ACCESSING KNOWLEDGE FOR PROBLEM SOLVING

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ABSTRACT

This paper studies the modes of thought that occur during the act of solving problems in mathematics. It examines the two main instantiations of mathematical knowledge, the conceptual and the structural, and their role in the afore said act. It claims that awareness of mathematical structure is the lever that educes mathematical knowledge existing in the mind in response to a problem-solving activity, even when the knowledge evoked is far from being evidently connected with the activity. For didactical purposes it proposes the consideration of mathematical techniques to facilitate the accessing of pertinent knowledge. All the assertions above are substantiated by close examination of some exemplars taken from various mathematical topics, and the presentation of some recent fieldwork results.

Introduction

Let us set the scene by immediately referring to a particular problem.

Example 1

Show that k! divides the product of any k consecutive positive integers.

The most efficient way to argue for this problem is the following. Consider for any positive integer n

$$\frac{n(n+1)(n+2)...(n+k)}{k!} \quad .$$

If we show that this fraction represents an integer we are done. However we notice that the above expression is in the form of a binomial coefficient, and so is guaranteed to be an integer.

In the approach above we have introduced and applied some knowledge that was not evidently relevant to the initial context of the question. The role of accessing appropriate knowledge here is decisive, as it is of course important generally in problem solving in Mathematics. It is essential for a solver to be able to transfer ideas from one context to another. To promote such an ability, there would seem to be two fronts that have to be nurtured. The first is to mentally organize mathematical knowledge as it is learned and develop it in a way that is conducive for application in problem solving. Indeed our example could even be regarded as a fact that could have been assimilated previously when learning about binomial coefficients. Such broader knowledge accumulated about a certain notion will be called a 'schema'. The second front is how the practitioner becomes skilled in making the connections she/he needs whilst working on non-routine mathematics. Are we simply reduced to say one just happens to notice something as in the example above, or can we analyze the process further? We shall consider awareness of mathematical structure as a possible way to achieve this.

The importance of accessing knowledge for solving activities and the creative challenge it demands means that it is natural to try to systematize the ways to cope with this mental action as far as possible. One way to effect this systemization is through techniques. (We shall specify exactly what we mean by a technique later in the paper.) In creating techniques we are often cementing interactions of different entities or systems, hence strengthening schemas. We feel, then, that students' acquisition of techniques is crucial for them to become efficient problem solvers. Some techniques are taught explicitly in the curriculum, but many others have to be garnered by the students themselves. Quite a few require only a slight shift in perspective in looking at acquired knowledge, but cognitively speaking we should not assume that such shifts would be easy for the student to accomplish on her/his own. Potentially anyway, yet further techniques would be gleaned from the students' experiences whilst occupied with their exercises, in drawing together similarities with previous work. However the required assimilation in order to process such perceived parallelisms into clear descriptions, as techniques would no doubt require certain maturity. There is a common saying in the professional community of mathematicians that "a trick met twice becomes a method". This disregards, though, the problems involved in identifying and extracting your method from the (possibly very different) contexts encountered.

This paper will address in more detail the issues raised above, and will discuss some pedagogical implications. In particular we will consider techniques that really only comprise simple reformulation of known material, as this class may be the most realistic to take in order to positively influence students' thinking patterns. In this context, I shall describe briefly some fieldwork that I conducted involving one such technique, employing bijections for the purposes of enumeration.

Knowledge Acquisition and Retention

It is plain that if we wish to access knowledge, we are first assuming that that knowledge is present. Hence knowledge acquisition and retention are relevant topics for our theme.

Here we shall be thinking only about mathematics content knowledge. (This excludes then knowledge of heuristics such as identified by Polya (1945) and metacognition as espoused by Schoenfeld in Schoenfeld (1992) for example.) There has been a tradition in mathematics education literature to compare 'conceptual knowledge' with 'procedural knowledge', see e. g. Hiebert & Lefevre (1986), however we shall add another category that we shall call 'structural knowledge'.

The conceptual, for us, concerns some sort of issue, circumstance or entity that can be modeled mathematically but may be also manipulated mentally to some degree independently of the mathematical model. Conceptual mathematics always in this way refers to a cognitive environment where the mind can process ideas that should be readily transferable to the mathematics. The part of the environment that supports these ideas is often referred as the concept image in the educational literature, see for example Tall & Vinner (1981). The concept image may take many forms, such as descriptive wording or use of diagrams. The concept image should be thought of as being much more than an informal representation; cognitively the concept image is more or less identified with the 'working' of the mathematics that it parallels. This strong identification between a mathematical system and a more intuitive realm means that a concept has the potential to convince the practitioner of the truth of some related proposals without having to make recourse to formal proof. Any known result that is at least partially understood via the concept will be termed 'conceptual knowledge'. It should be remembered that often the act, or we might say the art, of forming definitions must necessarily compromise the original concept image. [This is amply shown with Lakatos' work, as in Proofs and Refutations (1976)]. If the image is not adapted accordingly, there will be clashes between the image and the mathematical system leading to possible dysfunction in performance. In tertiary level mathematics, at least, images are not often induced within taught curricula, so this problem is usually never quite resolved completely. Even when they are 'officially' introduced, images may not capture every feature or special case involved in the mathematical system. [E.g. in Pinto & Tall (2001) it is remarked how a student could not reconcile the convergence of a constant sequence in the standard 'dynamic' graphical depiction for limiting properties of sequences often shown in text books.] The above suggests that conceptual knowledge may not be so easily assimilated or retained as one might have believed; and it is likely to be mentally processed inflexibly.

When mathematics education was still quite young as an autonomous discipline, Skemp (1978) emphasized the difference between 'to know how' (instrumental understanding) and 'to know not only how but also why' (relational understanding). Ever since the same concern has been voiced dressed in various guises and perspectives. Jones & Bush (1996) suggested that the notion of mathematical structure is a good medium to explain the state of 'comprehending the why'.

Following Rickard (1996), we describe a (mathematical) structure as a set of objects along with certain relations among those objects. Rickard's paper continues to further define structure abstractly (via the notion of isomorphism), but we shall not follow this here. As far as we are concerned, even though structure may be highly abstractly represented in axiomatic systems, it may also be identified locally within a given context. If the structure has to be analyzed, it must be to some extent extracted from the context, but this can be done in such a way where the contextual referents are always at hand. [As Mason (1989) points out abstraction involves ‘drawing away’, or ‘divorcing’, rather than just extraction.] Our perspective of structure, then, is to strip away all the intrinsic features and properties that are not relevant to a certain coherent means of manipulation of a system. In this kind of analysis, then, a sense of what is essential and what is not is built up, which surely contributes to an enhanced understanding of why approaches developed from the said means of manipulation should work.

Though we will not claim that conceptual knowledge is disjoint from structural knowledge (i.e. knowledge that is accrued from structural considerations), in essence the two are different in character. Structural knowledge is based on analysis or at least on reflection on connections and (inter-) relations (see Mamona-Downs & Downs, 2002), whereas conceptual knowledge depends on holistic mental images where structure should be implicitly represented but its presence not necessarily realized. However structural knowledge is meaningful; as a collorary, we contend that not everything that makes sense in mathematics is due to it being somehow ‘conceptual’!

Structural knowledge is more flexible than it might at first seem. First, parallel structure may be identified in different contexts and so associations are made between diverse mathematical topics. If you do allow the notion of abstract structure, then these concrete manifestations may be regarded as the various **representations** of the structure (again following Rickard.) Second, new perspectives of structure or connections between non-parallel structures may be made by considering (for example) different approaches of solving the same question. An important facet here is that proofs often 'import' structure that is not explicitly present in the context of the proposition to be demonstrated. Taking these two notes together, we claim that thinking in structural terms is highly beneficial in forming schemas, which in turn contributes to the range, depth and linkage of the knowledge that is available for accessing.

Perhaps the role of representations deserves a little more explanation. Note now that we have both concept images and representations as some sort of description of a mathematical entity; how do these differ? Well, the difference is perhaps a matter of perspective, and may be best understood by contrasting the following two casual phrases: ‘you can see it **as**’ for the image and ‘you see it **in**’ for a representation. A representation then can have features that may be exploited that would not be available from a concept image. (For example, a graph as a concept image may be taken as a way of understanding functions, but as a representation it may introduce notions like slope, not integral to the abstract function definition.) In fact, because an image is identified with the entity, a more relevant issue seems to be whether cognitive images can have representations (rather than to ask how the two differ). Although we base representations on an abstract structural basis, we do not want to give the impression that it is not appropriate to talk about a representation of a concept image. But when we do refer to such a thing, we shall assume that the image is robustly consistent to the structure of the mathematics that models the concept (so, if need be, the representation may be put onto a structural footing).

Structural knowledge, being well suited to explain why things work, should be conducive for acquiring and retaining knowledge. However neither traditional methods of teaching nor indeed many reform or innovative pedagogical approaches put much emphasis on fostering structural appreciation, so this potential source of cementing knowledge is largely not available for the average student.

Because of the reasons given above, the typical student can have an impoverished stock of knowledge compared to what could be hoped for from the curricula. As much of the information received is not backed up with a sufficiently secure sense in meaning, either at the conceptual or structural level, students will not retain much of the mathematical content to which they are exposed to, and also much of the knowledge of certain powerful trains of thought needed in successfully working in mathematics. True, procedural knowledge quite often can be memorized through repeated use, but this knowledge is not valuable as a tool in problem solving unless some of its structural underpinnings are appreciated. (We characterize procedural knowledge as knowledge that is mentally held with little meaning or significance. Typically procedural knowledge is the result of rote memory or results from procedures that were not comprehended or appraised.) Hence often a student's mathematical knowledge is 'frail', a term used by Steiner (1990), and as such must be largely 'inert', as put by Whitehead (1929). The perspective of this paper will be to concentrate on how to make inert knowledge into a more 'active form'; we will not take into account the possibility that the relevant knowledge might not be registered in the individual's mind in any form. Largely we will employ examples where the knowledge 'prerequisites' are not demanding. However this activity should in itself enrich and reinforce the way that the underlying knowledge is understood, which, in turn, should strengthen its retention in the mind.

Knowledge Schema Building – An Example

In order to maximize possibilities for applications of some particular knowledge to be made available it is highly desirable to explore the knowledge from different perspectives and to seek for linkages with other bodies of information. Doing this we say that we are forming a schema centered around this knowledge. The notion of 'schema' has been given different interpretations in the cognitive and educational literature. We mention the following three exemplars: (a) in the Piagetian theory adaptation of knowledge occurs through the construction and modification of schemata that constitute sequential manifestations of knowledge at different levels of mental maturation, see for example Flavell (1963), (b) the schema based mathematical performance, as analyzed by Hinsley, Hayes and Simon (1977), where it is argued that the student deals with a problem by placing it in a broad category often from the statement (or parts of it) of the problem, (c) in the APOS (Action – Process – Object – Schema) framework the schema associated with a mathematical object encapsulates the building up and expresses the connections that relate actions, processes or different protogenic objects to this particular mathematical object, Dubinsky (1991). Analyses in these traditions tend to be either psychologically dominated, or if mathematical content is a focus (as in APOS) the schema tends to be fairly 'closed' (self-referential to a single conceptual source). An exception to this can be found in some strands of the epistemological tradition in mathematics education, epitomized by the work of Anna Sierpinska. Here care is taken to compare related concepts in order to enhance the structural appreciation of a core concept. A good exposition of this approach is to be found on Sierpinska's work on limits of sequences, Sierpinska (1990). In this section we are not concerned in analyzing schemas per se, but we will illustrate the kinds of dynamics of thought that may be involved in the actual process of building up some strands of a schema.

Example 2

Consider the number of ways of selecting r things out of n things ($r \leq n \in |N$). Denote this number by $C_{r,n}$. We shall call $C_{r,n}$, as r and n vary, as **choice numbers** (rather than binomial coefficients as not to anticipate the course of the exposition.) Cognitively, a choice number has been assigned a certain significance in meaning apart from the fact that it represents an integer. This meaning may be thought of as a conceptual counterpart of the following more formally stated problem: calculate the number of subsets of order r in a set of order n. The words "selecting r things out of n things" then qualify as a concept image. This image is stable enough to allow some mental manipulation. For example, apart from informally arguing to obtain its standard algebraic expression, we may further convincingly argue the identity

$C_{r-1,n-1} + C_{r,n-1} = C_{r,n}$. All that has to be done is to pick out one thing A, and consider two cases; in the first case we consider all choices of r things including A, in the second all choices excluding A. This partition yields the result. Of course to accomplish this train of reason needs a certain mental agility. Note that the reasoning involved is completely parallel with that which would have been used had we attacked the counterpart problem instead. In general, it has been noted often that different formulations of essentially the same problem can cause considerable change in solving performance. (One might recall the famous experiment made by Simom and his colleagues, Simon (1989) which showed that most people took significantly less time to 'solve' an Hanoi Tower problem compared to an exactly analogous task where the discs used in the Hanoi Tower puzzle were replaced by acrobats of varying size, jumping off and on each others shoulders.) In the case of comparing arguments afforded by the concept image with the parallel ones afforded by the corresponding mathematically defined system, perhaps it is not so much appropriate to say that the former will be the 'simpler'. Rather they will tend to be the more transparent, whereas the arguments from the formal system will be more concrete in the sense that one has the access to the structure that the system avails.

We proceed now to describe two further ways of obtaining the identity

$C_{r-1,n-1} + C_{r,n-1} = C_{r,n}$.

(a) If you expand out $(1+x)^n$, there are 2^n terms depending on whether you pick 1 or x in each of the factors (1+x). For any one of these terms, if you have selected x in exactly r out of n factors, the term equals x^r. Collecting like terms, we obtain the result that the coefficient of x^r must equal the number of ways of choosing r things out of n, i.e. is $C_{r,n}$.

Consider now the reformulation below:

$(1+x)^n = (1+x)^{n-1}(1+x)$.

If a choice of r x's is made such that the choice for the isolated factor (1+x) is 1 (x resp.), then a choice is induced of picking r (r-1 resp.) x' s out of n-1 for $(1+x)^{n-1}$. The identity

$C_{r-1,n-1} + C_{r,n-1} = C_{r,n}$ follows.

(b) Imagine that you have an $a \times b$ rectangular array of squares ($a, b \in |N$) and denote the extreme bottom left square by L and the extreme top right square by R. Placed on L is an object that can be moved around the array only by making successive moves either going some spaces to the right along a row or going up some spaces along a column. The number of routes that the object can take to arrive at R is $C_{a-1,a+b-2}$. This is because necessarily each route must involve exactly a+b-2 'crossings' from one square to another; each crossing can be done either vertically or horizontally, but in total for the route to end at R we must have exactly a-1 of the crossings done vertically. By setting

$a = r+1$ and $b = n-r+1$, we may identify $C_{r,n}$ with the number of paths as described above. Now all paths ending at R must either pass through the square immediately to its left or the square immediately below. Knowing the number of routes going to these two squares as $C_{r,n-1}$ and $C_{r-1,n-1}$ respectively, we obtain our identity again.

Notice that both (a) and (b) constitute representations of the basic concept of enumerating ways of choosing r things out of n. Apart from varying terminology due to contextual differences, the argument to justify the identity $C_{r-1,n-1} + C_{r,n-1} = C_{r,n}$ is exactly the same in (a) and (b) as for our initial concept image processing. As the identity on its own is clearly sufficient to calculate $C_{r,n}$ for any r, n by assuming appropriate initial values (basically the Pascal triangle represents the identity), any relationship involving the choice numbers $C_{r,n}$ that can be shown in one situation may be shown analogously in the other two. However this misses some important points; the context that the representations provide can either contribute to providing cognitive tools or can actually afford techniques that otherwise would not be available as we are now going to illustrate.

The choice numbers $C_{r,n}$ are well known to provide a very rich system of formulae. (See e.g. Anderson (1989), Chapter 2.) We will take the representations (a) and (b) and illustrate how having experience with them could help a practitioner to procure some of these relationships. We shall start with (b). This representation has the special feature of being able to be treated visually, and it is the availability of diagrams that lead us quite naturally to obtain some results. We shall give just one example. Any route from L to R must necessarily enter the top row at some column; once the route has reached the top row, its path is determined. Hence the number of routes must equal the sum (over all squares S in the second top row) of the routes starting from L and ending at S. This yields the identity: $C_{r-1,r-1} + C_{r-1,r} + C_{r-1,r+1} \ldots + C_{r-1,n-1} = C_{r,n}$.

Now this result simply could be attained by recursive use of $C_{r-1,n-1} + C_{r,n-1} = C_{r,n}$, but by the time that we start long summations it is difficult to maintain the original concept image in terms of numbers of selecting things. Routes on arrays provide us with an alternative way to describe the numbers $C_{r,n}$, where now interpretations of sums and products can be made. Because of this new representation, certain identities become particularly significant and natural to extract (some of which are not so obvious as the one that we have employed).

However, although (b) may guide or inspire direction for posing and solving, essentially it does not offer any new methodology. The representation (a) is very different in this matter. The fact that we are now imbedding the set of choice numbers into the system of polynomials brings in a much more elaborate structure available for exploitation. For example, differentiation is now a device we can utilize. The following formula is just an expression of the representation (a):

$$C_{0,n} + C_{1,n}\, x + C_{2,n}\, x^2 + \ldots + C_{n,n}\, x^n = (1 + x)^n$$

By substituting $x = 1$ does not give us anything that is not already conceptually clear. However by the simple action of differentiating both sides and then substituting $x = 1$ we obtain a relationship which is far from being intuitive:

$$C_{1,n} + 2\, C_{2,n} + \ldots + n\, C_{n,n} = n\, 2^{n-1} .$$

In this example, we have introduced a couple of representations of the set of choice numbers that allow certain relationships between the choice numbers to be more easily and naturally formed. However, in a way this situation can also be reversed. Many combinatorial problems yield answers in terms of choice numbers, and it is important to simplify the resultant expressions if possible. In this activity, we might well want to make use of the cognitive or structural tools that our representations can offer. Hence the representations do not act only as platforms to inspire problems and results, but they can also be invoked in a solving activity.

In general, representations, as well as weaker associations, provide the kind of net of connections that would form a schema of a type likely to promote the application of knowledge to problem solving. Representations are particularly potent components in the fabric of a schema because of the closely drawn structural ties they have with the central concept image. What is even more important is that representations often offer quite powerful and novel ways of thinking about a theme, as we have illustrated in this section. If we can instill within the student an appreciation of 'neat' arguments, a representation that contributed to forming one is likely to be remembered. A schema critically depends on memory enforced by structural awareness.

Forming broad schemas will significantly increase the chance that problem solving triggers links with knowledge resources. However to take advantage of this fully the student must actively seek out potential applications; this will be explained further in the next section.

The Process of Identifying Applications of Knowledge in Problem Solving

To start with, we will consider a problem that needs little demand on the knowledge base. The style of writing dealing with its solution is meant to highlight the role of the educational research notion of control in problem solving, as explained in the book of Schoenfeld (1985). Indeed the problem we use is taken from this book (p.94).

Example 3

Let $\{a_1, a_2, \ldots , a_n\}$ and $\{b_1, b_2, \ldots , b_n\}$ be given sets of real numbers. Determine necessary and sufficient conditions on $\{a_i\}$ and $\{b_i\}$ such that there are real constants A and B with the property that

$$(a_1 x + b_1)^2 + (a_2 x + b_2)^2 + \ldots + (a_n x + b_n)^2 = (A x + B)^2$$

for all values of x.

We will write down a solution, but not as you would expect it to be presented in a text, but to reflect a plausible line of thought which could guide you to obtain an answer. You might start off to see whether you can gain some conceptual image for this expression. For this problem, obtaining such an image is unlikely; and it would be an act of control to come to this realization. Hence it would seem a structural approach is needed. A first perusal of the situation might draw your attention to the variable x, and lead you to an assessment that the expression having to hold for all x is a strong condition. Because of this it would seem to be a good idea to try out some particular values of x. Are there 'special' values that would be especially useful to employ? With this question in mind you review the expression again. You note the special feature of the expression that the left hand side is a sum of squared terms; if we drive this sum to zero then each term must also become zero. This can be done by setting x = - B/A on the right hand side of the expression. This breaks the back of the problem; necessarily all the quotients $b_i : a_i$ must be equal, and then it is straightforward to show that this condition is also sufficient.

Although the argument is not difficult, none of the students involved in the relevant fieldwork in Schoenfeld (1985) were able to solve the task. The problem may be due to the very structural level in which the strategy lies. When you are operating structurally, the main things concerning you are to regard the variable x as a degree of freedom or choice, and to access basic knowledge of which the most sophisticated is that a square is always non-negative. The combination of these two things then might in fact be challenging for students to achieve. How can we help them? Well the core of the strategy of our solution is to force the system into a special state of an often-used form; if the sum of terms squared equal 0 then all the terms are zero (in |R). If this becomes a part of the students' knowledge together with a habit to recall this knowledge whenever s/he meets a sum of squares, the student would be in a much better position to answer. True this kind of 'cueing' of knowledge does not represent the most creative thought, but we believe that it does play a very important role in doing mathematics at any level. We shall resume this theme by discussing techniques in the next section.

Example 4

Is there a partial sum of the harmonic series that is an integer, apart from 1? *

A sketch solution: Rewrite the n^{th} partial sum as

$$\frac{\sum_{k=1}^{n} 1\times 2\times\ldots\times(k-1)\times(k+1)\times\ldots\times n}{n!} \qquad (A)$$

For n≥2 an analysis of the numerator would reveal that the highest power of 2 dividing the numerator is less than the highest power of 2 dividing n!. Hence none of the partial sums for n≥2 is integral.

We shall flesh out this solution, but (as in the previous example) in such a way to represent some of the ‘background’ thoughts that would enable the forming of the argument. The philosophy in doing this is to point out possible difficulties that students might have in obtaining this approach by themselves, not necessarily to present a typical way that an expert would tackle the problem cognitively. We proceed to consider some different stages of the solving process.

(1) Would students necessarily rewrite the n^{th} partial sum as done above? If they want to ‘size up’ the problem to start off with, they might first be wanting to link up the issue raised in the question with their knowledge of the harmonic series. The basic information they are likely to bear

* We were introduced to this problem by S. J. Hegedus

on the issue is the fact that there always is a partial sum of the series which surpasses any particular integer, and that the terms of the series tend to zero. This knowledge would seem inviting to accommodate in a real number line image. The focus would then be naturally drawn to how successive partial sums 'jump over' integers and how long these jumps are. If the students believe that the answer to the question in the task is no, the natural strategy would be to try to construct around each integer an interval such that no partial sum could be contained in the interval. Clearly the lengths of these intervals would have to tend to zero. If, on the other hand, the students believe that the answer to the question is yes, then they might be tempted to try to justify this existentially on the following basis. As the closest partial sum to an integer gets arbitrarily close to the integer as the integer becomes arbitrarily large, the expectation would be for the two to coincide eventually. The first argument is not plausible, the second is an instance of a common misunderstanding that students show for sequences and series, see for example Mamona-Downs (2002).

Hence the knowledge that would seem the most pertinent to the problem because of the setting of the question does not seem to help us much. If students had started thinking in the ways described above, they would likely to have to abandon it soon. It would then be an act of self-regulation to decide to seek for alternative ways of approaching the question. Conceptually there doesn't seem to be much else to hold on to, but…

(2) there is a natural algebraic maneuver to make, the one taken in our sketch solution. It is motivated more by a practice (i.e. if you have a sum of fractions what you 'normally do' is reformulate it into a single fraction) rather than a conscious shift in strategy. The act performed here is quite modest, but what is impressive is how this small move has opened up a very different realm for the mind to explore compared to the one offered in (1) above. Students are now presented with a quotient of two integers with the issue whether that quotient can represent an integer. Now connections should be coming through from a completely different source of knowledge, including fractions in lowest terms, highest common factors, the Euclidean algorithm, and prime decomposition. Because of the algebraic form of the quotient, it is not likely to be able to carry out the steps of the Euclidean algorithm. What seems to be the most propitious tool available is prime decomposition. Up to now what has been employed is a global viewpoint of the question that did not prove fruitful; prime decomposition offers a way to look at the present state of the solving by local analysis (i.e. to consider prime power divisors for any prime independently from other primes) and hence promises to be flexible. The processing of the knowledge of the unique factorization theorem to suit the issue would be to check whether the greatest prime power divisor of the numerator is greater or equal to the greatest prime power divisor of the denominator for each prime. If for any $n\geq 2$ this is so, our question will be answered in the affirmative; if none of $n\geq 2$ satisfy it, our answer will be negative. The issue is now set into a particular milieu.

(3) Now it is a good time to pause and take stock of the new issue and perspective. Some structural reflection would reveal that for any given prime the greatest prime power divisor (GPPD) of n! may be feasibly found; similarly for the separate terms in the summation of the numerator (this information is not crucial anyway). What really should be of a concern is how to tackle the additions in the numerator. For this we might step away from the context and consider this local issue in the theoretical milieu, i.e. to work within the schema of the unique factorization theorem. What readily available information is there about GPPDs over addition? Suppose that a and b are positive integers, p is a prime and $p^r \| a$ means that r is the greatest power of p dividing a. Then

$p^r \,||\, a$ and $p^s \,||\, b$ with $r<s \;\Rightarrow\; p^r \,||\, (a+b)$ (B)

This result is straightforward; however no such universal results will be available in the case when

r = s. The only elementary fact that can be deduced is:

$p^r \,||\, a$ and $p^s \,||\, b$ with $r = s \;\Rightarrow\; p^r \,|\, (a+b)$ (C)

Hence in the first case (i. e. $r \neq s$) there is perfect control of the GPPD, whereas in the second (r = s) only little. What significance do these results have for the problem?

(4) As the main considerations about knowledge access for the problem are now covered, the exposition we give shall be briefer from now on. We return to the present solving situation with the attention on applying the knowledge given in (3) above in an efficient way, i.e. loosely speaking to arrange things such that case (B) is used rather case (C) as far as possible. To prove that the answer of the question is 'no', there are two working variables at hand; a particular prime p for basing the GPPDs, and the order in which the summation of the terms of the numerator of (A) are to be taken.

It happens that if we choose p = 2 (for whatever partial sum considered) and take the natural order of summation as suggested by the algebraic form of (A), then whenever we add the next term to the aggregate presently considered we always are encountering case (B) rather than case (C). (We leave the reader to explore this situation to understand why this happens.) This means that the GPPD of the numerator for 2 equals the lowest GPPD of any term of the numerator for 2. If $n>1$, the second term of the numerator is $n!/2$, which has a lower GPPD for 2 than does the denominator $n!$. Thus it is established that there are no partial sums of the harmonic series that equal an integer greater than 1.

Comments on educational issues concerning application of knowledge in example 4.

Despite the tools employed in this task are elementary, we feel that most mathematicians would agree with us in saying that this approach would be understandably difficult for students to create on their own. This can be partially explained by some of the classic themes espoused in the problem-solving tradition. For example, self-regulation to decide when to change tactics or focus, usage of explorative work, identifying patterns and extracting the right structure to construct proofs are all likely to have their roles for anyone adopting the approach. All these types of activities require skills involving flexible and individual thought. But on top of these there are further demands on the students, in accessing knowledge. It is on this facet we will concentrate on.

The first thing to note is that the context of the question could lead a student to follow an unpromising direction. The explicit mention of the harmonic series rather than just writing the mere algebraic expression $(1+ 1/2 + \ldots + 1/n)$ would in itself encourage dynamic imagery related to limiting properties. Even if the terminology was avoided in the presentation of the question, the student is quite likely to make the association with the harmonic series anyway. What this illustrates is that automated triggering in recalling knowledge may be misleading unless it is accompanied with a sense of criticism. Given the observation that if students fail to succeed in obtaining a solution using one argument they tend to give up rather than trying to find another, quite a few students would be frustrated in this question because they happened to follow this line.

When we gather all the fractional terms of the partial sums into one fraction, we are entering a mode of algebraic manipulation. Once students are in such a mode it seems very difficult for many of them to get out of it again. They might have some insight how to handle symbolism to guide it into some desired form, but it seems a rather foreign practice to impute meaning or intuitive significance whilst working algebraically. Without extracting meaning or significance, we are unlikely to link our work with our (long-term) knowledge. The seemingly simple act of mentally processing (A) as a quotient of two integers may well not be a natural one for students to perform. Students' behavior in this way could be enhanced in regular problem-solving courses.

The task in forming the connection between the situation of when a quotient of two integers yields another and prime decomposition was underplayed in (2) above. Really would this connection occur to a student? In general, the difficulty about accessing knowledge when it is not triggered automatically is that the application of the knowledge has to be anticipated at the same time as it is being accessed. In our case we might have to have an inkling how the fundamental theorem of arithmetic will help before being motivated to recall it. This kind of impasse perhaps might be avoided in our particular problem more than in others; triggering attention to prime decomposition likely may be achieved by deliberately seeking for hints how to proceed. A reflection that the present processing of the question is just an issue involving integers, and a recollection that an important tool in analyzing integers are GPPDs would seem enough to make the connection open for consideration. However how many students would make both the reflection and the recollection would be debatable.

Another feature of our approach is how it illustrates how knowledge interacts with problem solving. In (3) an issue raised on the level of specifically working on a particular problem was 'lifted' to the environment of the knowledge supporting that issue. Doing this helps to deliberate the issue in its full generality, and having done this the resultant expanded knowledge is pumped back into the solving environment to guide further strategy. Hence, in a sense the knowledge is responsive to the working as well as vice-versa. This though represents another switch of mode in thinking, and so comprises yet another challenge to the student.

Techniques

We regard a (mathematical) technique as a (mathematical) method with the following characteristics:

(I) There is a recognizable structural cue that suggests that the technique may be applied.

(II) There are one or more standardized steps or sub-goals to achieve, but typically there may be substantial problem solving involved in attaining these goals.

(III) The final step will yield some information of an identifiable type.

Perhaps any method might satisfy the above traits to a degree, but we think of a technique of being quite tightly constrained by them. In general, we consider methods to be less explicit and more flexible than techniques.

Techniques are associated with certain structural references, and as such are very different from heuristics, which tend to act as general advice in setting up strategy in problem solving. However there are some similarities between the two, in particular in the way that both can be rather speculative ways of working. (The problem solving aspect of a technique means that we are not assured to be able to carry out the technique even if it is suitably applied.) Because of this it is quite useful to think of a technique as, loosely speaking, lying between algorithms and heuristics as suggested by Schoenfeld (personal communication).

It is the feature of the cue that makes techniques highly significant in the process of accessing knowledge in problem solving. This feature means that whenever the relevant structural pattern is recognized, the student should be triggered to think about the technique. The technique itself comprises a rather specialized processing of knowledge. Hence the technique automates the (otherwise cognitively difficult) act of retrieving pertinent knowledge.

When the structural cue is strongly associated with some particular conceptual imagery then the application of the technique usually becomes habitual after some experience. For example students soon familiarize themselves with the standard techniques of optimization of (smooth) real functions using calculus tools. (Note that such techniques are not algorithmic, as finding roots is not necessarily easy.) However when the structure implicit in the cue is not identified with a single specific mathematical context then we find that techniques are usually not taught nor consciously held in the mind of the students. As a consequence, the tendency is that the broader a technique is, the less it is appreciated.

One broad technique that definitely is usually taught though is induction. Note that although the technique itself can be supported by fairly evocative imagery (e.g. a line of dominoes placed in a line in such a way that the knocking down of the first will cause all the others to fall in sequence), the description of the cue must be very general and may not seem very concrete. Perhaps it could be characterized by the identification of a family of objects indexed by the positive integers together with an explicit hypothesis about a property of the objects. This encompasses a much more extensive vista of applications of induction than those typically 'registered' by the student that might only stretch to proving algebraic identities. (And even in this case students may only use induction when directed to do so.) Another facet that further restricts students' vision about induction is that usually their experience with the technique is limited to situations where the 'hypothesis' to use is more or less given to them. A more creative situation (and one that would be more true to research work) would be for the students to provide the hypothesis themselves. One way of attempting to do this would be to do some experimental work by examining the property for some specific members of the family of objects and to try to discern a pattern as a basis to forming a hypothesis. In this way we have added a new constructive first step to our original technique (i.e. to develop a hypothesis), and as a result the cue widens even more. We shall call such extensions as **constructively widened techniques**.

Another hugely important technique that also admits a constructively widened technique is the use of 1:1 correspondences for enumeration purposes. In the basic form of the technique the cue is the situation of having two (finite) sets, A and B say, for one of which (say B) we know the order (i.e. the number of elements) and for the other (A) we wish to find the order (or a bound to it). The task involved in the technique is to construct a 1:1 correspondence from a certain set of subsets of B into or onto A, and then deduce some information about $|A|$. In the constructively widened form the cue becomes simply a set A about whose order we want some information. The first stage of the technique now is to identify or construct a second set B for which it would seem propitious to form a 1:1 correspondence with A.

Even though the knowledge on which this technique is based on is both elementary and fundamental (i.e. a bijection preserves set order), students might well not be able to utilize it as suggested in the technique above. It is important that the students have processed the knowledge exactly into the context of the cue. Then whenever instances of the cue are recognized, there should be awareness on the part of the students that the technique is available. (Of course they might choose not to pursue it because they can foresee difficulties or an alternative approach that they prefer.) There are two ways of instigating such awareness; first to explicitly introduce a description of the technique and its cue in class, and second to give the students a sequence of relevant tasks, starting with those yielding the most transparent applications. An important technique deserves some focused pedagogical attention.

To illustrate the points I have just made, I shall briefly describe some fieldwork that I have recently conducted with the collaboration of M. Downs on the technique of employing 1:1 correspondences for enumeration. The participants were volunteers from a 'proof' course that is mainly directed towards students contemplating to take a major in Mathematics. They all had similar tertiary-level mathematical background; each had passed a couple of courses in calculus and one in linear algebra. The institution involved is the University of California, Berkeley. The fieldwork comprised two stages. In the first the six participants worked on a problem sheet on their own. The second was a teaching experiment; it consisted of an open discussion between the participants (four students) about the same problems, with the researchers sometimes prompting its direction. The tasks were designed so that each could be solved by constructing a suitable bijection; however some afforded alternative approaches, but these would always be tedious and more 'messy' in comparison. In their proof course, the students had just been exposed to a short formal treatment of bijections.

The motivation behind the 'written' stage was to see how well the students were already equipped for applying the technique. The responses indicated that on the main the students did not exploit the bijections that were fairly natural to invoke. In one problem one student did give a correct answer by an informal bijective argument, but it transpired that that student had met the question and the approach before. Otherwise the students either did not progress, or opted for the more tedious methods available or worked experimentally. These results would strongly suggest that this population was not able to apply the technique. But what was interesting is that at several places the students wrote notes as asides to their main argument that expressed the basic idea that would have supported the construction of a bijection had the technique been followed. The students seemed not to have the means or confidence to develop the ideas. The mere awareness of the technique as a mentally registered entity probably would have been sufficient to allow the students to utilize these ideas to promote complete arguments.

We attempted to test the validity of this conjecture in the 'teaching experiment' part of the fieldwork. Once we had introduced a background of employing bijections into the session, we wanted to see how easily the students would construct the appropriate correspondences and to observe any ways that they seemed not to be at ease. We illustrate the results by summarizing what happened with one task considered in the session.

Fieldwork Question

Let C be a circle, and suppose that $p_1, \dots, p_n$ are n points on C. Construct all chords of C connecting 2 points from $p_1, \dots, p_n$. A crossing is a point strictly inside C that is an intersection point of the constructed chords. What is the maximum number of crossings? (That is find the number of crossings with the assumption that each crossing lies on only two chords.)

In order to start the discussion for this question, the researchers drew two simple diagrams on the blackboard, both showing a circle. Picture 1 further indicated a single chord, picture 2 one crossing and the two chords that intersect there. Picture 1 was meant to act as a prompt towards an analysis via considering the number of crossings on a chord. As this number is not constant, this approach is involved though still viable. Picture 2 was meant to hint a neat way of solving the problem using a bijection in the spirit of our technique; correspond any crossing with the set of four boundary points formed by the end points of the two chords passing through it. We may then deduce that the number of crossings is $C_{4,\,n}$.

After agreeing early on that the number of crossings on a chord is not constant, the students' attention was solely caught on figure 2 rather than figure 1. Almost immediately one student put forward the bijective argument that allows you to equate the number of crossings with the number of subsets of order 4 of the set $\{p_1, \ldots, p_n\}$. For this student, though, this action had merely transposed the original problem to a new one, because he was not familiar with choice numbers. Another student who had studied combinations before helped out, so the participants could at least understand that the new form of the problem was now a standard one. However this is rather a side issue in respect to the application of the enumeration technique. Two out of the four students showed themselves very comfortable with the bijective argument; even though it was understood on the intuitive level, it proved quite robust when these two students were asked to justify why the relation is 1:1 and onto. The other two students though obviously had misgivings. One of these students consistently showed a dislike or mistrust of the technique in general. She preferred alternative approaches such as breaking the problem down into stages or cases, or employed experimental examination. These procedures seemed a lot more secure and concrete to her than the highly constructive aspects of the technique. The remaining student though had shown himself receptive to the technique in other tasks, his qualms were more local to this particular question. He seemed to appreciate the bijective argument but he appeared not to believe that the simple local structure (as suggested in figure 2) can possibly represent the complicated looking structure of the whole system. In a way, his wish to reconcile the local structure with the global is to be applauded, but it put him to some disadvantage compared to the students who did not feel the cognitive need to attempt such assimilation.

Let us now try to draw together our thoughts about how techniques affect the process of accessing knowledge for problem solving purposes. Techniques that are intimately tied with a certain closed content domain should largely become part of the schema centered around the relevant concept image; the linkage of problem solving in this case would likely first pass through the image and then (if appropriate) to the technique. (For example problem solving might reveal an issue on optimization that leads you to use the standard calculus techniques.) This kind of circumstance is not so significant to our theme. However the situation where the technique is broader and can be applied in many mathematical contexts is different. Indeed we do not suppose that these contexts have been identified or listed. Whenever the problem solving activity happens to wander into any one of these contexts and the present state of the system reveals (or brings up an issue concerning) the particular type of structure as described in the cue, then the technique is available for application. This needs both an awareness of the technique and a general alertness in 'spotting' the cue. However what is provided by the technique is a standardized way of channeling a common structural feature or issue into knowledge processed in a particular way likely to advance a solution. We believe that this role of techniques is vitally important in rendering some of the more creative aspects in mathematics somewhat more routine and accessible. The most general techniques, such as employing correspondences, could truly be considered as very potent universal lines of thought in doing mathematics. However we should not forget about techniques of more modest significance; these are often under-employed because they have been given dominant associations, so that the technique tends to be used in limited contexts. (For example the technique of partial fractions is likely to be used only for solving integrals.)

Our fieldwork on enumeration via bijections suggests several educational issues. Firstly, because broad techniques do not usually have an identity for students, even if students have an intuition about a relevant relationship they may lack the framework to develop it. Hence it seems important to teach students some of the most consequential techniques, just as induction is taught. Doing and discussing a sequence of tasks pinpointing applications of the technique seems an effective way to achieve this. In our fieldwork, three out of four of the students seemed to come out of the discussion stage with a fair appreciation of the enumerative technique; one student at the end of the session said: "*I learned a lot and had never thought of bijections in this way before*". However there are caveats. The fourth student did not seem to get on with the technique at all. From the constructivist perspective of mathematics education we might be criticized in trying to impose methodology. However we feel that this may be countered by the argument that basic techniques form such vital ways of thinking that we cannot afford to let students believe that they can bypass them by inventing their own methods each time. The student would risk lacking the possession of essential problem-solving tools.

A second consideration is that although a technique has its problem-solving aspects, it also has procedural aspects. The latter means that an application of a technique may not elucidate its role within the global structure. Hence a reliance on a technique may represent an undue restriction in thinking about a system. This problem, though, is really a question concerning self-regulation.

Epilogue

The main pioneer of problem solving as a discipline in mathematics is generally considered as being Polya. His work on heuristics, especially the book "How to solve it" (1945), on the main received a good reception from mathematicians. However subsequent fieldwork based on his philosophies did not live up to expectations. Later, educational researchers such as Schoenfeld attempted to find the cause of these disappointments. What was decided was that Polya had succeeded to lay down a tactical base for problem solving, but had left out a managerial aspect. This led to mathematics educators to adopt the psychological notion of metacognition (roughly speaking, self-consciousness of your own cognitive processes). This is split into four main categories: resources, control, belief systems and classroom community influences (see Schoenfeld, 1985). It is in the category of resources that knowledge is treated; Schoenfeld summarizes it thus, p.44 ibid:

Resources are the body of knowledge that an individual is capable of bringing to bear in a particular mathematical situation. They are the factual, procedural, and propositional knowledge possessed by the individual. The key phrase here is "capable to bear"; one needs to know what an individual might have been able to do, in order to understand what the individual did do.

Clearly the topic of resources pertains a lot to cognitive science, as it is the human brain that is storing and processing information. However scientists in this field have only been able to model mental operations relevant to mathematical knowledge where linkages occur spontaneously and are “nearly automatic”. [For a comprehensive account of this work see Silver (1987).] Mathematics educators in problem solving have noted these limitations, but without the backing of cognitive theory for more sophisticated channels of accessing knowledge they have preferred not to expand so much on the knowledge base, but to concentrate on control with which "*... solvers can make the most of their resources*" (Schoenfeld, 1985). From this standpoint problem solving then seems to depend on triggering associations with the available resources. The perspective of this paper is how to make these triggering processes more effective, and to stress that the act of knowledge accessing for problem-solving purposes can be far from being mechanized in contrast to what the psychological literature seems to suggest.

In this regard, we are guided by a naive metaphor where we imagine knowledge providing 'hooks' and problem solving situations as providing 'loops'. By increasing the number and size of the hooks and loops we increase the chance that a pair will clasp. Augmenting the size of a hook involves securing and enhancing a reliable concept image, and processing it in a convenient way for its application. Creating new hooks, in the context of a fixed body of definitional knowledge, is done through making connections and forming schemata. By enlarging a loop we mean that we become more aware of the structural aspects of the present state of the working system. Finally we may guide our system into another state, perhaps motivated by a realization that a linkage with some knowledge is imminent, to make further 'loops'.

Poincaré in his essay Mathematical Creation (Poincaré, 1913) made a similar metaphor, for knowledge interactions in the context of unconscious incubation preceding a sudden inspiration; *“the future elements of our combinations are something like the hooked atoms of Epicurus”*. However our use pertains to a different circumstance; we are consciously attempting to let knowledge bear on our solving activities. But very often in order to do this we have to simultaneously anticipate what knowledge is required and consider how to manipulate the system into a state that affords an application of that knowledge. Cognitively this is a difficult demand on students, and the situation is worsened by the fact that the issues overcome in such situations are completely lost in standard style presentation. In general, we advise that some account of the 'thought behind' a solution appears in its exposition, in the same sort of spirit Leron (1983) recommended that the rationale of the constructions made in a proof be informally explained.

However what was said above might suggest a picture that for every individual problem-solving scenario considered we are creating essentially a novel set of ideas, connections and strategies. This clearly is misleading. Mathematical arguments in detail have a bewildering variety, but in outline there seems to be a relatively few types of central features that support them. Taking advantage of these common characteristics is of huge importance; it allows mathematicians to identify types of arguments that can be treated in a similar way. This factor has to be accounted for in our thesis. In this paper we restricted ourselves to what might be the most tangible form of unifying argumentation, that is through techniques. Our description of techniques is such that some very fundamental ways of thinking in mathematics are represented. These require on the face of it only slight re-processing of basic knowledge, but a fieldwork we conducted suggested that students were not alert to the particular technique involved. We propose that some important broad techniques should be explicitly taught. Here we acknowledge the difficulty of already crowded syllabi. However we believe if we were able to accustom students not only to interiorize arguments together with the mathematical facts they provide, but also to take in some of the structural features of the arguments divorced from the facts then the exercise would be justified. For in this circumstance students will start to get in the habit of developing working techniques for themselves.

We note that techniques help in the problem of 'transfer' often referred to in the educational literature. The problem of transfer is about the common phenomenon that students (at all ages) often behave as if they do not recognize analogous problems set in different contexts. Silver's research, Silver (1979), showed that this was mainly due to students not being aware of the underlying mathematical structure. Sierpinska (1995) elaborated this theme, claiming that a present trend in mathematics education at school level suggesting that task contexts should emulate as far as possible 'real-life' situations is detrimental to the transferal of problems. The main message in her paper is that school tasks should be concerned about 'applications', without worrying too much about the applications' status of being either abstract or authentic to reality. She states:

We need 'contexts', but only in the sense of problems that give meaning and sense to what students learn: ***knowledge is always an answer to a question***.

Silver's and Sierpinska's position for school mathematics is somewhat similar to ours for tertiary level problem solving. What we can expect of more mature students is to develop a sense of structure. We believe that **meaning** in mathematics has both conceptual and structural aspects. Conceptual thinking can be both limited and unreliable without accompanying structural appreciation. Structural considerations do not have to be regarded as being abstract, but we conjecture that it is mostly at the level of recognition of parallel structure that the transfer problem is to be resolved. The parallel structure 'connects' with the same knowledge basis, which then is 'applied'.

The theme of reflecting on structure is a recurring one in our paper. The word structure is one that is commonly employed in mathematics education literature but it rarely forms a focus for analysis. Picking up from Sierpinska's assertion that "knowledge is always an answer to a question", we note that this does not seem to represent well how most students retain their knowledge. In truth, the typical student is not often engaged in pondering about mathematical issues but usually is immersed in tackling a mass of exercises. For this reason the student's appreciation of her/ his mathematical knowledge will usually be superficial, and not very effective for use in problem solving. To learn through issues typically requires the unraveling of a rich mixture of motivations both at the structural and the conceptual level. Much the same combination is required for problem solving itself; sometimes informal arguments based on the conceptual image suffices, sometimes arguments are made completely from structural considerations, but most problem-solving tasks involve a blending of the two. But thinking conceptually and thinking structurally seem to form disparate modes. On the metacognitive level we feel that it is important for students to be aware of these two modes of thought. This would represent an important aspect in control; that is, taking the decision about which mode would be the more profitable to assume at any particular time in a solving path.

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