualization of the real numbers on the real line or that of the complex numbers by means of the points in the plane not only made its incursion in mathematics without resistance, but in the case of the complex numbers (Argand, Gauss), it was the means that made possible the general acceptance of this expansion of the number system against the resistance to admit complex or imaginary numbers as decent and honest mathematical objects.

In any case one has to be aware that our visualizations contain many aspects that have to do with tradition, tacit agreements, consensus and this makes them dependent in their use of a whole code to understand them that has to be transmitted, acquired and made sufficiently familiar to each one of their users. It is true that *"an image is worth a thousand words"*, but this presupposes an important condition, that the image comes to be correctly deciphered and understood. Otherwise an image is worth nothing.

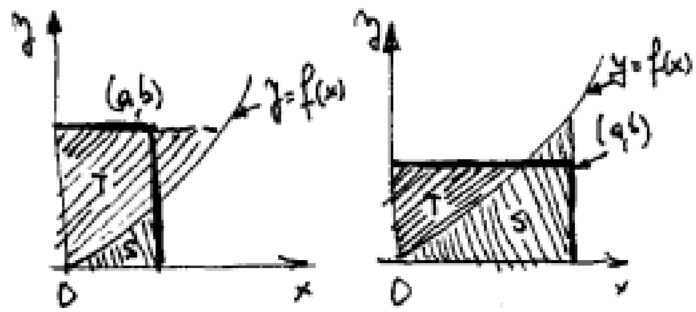
Another example of isomorphic visualization: Young's theorem.

Young's theorem, an inequality with plenty of important applications in analysis affirms the following.

Let y=f(x) be a real function defined on [0,inf) such that $f(0) = 0$, $f(x) > 0$ for each $x > 0$, f is continuous and strictly increasing on [0,inf) and $f(x)$ tends to infinity when x tends to infinity. Let y=g(x) the inverse function of f, i.e. for each x in [0,inf) we have g(f(x))=x. Then, for each pair of positive numbers a and b, one has

$$ab \leq \int_0^a f(x)dx + \int_0^b g(x)dx$$

The proof of this interesting result becomes obvious by merely inspecting the following figure



The inequality stated above simply affirms that the area of the rectangle with opposed vertices at the points (0,0) and (a,b) is less than or equal than the sums of the shaded areas S and T of the picture. The equality is exactly obtained when b=f(a), i.e. when the point (a,b) is a point of the graph of y=f(x). It would not be difficult at all to translate this into a completely formalized proof, if one has to content somebody with a especial desire of rigor.

Homeomorphic visualization

In this kind of visualization that I am calling "homeomorphic" some of the elements have certain mutual relations that imitate sufficiently well the relationships between the abstract objects and so they can provide us with support, sometimes very important, to guide our imagination in the mathematical processes of conjecturing, searching, proving,... Let us analyze an example that might be useful in order to make clear the nature of the homeomorphic visualization.

The Schröder-Bernstein theorem

Let A and B be two sets. Assume there exists an injective function f (i.e. a one-to-one mapping) from A to B and another injection g from B to A. Then there is a bijection h from A to B, i.e. an injection h such that h(A)=B. The following simple and elegant proof which appears in the classical and well-known textbook *Modern Algebra*, by Birkhoff and MacLane, is based on a convenient visualization of the sets and mappings of the statement. The presentation will be very succinct but, I hope, sufficiently clear.

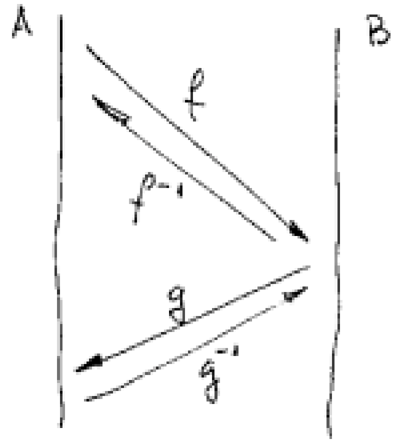
We start by representing the two sets A and B by the two straight lines of the figure above and the functions f and g by the descending arrows of the figure. We consider their inverse functions f^{-1} and g^{-1} and represent them by the corresponding ascending chains (we shall also consider as a chain a point in A or B that has no ascending arrow starting from it). We consider the ascending chains of linked arrows and classify them in the following way:

Class 1: ascending chains that end in A

Class 2: ascending chains that end in B

Class 3: chains that never end, i.e. chains that either are cyclic or pass through infinite points.

It is easy to see that this classification of the chains induces a classification of the points of A (and of B) into three disjoint sets according to the type of chain that goes through it.



And now we can easily define the bijection $h(x)$ we are looking for:

$$h(x) = \begin{cases} f^{-1}(x) & \text{if } x \text{ is of type 2} \\ g^{-1}(x) & \text{if } x \text{ is of type 1} \\ f(x) & \text{if } x \text{ is of type 3} \end{cases}$$

To check now that h is a bijection is an easy matter.

Here it is quite clear that our sets A and B may have nothing to do with straight lines, that our reference to the "ascending" and "descending" chains in the proof of the theorem is totally arbitrary, but they give us a very useful mental support for the key idea of "inverse image of a mapping" that is here the key for our proof.

And it is also quite clear that we could efface any visual connotation and write a completely formal proof that could astonish our reader who would keep wondering where our magnificent ideas could come from. Unfortunately this has been the prevalent fashion for quite a long time in papers, textbooks, lectures... inspired in such a style of mathematical miscommunication.

In this example it becomes manifest the power of this type of homeomorphic visualization that in many cases can become a quite personal and subjective process, perhaps often not easily communicable, but in any case the effort to hand it over to our students is worth doing.

Analogical visualization

Here we mentally substitute the objects we are working with by other that relate between themselves in an analogous way and whose behavior is better known or perhaps easier to handle, because it has been already explored.

This kind of visualization or analogical modelization was one of the usual discovery methods used by Archimedes, according to what he tells his friend Eratostenes in the famous letter which is known by *The Method*. There are many spectacular discoveries by Archimedes, for example his calculation of the volume of the sphere, which was first obtained by following this way of analogies and thought experiments of mechanical nature.

The following example, which arose in a workshop on solving problems with university students, can illustrate the way of proceeding.

The problem is the following: *we are given four segments of lengths a, b, c, d, with which one can form a convex quadrilateral in the plane of side lengths a, b, c, d, in this order. It is clear that if we can form one, then we can form many different convex quadrilaterals. Among them find the one enclosing the maximal area.*

The mechanical problem that can provide the adequate analogy leading to the solution is the following. We are given four thin rods forming an articulate plane and convex quadrilateral. We enclose it in a big soap film that contains the quadrilateral in its interior. We puncture the film at a point inside the quadrilateral. The equilibrium position of the rods will be such that that the tension of the soap film outside is minimal, i.e. the area of the quadrilateral is maximal.

Therefore our problem is reduced to find the equilibrium position of the rods in this situation. The forces acting on our system are reduced to four perpendicular forces to the sides applied at their midpoints, directed towards the exterior of the quadrilateral and each of magnitude proportional to the length of the corresponding side. It is easy to see that the equilibrium is obtained when the four perpendiculars at the midpoints of the sides concur, i.e. when the quadrilateral can be inscribed in a circle. This solves our original problem.

The use of the analogical method should not surprise any mathematician. It has been very often put to work in mathematics, not only by Archimedes but also, for instance, by Johann Bernoulli in his analogical solution to the brachistochrone problem proposed by him in the *Acta Eruditorum* "to the most acute mathematicians of the whole world". In this case an analogy with the behavior of the light rays was the guide towards his solution.

Even the most ingrained formalist should consider that the fields on which such analogies are based are capable of the most rigorous development, if this is what one should strive for.

Diagramatic visualization

In this kind of visualization our mental objects and their mutual relationships concerning the aspects which are of interest for us are merely represented by diagrams that constitute a useful help in our thinking processes. One could say that in many cases such diagrams are similar to mnemotechnic rules.

The tree diagram we use in combinatorial theory or in probability and many others each mathematician develops for his or her own use, of a very personal nature, are of this type. Such symbolizations and diagrams become in some cases of generalized use, but in many cases they are of a very personal, individual use, and cannot be easily shared with others.

But in many cases they could be communicated with little effort to many others that would find them extremely useful. However sometimes people think that such images, diagrams,... constitute a real obstacle for the development of the individual in mathematics, since what matters, they say, is only the formal justification of our arguments.

It is my opinion that the success that is experimented by the great teachers in mathematics is very often due to the efforts they make to transmit to others and to share with them not only the results of theirs and others researches, but also the processes by which somebody somewhere was able to obtain such results.

When one examines the mathematical writings of Euler, *the teacher of us all*, one perceives this expositive quality of one of the great geniuses of mathematics.

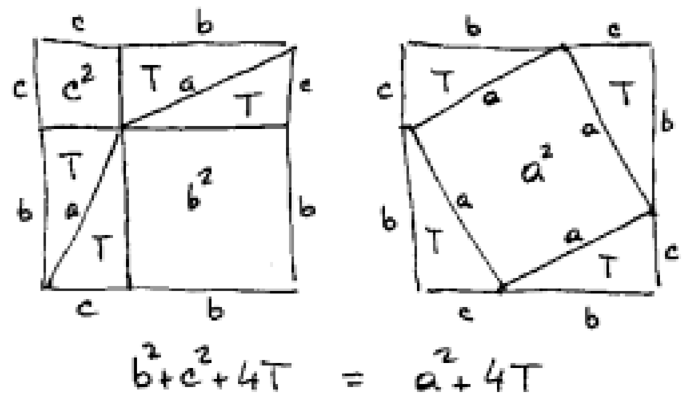
It is clear that the classification of the possible types of visualization we have seen here is neither exhaustive nor a clear cut one. There will be obviously many cases which cannot be enclosed in anyone of the types we have described here.

3 Visualization over the centuries

What has been the role of visualization along time? We shall briefly examine some of the most significative points.

The visualization at the origin of modern mathematics

The Greek word *theorein* means "to contemplate" and *theorema* is what is contemplated and not, as we now understand it, what is proved. In particular, among the early Pythagoreans who first cultivated mathematics in our modern sense, the study of the numbers and the relationships among them was performed by means of different configurations done by means of pebbles, small stones, *psefoi*, in Latin *calculi*. As a token here we can see above two of their most simple theorems.



For the Pythagoreans visualization was something connatural to the exercise of mathematics. In Plato the specific role of the image in the mathematical construction is more explicit and strongly emphasized. The image evokes the idea as the shadow evokes the reality. The drawn circle is not the reality. The real thing is the idea of the circle, but its image plays a very important role as evocative element of the idea. The way of knowledge he calls *dianoia* is very specific of the mathematical knowledge. The mathematician gets close to the intelligible through the reference to the sensitive.

The Elements of the mathematicians preceding Euclid probably contained, as Euclid's Elements do, many references that form an indispensable part of the text. But one can venture that it was probably in Euclid's lost *Book of Fallacies* where the references to geometrical paradoxes and fallacies had a especially important role. One could guess that this book rather than the *Book of Elements* could have been the one that was used by Euclid and his pupils in his learning practice.

As we have already seen, Archimedes used with advantage his analogical method as a very fundamental tool for his mathematical discoveries, although, one has to add, with a certain sense of embarrassment.

The modern classics

Descartes, in his *Regulae ad directionem ingenii*, has several rules that directly involve visualization processes. He strongly emphasizes the different roles of images and figures in the mathematical thinking.

Here one can see three of the most significative rules in this context:

REGULA XII.

Denique omnibus utendum est intellectus, imaginationis, sensus, et memoriae auxiliis, tum ad propositiones simplices distincte intuendas, tum ad quaesita cum cognitis rite comparanda ut agnoscantur, tum ad illa invenienda, quae ita inter se debeant conferri, ut nulla pars humanae industriae omittatur.

(Finally it is necessary to make use of all the resources of the intellect, of the imagination, of the senses and the memory: on the one hand in order to distinctly feel the simple propositions, on the other hand in order to compare that which we are looking for with what is already known, in order to recognize those; and also to discover those things that must be compared to each other in such a way that no element of the human ability is omitted).

REGULA XIV.

Eadem est ad extensionem realem corporum transferenda, et tota per nudas figuras imaginationi proponenda: ita enim longe distinctius ab intellectu percipietur.

(This rule must be applied to the real extension of the bodies. It all must be proposed to our imagination by means of pure figures. Since in this way it will much more distinctly perceived by the intellect).

REGULA XV.

Juvat etiam plerumque has figuras describere et sensibus exhibere externis, ut hac ratione facilius nostra cogitatio retineatur attenta.

(It is also useful in many occasions to describe these figures and to show them to our external senses, so that in this way our thought might maintain more easily its attention).

It seems also clear that the original idea driving Descartes to the development of the analytic geometry arose as an attempt to combine the geometric image of the ancient Greeks with the already at his time sufficiently well structured algebra.

The calculus of the seventeenth century is born with a very strong visual component and remains so in the first centuries of its development, in continual interaction with geometrical and physical problems. The following words of Sylvester may summarize and represent the feeling of some of the great classics of mathematics about visualization: *"Lagrange... has expressed emphatically his belief in the importance to the mathematician of the faculty of observation. Gauss has called mathematics a science of the eye..."* (The Collected Works of James Joseph Sylvester, Cambridge University Press, 1904-1912, quoted by Philip J.Davis (p.344) in *Visual Theorems*, Educational Studies 24 (1993) 333-344).

Visualization, as we see, has been a technique generally used by the most creative mathematicians of all times. One or other type of image accompanies their mathematical lucubrations, even the most abstract, although the nature of these images presents a difference from person to person much greater than we suspect.

Visualization, as we can see through these small samples extracted from the history of mathematics, has played a very important role in the development of mathematics. And so it had to be, given the peculiar structure of human knowledge, very strongly conditioned by visual, intuitive, symbolic, representative elements, and given the nature of mathematics and its purposes of obtaining an image, as accurate as possible, of the world around us.

The formalism of the 20th century and the visualization

In spite of the role played traditionally by visualization, the formalistic tendencies prevailing during a good part of the 20th century, as we shall see in a moment, had as a consequence a sort of demotion of visualization to an inferior position. Visualization was looked upon with mistrust and suspicion. It would take too long to analyze the reasons that may cause this situation, but I try to schematically pinpoint some of them.

The rational status of the Calculus in the 17th century was beset by doubts and confusion and it was not until the end of the 19th century, with the arithmetization of analysis, that became free of any doubt.

The non Euclidean geometry's in the middle of the 19th century lead many persons to be highly diffident of intuition in mathematics.

The initial polemic against Cantor's set theory at the end of the 19th century and the paradoxes around the foundations of mathematics drove many mathematicians to emphasize the formal aspect in the structure of mathematics, trying to achieve in them a solid basis for the mathematical edifice.

The results falsely or incompletely proved (for instance, of the four-color theorem or the Jordan closed curve theorem) based on a naive confidence in certain intuitive elements contributed to foster a more rigorous attitude towards the intuitive proofs, looking with distrust the merely intuitive arguments.

All these facts lead to create a trend towards the strict formalization, not only in what is related to the foundations of mathematics, what seemed to be amply justified, but also in what relates to the normal intercommunication among within the mathematical community and even, what is still much worse, in what attains the mathematical teaching and learning processes at every level. The consequences were very serious in what visualization concerns. The atmosphere of mistrust so created lead some mathematicians to aggressively advocate a more or less complete abandon it. The influence of formalism in the presentation of new results and theorems in the journals was the unavoidable norm. Even the structure of text books at the university level, and sometimes even at secondary and primary levels (*"modern mathematics"*) tended to conform to the same standards.

As a sample of such attitude one can read a couple of sentences in the introduction of a text book by Jean Dieudonné on linear algebra and elementary geometry: "I have decided to introduce not a single figure in the text... It is desirable to free the student as soon as possible of the straitjacket of the traditional "figures" mentioning them as scarcely as possible (excepting, of course, point, line, plane)..."

The model for the mathematical activity for long time was the formalist model, and even the teaching at the secondary level in many countries was contaminated by such tendencies.

One can find a clear testimony of such tendencies together with a brief attempt to explain it in the work *A Mathematician's Miscellany* by J.E. Littlewood, where he openly acknowledges the many benefits of visualization in his own research work.

"My pupils *will* not use pictures, even unofficially and when there is no question of expense. This practice is increasing; I have lately discovered that it has existed for 30 years or more, and also why. A heavy warning used to be given (*footnote*: To break with 'school mathematics') that pictures are not rigorous; this has never had its bluff called and has permanently frightened its victims into playing for safety. Some pictures, of course, are not rigorous, but I should say most are (and I use them whenever possible myself). An obvious legitimate case is to use a graph to define an awkward function (e.g. behaving differently in successive stretches): I recently had to plough through a definition quite comparable with the "bad" one above, where a graph would have told the story in a matter of seconds."

(In *Littlewood's miscellany*, edited by Béla Bollobás (Cambridge University Press, Cambridge, 1986), p.54)

Towards a return of visualization?

What is the present situation? It seems that in the last decade or so one can perceive a much more flexible attitude and a certain tendency toward a renewal of the influence of visualization in the mathematical activity, teaching, learning, doing research and publishing it. With decision, especially among many of those who do research in mathematics education. With many different attempts, not always very successful, among those who have tried to explore the possibilities of the computer for the mathematical tasks. And also with certain inertia, if not opposition, of a good part of the mathematical community.

4 The role of visualization in Mathematical Analysis

The image, as we have seen, has very important uses in many different types of mathematical activity. The image is frequently the matrix from which concepts and methods arise. It is a stimulating influence for the rise of interesting problems in different ways. It often suggests relationships between the different objects of the theory which are in a way somewhat difficult to detect by just logical means. It suggests in subtle ways the path to follow in order to solve the most intricate problems of the theory and even those connected with the development of the theory itself. The image is also a very powerful tool to grasp in a unitary and holistic way the different contexts constantly arise in the different task connected with the theory. It is also a rapid vehicle for the communication of ideas. It is also an auxiliary tool for the unconscious activity around the most obscure problems connected with it.

Visualization is therefore extraordinarily useful in the context of the initial process of mathematization as well as in that of the teaching and learning mathematics. All this makes very clear the convenience of training our own visual ability and to introduce to it those whom we are trying to introduce to mathematics. This applies not only to geometry, where all these considerations are quite obvious, but also to, for instance, mathematical analysis. The ideas, concepts, methods of analysis have a great richness in visual, intuitive, geometrical contents, that are constantly arising in the mental workings of the analyst. It was not in vain that mathematical analysis arose as a need to quantitatively mathematize in the first place the spatial relationships of the objects of our ordinary life. These visual aspects are present in all kinds of activities of the mathematician, in the presentation and handling of the most important theorems and results as well as in the task of problem solving. They seldom pass over to the written presentation, perhaps partly because of the difficulty inherent to this task, and in some other occasions because of the adherence to the most fashionable form of presentation *"the more formal, the better"*.

In fact the experts in a field of mathematical analysis have visual images, intuitive ways of approaching certain usual situations, imaginative ways of perceiving concepts and methods of great help for them and that would be of great value also for others in their own work. The experts, through the assistance of such visual tools, are able to relate, in a very versatile and flexible way, constellations, frequently very involved, of facts and results of the theory and through such relationships they are able to select in a co-natural way and without effort the most adequate strategies for solving the problems of the theory.

These images are able in many cases to offer all the necessary elements to build, if one so wish, the whole formal structure of the corresponding theoretical context or the problem. The expert knows, even without having done it so, that just by investing the necessary amount of time and by accepting to suffer the corresponding boredom inherent to the task, they, or any other, would be able to afford all the necessary ingredients to build up a proof capable to satisfy the most exacting appetite of rigor.

The following testimony of Hadamard on the role of visualization is quite representative of the influence of the image in the mathematical processes of an analyst:

"I have given a simplified proof of part (a) of Jordan's theorem [*that the continuous closed curve without double points divides the plane into two different regions*]. Of course, my proof is completely arithmetizable (otherwise it would be considered non-existent); but, investigating it, I never ceased thinking of the diagram (only thinking of a very twisted curve), and so do I still when remembering it. I cannot even say that I explicitly verified or verify every link of the argument as to its being arithmetizable (in other words, the arithmetized argument *does not* generally appear in my full consciousness). However, that each link can be arithmetizable is unquestionable as well for me as for any mathematician who will read the proof: I can give it instantly in its arithmetized form, which proves that that arithmetized form is present in my fringe-consciousness." (Jacques Hadamard, *The Psychology of invention in the mathematical field*, p.103, footnote).

My opinion is that one of the important tasks of the expert in analysis in his intention of introducing the young students to his or her field should be to try to transmit not only the formal and logical structure of the theorems in this particular area, but also, and probably with much more interest, to offer them these strategical and practical ways of the profession with which he or she has perhaps learned and become familiar with much effort through the passage of the years. They are probably much more difficult to make explicit and assimilable to the students, precisely because they are often located in the zones less conscious of the activity of the expert. It is quite clear that this task is going to present many aspects that are strongly subjective and that they are much more difficult to make explicit and assimilable for our students, precisely because of the fact that they are situated in the zones less conscious of the own activity of the expert.

By its own nature this task is going to involve many elements that are strongly subjective. The ways to visualize and to make more close and intuitive the ideas of mathematical analysis to make them work in certain concrete problems and situations are going to depend in an intense way of the mental structure of each one. The degree of help the visual support affords varies, with certainty, in a strong way, from individual to individual. What for one helps perhaps may be a hindrance to some other person. But these differences should not represent an obstacle in our attempts to offer with generosity to other those instruments that for us have resulted quite useful in our work to such a point that this work without them would be much more difficulty, abstruse and boring.

5 Difficulties around visualization

Obstacles and objections

There are many obstacles and objections that hinder a more decisive progress in order to put visualization in the right place it deserves in the job of communicating and transmitting mathematics at the educational level and also to restore its status in the tasks concerning research. Here we present some of them.

"*Visualization leads to errors*"

It is quite true that an incorrect use of visualization can lead us to errors in different ways. Sometimes because the figure we rely upon suggests a situation that in fact does not take place. This is the case of many geometrical fallacies like the ones to be found in the classical book by W.W.Rouse Ball *Mathematical Recreations and Essays,* Chapter III. An efficient way to get rid of such false arguments that seem to originate in an incorrect interpretation of the figure is to consider a figure similar to one proposed but in an extreme position of its elements. It often happens that our intuition leads us to a false conclusion because the figure in question approximates the one that in fact takes place. When we take a similar figure in a limit position, the truth shows up.

In some other cases the visual situation misleads us to accept certain relationships that appear so highly obvious that never comes to our mind the need or the convenience to justify them more rigorously. Euclid's axioms, for instance, with all its astonishing maturity, are not exempt from some very subtle gaps coming from this type of geometrical situations that had to be corrected by Hilbert in his *Grundlagen der Geometrie* (1902).

The *"proof"* built up by Arthur Kempe in 1879 of the *"four-color theorem"* was based in a geometric relationship that, although false, seemed so clear that it was accepted by the mathematical community of the moment until 11 years later when Heawood became aware of the fact that the proof was incomplete. By the way, it was Kempe's attempt the one that inspired the strategy, more than one century later, that lead Appel and Haken to a successful proof of the fact that four color suffice to appropriate color any particular map.

The *"proofs"* by Jordan (1893) and by other mathematicians of the visually obvious fact that a plane simple closed curve divides the plane into exactly two regions, the interior and the exterior, were not rigorous, since they contained assertions without rigorous justification based on intuitive relationships. Later on such assertions were established with a considerable effort. The first correct proof came 20 years after the statement of the theorem by Jordan in the work of O.Veblen (1913).

But the possibility that visualization can lead to error should not be a valid argument against its efficiency in the different processes of the mathematical activity, as well in the creative tasks it entails as in the processes of communication and transmission. Even the most formal techniques are open to errors, incomplete reasoning, fallacies,... And one should take this fact as something quite natural.

Mathematical thinking is not normally presented through a completely formalized exposition that could be automatically checked and controlled in each one of its steps. The communication style of mathematicians is at the moment rather far from that stage and it is probable that it will keep for some time. On the other hand it is debatable whether it would be convenient to adopt such a style of communication, if it becomes possible. The mathematical language is today a sort of mixture halfway between the natural language and the formalized language, a rather bizarre jargon consisting of elements of the natural language, some esoteric words, and logical and mathematical symbols. And in this curious mixture mathematicians are constantly alluding, in a more or less explicit way, to certain tacit agreements of the mathematical community of the time, which are loaded with intuitive, visual connotations, which each one presupposes to be known by the others.

In my opinion, it is not very surprising that such a language, especially in rather elaborate contexts, may be open to ambiguities, mistakes and obscurities. To illustrate this fact let us consider a rather recent example. The *"proof"* of Fermat's theorem solemnly presented in June 1993 by Andrew Wiles was able to convince the experts in the field for several months before they detected a rather serious gap. Some thought that to fill it could take another couple of centuries. The work of Andrew Wiles and Richard Taylor for a year was again successful. In 1995 the proof met with the approval of the experts and was published in the *Annals of Mathematics*.

"And now, please, give us a mathematical proof"

I imagine that a multitude the teachers share more or less the same experience. After having made a strong effort to make quite obvious to our students of a mathematical situation by means of a visual argument, we hear: "Now, please, give us a truly mathematical proof"

What is a proof? For the Pythagoreans working at the seashore with their pebbles it would be: *"Just look!"* For Littlewood: *"A proof is just a hint, a suggestion: look in this direction and convince yourself"*. For René Thom: *"A theorem is proved when the experts have nothing to object"*.

Should we say that an assertion is only proved when it comes at the end of a more or less lenghty chain of logical symbols? Maybe it is so in the paradise of the imagination of the formalists or logicians, but certainly not in the real world of the mathematician. He or she is already satisfied with a more reasonable degree of rigor. An isomorphic visualization, for instance, with well identified rules of codification and decodification that make it clear how to go from the image to the formal argument, is sufficient for the ordinary mathematician. It could be converted, with some effort in cases, in the most rigorous proof in order to content the most entrenched of the formalists.

Some other types of visualization, homeomorphic, diagramatic, are able to smooth out the path of other mathematicians, experts or students in order for them to explicitly construct a rigorous proof, if it is necessary, much more easily than with the terse, pedantic and often unintelligible kind of proof that the fashion has imposed for already too long time in our mathematical communication.

Of course the student that asks for a "real proof" after having been offered a faultless visual one has possibly in mind the bias, often transmitted by his teacher, that only that assertion which results after some logical quantifiers deserves the name of a proof. And this happens, it seems to me, because in our mathematical education we seldom have had into account the importance of the habit of correctly interpreting our visualizations, translating them, when it seems adequate, into a more formal language.

Visualization is difficult

Theodore Eisenberg and Tommy Dreyfuss have written an interesting paper with title *On the Reluctance to Visualize in Mathematics*. In it they try to analyze the different obstacles that one encounters in the visualization processes in mathematical education.

As I have said before, visualization is an intellection process which is direct and effortless, but only for the one who is sufficiently prepared to perform it in an efficient way. This preparation implies an immersion and familiarization with the task of decodification of the image. When such a preparation is absent, what for others might be an effortless and pleasant exercise can become a worrying and absolutely incomprehensible hieroglyph. It is true that *an image is worth a thousand words,* but one forgets to add the all-important condition that the image is understood. Otherwise it is worth nothing.

A road map, for instance, is not the reality of what is represented. It is just a set of symbols and codes one has to learn to interpret. The correct performance of a visualization requires a previous preparation, an education that not many mathematicians are able to transmit because they are not conscious of what it presupposes of convention, of tradition, of familiarity with certain codes nowhere explicitly written. And this is one of the aspects which our mathematical community, and especially our educational community, should emphasize.

On the other hand there are also difficulties which come from the low status that visualization has in our mathematical community. Our researchers make a continuous use of visualization, but their use is timid, half-hearted, something they seem to be ashamed of. No prestigious journal would admit for publication a paper in which the arguments and the proofs of the theorems would not be presented in the more or less formalized language in vogue, even if any other mathematician could recognize through them their validity. It is a question of observance to the prevailing norms. One often hears with scorn many people speaking about proofs presented "waving hands", when it is a fact that an adequate gesture can often open the minds of our audience.

Our students suffer of a certain distortion with respect to visualization and this is the origin of their attitude with respect to it and also of the following phenomenon rather frequent in our mathematical courses. We start by trying to explain for them the intuitive meaning of a theorem, what perhaps is for us the most important portion of our intervention in the hour. In the most favorable cases they will look at us with a certain attention, but without writing down a single word in their notebook. Just when we start writing down on the blackboard what is going to be a formal proof, i.e. what probably is already carefully written in their textbook, they start trying to get down in their notebooks "black over white" what seems to be for them the essential part of their work in class.

Visualization is also difficult for some other reasons of a practical nature and that become especially apparent at the level of the written, non-direct communication. Visualization is a dynamical process. The transmission means until now used in articles and in the textbooks that our students use is, basically, the written word, a statically vehicle that is not well adapted to the needs of the visualization processes. In the direct, oral presentation of a visualization its different elements start to appear little by little, rounding off an image that starts being rather simple and possibly finishes by appearing extremely complicated. In a book or article one presents usually the final image with all its elements and this becomes quite difficult to interpret. In order to show in the textbook something which would be near to the oral presentation of the same fact one would need perhaps six different figures. No editor would allow such a waste, insisting that the space is expensive and so everything has to be in a single figure.

Probably the communication means of the near future, especially for textbooks, will be something similar to the CD-ROM that allows one to mix in an interactive form text, dynamical images, computer programs that are

adequate for the field one is dealing with...

Some of the tasks ahead

I shall list some lines along which we could start working in order to put visualization in the place that corresponds to it according to its usefulness and to mathematical tradition.

Prevent possible deviations. We should try to explicitly teach to perform correctly the processes of visualization. We should pay special attention to the different types of visualization and to their especifi usefulness in the mathematical teaching and learning. We should try to be aware of the process of codification and decodification implied in the practice of visualization and trying to make them explicit for our students.

We should try to stress in our teaching the habits of visualization, trying to make very explicit their value in the practice of mathematics.

We should hold visualization in high esteem. We should insist in visualizing and, from time to time, we should transcribe our visualization into formal expressions in order to put it out of doubt that what we are doing is "real mathematics" and that what we explain by visualizing it can be also written in formal language.

We should appreciate the value of visualization not only in our frequent use of it but also in our evaluation of the uses our students and others make of it and of the different skills which visualization involves.

6 Visualization with and without the computer

The few examples we have proposed in the preceding pages have not needed at all the help of any sophisticated tool. A great part of the visualization that we advocate can be performed as it always has been done, by means of our imagination and representative ability, with the help of the normal tools at hand, paper and pencil, chalk and blackboard.... In general it is not even necessary to resort to straightedge and compass, since the main objective of our drawings is to help our intuition which is able to think correctly with the help of incorrect figures. The accuracy and precision of our drawings should be proportionate to what we expect from the type of representations we are using. It is of very little use to draw with straightedge and compass when a hand made figure is more that sufficient to suggest the relationships that are important for us. In most occasions the drawings are mere auxiliary tools of our imagination helping it to get a better grasp of the relations that help us towards the comprehension of subjects we are dealing with.

But it is quite clear that at this moment many technological tools are at hand that can help us in some circumstances when a simple hand made drawing is not satisfactory. The practice of visualization can be now importantly enhanced with the help of these tools in many different ways.

In what attains mathematical analysis one can say that the existence of symbolic calculus programs, such as MAPLE, MATHEMATICA, DERIVE, and many others, with their versatile representative abilities, with their capacity for interaction in every field of mathematics is already producing deep transformations in the new ways of doing research, teaching and learning mathematics. And this tendency seems to show no limitations.

Let us just consider a simple example. Some years ago, in order to represent a curve in the plane given by a not too simple equation f(x,y)=0, one used to advise the student to plot first a few elements of easy computation in order to get an initial feeling about the curve (intersections with the axes, possible horizontal and vertical asymptotes ....). Today almost any symbolic calculus program, even those incorporated into many pocket calculators, allow our students, given a rather sophisticated function, to obtain a graph of it and so to have an immediate grasp of many of its most important features. This already helps them to look in the right direction towards the solution of many problems that curve might offer. The student who is able to establish an intelligent dialogue with the machine through its representation capacities is in much better position to understand all the problems that might be proposed.

The new tools that are now in the hands of most of our students have opened quite new worlds to exploration that a few years before were closed to our view. To obtain 200 iterations of a simple function like 4x(1 -x) with 12 exact decimal digits starting with x=0.7, for example, was a gigantic task some years ago. Not so anymore. Now it may be made in a fraction of a second. Such capacities have opened new worlds for exploration on different topics such as dynamical systems, mathematical chaos, fractal geometry, and many others. On the other hand we have today many programs which are specifically destined to promote the visualization in different fields of mathematics, multidimensional analysis, geometry of different types.... All this is going to contribute to stimulate the current trend towards revitalization of the visual aspects of mathematics in many different areas.

In what follows I shall present a few examples that help to perceive how visualization may be of great help in the teaching and learning of some of the most basic aspects of mathematical analysis and later on I add also some other which are already a little more sophisticated. The images I introduce here are mostly handmade, in order to emphasize that the visual help one can obtain from such representations does not depend on the accuracy and precision of the pictures.

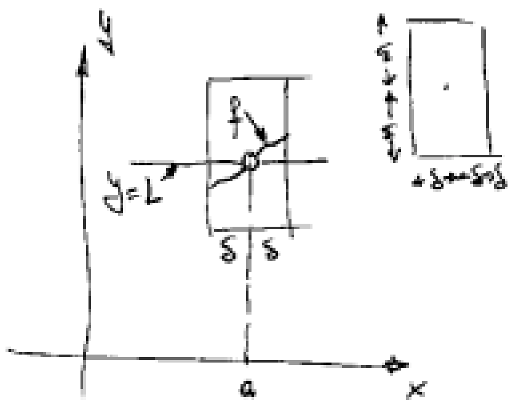
7 Samples of visualization in basic Real Analysis

One could easily present a whole course of introductory real analysis with the concrete goal of giving a visual slant to the most basic notions and results of the field. In my opinion it would contribute to balance the still prevailing bias toward explicit logical rigor and formalization.. I myself have written a small work with this orientation entitled *El rincón de la pizarra* (Pirámide, Madrid, 1996). But I think that in a normal situation it is healthier to make use in our teaching and learning of all possible recourses.

On the one hand the best advice to correctly choose our ways to confront a specific problem should come from the inspection of the features of the problem, and on the other hand a permanent bias in favor of the visualization processes could become also harmful for our students. Also one should take into account that each one of our students has its own peculiarities concerning the ways (logical, formal, intuitive,...) to attack a problem. In any case it seems very convenient to show the different possibilities that are available when one tries to introduce them to a particular field.

In what follows we shall explore the possibilities of a visual approach in order to get an adequate comprehension of the main concepts and results of introductory real analysis. We do it by offering some pictures accompanied by a few sentences in order to convey the meaning of them. The drawings are going to be handmade and rather rough, but, I hope, intelligible. I proceed so in order to make clear that the precision and accuracy of our pictures is not very important in order to achieve the goal we aim for.

Continuity of a function at a point

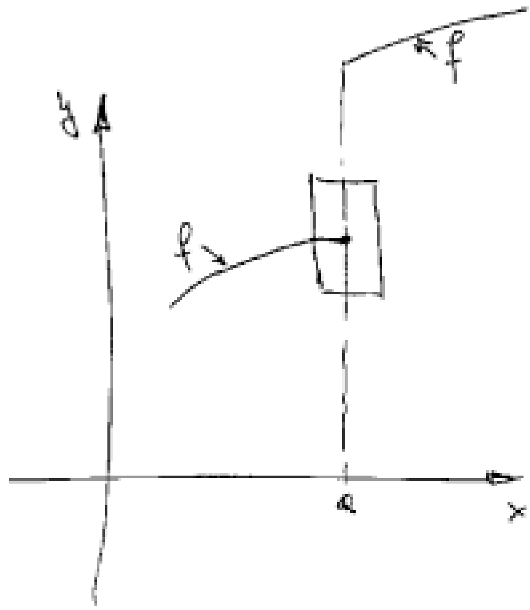


A function $f:\mathbb{R}\to\mathbb{R}$ can be isomorphically visualized by its graph.

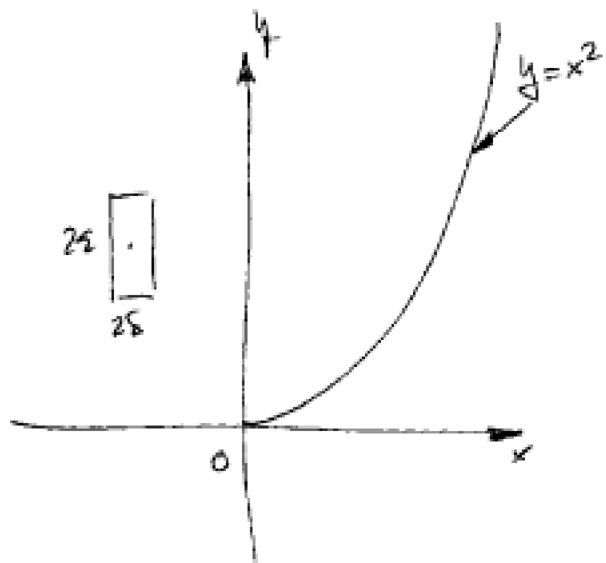
In order to deal with the notion of continuity of f we are going to introduce rectangular windows of height 2ε and width 2δ (of sides parallel to the axes Ox, Oy) centered at the points of (the graph of) f.

A function f is continuous at the point $a\in\mathbb{R}$ when the following happens: no matter how small we fix the height of a window centered at the point $(a, f(a))$ we can choose its width conveniently so that we can see the graph of f going from the left side of the window to its right side without going across its lintel (upper side) nor its threshold (lower side).

The function f of the picture in the next page is clearly not continuous at the point a.



It is clear that if we consider a different point, it may be possible that for the same height of the window we have to choose a different width in order to see the graph inside the window. An example follows



If we consider the function $y = x^2$, it is clear that we if take a point far to the right the slant gets more pronounced and the width which was adequate for points close to 0 is not any more valid.

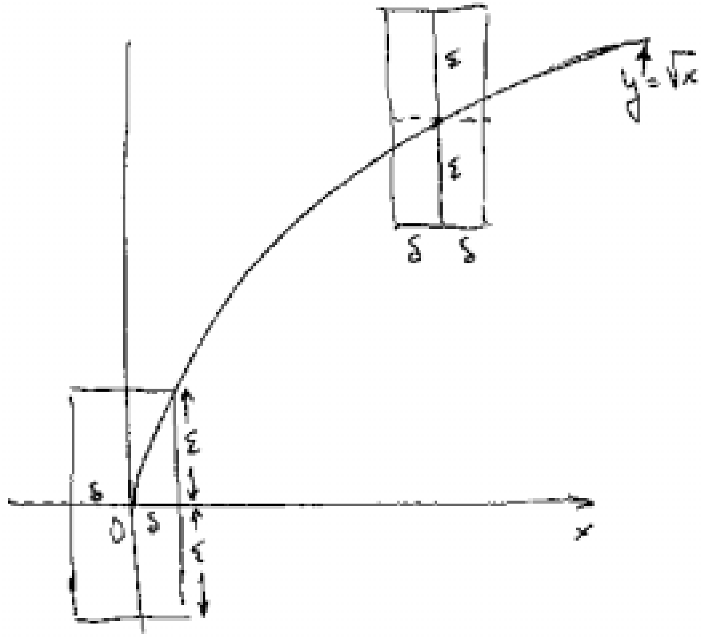
Uniform continuity

The motivation for this notion comes from the final remark of the preceding paragraph.

The function f will be said uniformly continuous on $\mathbb{R}$ when given a window height we can choose a window width such that this window centered at *any* point of the curve allows us to see the curve inside it.

An example follows

The function $f(x) = \sqrt{x}$ for $x \geq 0$ is uniformly continuous. The window which is adequate when

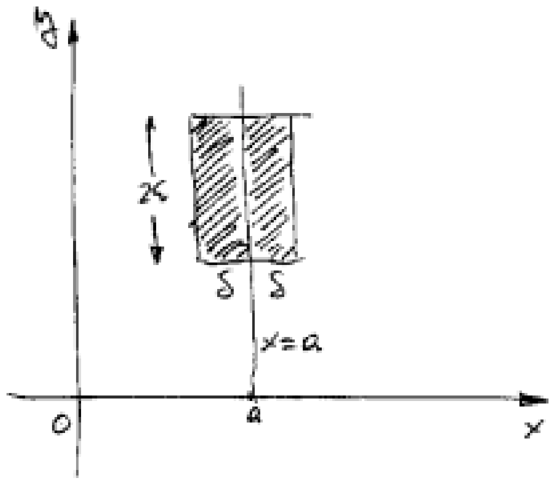


centered at $(0,0)$ is also good at any other point (one sees that the slope of the function decreases and this is what makes the window appropriate in this case).

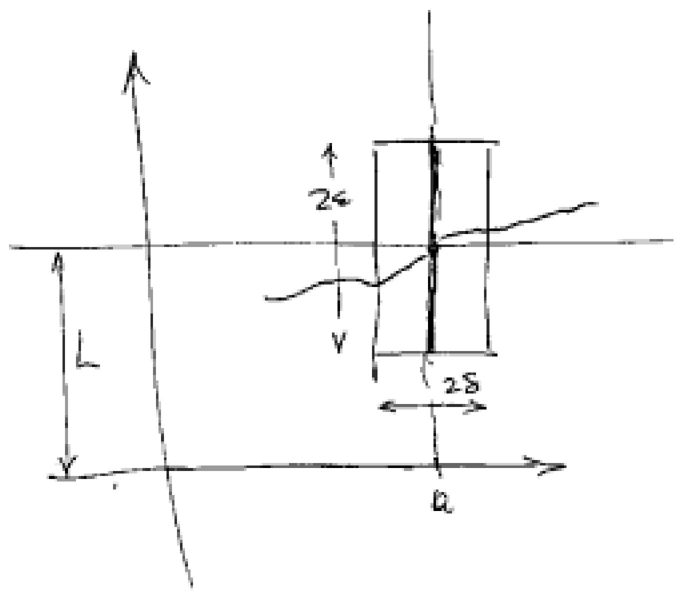
After considering this example one can easily conclude: *if the absolute value of the slope of the curve is always below a fixed finite value k, then the function is uniformly continuous*, since for each height 2ε we can choose a width $2\delta = 2\varepsilon/k$ so that when this window is centered at any point of the graph we can see the curve inside. More precisely: if $f : \mathbb{R} \to \mathbb{R}$ has a derivative at each point and $|f'(x)| \leq k$ at any x, then f is uniformly continuous on $\mathbb{R}$.

Limit of a function at a point

Now we consider windows as above, but we are going to disregard what happens along the vertical segment splitting it in two equal portions. To be more clear, we shall be interested in what happens in the shaded portion of the window in the figure in the next page (we shall call it *a split window*)



And now we can say that a function f has limit L at the point a , when for each window height there we can choose a width such that the corresponding split window centered at (a, L) lets us see the graph of the curve. The following figure will make it more clear.



Through it we try to suggest that what happens at the line $x = a$ does not matter.

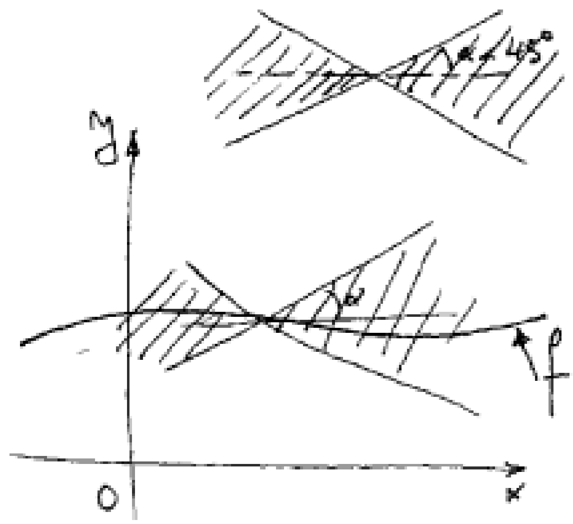
It is not my intention here to do so, but it would not be a difficult exercise to use the notions we have introduced in order to visually deduce the main properties of the real functions related to continuity and limits.

Contractive functions

The notion of contractive function on the real line is easily visualizable in a very interesting way. We are going to introduce now *"angular windows"*.

An angular window of angle $\alpha \in [0, \pi/2)$ centered at the point (a, b) is the portion of the plane enclosed by the two straight lines passing through (a, b) and forming angles with Ox of magnitudes α and $-\alpha$ containing the horizontal line through the point (a, b).

In the figure below the angular window is the shaded zone, corresponding to an angle $\alpha < 45$



A function $f:\mathbb{R}\to\mathbb{R}$ is a Lipschitz function of constant $k\geq 0$ when the angular window of angle $\alpha=\arctan(k)$ centered at any point of f contains the graph of f.

The translation of this definition into analytical terms is, of course:

A function $f:R\to R$ is a Lipschitz function of constant $k\geq 0$ when for each $a,b\in R$, $|f(b)-f(a)|\leq k\,|b-a|$.

When the constant k is less than 1, i.e. when the angle is less than 45^o, then the function is called a contractive function.

From the visual definition it easily follows that any Lipschitz function is uniformly continuous (for any window height 2ε one chooses the width $2\delta=2\varepsilon/k$ corresponding to the window with that height and whose diagonals have slope k and $-k$.)

The iterations of a function

As we shall see, it is often useful, given a function $f(x)$, $f:\mathbb{R}\to\mathbb{R}$, to consider the iterated values starting from $x=a$, i.e. the values, $f(a), f^2(a)=f(f(a)), f^3(a)=f(f(f(a))), ..., f^n(a), ...$ The visual determination of these values starting from the graph of the function is interesting:

from the point $(a,0)$ one draws a vertical segment to the curve and one obtains $(a,f(a))$

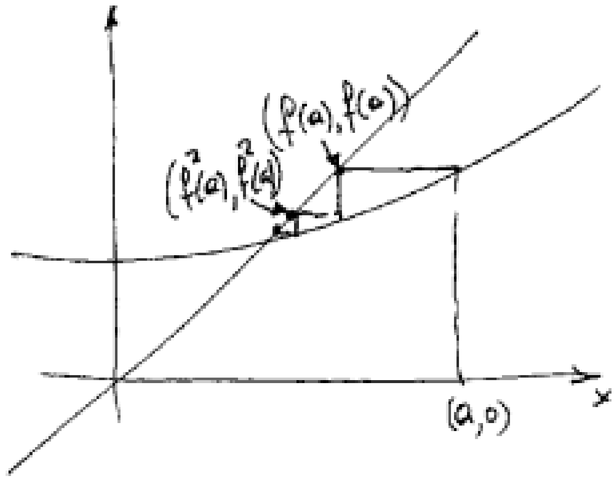
from $(a,f(a))$ one draws a horizontal segment that intersects the bisector $y=x$ at the point $(f(a),f(a))$

from this point one draws a vertical segment to the curve and one obtains $(f(a),f^2(a))$

....

In this way we obtain the different values $f(a), f^2(a), f^3(a), ...f^n(a), ...$

The following figure makes the process clear



It also suggests that when the function is contractive, the sequence of points on the bisector line are going to converge to a point which belongs both to this line and to the curve, i.e. it is a point $(p, f(p) = (p, p)$. This means that $f(p) = p$. We shall visually prove this property in detail.

Fixed points

Given a function $f : \mathbb{R} \rightarrow \mathbb{R}$ we say that $p \in \mathbb{R}$ is a fixed point for f when $f(p) = p$. The visual translation is: *a fixed point p for the function f is any of the abscissae of the intersections , if they exist, of the graph of f with the line $y = x$.*

Fixed points are extremely important in modern analysis and for this reason the following theorem is at the center of the theory.

A visual proof of the fixed point theorem for contractive functions

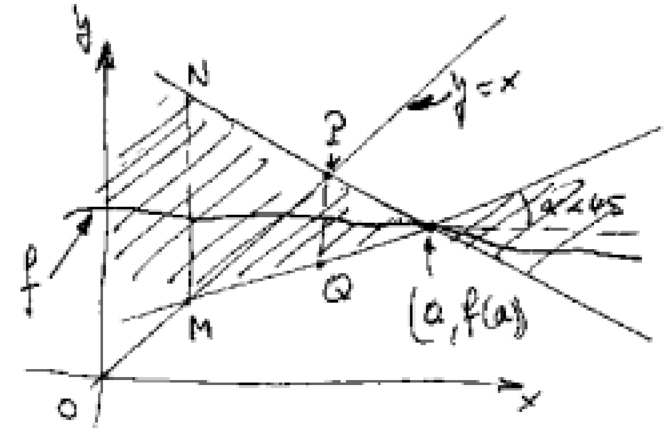
After the exploration we made above, when dealing with the iterations of a function, the following theorem should not be a surprise.

If $f : R \rightarrow R$ is a contractive function then there is a unique fixed point p for f that can be obtained by choosing any $a \in R$ and determining $p = \lim_{n\rightarrow\infty} f^n(a)$.

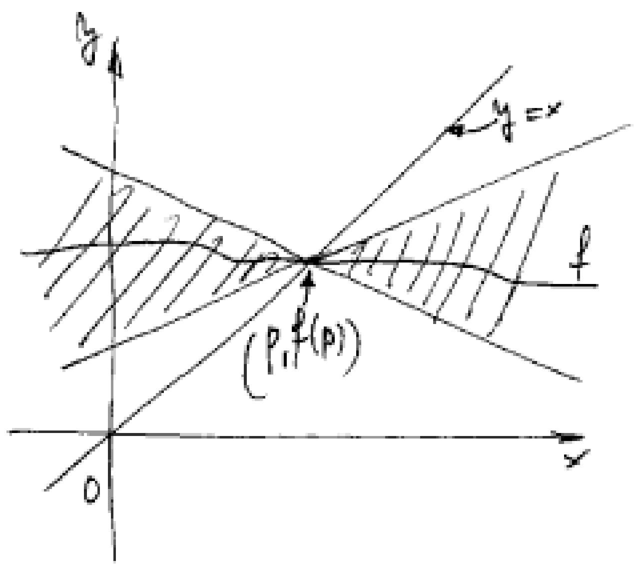
The existence of at least one fixed point is visually proved in the following way.

If we take any point $(a, f(a))$ of the graph of f and we center on it the corresponding angular window, as in the figure, it becomes clear that the sides of this window (since $\alpha < 45^o$) intersect the line $y = x$ at two points P, M (unless $a = f(a)$, but then we already have our fixed point a).

Since the graph of f is enclosed in the window we have drawn, it is obvious that it has a point on the segment PQ , below the line $y = x$ and another one on the segment MN , above the line $y = x$. Therefore the continuous curve f has at least one point of intersection with $y = x$, i.e f has at least one fixed point p.



The fact that this point is unique follows visually by just centering the angular window at the point $(p, f(p))$ as indicated in the figure below



Since $\alpha < 45^o$ and the graph of f is in this angular window, it cannot intersect again the line $y = x$. This means that f has a unique fixed point p.

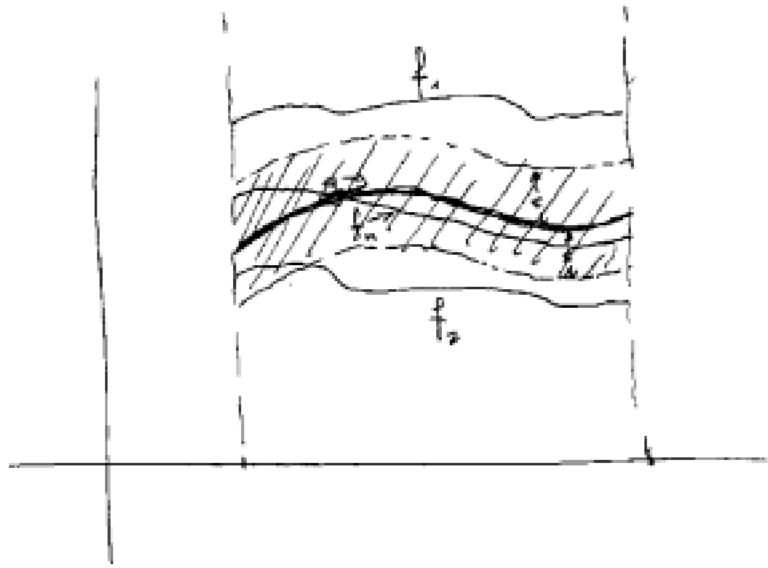
From the analytic characterization of the contractive function we have

$$|f^n(a) - p| = |f^n(a) - f^n(p)| \leq k \left|f^{n-1}(a) - f^{n-1}(p)\right| \leq k^n |p - a|$$

and, since $k < 1$, we obtain $p = \lim_{n\to\infty} f^n(a)$ and so the theorem is proved.

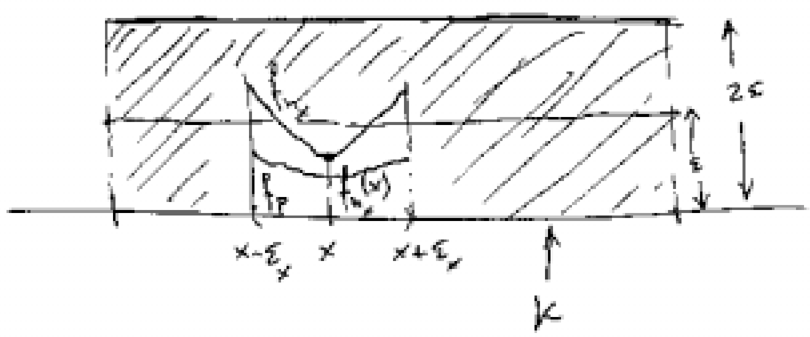
Sequences of functions and uniform convergence. Dini's theorem.

Let f_n and sequence of functions from $K \subset \mathbb{R}$ to K. That f_n converge uniformly on K to another function g means visually that for any plane strip of width $\varepsilon > 0$ around g we can choose a subindex m such that for each $n \geq m$ the function f_n is inside that strip as the figure below suggests. One can check that this is the exact translation of "for each $n \geq m$ and for each $x \in K$ one has $|f_n(x) - g(x)| \leq \varepsilon$.



Dini's theorem asserts that *if K is a compact set and if the sequence f_n of continuous functions converge montonically at each point $x \in K$ to $g(x)$, g being also a continuous function on K, then the convergence of f_n* to *g is uniform on K.*

The visual proof of this theorem is interesting.First one can assume that $f_n(x)$ decreases at each point and and one can reduce the theorem to the case where g is 0 on K by considering the functions $f_n - g$.



We fix a strip of width $\varepsilon > 0$ around the axis Ox. For each x in K there is an n_x such that for each $p \geq n_x$ one has $0 \leq f_p(x) \leq f_{n_x}(x) \leq \varepsilon$. Therefore for each $x \in K$ there is an open interval $(x - \varepsilon_x, x + \varepsilon_x)$ such that for each point $p \geq n_x$ and each t in the interval one has $0 \leq f_p(t) \leq f_{n_x}(t) \leq 2\varepsilon$.

Since K is compact we can choose a finite number of such intervals covering K. If N is the greatest n_x corresponding to these finite collection of intervals we see that for each $n \geq N$, f_n is in the 2ε -strip of the function g. This concludes the proof of Dini's theorem.

As an exercise I would like to suggest a visual proof of the following theorem related to the one by Dini: *if K is a compact set and if the sequence f_n of monotone continuous functions converge at each point $x \in K$ to $g(x)$, g being also a continuous function on K, then the convergence of f_n* to *g is uniform on K.*

I think the proof that results becomes significatively more transparent than the one usually offered.

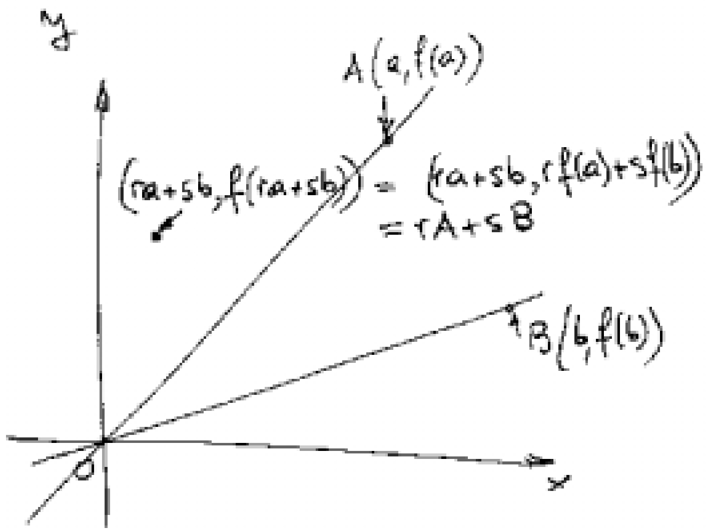
A theorem made simple by means of a visualization

An additive function $f : \mathbb{R} \to \mathbb{R}$ is a function such that $f(a+b) = f(a) + f(b)$ for each $a, b \in \mathbb{R}$.

It is then easy to see that $f(0) = 0$, that for each $m \in \mathbb{Z}$ we have $f(mx) = mf(x)$ and that for each $r, s \in \mathbb{Q}$ we have $f(ra + sb) = rf(a) + sf(b)$.

The following interesting fact has an immediate visual proof: *the graph of any additive function f is either a line through the origin or else is a set of points dense in the plane.*

Assume that the graph has two points $A(a, f(a))$ and $B(b, f(b))$ such that the straight line AB does not go through the origin.



The set of points $\{rA + sB : r, s \in \mathbb{Q}\}$ is obviously dense in the plane.

From this fact one can easily conclude that any function g which is additive and continuous, then it is of the form $g(x) = \lambda x$ for $\lambda = g(1)$. It is not difficult to see that any additive and measurable function has also to be of this same form. If we determine a function which is additive and not of this form we deduce the existence of non-measurable functions.

Such an additive function not of the form $g(x) = \lambda x$ is determined in the following way. Let us consider the vector space $\mathbb{R}$ over the field of rational numbers $\mathbb{Q}$. We determine a basis of this vector space by taking first the elements 1 and $\sqrt{2}$, which are clearly linearly independent over the rationals, and completing this set to a basis in an arbitrary way. Let this basis be $\{1, \sqrt{2}, e_3, e_4, ...\}$. Let us now define $g(1) = 1$, $g(\sqrt{2}) = 2$, and for any element $\alpha \in \mathbb{R}$, $\alpha = r_1 1 + r_2\sqrt{2} + r_3 e + r_4 e_4 + ...$ we set

$$g(\alpha) = g(r_1 1 + r_2\sqrt{2} + r_3 e + r_4 e_4 + ...) = r_1 g(1) + r_2 g(\sqrt{2}) + r_3 g(e_3) + ...$$

In this way g is additive and obviously the line passing through $(1, g(1))$ and $(\sqrt{2}, g(\sqrt{2}))$ does not go through the origin. The function we have so defined cannot be measurable.

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