"RATIO": RAISING TEACHERS' AWARENESS OF CHILDREN'S THINKING

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ABSTRACT

We believe that the problem of teacher preparation is one of development of "pedagogical content knowledge" rather than "subject knowledge" per se. It has been found that even experienced teachers may not be aware of the misconceptions that learners tend to exhibit, or at what stage of development and in what areas of the curriculum these are likely to be manifested. This pedagogical knowledge is, we believe, important to teachers' mental models of their learners, and hence their teaching effectiveness.

In this study, we aim to contribute to teachers' awareness of their pupils' strategies and misconceptions in the field of "ratio": a topic that is difficult to teach and learn in the middle school years.

Towards this aim, we constructed a diagnostic instrument which reveals children's proportional thinking. Our instrument contains two versions, one with "models" thought to be of service to children's proportional reasoning and one without. It is also designed to function as a questionnaire for assessing teachers' pedagogical content knowledge. We use the same items that form the children's diagnostic instrument, but we ask the teachers to predict the children's errors and likely explanations and to comment on the difficulty of the item.

We present data on Year 6,7,8 and 9 (aged 10 to 14) children's performance at three items of our tool and we compare them with data on trainee teachers' pedagogical content knowledge with respect to children's thinking in these particular items. We also present the trainees' perception of difficulty hierarchy of our instrument as a whole and contrast it with the learners' difficulty hierarchy.

Our data indicate a gap between pupils' strategies and errors and their future teachers' perception of those. Further research is needed to investigate the use of such an instrument in teaching and in teacher education.

Key words: Mathematics Education, Ratio and Proportion, Misconceptions, Teachers' Awareness, Teachers' Preparation.

1. Introduction

Extended research from as early as 1966 until now (Lunzer & Pumfrey 1966, Hart 1981, Hart 1984, Tourniaire & Pulos 1985, Singh 1998) in the field of proportional reasoning reveals that solving ratio and proportion problems is a very difficult task for most pupils in the middle school years throughout the world. The above research studies identified common errors and misconceptions in pupils' proportional reasoning which affect their learning.

We believe that a starting point for the effective teaching of the topic of ratio is the teachers' awareness of these misconceptions. In previous work we have found that even experienced teachers may not be aware of the misconceptions that learners tend to exhibit, or at what stage of development and in what areas of the curriculum these are likely to be manifested. This knowledge is, we believe, important to teachers' mental models of their learners, and hence their teaching effectiveness (Williams & Ryan 2000, Hadjidemetriou & Williams, 2001).

Thus, a significant aspect of teacher preparation is one of development of what Shulman (1986, 1987) calls "pedagogical content knowledge" rather than subject knowledge per se. "Subject matter content knowledge" refers to "the amount and organization of knowledge per se in the mind of the teacher" (Shulman 1986, p.9) whereas pedagogical content knowledge refers to "subject matter knowledge for teaching" (p.9) and includes "an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons. If those preconceptions are misconceptions…teachers need knowledge of the strategies most likely to be fruitful in reorganizing the understanding of learners" (Shulman 1986, p.9-10)

In this study we aim to contribute to teachers' awareness of their pupils' strategies and misconceptions by developing an assessment instrument for proportional reasoning. This instrument was designed to assess pupils' performance at simple ratio and proportion tasks: to reveal their strategies and to locate significant misconceptions that need to be addressed in teaching. We also aim to explore whether this instrument would be suitable for assessing this aspect of teacher's pedagogical content knowledge. Particularly we were interested in the function of our instrument as a tool for teachers' training in mathematics.

Twenty-four, "missing-value" type, items were used to construct the instrument. All the problems were selected having as criterion their "diagnostic value", their potential to provoke a variety of responses from the pupils, including errors stemming from misconceptions already identified in the literature. As a result of this selection, errors indicative of common and frequent misconceptions such as the "additive strategy" (which will be described later) were expected to occur.

On the other hand, since it is recognised that children's methods differ in varying circumstances, we tried to use a variety of problems as far as "numerical structure", "semantic type" and "local context" is concerned. Thus, we hoped that less frequent misconceptions or even ones that are not mentioned in the research literature would also occur.

Some of the items have been adopted with slight modifications of those used in previous research and others have been created based on findings of that research. (CSMS 1985, Lamon 1989, Lamon 1993, Tourniaire 1986, Cramer, Bezouk & Behr 1989, Resnick & Singer 1993, Kaput & West 1994, Ryan & Willliams 2000, Singh 1998)

Finally, two versions of the instrument were constructed (both of these versions can be seen in full on the web at http://www.education.man.ac.uk/lta/cm/index.htm). The first version ("W Test") contains all the 24 items presented as mere written statements. The second version ("P Test") contains the same items supplemented by "models" thought to be of service to children's proportional reasoning. These models involve pictures, tables or double number lines, which can be used in modelling ratio problems. Lamon (1993) advocates the use of pictures, Middleton and Heuvel-Panhuizen van den (1995) support the use of ratio tables and Streefland (1984) suggests the use of double number lines. Our purpose was to compare the difficulty of the parallel items for the children and test the awareness of future teachers' of mathematics of such models.

2. Method

In order to be able to administer more items to the same sample of pupils, each version of the test consisted of two separate test forms with common linking items. Thus, Test W was divided in Test W1 and Test W2. Test W1, designed to be easier, consisted of sixteen items and Test W2 has the same number of items, but was designed to be more difficult. Eight of the items were common for both tests. Exactly the same pattern applies for tests P1 and P2 into which Test P was divided. Finally we equated Test W1 and P1 through common items and we did the same for Test W2 and Test P2 in order to be able to compare the difficulty of the parallel items for the children.

The pupils' data presented here come from a sample (N=232) of Year 6,7, 8 and 9 pupils (aged 10 to 14) from 4 schools in the North West of England.

Before administering the tests to the pupils, their teachers were asked to comment on the suitability of the test items for their classes. They found that although they differed in difficulty the items were generally acceptable for the pupils' age. They viewed them as valid assessment of the curriculum they are teaching.

Nine trainee teachers of mathematics participated in this study. These are people that have already obtained a university degree in mathematics and are trained in order to work as mathematics teachers at schools. In order to assess their pedagogical content knowledge the form W of the test (all the 24 items) was administered to them. They were asked to complete it and to provide additional information: to predict possible correct and erroneous strategies at each item and to suggest on tools, methods or activities that could help the pupils overcome their difficulties.

Firstly a qualitative analysis of the tests' results was conducted. For each item, all the pupils' answers, correct and erroneous, were recorded. Each answer in the list was accompanied were possible, by the strategies that pupils followed to obtain it. Then these answers and strategies were cross-examined with the ones that were suggested by the trainee teachers for the corresponding items.

The qualitative data provided interesting indications concerning the trainees' pedagogical content knowledge. In illustrating the essence of these data, we decided to present in detail one item, the one we named "Paint 1" and then present in summary the results from two more items, which we named "Mr Short and Mr Tall" and "Printing Press". Finally, we present a comparison between teachers' estimates and actual pupils' difficulty for all the items.

3. Results

Item: "Paint 1"

Presentation of the item

The "Paint 1" item was presented in the Test W1 as follows:

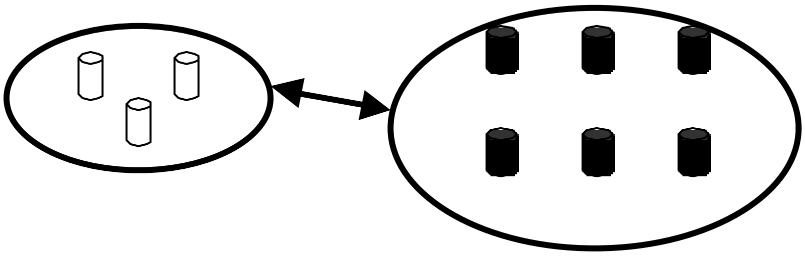
Sue and Jenny want to paint together.
They want to use each exactly the same colour.

Sue uses 3 cans of yellow paint and 6 cans of red paint. Jenny uses 7 cans of yellow paint. How much red paint does Jenny need?
Answer:

How did you find this answer? Please show your working out below.

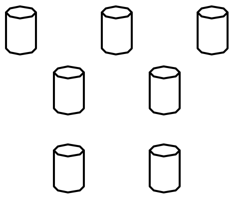
The presentation of the same item in the Test P1 is given below:
Sue and Jenny want to paint together.
They want to use each exactly the same colour.
Sue uses 3 cans of yellow paint and 6 cans of red paint.

3 cans of yellow paint *6 cans of red paint*



Jenny uses 7 cans of yellow paint.

7 cans of yellow paint *How many cans of red paint?*



?

How much red paint does Jenny need?
Answer:

How did you find this answer? Please show your working out below.

Pupils results

The qualitative analysis of the pupils' data yielded the following list of pupils' answers and strategies (all the percentages for the correct and incorrect strategies refer to the W form of the item):

Correct strategies (Correct answer: 14)

1. "Doubling" and "For every" strategy (Tourniaire, 1984).

These, multiplicative in essence, strategies were used by 17.2% of the pupils. The doubling method can be applied simply as: 3x2=6, therefore 7x2=14. Employing the "for every" strategy means finding the simplest ratio that expresses the relationship of the problem. In the case of an integer ratio this method is equivalent to the "unit value" method. In the "Paint 1" item the simplest ratio that expresses the relationship of the problem is the ratio 1:2 and by multiplying both of its terms by 7 the answer can be found

Incorrect strategies

1. "Constant Sum" strategy (Mellar, 1987) (Answer: 2)

This was the most common pupil strategy since it was used by 34.5% of the pupils. In this item, the pupil who applies the constant sum strategy thinks that the sum of Sue's cans should be equal to the sum of Jenny's cans: 3+6=9 therefore 7+2=9 and so the answer should be 2.

2. "Constant difference" or "Additive" Strategy (Tourniaire & Pulos, 1985)(Answer: 10)

This is a frequently used error strategy that has been mentioned by Inhelder and Piaget (1958) and has been widely observed ever since (Hart 1981, Hart 1984) "In this strategy, the relationship within the ratios is computed by subtracting one term from another, and then the difference is applied to the second ratio." (Tourniaire & Pulos 1985, p.186)

Here, this was the second most common strategy employed (20.7%).

In this particular problem, the answer 10 can be obtained either by thinking that 3+4=7 so 6+4=10 or by thinking that 3+3=6 and so 7+3=10.

3. "Incomplete Strategy" (Karplus, Pulos & Stage, 1983) (Answer: 6)

This strategy was used by 3.4% of the pupils. For them, the number asked should be the same as the one given from the same measure space: that is 6, since 6 are the cans of red paint given.

4. Incorrect application of build up method

3.4% of the pupils could not apply a build up method correctly.

For example, the answer "13" was obtained as follows:

" 3 yellow   6 red
6 yellow   12 red
6+1=7   12+1=13"

The rest of the pupils either gave answers that derived by strategies that we recorded as "random operations" because they were not justified properly or did not answer at all.

A tool that could facilitate pupils' thinking.

The pupils' performance on the W form of the item was compared with the pupils' performance on the P form using the data from the overall Rasch analysis of the items. The percentage of correct answers on the W form was 17.2% whereas this percentage for the P form was 55.2% which seems definitely higher. We believe these data are enough to hint that a pictorial representation of a ratio problem might influence positively pupils' proportional reasoning.

Trainee teachers' results

All the teachers provided the correct answer "14" to the "Paint 1" item, except one who wrote down as an answer the phrase "Depends on the size of Jenny's room".

They offered as the **correct strategies** that pupils would use the following:

1. Doubling strategy

Three of the student teachers predicted that a possible correct strategy used by pupils would be "doubling"

2. For every strategy

Only one trainee suggested that this problem could be solved by "noticing that the ratio of red paint to yellow paint is 2:1"

3. Multiplicative (within measure space approach)

One trainee teacher offered as a second possible strategy apart from doubling a multiplicative, within measure space, approach. In his own words: "Jenny used 7/3 x as much paint as Sue therefore red=7/3 x 6 = 14"

4. Cross multiplication method

One predicted as a probable strategy “setting up a proportion 3/6=7/x and then 3x=42 so x=14.”

The **incorrect strategies** that the trainees predicted where the following:

1. Additive Strategy (Answer: 10)

Only two teachers suggested that an incorrect strategy that would be used for this item would be the additive strategy.

2. Incomplete Strategy (Answer: 6)

One wrote that an erroneous approach would be “being unable to recognise the ratio of red to yellow can be used to find the answer”. We presume that she had in mind the incomplete strategy.

No one could predict the constant sum strategy and finally one wrote “perhaps they would reverse one part of the proportion”.

A tool that could facilitate pupils’ thinking.

Just one teacher suggested the provision of pictorial help as a tool that would facilitate pupils. She suggested that “drawing the problem out” could help the pupils find the correct answer.

Comments on the results for the item “Paint 1”

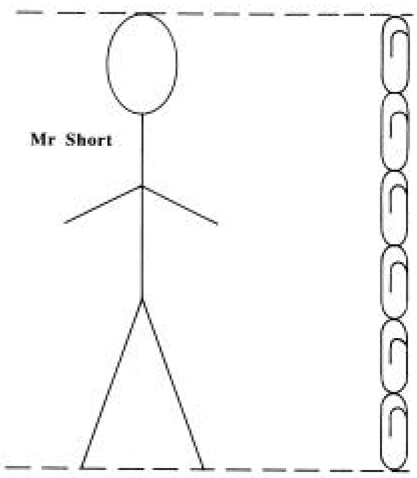
Only two of the trainee teachers could predict the well documented and many times replicated in the research literature additive strategy. No one could predict the most common incorrect strategy for this item, which was the constant sum strategy and all but one, had no suggestions about tools or activities that could aid pupils’ thinking.

Item: “Mr Short and Mr Tall”

Presentation of the item

The “Mr Short and Mr Tall” item was one of the items that linked the P and W forms of the test and was presented in both versions as follows:

You can see the height of Mr Short measured with paper clips.



Mr Short has a friend Mr Tall.

When we measure their heights with matchsticks:

Mr Short’s height is four matchsticks

Mr Tall’s height is six matchsticks

How many paper clips are needed for Mr Tall’s height?

A summary of trainee teachers and pupils’ results

All the trainees gave the correct answer “9” and the rest of the data are summarised in the table below:

	Correct Strategies	**Incorrect Strategies**	**Appropriate Tool**
Pupils	1. For every and multiplicative strategy (11.2%) 2. Build up method (4.3%) 3. Unit value method (1.7%)	1.Additive strategy (38.8%) 2. “Magical doubling” (6%) 3. Incomplete strategy (4.3%)	The use of actual models(paperclips and matchsticks) +appropriate teacher intervention
Trainee Teacher 1	Cross multiplication method	Did not mention any	Did not mention any
Trainee Teacher 2	For every method	Did not mention any	Did not mention any
Trainee Teacher 3	Did not mention any	Did not mention any	Did not mention any
Trainee Teacher 4	Multiplicative (within measure space) approach	Additive strategy	“Drill them with lots of unitary proportion sums so that they always find what the ratio for 1 unit is”
Trainee Teacher 5	Multiplicative (within measure space) approach	Additive strategy	Did not mention any
Trainee Teacher 6	Did not mention any	Did not mention any	Did not mention any
Trainee Teacher 7	Did not mention any	Did not mention any	Did not mention any
Trainee Teacher 8	Multiplicative (within measure space) approach	“Not recognizing that it is necessary to calculate the ratio between Mr Short and Mr Tall’s height in matchsticks and then applying that ratio to the paperclips.”	Did not mention any
Trainee Teacher 9	**Cross multiplication method**	1.“Miscounting paperclips” 2. “Setting up the proportion wrong”	Did not mention any

The “magical doubling” method (Mellar, 1987) mentioned in the table means that the pupil doubles (when doubling is inappropriate) one of the data of the problem in order to find an answer. In this case, the answer obtained was “12”.

Comments on the results for the item “Mr Short and Mr Tall”

A characteristic of this item is that it provoked the highest occurrence of the incorrect additive strategy compared with all the other items of the test. This strategy was mentioned by only two of the trainees. It is also notable that none of the pupils used the cross multiplication algorithm whereas two of the trainees suggested it as a possible correct strategy.

Item: "Printing press"

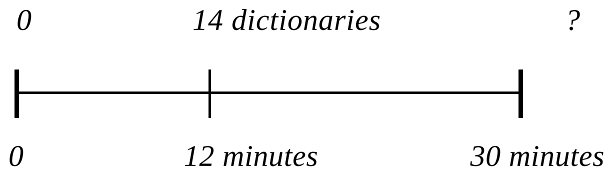
Presentation of the item

The "Printing Press" item was presented in the Test W2 as follows:

A printing press takes exactly 12 minutes to print 14 dictionaries.
How many dictionaries can it print in 30 minutes?
Answer:

How did you find this answer? Please show your working out below.

The presentation of the same item in the Test P2 is given below:
A printing press takes exactly 12 minutes to print 14 dictionaries.
How many dictionaries can it print in 30 minutes?
(You may use the figure below to help you find the answer)



Answer:

How did you find this answer? Please show your working out below.

A summary of the trainee teachers' and pupils' results

All the trainees gave the correct answer "35" and the rest of the results are presented in summary, below:

	Correct Strategies	**Incorrect Strategies**	**Appropriate Tool**
Pupils	1.For every strategy (6.9%) 2. Build up method (5.2%)	1. Additive strategy (15.5%) 2. Magical doubling (13.8%) 3. Using as a unit value the value of the quantity the problem starts with (3.4%) 4. Incorrect application of build up method (3.4%)	Maybe the use of the double number line (correct answers at Test W2=15.5% whereas correct answers at the Test P2=20.7%)
Trainee Teacher 1	Cross multiplication method	Did not mention any	Did not mention any
Trainee Teacher 2	Multiplicative, within measure space approach	Did not mention any	Did not mention any
Trainee Teacher 3	Multiplicative, within	Did not mention any	Did not mention any

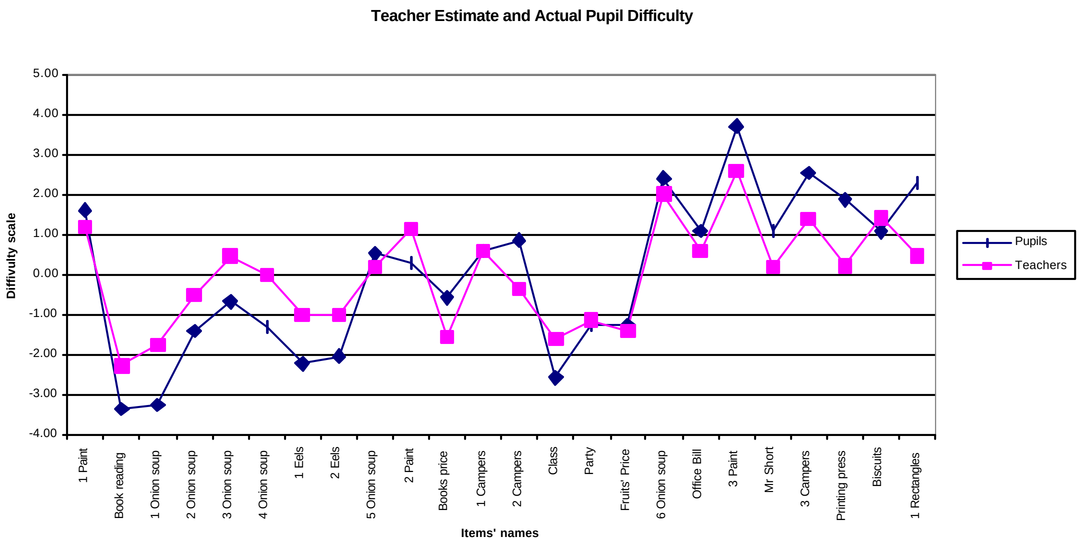
	measure space approach		
Trainee Teacher 4	Multiplicative, within measure space approach	Additive strategy	Did not mention any
Trainee Teacher 5	Unit value method	Additive strategy	Did not mention any
Trainee Teacher 6	Multiplicative, within measure space, approach	Did not mention any	Did not mention any
Trainee Teacher 7	Multiplicative, within measure space approach	Did not mention any	Did not mention any
Trainee Teacher 8	Multiplicative, within measure space approach	Did not mention any	Did not mention any
Trainee Teacher 9	1. Cross multiplication method 2. For every strategy	Did not mention any	Did not mention any

Comments on the results for the item "Printing Press"

Again, only two of the teachers mentioned the occurrence of the additive strategy, none of them predicted the incorrect strategy "magical doubling" and none of them mentioned any tools that could help pupils perform better.

Comparison between teachers' estimates and pupils' difficulty for all the items.

The trainee teachers recorded their perception of the difficulties of the items on a five point Likert scale. Their data were subjected to a rating scale analysis and the results were correlated with the children's difficulty estimated by the test analysis. The results can be seen at the figure below:



Although there are some discrepancies the high correlation (rho=0.88) is encouraging since it shows that the trainees were able to predict in general the difficulty hierarchy of the items.

3. Conclusion

Due to the small sample of pupils and trainees examined so far, the aim of this paper is not to generalise about teachers' pedagogical content knowledge. Instead, it aims to suggest a tool for evaluating and even developing this knowledge.

The data that were presented here showed that these nine "teachers to be" do not possess integrated mental models of the pupils' learning about ratio and proportion. There seems to be a gap between pupils' strategies and errors in proportional reasoning tasks and their future teachers' knowledge of these. The existence of this gap gives us reason to believe that a well-designed diagnostic instrument may be a tool that will help the training of future teachers of mathematics in two ways. First, they can be informed on their pedagogical content knowledge about ratio and proportion by trying the teachers' version of such an instrument themselves. Then, they might be able to enhance that knowledge, by delivering the same instrument to pupils and by comparing the actual data with their previous predictions.

Consequently, the next stage of the research should be to try and provide robust research findings about the use of the instrument in teacher education and in teaching in general.

Acknowledgement: We gratefully acknowledge the financial support of the Economic and Social Research Council (ESRC), Award Number R42200034284

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