SOCIOCULTURAL FACTORS IN UNDERGRADUATE MATHEMATICS: THE ROLE OF EXPLANATION AND JUSTIFICATION

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ABSTRACT

This paper extends the study of social interaction patterns as a means to characterize mathematics learning to the learning and teaching of mathematics at the undergraduate level. We present here the analysis of teaching episodes from a discrete mathematics course to document the change in social and sociomathematical norms over the course of one semester. First, the instructor established the social norm that students justify, explain and share with their peers their thinking and solution processes. We show how the instructor of the course established an expectation for explanation and justification, and how students' interactions developed in accordance to this normative understanding through the semester. That is, we trace students' development from the passive acceptance of the instructor's authority to the expectation that students become contributors to the class and that they all share common understandings. We then shift our focus to the sociomathematical norms – normative interactions specific to mathematics. We discuss the development of students´ explanations from the procedural level to ones that are grounded in deeper conceptual understandings. We finally link the shift in the aforementioned social and sociomathematical norms in students' interactions to the development of students' ability to reason deductively.

Introduction

Schoenfeld argued that “mathematics is an act that is socially constructed and socially transmitted” (1992, p. 335). As such, it is governed by a set of norms; an etiquette for what is deemed appropriate behavior by members of the mathematics community. These can be *social* rules, that is, the ways in which members of the community interact and exchange ideas – rules that are not specific to mathematics but may characterize the behavior of members of other fields (e.g., historians). There is also a set of mathematical, or *sociomathematical* rules, that is, rules that are specific to the field of mathematics, such as what constitutes a proof (Cobb, Wood, Yackel, McNeal, 1992; Yackel & Cobb, 1996). While the second set of rules are explicit in the field, the first set can be very implicit. And yet, one may argue that social norms constitute the broad basis upon which the mathematical norms are constituted.

In recent years, we have witnessed a renewed interest in this social facet of mathematics and a growing tendency in studying social interaction patterns as a means to characterize mathematics learning (e.g., Yackel, 2001). Yet, little work has been done at advanced levels; the bulk of the research in this area has been conducted in elementary and secondary school classrooms (Cobb & Bauersfeld, 1995; Cobb, Yackel & Wood, 1992). In this paper we join the efforts of Yackel, Rasmussen and King (2000) to extend these analyses to the learning and teaching of mathematics at the undergraduate level using data from a classroom teaching experiment in discrete mathematics. We document the development of social and sociomathematical norms regarding explanation and justification over the course of one semester, and we discuss how these norms were constituted in this specific case.

First, we focus on the social norm that students publicly explain their thinking and solutions and try to make sense of other students’ thinking. We show explicitly how the instructor of the course established an expectation for explanation and justification, and how students’ interactions developed in accordance to this normative understanding through the semester. That is, we trace students’ development from the passive acceptance of the instructor’s authority to the expectation that students become active contributors to the class and that they all share common understandings. We then shift our focus to the sociomathematical norms – interactions specific to mathematics. We discuss the development of students’ explanations from the procedural and empirical level to ones that are grounded in deeper conceptual understandings within the context of the course. Finally, we discuss the social interactions with respect to reformed instruction in advanced mathematics classrooms.

Methodology

Participants for the study were a group of 50 undergraduate mathematics students enrolled in a two-semester, first-year course on discrete mathematics emphasizing mathematical argumentation and proof. The course was taught by one of the two investigators, while the other investigator collected data. For homework assignments and reference purposes, the course used a broad text on discrete mathematics (Grimaldi, 1999). A typical class section begun with a problem introduced by the instructor followed by student group work. Students were encouraged to ask each other questions and help each other clarify concepts and problem requirements. The small group work was usually alternated with whole class discussion of students’ approaches, thinking and questions. Throughout the course there was a concerted focus on both written and verbal expression of student thinking. Implicitly the instructor worked towards establishing the social norms that students are expected to explain their reasoning, to try to make sense of each other's explanations, and to challenge each other's reasoning and justifications.

Each class was videotaped and attention was paid to both the instructor's actions and the students' reactions, including the students' interactions when working in groups. In analyzing the data, our first goal was to demonstrate the use and change of social and sociomathematical norms in the classroom over time. In order to document these, we analyzed transcripts of classroom discourse data according to its function and pattern (Potter & Wetherall, 1987), using each speaker's turn as the basic unit of analysis. We focused our coding on the forms of explanation and justification used by students. This required detailed coding of verbatim transcripts, with the meaning of each speaker's turn interpreted within the context of the larger conversation. Additionally, students were presented with written assessments, at the beginning and end of each semester. These assessments were analyzed to identify shifts in students' proof schemes (Harel & Sowder, 1998) and each student's own competency in justifying and proving.

Social Norms

Students were initially surprised by and even resistant toward the social norm that they explain their thinking and try to make sense of other students' thinking. It became apparent that the instructor's expectations that students explain publicly their thinking ran counter to the students' earlier experiences of mathematics work. Students felt uncomfortable engaging in explanations of their thinking and even lacked the language to do so. They were initially hesitant to challenge their classmates' thinking and acknowledged that they did not know how to explain why their solutions worked. However, as the semester progressed, students got accustomed to engaging in explanations and justifications. Here, we present excerpts from two different episodes in the course from two different points in time, that sharply contrast social norms regarding student explanation and making sense of each other's thinking. In the first case that took place during the second week of the semester, students did not feel the obligation to explain their thinking nor did they expect to make sense of other students' explanations, despite the instructor's urge to do so. In the second case the students felt obliged to do so, without prompting from the instructor.

First episode (second week). The class was introduced to combinations and permutations – students were asked to find the number of different combinations of pastries one can purchase from a bakery. The instructor prompted students to "think of their thinking" and to question each other's approaches and arguments implicitly letting students know that there is an expectation that they will engage in this question and share their reasoning. Further, students were asked to work in groups. The following was the interaction among Isabelle and Josh:

Isabelle: What did you do?
Josh: You multiply them all out and you get $10 \times 9 \times 8 \times \ldots$.
Isabelle: Oh, OK.

The level of discussion described in this short episode among Isabelle and Josh is illustrative of the discussions that took place among almost all groups; when students were asked to work in groups and to collaborate in solving the problem while making sure they question each other's thinking, they, instead, tended to ask each other (or the instructor) for a method to solve the problem or for an answer–a procedural approach to problem solving–and accepted each other's solutions without further questioning. In the few cases where a student asked another student for further clarification or explanation for his answer or approach, the response often was "it worked for me!"

Second episode (seventh week). The class was discussing rational and irrational numbers and students were asked to consider the square root of 2, and to show it is irrational. As usual, the instructor prompted students to question each other's approaches and arguments. The following is the interaction among Jared, Daniel, and Mike while thinking about the problem in a whole class discussion:

Jared:	I set $\sqrt{2} = p/q$. Then I…
Daniel:	What are p and q?
Jared:	Two integers
Daniel:	Any integers?
Jared:	Two integers
Daniel:	If it's not *any* integers, then it's not true for all cases, and then someone can come up with a case where it fails and your argument is gone.
Mike:	To me, the important thing to remember is that $\sqrt{2}$ is written as a specific ratio, not any p/q. We are trying to show it can't be rational….

In contrast to the first episode where students hesitated to challenge each other, during the second episode, students expected their classmates to explain their reasoning. After Jared started sharing his thoughts, Daniel, without prompting from the instructor, asked for further explanation – what numbers was Jared considering in his proof. Jared clarified, but Daniel prompted for more – a sincere attempt to understand Jared's reasoning. Notice, however, that the interaction was not a dialogue among two naturally inquisitive students; Mike joined the discussion in an attempt to clarify the argument further. Mike's language further suggested that the argument was a collective one; he pointed that "*we* are trying to show it can't be rational" (emphasis added), it was no longer Daniel's attempt to show that the square root of 2 is irrational, it was an argument embraced by the class. It was, from that point on, the class' responsibility to clarify for each member of the community and to ensure that each member shares the ownership and understanding of the argument.

Overall, this episode illustrates how the students had advanced during the course of semester in their ability to debate with their peers. Furthermore, they had overcome their initial resistance towards public argumentation and had developed the expectation that others explain their reasoning to the class. The two social norms, that students were expected to explain their reasoning and that they were expected to try and make sense of other students' thinking were gradually constituted throughout the semester. Such discussions are essential in students' mathematical development and in the development of the classroom as a community of learners.

Sociomathematical Norms

We showed that as the semester progressed, students got accustomed to engaging in explanations and justifications. Furthermore, the *quality* in students' explanations and their capacity to express their mathematical thinking in increasingly formalized ways changed substantially over time. Students' arguments gradually shifted from empirical and procedural to deductive and conceptual. As students advanced in their ability to argue, they also raised their expectations as to what counts as a strong mathematical argument; while during the first weeks of class the instructor's request for explanation often resulted in a description of the procedure that a student used or the listing of several examples, a few weeks later students attempted to explain the generality of the argument. We argue that students acted in accordance with the normative understanding that they were expected to explain, but they also established sociomathematical norms that are very specific in the mathematics community as to what constitutes an acceptable explanation in mathematics

Once again, we discuss the two episodes in the course that were presented in the previous section to contrast social norms regarding student explanation. We now discuss these same episodes from a different standpoint; students' growth in their use of *mathematical* arguments.

First episode (second week). In the first episode presented in the previous section, Josh shares his solution with Isabelle. Isabelle's question "what did you do" is a prompt for a procedure that will produce an answer and that is precisely what Josh has to offer – a guide that will lead her to the correct solution to the problem. Josh did not see the need to give a conceptual explanation (why one should multiply out all the numbers) and Isabelle, in turn, was satisfied with the procedure and did not see the need to prompt for an explanation. The discussion among Isabelle and Josh once again illustrates the quality of the arguments that were exchanged among students during the first few weeks of the semester – students exchanged procedural explanations and recipes for solutions that appeared to produce correct answers.

Second episode (seventh week). In this episode, Jared started to share his approach to showing that the square root of 2 is irrational, but was interrupted by Daniel who questioned the generality of Jared's use of integers. Finally, Mike attempted to help Daniel in understanding the proposed solution. Their mathematical argument seems to be in determining the meaning of 'p' and 'q', specifically, whether they represent a fixed but unknown pair of integers, or whether they represent any two arbitrary integers. The excerpt shown in the previous section illustrates that the students acted in accordance with their own understanding in explaining their thinking and making sense of each other's thinking and it attests to the existence of classroom and sociomathematical norms by which such conversations can occur. Such discussions are essential in students' mathematical development and in the development of the classroom as a community of learners. In particular, we claim that classroom discussions such as this helps to build a habit of mind whereby students internalize public argumentation in ways that facilitate private proof construction.

We take shifts in student responses to one proof problem given on two occasions as part of the evidence for this claim. The problem was administered on the first day of class, before any instruction occurred, and 9 weeks later on a mid-term assessment. For purposes of interpreting the quantitative results, we provide one possible "correct" solution.

> **PROBLEM:** *Prove that the sum of an even number and an odd number is always odd.*
>
> **POSSIBLE SOLUTION:** *Let x be even and y be odd. They x = 2m and y = 2n+1, for integers m and n. Then x + y = 2m + 2n + 1 = 2(m+n) + 1 = 2k + 1, where k=m+n is an integer. But 2k+1 is odd, by definition, so x + y is odd. Thus the sum of an even number and an odd number is always odd.*

The problem described above was given to students as part of an individual pre-assessment at the beginning of the semester, and as part of a mid-term assessment. We do note, however, that students worked *in pairs* during the mid-term assessment. While this arrangement certainly contributed to the success students had with this problem, it also illustrates the type of socio-mathematical norms that had evolved in the classroom, norms that we take to be critical for the development of students' capacity to build proofs. Even so, the results clearly indicate significant gains in students' responses. Table 1 shows a summary of student solutions to this problem.

	Pre-Test (individual) (50 responses)	**Mid-Test (paired)** (51 responses)
Correct proofs		
completely correct	1 out of 50 (2%)	34 out of 51 (67%)
almost correct (minor error)	---	13 out of 51 (25%)
Total correct proofs	1 out of 50 (2%)	47 out of 51 (92%)
Incorrect proofs:		
Used examples as a "proof"	26 out of 50 (52%)	2 out of 51 (4%)
Used illogical reasoning	10 out of 50 (20%)	2 out of 51 (4%)
Looked at a narrow case	10 out of 50 (20%)	2 out of 51 (4%)
No attempt made	7 out of 50 (14%)	0 out of 51 (0%)
Total incorrect proofs	49 out of 50 (98%)	4 out of 51 (8%)

Table 1. Summary of student responses.

Only one person (2%) gave either a correct or essentially correct proof on the first attempt, while 92% of the class gave correct (67%) or essentially correct (25%) proofs on the second attempt. In addition, 52% of students on the first attempt used examples to 'prove' the conjecture, while only 4% of students used this as a strategy on the second attempt. Moreover, there was a significant increase in students' level of formalization, particularly, their capacity to express their thinking in increasingly formal ways via symbolic language. Only 16% of respondents on the pre-test used some form of symbolization, whether correctly or incorrectly (otherwise, if students attempted a proof, they used everyday language). On the mid-test, 94% of students expressed their proof or proof attempts symbolically in a manner similar to the possible solution given here.

Concluding Remarks

This study adds to the literature on the nature of cognitive and social dimensions of mathematics instruction and learning at the university level. We presented here some examples on the nature of social and sociomathematical norms that support the learning of mathematics in a discrete mathematics classroom. Our results suggest that college mathematics classrooms can potentially function as communities of learners, in which students engage in sense-making and meaning-making. In this respect, this study supports the work of Yackel, Rasmussen and King (2000), in that, over time, students' attitudes develop from the passive acceptance of the instructor's authority to the expectation that students become active contributors to the class and that they all share common understandings.

The significance of this work for mathematics reform at the university level is that it provides a different perspective to view and analyze mathematics learning that complements the work of mathematicians and mathematics educators who have focused primarily on the individual cognitive aspects of advanced mathematics learning (e.g., Dubinsky, 1992; Harel & Sowder, 1998). We are suggesting a shift towards the study of social processes as a way to understand students' development. As we continue to examine the cognitive and social dimensions of mathematics learning in college classes, it is important that we look deeper into the interconnections of social and cognitive development.

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