# HOW PEOPLE LEARN … MATHEMATICS

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## ABSTRACT

I address four of the seven themes of the 2nd International Conference on the Teaching of Mathematics – research, technology, pedagogical innovation, and curricular innovation – from the point of view that learning mathematics is, first of all, *learning*. Research from a variety of fields – education, neurobiology, cognitive psychology – provides a consistent set of messages about what learning is, how learning takes place, and how teachers can facilitate learning.

I offer necessarily brief surveys of research on the main themes, and then I describe how my understanding of this research has led to the design of a learning environment (a combination of an interactive classroom, an online delivery system, a rich set of tools, demanding course requirements, innovative course materials, effective in-class and assessment practices, and intangibles) that is radically different from my practice of, say, 20 years ago. I also provide an example of a research-based design for a single lesson.

My conclusions touch on the need for continuous curriculum renewal, for effective strategies to stimulate deep learning, for goal-directed assessments, for addressing the needs of a would-be mathematically literate public, and for preservice and inservice professional development.

# 1. Introduction

The 2nd International Conference on the Teaching of Mathematics intends to address new ways of teaching undergraduate mathematics. The first four of seven conference themes (slightly abridged) are

- **EDUCATIONAL RESEARCH**: Results of current research in mathematics education and the assessment of student learning. …
- **TECHNOLOGY**: Effective integration of computing technology…into the undergraduate curriculum
- **INNOVATIVE TEACHING METHODS**: … cooperative and collaborative teaching, writing in mathematics, laboratory courses.
- **CURRICULA INNOVATIONS**: Revisions of specific courses and assessment of the results … innovative applications, project driven curricula.

This paper cuts across all four of these themes – and has some implications for the other three as well – professional development, relationships to other disciplines, and distance learning technologies.

I write from the perspective of a 40-year teaching career at Duke and other universities, including many attempts at innovative curriculum development and incorporation of technology into the learning process. To be candid, for the first half of my career I mostly failed to have any significant impact on my students, at least in the sense of stimulating sound knowledge and understanding of mathematics. My truly successful students were few enough in number that I still remember their names – and I have always suspected that they would have succeeded just as well without me.

I'm obviously a slow learner, but frustration is a powerful motivator. A series of opportunities in the 1980's and since has permitted me to learn a good deal about my profession that I should have learned much earlier, and to put that learning to use as a teacher and curriculum developer. At first my learning was experiential (that is to say, *ad hoc*), trying things in the classroom, rejecting what did not work, and reinforcing what did. One might describe this as "natural selection" in the evolutionary sense. Later I began to study the research literature – not just in mathematics education, but also in cognitive psychology and neurobiology – to find reasons for my successes and failures. It probably would have been more efficient to proceed in the other order – as I said, I'm a slow learner. In this paper I share some of what I have learned, along with connections to the conference themes.

# 2. Research

The first part of my title comes from the book *How People Learn: Brain, Mind, Experience, and School*, a (U.S.) National Research Council study (Bransford, *et al.*, 1999) that summarizes the very substantial body of research on learning, especially that of the past 30 years. Here is the start of the Executive Summary (p. xi):

> "Learning is a basic, adaptive function of humans. More than any other species, people are designed to be flexible learners and active agents of acquiring knowledge and skills. Much of what people learn occurs without formal instruction, but highly systematic and organized information systems – reading, mathematics, the sciences, literature, and the history of a society – require formal training, usually in schools. Over time, science,

> mathematics, and history have posed new problems for learning because of their growing volume and increasing complexity. The value of the knowledge taught in school also began to be examined for its applicability to situations outside school.
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> "Science now offers new conceptions of the learning process and the development of competent performance. Recent research provides a deep understanding of complex reasoning and performance on problem-solving tasks and how skill and understanding in key subjects are acquired. … "

My point in citing this and other works on learning research is that learning mathematics is, first and foremost, *learning*. Our subject is not exempt from what others have learned about learning, and indeed our curricula and pedagogy, to be successful, must be informed by research on learning. Readers of this paper will probably not be surprised by any of the findings in the NRC study – but may be surprised to learn the strength of the research base underlying the strategies we have come to associate with the words "reform" and "renewal."

The 1990's have been described as "The Decade of the Brain," a period in which the study of live, functioning, normal brains has come into its own through non-invasive technologies, such as positron emission tomography (PET) and functional magnetic resonance imaging (fMRI). This research will continue for many decades, of course. As the NRC study states (p. xv), "What is new, and therefore important for a new science of learning, is the *convergence* of evidence from a number of scientific fields." (Emphasis in the original.) That is, the messages from neuroscience are entirely consistent with and supportive of what we have learned from developmental psychology, cognitive psychology, and other areas of research.

There is one sense in which learning mathematics is different from learning many other things, such as speaking our native language, remembering visual and aural images of familiar people and places, and driving a car. The first and most fundamental biological fact about our brains is that they have not evolved significantly from the brains of our hunter-gatherer ancestors. Thus, we are superbly adapted – or would be if it were not for environmental influences – for fight-or-flight decisions and other survival tactics. As Dehaene (1997) has so beautifully documented in *The Number Sense*, this means that humans (and other species as well) are practically hard-wired to do arithmetic with small integers – but everything else in mathematics is *hard*, because it doesn't come to us instinctively. On the other hand, we learn many things that are not instinctive in an evolutionary sense, such as history, philosophy, foreign languages (beyond infancy), music, and neurobiology. One might say the Education is about *learning the things that hard to learn* – of which mathematics is just one example. [Exercise for the reader: Why is "driving a car" – clearly not an evolutionary adaptation – a relatively easy task for adolescents and adults in a developed society?]

We summarize here some of the key findings from the NRC study (Bransford, *et al.*, 1999, pp. xii-xviii) that are relevant to collegiate education, in particular, to undergraduate mathematics.

- **Collateral Development of Mind and Brain**
  - "Learning changes the physical structure of the brain."
  - "Structural changes alter the functional organization of the brain, [i.e.], learning organizes and reorganizes the brain."
  - "Different parts of the brain may be ready to learn at different times."
- **Durability of Learning and Ability to Transfer to New Situations**
  - "Skills and knowledge must be extended beyond the narrow contexts in which they are first learned."

- "…a learner [must] develop a sense of *when* what has been learned can be used …. Failure to transfer is often due to … lack of … conditional knowledge."
- "Learning must be guided by general principles …. Knowledge learned at the level of rote memory rarely transfers …."
- "Learners are helped in their independent learning attempts if they have conceptual knowledge. …"
- "Learners are most successful if they are mindful of themselves as learners and thinkers. … self-awareness and appraisal strategies keep learning on target … . … this is how human beings become life-long learners."

♦ **Expert vs. Novice Performance**

- "Experts notice … patterns … that are not noticed by novices."
- "Experts have … [organized] content knowledge …, and their organization … reflects a deep understanding of the subject matter."
- "Experts' knowledge cannot be reduced to sets of isolated facts … but, instead, reflects contexts of applicability …."
- "Experts have varying levels of flexibility in their approaches to new situations."
- "Though experts know their disciplines thoroughly, this does not guarantee that they are able to instruct others …."

♦ **Designs for Learning Environments**

- "*Learner-centered environments* … Effective instruction begins with what learners bring to the setting … learners use their current knowledge to construct new knowledge … what they know and believe at the moment affects how they interpret new information … Sometimes learners' current knowledge supports new learning; sometimes it hampers learning."
- "*Knowledge-centered environments* The ability to think and solve problems requires knowledge that is accessible and applied appropriately. … Curricula that are a 'mile wide and an inch deep' run the risk of developing disconnected rather than connected knowledge."
- "*Assessment to support learning* … Assessments must reflect the learning goals …. If the goal is to enhance understanding and applicability of knowledge, it is not sufficient to provide assessments that focus primarily on memory for facts and formulas."
- "*Community-centered environments* [An] important perspective on learning environments is the degree to which they promote a sense of community. …"

♦ **Effective Teaching**

- "Effective teachers need 'pedagogical content knowledge' – knowledge about how to teach in [the] particular [discipline], which is different from knowledge of general teaching methods."
- "Expert teachers know the structure of their disciplines and [have] cognitive roadmaps that guide the assignments they give …, the assessments they use …, and the questions they ask in the … classroom …."

♦ **New Technologies**

- "Because many new technologies are interactive, it is now easier to create environments in which students can learn by doing, receive feedback, and continually refine their understanding and build new knowledge."
- "Technologies can help people visualize difficult-to-understand concepts …."

- "New technologies provide access to a vast array of information, including digital libraries, real-world data for analysis, and connections to other people who provide information, feedback, and inspiration, all of which can enhance the learning of teachers and administrators as well as students."
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# 3. Technology

There has been a great deal of controversy over the past two decades about the presumed effects, good and bad, of using technological tools (calculators and computers) in teaching and learning mathematics. The debate is beginning to be informed by a substantial and growing body of research, which one hopes in time will replace strident assertions of deeply held opinions. The NRC report cited in the preceding section highlights the positive features, particularly of interactive technologies, for learning in general. A forthcoming volume (Heid and Blume, to appear) surveys research on the role of technology in teaching and learning mathematics at all levels. As a co-author of one of the chapters in that volume (Tall, *et al*., to appear), I have had an opportunity to learn more about this research as it relates to college-level mathematics. Our paper includes an analysis of a large number of recent research papers and Ph.D. theses in mathematics education that focus on technology in calculus and related subjects. In simplified form, the key messages are

1. *Technology used inappropriately makes no significant difference.* In particular, adding calculators and/or computers to a traditionally taught and assessed mathematics course may make it marginally better or worse, but there won't be much change. "Better" is likely to be associated with students finding ways to use the technology that are not necessarily planned by the instructor. "Worse" is likely to be associated with time and effort devoted to yet another task, particularly if it is seen as disconnected from all the others.
2. *Technology integrated intelligently with curriculum and pedagogy produces measurable learning gains.* It may be impossible to tease out whether the gains are the direct result of the technology or of the rethought curriculum and pedagogy. (Do it matter?)
3. *There is little evidence that one technology is "better" than another*. What matters is how the technology is used.
4. *There is substantial evidence that using computer algebra systems for conceptual exploration and for learning how to instruct the software to carry out symbolic calculations leads to conceptual gains in solving problems that can transfer to later courses.* In comparison, students in traditional courses tend to use more procedural solution processes that do not easily transfer to new situations.
5. *Technology enables some types of learning activities* (e.g., discovery learning) *and facilitates some others* (e.g., cooperative learning) *that are harder or impossible to achieve without technology*.

These results are completely consistent with what is known about learning in general – which reinforces my point that learning mathematics is, first of all, *learning*, and only secondarily about mathematics.

One of the more interesting points in the research on technology in mathematics courses is the role of the teacher in influencing the outcome. Keller and Hirsch (1998) found that students' preferences for numeric, graphic, or symbolic representations reflect in part the teacher's preference. Kendal and Stacey (1999) studied three teachers who taught the same calculus syllabus using TI-92 symbolic calculators. Teacher A enthusiastically used the computer algebra system at every opportunity, while Teacher B was more reserved and underpinned the work with paper-and-pencil calculations. Teacher C was enthusiastic about the graphing abilities of the calculator and used it more often for graphical insight than for symbolic calculation. The three teachers also had different predictions about their students' algebraic competence, geometric competence, and likelihood of success while using the technology. Mean scores on the common end-of-course assessment were essentially the same for the three sections, but students in each of the sections were successful on different questions, more or less in accord with their teacher's expectations and privileging of specific uses of the technology.

# 4. Curriculum

What do we really want to teach, and why do we want to teach it? Are the important topics in mathematics essentially unchanged over time, or should the curriculum be viewed as something like a living organism – perhaps as a species of organisms, with births, deaths, evolution?

Whenever I think about these questions, I am reminded of our sister sciences, for which the answers are much more obvious. For example, when I was a student, continental drift was considered a heretical theory – not just wrong but wrong-headed, not worth serious scientific discussion. One could easily list several dozen significant paradigm shifts in science over the past 50 years, most of which have been reflected in science curricula at some level.

Over the same period of time, mathematical knowledge has literally exploded, both in its pure sense and in its relationship to science and technology. And yet we tend to think of the academic content of our discipline (at least K-14) as essentially static. We know better, of course. When I was a student, the list of important skills (necessarily paper-and-pencil skills, except for occasional use of a slide rule) included calculation of square roots, interpolating in trig and log tables, and polar and logarithmic graphing, along with others that subsequently disappeared from the "standard" curriculum. It is very rare now to encounter a student who has ever calculated a nontrivial square root by hand or who has ever seen a log table or a slide rule (never mind knowing what to do with them). The non-Cartesian graphing techniques disappeared because the presumed benefits were not commensurate with the intellectual demands of learning how to do them (not to mention the cost of special graphing paper). But now those techniques are back in our curricula (or should be), because they have important conceptual content and modeling significance, and because our modern technology makes them easy, cheap, and accessible to all.

So why do some of our colleagues continue to insist on advanced factoring techniques as a prerequisite skill for calculus, when the original reason they were in the curriculum was to be able to solve carefully contrived max/min problems? And why do we assume that essentially all of single-variable calculus is a prerequisite for differential equations – or that the really important techniques in differential equations are the purely symbolic ones? Any problem that has been reduced to a button on an omnipresent calculator – such as square root, log function, max/min, or graphical-numerical solution of a differential equation – can no longer be considered a difficult or inaccessible problem. Now that many of these formerly difficult problems have been rendered easy, we have to confront the fact that solving the problems does not imply understanding of the conceptual content.

Much of our profession continues to resist research-based calls for curricular (and other) changes, such as the NCTM *Principles and Standards* (NCTM, 2000). The current *Standards* are themselves the product of extensive debate, development of curricular materials, trial, research, and revision since publication of the predecessor document in 1989. And yet many academic mathematicians cannot conceive of a successful secondary curriculum that is not organized by presumed precursor topics for calculus, organized into courses titled Algebra I, Geometry, Algebra II, Trigonometry (perhaps in combination with, say, Analytic Geometry), and Precalculus.

The calculus reform initiative in the U.S. (see Roberts, 1996, Ganter, 2000) has more or less coincided with NCTM efforts to reform school curricula and has been the driving force in reform of collegiate curricula at all levels. Successes and failures of this initiative have to be viewed against the backdrop of an established system in which the table of contents of a textbook was seen as a complete description of a course. Thus, among the early "reformers" were some who saw their task as grafting technology onto an unchanged (unchangeable?) syllabus. (We have already noted in the preceding section the failure of these efforts to produce significant learning gains.) Others saw their task as creating the next best-selling calculus textbook – or, in some cases, grudgingly accepted commercial publication of a textbook as the primary means of dissemination of their good ideas for reform. Only a relative handful of these curricular efforts ever made it to commercial publication, and, for a number of reasons, only one (Hughes Hallett, *et al*., 2001) was ever a true commercial success. Each subsequent edition of this work looks more "traditional" but still retains the creative problems and other tasks that set it apart from a traditional text. Meanwhile, the commercially successful traditional calculus books are taking on a more "reformed" appearance without a significant change in real content or approach.

Over the next few years, and perhaps beyond, we will see growing use of the World Wide Web for dissemination of innovative curricular materials, both commercial and free (or grant-supported), bypassing the traditional publishers and enabling direct access to interactive materials that cannot reasonably be reduced to print. One example of this is the Web publisher Math Everywhere, Inc. (http://matheverywhere.com/), an enterprise created by Bill Davis and colleagues to market interactive courseware, including *Calculus & Mathematica*® (1994), one of the most successful products of the calculus reform initiative. By "successful," I do not mean in the commercial sense – it's not clear to an outside observer that Addison-Wesley's marketing attempts were ever successful. On the other hand, a number of the research studies cited by Tall, *et al*. (to appear) compared *C&M* to traditional courses and found significant learning gains for the *C&M* students. In addition to the "classic" *C&M*, the MEI Web site now offers a range of similar courses, in various stages of maturity, addressing much of the lower-division college curriculum.

The *Connected Curriculum Project* (http://www.math.duke.edu/education/ccp/), in which I am a principal, is an example of free distribution (supported by a National Science Foundation grant) of materials that grew out of an earlier calculus reform project (Smith and Moore, 1996), another commercial failure for which the research studies generally showed significant learning gains. The *CCP* materials are not entire courses – rather they are modular, interactive units that lead students through important concepts and applications throughout the lower-division curriculum.

There are a number of free Web sites offering peer-reviewed college-level curriculum materials in a variety of disciplines, including mathematics. Among these are the *Mathematical Sciences Digital Library* (MathDL, http://www.mathdl.org/), *MERLOT* (http://www.merlot.org/), and *iLumina* (www.ilumina-dlib.org/). I am affiliated with the first of these – an NSF-funded project of the Mathematical Association of America – as Editor of the *Journal of Online Mathematics and its Applications* (JOMA, http://www.joma.org/). JOMA is a peer-reviewed academic journal that includes, among other things, high-quality, innovative, and class-tested curricular materials, as well as user and research articles about these materials.

## 5. Pedagogy

The NRC study (Bransford, *et al*., 1999), while extensive, does not encompass all of the important research threads in the study of higher education. For example, researchers in Scotland, Australia, and Sweden (Entwistle and Ramsden, 1983; Entwistle, 1987; Ramsden, 1992; Bowden and Marton, 1998) have studied student *approaches* to learning, with a focus on approaches that lead to deep vs. surface learning. (See also Rhem, 1995.) Deep learning approaches are quite different from surface learning approaches, and a given student – whatever his or her "learning style" – may exhibit different approaches simultaneously in different courses. These student-selected "coping strategies" are often influenced by expectations set by the instructor, consciously or unconsciously.

In particular, surface learning is encouraged by

- excessive amounts of material to be covered,
- lack of opportunity to pursue subjects in depth,
- lack of choice over subjects and/or method of study, and
- a threatening assessment system.

On the other hand, deep learning – the organized and conceptual learning described in the NRC study – is encouraged by

- interaction with peers, especially working in groups,
- a well-structured knowledge base with connections of new concepts to prior experience and knowledge,
- a strong motivational context, with a choice of control and a sense of ownership, and
- learner activity followed by faculty connecting the activity to the abstract concept.

These are especially important aspects of pedagogy for those of us whose goals include teaching mathematics to a much broader audience than just those who intend to replace us as mathematicians. Notice in particular, the similarity of the "surface" list to the way many mathematics courses are taught in many colleges and universities – with results that are almost universally considered unacceptable. And notice also that the "deep" list comprises principles that have been incorporated into all of the major "reform" efforts of the past 15 years or so.

Much of the reform has been carried out with scant or no knowledge of research – in some cases, even as the relevant research was under way. However, it is no accident that the strategies we found empirically to be effective are the same as those that have been shown by research to be effective. Perhaps the most significant aspect of the reform efforts has been the near-universal realization that revision of curricula is not enough, that decisions about topics are not enough, that inclusion of technology is not enough – that none of this matters unless our pedagogical strategies are also effective.

## 6. Putting it All Together: Research, Technology, Curriculum, Pedagogy

In a recent paper (Smith, 2001) I wrote about the Web-supported classroom environment in which I have taught for the past three years. The courses I teach now are the product of what I have learned over the past two decades about research on learning (in neurobiology, cognitive psychology, and empirical educational studies), supported by modern computer technology, carefully designed curricular and assessment materials, and active-learning strategies in and out of the classroom. My students and I benefit from Duke University's commitment to quality education in the form of an Interactive Computer Classroom, Web delivery support via Blackboard 5.5, an extensive array of site-licensed software, and excellent staff support. Unfortunately, one of the disadvantages of committing a classroom or course description to paper is that it quickly goes out of date, expecially if Web resources are involved. There is an online version of my 2001 paper at http://www.math.duke.edu/~das/essays/classroom/ in which I have kept the links to classroom and course resources current.

Key features of my courses include

- articulated goals and assessments directed toward achieving the goals;
- a goal-setting exercise at the start of each term to give students a sense of common purpose and joint ownership;
- weekly plans that spell out the objectives, activities, readings, and problem assignments;
- a carefully cultivated sense of *community* in which students see each other and me as partners in their learning enterprise, not as competitors or adversaries;
- an online discussion board, plus easy access to e-mail for all course participants, to facilitate the sense of community;
- a mix of in-class activities – lecture supported by online interactive "notes" in a computer algebra file, informal group activities in teams of two to four (with or without use of a computer), structured lab activities using *Connected Curriculum Project* materials, and online use of resources from remote sites;
- challenging take-home open-book tests with all resources available;
- regular homework graded assignments on a weekly cycle, with a requirement that all submitted solutions be accompanied by a check and/or a correctness argument;
- campus-wide access to a computer algebra system (currently *Maple*® 7);
- use of every learning task as an assessment (formal or informal) for which feedback is given, and conversely, use of every assessment as a learning opportunity;
- a non-threatening distributed grading system among a range of different activities, roughly half with group grades and half with individual grades;
- a weekly electronic journal submission with a paragraph or two of reflection on the week's work;
- team projects with classroom presentation and multiple-submission papers;
- Web delivery of all important course documents and online submission of most student work;
- emphasis on realistic or real-world problems that are meaningful to students on their own terms and that serve as motivators and scaffolding for the mathematical concepts

Without my belaboring the point, the reader should find many points of contact between this list of strategies and the research findings cited earlier.

To illustrate the construction and use of research-based materials, I will give one example of a module (Moore, *et al.*, 2001) that I use early in a multivariable calculus course. This module could be used with any students who have had some exposure to polar coordinates, parametric representations, logarithmic graphing, and the relationship between tangent lines and derivatives.

The module, which may be seen at the URL given in the References, starts with a background page on spirals in nature, in particular, the spiral shell of the chambered nautilus (*N. pompilius*). This page is linked to other sites for information about Aristotle, who studied gnomonic growth, and D'Arcy Thompson, author of the 1917 classic *On Growth and Form*, from which some of the content of the module is taken. There are also links to other sites with information about spirals in nature (seed patterns, nebulae, etc.) or related mathematical topics (Fibonacci numbers, evolutes of curves, etc.). My observation has been that students seldom follow any of these links – that they may do no more with the background page than look at the pictures, because it doesn't appear to contribute anything to completion of their assignment. However, part of the richness of the Web is that one can provide alternate learning paths for those who choose to take them – and without interfering with those who want to follow a straight line toward a specific goal.

The "business" of the module starts on the next page, where students are shown an enlarged cross-section of the nautilus shell superimposed on a polar grid and are challenged to reproduce the spiral shape. Their first step is to make a list of radial measurements (with a ruler), either on the screen or on a printed version of the picture. Thus we start with a tactile activity that leads to student ownership of the data from which the model will be derived. Students then test their data by logarithmic plotting for an exponential growth pattern, from which they can then derive a polar formula, $r = f(\theta) = Ae^{k\theta}$, and immediately test their model to see if the polar graph fits the data. They don't have to ask anyone "Is this right?" – they see immediately if they have made a mistake, and they have to get the formula right before they can move on.

On the next page, students link polar plotting to parametric plotting via the polar-to-Cartesian change-of-coordinate formulas and plot their spiral again in rectangular coordinates. They also use this representation to zoom in at the origin and discover the self-similarity of the exponential spiral – a rather different result from the local linearity they usually associate with "zooming in."

Finally, students use the power of the computer algebra system (CAS) to explain the name "equiangular" – that is, to show that the angle between radius vector and tangent line is constant. This calculation involves calculus and algebra steps that only a few students would complete successfully with pencil and paper. With the CAS, almost everyone can complete the calculation and at the same time keep their focus on the mathematical concepts involved.

At the end of the lab activity, each student team completes their CAS-based report by answering the following summary questions:

1. Describe in general terms the process of finding a polar formula from the radial measurements on a seashell picture.
2. What happens when you zoom in at the center of an equiangular spiral? The behavior you observed is called **self-similarity**. Explain the name.
3. What remains constant as $\boldsymbol{r}$ grows in an equiangular spiral?
4. Describe in geometric terms why the equiangular spiral has the name it has.
5. What is the shape of an equiangular spiral with $\boldsymbol{\beta = \pi/2}$? How is this reflected in the formula for $\boldsymbol{r}$ as a function of $\boldsymbol{\theta}$? How is it reflected in the relationship between $\boldsymbol{\beta}$ and $\boldsymbol{k}$?

The last question asks about a case not previously encountered in the module – that in which the "equiangle" β is a right angle and the "spiral" is a circle. Since the relationship they have found is $\tan \beta = 1 / k$, they have make sense of this formula when the left-hand side is $\infty$.

This module illustrates design that takes students through at least one complete Kolb learning cycle (see Wolfe and Kolb, 1984):

- Concrete experience: input to the sensory cortex of the brain in the form of seeing, touching, moving – e.g., taking measurements;
- Reflection and observation: mainly right-brain activity, reinforced by use of previous learning – e.g., logarithmic plotting);
- Abstract conceptualization: left-brain activity – e.g., finding the polar exponential growth formula;

- Active experimentation: often involves the motor brain, sometimes the sensory cortex as well – e.g., testing the conceptual model against the reality of the data.

If the testing phase does not show complete success, the cycle may start over with the same problem, now being viewed from a slightly enhanced knowledge base – at least with the knowledge that something they thought would work in fact did not. When students achieve success at one experimentation point, they are ready to move on to the next learning cycle.

This example links Kolb's research on experiential learning to the neurobiological evidence that *deep learning is whole-brain activity* (see e.g., Rhem, 1995, Zull, 1998).

# 7. Conclusions

Research studies on learning in general and on learning mathematics in particular (with or without technology), together with my teaching and development experiences of the last two decades, lead me to several conclusions:

1. Curricula need to be rethought periodically from the ground up, taking into consideration the tools that are available. It is not enough to think of clever ways to present mathematics as the content was understood in the mid-20th century, when the available tool set was quite different, as was the intended audience.
2. Much of the effort that goes into curriculum design can be squandered if one does not also rethink pedagogical strategies in the light of research showing the effectiveness of active-learning strategies and distinguishing between good and bad ways to stimulate deep learning approaches. It is not enough to adopt (or write) a new book or even a new book-plus-software package.
3. Our tools for assessing student learning – whether for purposes of assigning grades or for evaluating effectiveness of our curricula – need to be consistent with stated goals for each course and with the learning environments in which we expect students to function. It is not enough to continue giving timed, memory-based, multiple-choice, no-tech examinations.
4. If we are serious about mathematical understanding for everyone with a "need to know" – not just the potential replacements for the mathematics faculty – then we must plan our curricula, pedagogy, and assessments for effective learning of the skill sets and mental disciplines that will be needed by a mathematically and technologically literate public in the 21st century. It is not enough to keep using ourselves as "model learners."
5. Revision of curricula, pedagogy, assessment tools, and technology tools will accomplish little without concurrent professional development to keep faculty up to date with the required skills, knowledge, attitudes, and beliefs. It is not enough to continue acting as though an advanced degree in mathematics is evidence of adequate preparation to teach.

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