DIFFERENTIAL CALCULUS OF SEVERAL VARIABLES WITH MATHEMATICA OR MAPLE

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ABSTRACT

The use of Computer Algebra Systems (CAS) in Mathematical Laboratories at Engineering Schools is increasing. Therefore, it has become necessary to undertake a new educational approach in the teaching of many mathematical topics. Obviously it is not possible to give the same lectures as in the 1950,s. The CAS allows us to experiment and teach in "a different way". The students' mathematical knowledge at the beginning of their studies in the University is completely different. Moreover we have fewer credits assigned to Mathematical topics in engineering curriculum.

The objective of he authors, professors of different Spanish Universities, is to try to demonstrate a new way to use the CAS, concretely *Maple* and *Mathematica*, as a support in the teaching of Differential Calculus of Several Variables. They take advantage of the computer's graphical capacities to sketch the graphs of surfaces defined in explicit, parametric or implicit form. They survey the general topics: limits, partial derivatives, differentiability, chain rule, implicit functions and extreme values. For each topic, they give a few basic instructions of the CAS used; afterwards they give several examples, in increasing order of difficulty, and finally the students must solve exercises and try to establish a general procedure for a quite general problem.

In this paper we have included several examples used in this survey.

KEY WORDS: Computer Algebra Systems, Mathematical Education, Differential Calculus.

1. Introduction

Over the past 20 years Spain has seen the advent of CASs (Computer Algebra Systems) as a teaching complement for the mathematical training of engineering students. In general, however, this has not been accompanied by necessary changes in teaching methods to obtain optimum benefits from this tool. What usually happens is that the teacher merely adds to traditional teaching contents the required instructions to do with the computer exactly the same as can be done with a paper and a pencil. This leads to an overload of work, using the CAS in a mechanical fashion but not for actually teaching mathematics. The aim of CASs (that is, to offer a tool for doing and learning mathematics) is replaced thus by a situation in which students become experts in the CAS without understanding the mathematical issues being dealt with.

On the other hand, it is much more common to find CASs being used in courses on Elementary Calculus than for the teaching of Calculus of Several Variables.

In this paper we propose a more integrated and harmonious use of a CAS (*Maple*, *Mathematica* or any other) within daily teaching practice. We have honed this down to show the possibilities of using CASs for giving a course in Differential Calculus of Several Variables. The proposal is the result of our experience as math teachers in engineering schools in different Spanish Universities, where either *Mathematica* or *Maple* are used.

2. Our starting point

In Spanish Engineering Schools, Differential Calculus of Several Variables is generally taught after a single semester course on Elementary Calculus. The usual topics in this area are: Basic Notions of R^2 and R^3, Limits, Directional and Partial Derivatives, Homogeneous Functions, Chain Rule, Implicit Differentiation, Inverse Function, Taylor's Formula, Maxima and Minima, Lagrange Multipliers.

Usually there are 30 hours of class (including theoretical and practical ones) for teaching these contents and it is assumed that the personal work of each student will match the same number of hours.

The aim sought is that our students should be able to gain a reasonable mastery of the concepts, to acquire basic manual calculation skills, for later use in mathematical issues or in applications in subjects specific to engineering, and moreover to know when and how to use a CAS for solving a mathematical problem.

Finally, it should be stressed that our students have a basic initial knowledge of the CAS to be used (*Mathematica* or *Maple*, depending on the University).

Therefore, in view of the manifest lack of teaching time, if the above aims are to be achieved this overall picture requires a new methodological focus.

3. Our teaching material

To carry through our methodological proposal, we have compiled teaching material for a standard course on Calculus of Several Variables. Concretely this material consists of a text book that systematically includes theoretical concepts and solved and proposed problems (see García, A., López, A., Romero, S., Rodríguez, G., de la Villa, A., 1996), together with a CD that includes problems solved with the help of a CAS.

Here we describe the material featured in the CD; in particular, that relative to Differential Calculus of Several Variables, and at the same time explain the strategy followed to compile it.

We note that the CD includes two alternative directories with parallel contents: one elaborated with *Maple* and the other with *Mathematica*. Accordingly, the examples we offer here may be in one system or the other.

This teaching material gives exhaustive coverage to all the above issues and allows us a differentiated use by the teacher as a function of the particular needs of each case. Based on the teacher's own criteria, the material can be used as class material or as a guide for the personal work of the students, attending to the students' own characteristics and the time available for developing the different topics.

We believe that our work should help to palliate the scarcity of material for the teaching of Calculus of Several Variables, at least in Spanish, as compared with the vast amount of material currently available for the teaching of Calculus of a Single Variable.

4. Our methodological proposal

The methodological proposal is orchestrated on the basis of the integration of traditional teaching, lectures and problem classes (with the introduction of different concepts and blackboard exercises), with laboratory classes in which the corresponding CAS is used to reinforce the theoretical concepts and solve problems.

We offer a methodological proposal that involves the students dedicating approximately one third of the total work time programmed for the subject (both as regards class hours and personal work) to guided use of the CAS, following the teacher's instructions and pursuing the following aims:

4.1. Improving visualisation

The possibility to sketch the graph of a surface immediately gives us additional information in the study of a function of two variables, making conjectures about the properties of the function, extremes, bounds, etc. For example figures 1 and 2, both obtained with an elementary *Maple* input, help to understand concepts such as non-differentiability and relative extremes.

The graphical capacities can also be used by the teacher to offer presentations that will help to introduce different mathematical concepts. However, it should be noted that many instructions are often required to obtain "good" presentations of graphics and the teacher must take into account the cost of performing these.

4.2. Experimentation

The possibility of performing long tedious mathematical calculations very quickly enables the introduction of new strategies for problem solving, contrasting numerical, analytical and graphical techniques.

Thus, for example, with a CAS it is easy to know the variables that can be chosen as dependent in the implicit function theorem. To do so, assuming that the rest of the hypotheses of the theorem are satisfied, it is only necessary to check whether the determinant of the Jacobian matrix is null or not. So, even if the system of implicit functions is difficult, the problem is reduced to an automatic calculation.

Also, in order to get a first impression about the existence of a limit of a function it is possible to sketch a graph or construct tables of values for the function, near the limit. For example, to study the existence of the limit of the function $f(x, y) = \dfrac{xy}{x^2 + y^2}$ at the origin of the coordinates,

the Mathematica input

```
TableForm[Table[f[a + j h, b + k h], {j, 1, n}, {k, 1, n}]]
```
with a=b=0, h=0.001, k=0.001, n=5 generates Table 1. The dispersion of the data in this table leads one to suspect that the limit does not exist, which can be proved by using the technique of findings two subsets for which the limits will be different, for example, $y = x$ and $y = 2x$.

As part of the teaching strategy, when faced with any problem we should encourage students to experiment in a general sense. A crucial aspect is to avoid laziness in the use of different trials since the mechanical processes are performed by the computer.

4.3. Release from mechanical work

We should not overlook the need for students to acquire basic computational skills. Once it has been determined that the students have assimilated the concepts and that they are able to do "operations" manually in cases of calculus that are not too complicated, they can then use CASs as "advanced calculators" for the more tedious calculations. Thus, for example, in the Chain Rule the Jacobian matrix of the function g∘f is the product of the Jacobian matrices of g and f. When the component functions of f and g have complicated expressions, the Jacobian matrix of the function g∘f, with $f:\Re^5 \rightarrow \Re^6$ and $g:\Re^6 \rightarrow \Re^4$ demands a huge "manual" effort, which can be done automatically with a CAS.

We believe that CASs, understood as "advanced calculators", can be used in most issues topics of differential calculus.

The usefulness of such tools in intermediate calculations, relative to concepts that have already been analysed and that are necessary for later stages of learning, is also unquestionable. For example, the hypotheses of the existence of an inverse function can be analysed using a CAS, since the calculations involved in checking such hypotheses are routine. We therefore see that the use of a CAS allows to save time dedicated to routine calculations, and can be used to better model problems involving real situations and to interpret, at each step of the process, the results obtained.

As an example, let us consider the following problem (taken from García, A., López, A., Romero, S., Rodríguez, G., de la Villa, A., 1996):

The temperature of a plate at any point (x,y) is given by the function $T(x,y) = 25 + 4x^2 - 4xy+y^2$. A heat alarm, situated at points on the circumference $x^2 + y^2 = 25$ is triggered at temperatures higher than 180ºC or below 20ºC. Will the alarm be triggered?

To solve this, the students need to optimise T(x,y) subject to the constraint $x^2 + y^2 = 25$. Then we can use the method of Lagrange multipliers. Calculations can be done by computer and the points $\{-2\sqrt{5}, \sqrt{5}\}$, $\{2\sqrt{5}, -\sqrt{5}\}$, $\{-\sqrt{5}, -2\sqrt{5}\}$, $\{\sqrt{5}, 2\sqrt{5}\}$ are obtained as possible solutions. Then the students must evaluate the function T in those points to find the extreme values and to interpret the result. Since the maximum of these values is lower than 180 and the minimum higher than 20 the alarm will not be triggered.

Moreover, symbolic capacities of a CAS allow us to work efficiently with formal expressions, like those related to the properties between operators. For example it is trivial, using Mathematica, to prove $div(grad\,f) = \mathbf{D}\,f$. After loading the package **Calculus`VectorAnalysis`** the input

```
Div[Grad[f[x,y,z],Cartesian[x,y,z]]]-
Laplacian[f[x,y,z],Cartesian[x,y,z]]
```
produces the output 0.

We also can prove general properties in real problems as it is shown in the following example (taken from Marsden, J. and Weinstein, A., 1985):

The specific volume V, pressure P, and temperature T of a Van der Waals gas are related by the equation $P = \dfrac{RT}{V - b} + \dfrac{a}{V^2}$, where a, b, R are positive parameters. R is the universal constant

of the gases, b represents the volume of the gas molecules in liquid state (b<<V) and $\dfrac{a}{V^2}$

represents the inner pressure due to the interaction between molecules.

The students, using implicit function theorem, can prove that any two variables of V, P, or T can be considered independent. The students can also find $\dfrac{\partial}{\partial P}T$, $\dfrac{\partial}{\partial V}P$, $\dfrac{\partial}{\partial T}V$ and verify that

$$\left(\frac{\partial}{\partial P}T\right)\left(\frac{\partial}{\partial V}P\right)\left(\frac{\partial}{\partial T}V\right)=-1.$$

Here the CAS has been used for verifying the hypothesis of the implicit function theorem and for symbolic computations.

We can ask them for a generalisation for this formula when F is a function with n variables.

4.4. Distinction between algorithmic and non-algorithmic processes

The use of CASs is not a panacea that will allow the solution of any mathematical problem. In this sense, it is important to stress that students should distinguish between processes that are algorithmic and those that are not.

When a process can be "algorithmized", students should be encouraged to perform a simple procedure consisting of translating the mathematical process to the language of the CAS used.

Within algorithmic processes, it is appropriate to distinguish between two types:

● Algorithms with an ensured answer, in which information is always obtained.

For example, it may be very simple for a student to perform a Maple procedure that will return the tangent plane at the surface, implicitly defined by the equation F(x,y,z)=0, at a point (a,b,c). It suffices to know the definition and write the instructions for finding the gradient vector and the equation of the tangent plane. A basic form of this procedure could be as follows:

```
> Tang_Plane:=proc(F,a,b,c)
  local gr;
  gr:=subs({x=a,y=b,z=c}, linalg[grad](F(x,y,z),[x,y,z]));
  simplify(gr[1]*(x-a)+gr[2]*(y-b)+gr[3]*(z-c)=0)
  end:
```

To use this procedure in later problems, the student must previously verify that the hypotheses of the Implicit Function Theorem are satisfied. The procedure can also be improved by including, inside it, the instructions necessary to see if such hypotheses are indeed satisfied and ensuring that an error message will appear if they are not. Also, it is possible to add a graphical instruction to draw the surface, together with the tangent plane. Students generally try to improve the procedure until it is as complete as possible.

● Algorithms that may include computational problems.

In this type of algorithms, the answer is conditioned to intermediate processes that cannot be performed owing to difficulties such as the impossibility of solving an equation, excessive systems requirements, etc.

Thus, for example, for the calculation of constrained extremes, the Lagrange multiplier algorithm requires the solution of a set of equations, which is not always guaranteed.

There are also non-algorithmic processes, for which interactive use of the CAS must be controlled. For example, finding limits is a non-algorithmic process.

It is possible to design alternatives using negative criteria such as reiterated limits or some positive criterion, such as the change to polar coordinates, which must be used with care (see next section). Thus, the function f(x,y), after the changes $x = a + r\cos\phi$, $y = b + r\sin\phi$, is converted into the

function $F(r, \phi)$. If this function has a uniform limit in the variable ϕ when r→0, then there exists limit of f(x, y) at the point (a, b).

4.5. Fostering a critical spirit

It is clear that a CAS will enormously facilitate the task of checking results, since alternative methods can be easily used. Students should not develop a blind faith in the computer results, either intermediate or final; instead, they should at all times attempt to be in control of the situation in the sense that the results should be compatible with the context of the problem that is being solved in each case. It should be noted that sloppy use of CASs may give rise to errors with which the student is not very familiar, such as the errors due to previous assignations in the work session.

Additionally, the limitations shown by CASs can help us to foster students' critical awareness, presenting situations of calculations in which the answer of the CAS is unexpected.

For example, the expected output of the Maple instruction

$>$`mtaylor(sin(x*y)/(x*y), [x=0, y=0],10);`

should be $1 - \dfrac{x^2 y^2}{6} + \dfrac{x^4 y^4}{120}$ but the computer output is $1 - \dfrac{x^2 y^2}{6}$.

Morever a CAS can obtain a mistaken result due to program failure or due to the consideration of conditions on variables that the user has not taken into account (see Alonso, F., García, F., Hoya, S., Rodríguez, G., de la Villa, A., 2001).

Also, the little care that students sometimes take in formal calculus may be aggravated by the use of a CAS.

Thus, if to calculate the limit at the point (1,1) of the function

$$f(x, y) = \frac{3x - 3x^2 + x^3 - 4y + 6y^2 - 4y^3 + y^4}{-2 + 3x - 3x^2 + x^3 + 4y - 6y^2 + 4y^3 - y^4}$$

we do the change to polar coordinates (see 4.4), where $a=b=1$, the resulting expression is

$$F(r, q) = \frac{\cos(q)^3 + r \sin(q)^4}{\cos(q)^3 - r \sin(q)^4}.$$

Mathematica and Maple simplify to 1 the limit of this expression when r→ 0. Apparently, the limit does not depend on θ, but it is false, since if θ=π/2, then the limit is –1. Therefore, the double limit does not exist.

Students must see the need to control the results.

5. Conclusions

We would like to propose a way of using mathematical software as a pedagogical tool. The graphical capacities of software packages can help to understand some concepts related to Differential Calculus of Several Variables.

We also show another possibilities of use: release of the mechanical work, possibility of experimentation, distinction between algorithmic and non-algorithmic processes and mainly to always have a critical spirit about the obtained results. Let us remark on the importance of fostering a change in the mentality of the teachers teaching theses kinds of topics.

We suggest that the software package chosen should not be as important as the way in which it is used; the same ideas and strategies with respect to software in education can be put into effect using different packages.

The possibilities of using new technologies are not limited to "witnessed" teaching (i.e., with the physical presence of the students); the teaching material compiled can be used in "virtual"

teaching since there are no special problems involved in presenting the material online. In any case, this material can be completed with tutorials designed to facilitate its use.

REFERENCES

- Abell, M.L., Braselton, J.P., 1999, *Maple V. By Example*, Academic Press.
- Alonso, F., García, A., García, F., Hoya, S., Rodríguez, G., de la Villa, A , 2001, "Some unexpected results using Computer Algebra Systems ", *The International Journal of Computer Algebra in Mathematics Education*, **8**, 239-252.
- Cheung, C.K., Keough, G.E., Murdoch, T., 1996, *Exploring Multivariable Calculus with Mathematica,* John Wiley & Sons Inc.
- Coombes, K.R., Lipsman, R.L., Rosenberg, J.M., 1998, *Multivariable Calculus and Mathematica,* Springer-Verlag.
- García, A., López, A., Romero, S., Rodríguez, G., de la Villa, A., 1996, *Cálculo II. Teoría y problemas de funciones de varias variables*. Clagsa.
- Franco, A., Franco, P., García, A., García, F., González, F.J., Hoya, S., Rodríguez, G., de la Villa, A , 2000, "Learning Calculus of Several Variables with New Technologies", *The International Journal of Computer Algebra in Mathematics Education*, **7**, 295-309.
- Marsden, J. And Weinstein, A., 1985, *Calculus III*. Springer-Verlag.
- Rincón, F., García, A., Martínez, A., 1995, *Cálculo científico con Maple*. Ra-ma.
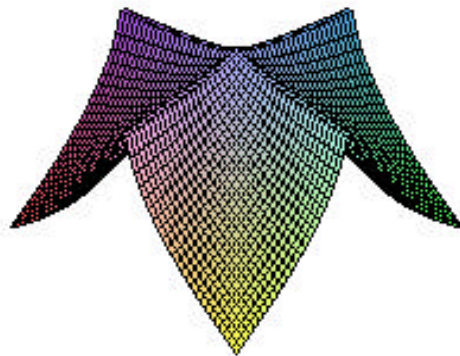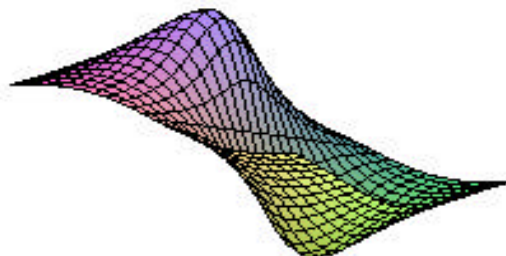
FIGURES AND TABLES

Figure 1

Figure 2

0.5	0.4	0.3	0.235294	0.192308
0.4	0.5	0.461538	0.4	0.344828
0.3	0.461538	0.5	0.48	0.441176
0.235294	0.4	0.48	0.5	0.487805
0.192308	0.344828	0.441176	0.487805	0.5

Table 1