MATHEMATICS FOR ELEMENTARY TEACHERS: Making Sense by "Explaining Why"

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ABSTRACT

In order for prospective teachers to develop the reasoning and sense-making abilities of their future students, the teachers themselves must make sense of and reason about the mathematics they will teach. However, many prospective teachers have only experienced mathematics as the rote following of procedures, and are not aware that reasoning can be used to solve problems in non-standard ways, or that reasoning underlies the standard procedures in mathematics. A way to help prospective elementary teachers make sense of and reason about mathematics is to engage them in explaining mathematics. This paper discusses obstacles that arise in doing so, and recommends ways to overcome these obstacles. The paper also describes desirable features of problems asking for explanations, and gives examples. Finally, the paper gives guidelines to help students write good explanations.

1 Introduction

Recent reform efforts in mathematics education emphasize that students should make sense of mathematics and engage in mathematical reasoning (NCTM, 2000). In order for prospective teachers to develop the reasoning and sense-making abilities of their future students, the teachers themselves must make sense of and reason about the mathematics they will teach. However, many prospective teachers have only experienced mathematics as the rote following of procedures, and are not aware that reasoning can be used to solve problems in non-standard ways, or that reasoning underlies the standard procedures in mathematics. How then can prospective teachers learn to make sense of and reason about mathematics in a way that will help them to enable their own future students to make sense of and reason about mathematics? This article addresses this issue for prospective elementary teachers.

Certainly, making sense of mathematics and engaging in mathematical reasoning are intimately connected to *explaining* mathematics. Every mathematics teacher knows that when we explain mathematics, we enhance and solidify own understanding of mathematics. And every mathematics teacher knows that when we explain (or prepare to explain) mathematics, we sometimes uncover our own lack of understanding. It is only when we can explain a piece of mathematics in a way that makes sense both logically and intuitively that we feel we understand the mathematics. Thus, prospective teachers should learn to explain mathematics not only because they will explain mathematics to their future students, but also because explaining mathematics enhances their own understanding of mathematics and their own mathematical reasoning abilities.

To be an effective tool in teacher education, we should choose the explanations that we ask prospective teachers to give deliberately. What features should we seek in the problems we ask prospective teachers to explain, and why? What should we expect or ask teachers to draw on in producing their explanations? What are ways to help teachers improve their ability to explain?

2 What Kind of Explaining?

What kind of explaining of mathematics should prospective teachers engage in? Starkly different choices can be made, even when the subject matter is centered on the mathematics the teachers will teach.

One choice is to give prospective elementary teachers axiomatic developments of numbers and of geometry, and to expect the teachers to establish various facts in arithmetic and geometry by giving rigorous proofs that refer to axioms and to previously established theorems. These are not bad objectives, and they may be reasonable in small amounts, but will the teachers be able to use this learning to help their own young students reason about and make sense of mathematics? Realistically, the connection may be too long for most teachers to bridge in practice.

The Mathematical Education of Teachers (2001) recommends the following:

> All courses designed for prospective teachers should develop careful reasoning and mathematical "common sense" in analyzing conceptual relationships and in solving problems. (Chapter 2)

This suggests an *intertwining* of logical reasoning with ordinary sense-making. Thus the explaining that I advocate here is more than just logical and convincing to a skeptic; it should be truly explanatory, and it should help to make sense of the related mathematics.

For example, we can use induction to prove that

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

The proof is logical and it will convince someone who understands induction, but it doesn't show where the simple formula $\frac{1}{2}n(n+1)$ comes from. In this sense, it doesn't really explain why the equation above is true. Instead, if we imagine a "step triangle" made of n rows of squares, with 1 square in the first row, 2 squares in the second row, 3 squares in the third row, and so on, then we can see visually why the $\frac{1}{2}n(n+1)$ formula makes sense: put two step triangles together to make an n by $n+1$ rectangle.

"Explaining why" is different from proving in several ways. When "explaining why", a careful examination of several important cases is often more illuminating than an argument that covers all possibilities. For example, how should we explain why the standard longhand multiplication procedure is valid? Instead of a proof, we can examine some special cases carefully, such as some 2-digit by 2-digit products. Also, although one proof establishes truth, when "explaining why" we should seek several explanations, and we should try to coordinate these explanations. To explain why longhand multiplication is valid, we can use the distributive property; we can also draw a rectangle and subdivide it into pieces corresponding to steps in the procedure. Best of all, we can link these two explanations.

Thus I propose the following as desirable features of explanations that prospective elementary teachers should engage in:

- The explanation is logical.
- The explanation explains in a common-sense way. It is convincing, both to the person who is explaining and to the intended audience (e.g., peers, the instructor, children).
- If possible, there are several explanations, such as one using equations and one using a picture, and the explanations are coordinated.

The literature includes examples of teachers and prospective teachers engaged in sense-making by explaining mathematics. For example, Schifter (1998) describes a teachers' seminar in which teachers worked with problems such as the following:

> Wanda really likes cake. She has decided that a serving should be 3/5 of a cake. If she order four cakes, how many servings can she make? (p. 67)

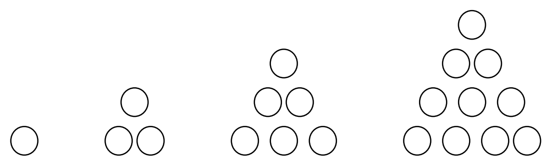
The teachers reasoned with the aid of pictures to explain why the solution made sense. Simon and Blume (1996) describe a course in which prospective elementary teachers worked on various explanations, such as explaining why the area of a shape can't be determined from its perimeter.

Prospective teachers should learn to "explain why", but what are some of the issues and obstacles we encounter in attempting to carry this out? The next two sections will address this.

3 Obstacles in Learning to "Explain Why"

Many prospective elementary teachers enter their mathematics training expecting to learn to give children clear directions for carrying out mathematical procedures. This creates an obstacle in a course that is about "explaining why", and not about "showing how". Therefore, when I teach our first mathematics course for prospective elementary teachers, I discuss carefully why we focus on "explaining why". I take students' questions of "why do we need to know this?" seriously, and address them in detail. Soon, most students see the wisdom in our approach. But this is not the only obstacle.

Initially, many prospective elementary teachers have a shallow conception of what it even *means* to explain why something is true. We often begin our first mathematics course for elementary teachers by considering triangular arrays of dots:



Every time, at least one student offers something roughly like the following to explain why the formula $\frac{n(n+1)}{2}$ gives the correct number of dots in the nth triangle:

> There is an $n+1$ in the formula $\frac{n(n+1)}{2}$ because you are adding 1 to each row in the triangle.

This "explanation" is really a mnemonic device that connects the formula to the problem in a superficial way. A student who offers it may not understand what explaining means.

A while ago, I assigned the following problem early in the semester:

> Mary says that $100 \times 3.7 = 3.700$. Why might Mary think this? Explain to Mary why her answer is not correct and why the correct answer is right. If you tell Mary a procedure, be sure to tell her why it makes sense!

Despite the instructions, and despite having discussed place value in class, most students simply told Mary that $3.7 = 3.700$ and that she should move the decimal point 2 places to the right. When I have asked students to explain why the standard multiplication procedure makes sense, some have responded with a clear explanation for how to carry out the procedure. Thus I now give my students more guidance in "explaining why" early in the semester.

Similarly, as reported in Ma's study (1999), when elementary teachers were presented with a hypothetical situation in which students mistakenly did not shift over the partial products when calculating

$$\begin{array}{r} 123 \\ \times 645 \\ \hline \end{array}$$

many American teachers suggested remedies that focused on clarifying the multiplication procedure, such as using lined paper sideways, or using whimsical placeholders to catch the students' attention (pp. 28–35). The American teachers tended to "show how" rather than to "explain why".

Thus instructors of courses for prospective elementary teachers should not assume that the prospective teachers know it is possible to give meaningful explanations for mathematical procedures and facts. Most students need time and practice to develop the notion that mathematics can be explained, and what it means to do so.

Another initial obstacle is students' beliefs about what constitutes mathematical activity. For some students, common-sense reasoning and pictures may not seem "mathematical" enough. I posed the following problem early one semester:

> Susan was supposed to use $\frac{5}{4}$ of a cup of butter in her recipe but she only used $\frac{3}{4}$ of a cup of butter. What fraction of the butter that she should have used did Susan actually use? Draw pictures to help you solve this problem. Explain your answer clearly. For each fraction in this problem, and in your solution, describe the *whole* that this fraction is associated with.

One student responded by drawing pictures to show $\frac{5}{4}$ and $\frac{3}{4}$ cups of butter, and then calculated:

$$\frac{3}{4} \div \frac{5}{4} \text{ or } \frac{3}{4} \cdot \frac{4}{5} = \frac{12}{20} = \frac{6}{10} = \frac{3}{5} \text{ out of } \frac{5}{4}.$$

She went on to explain as follows:

> ... To find the fraction of the butter that Susan used out of what she should have used you need to divide $\frac{3}{4}$ and $\frac{5}{4}$. When dividing fractions you can take the reciprocal of the second fraction and multiply it by the first fraction: $\frac{3}{4} \cdot \frac{5}{4}$. When you do that you find that Susan used $\frac{3}{5}$ of the $\frac{5}{4}$ of butter. ...

Despite the directions to use a picture to help solve the problem, the student showed (correct) calculations and discussed those calculations. Perhaps the student didn't know how to use a picture to solve the problem (although we had used both pictures and calculations in class, and the student was a consistently diligent worker), or perhaps the student didn't find a picture together with common-sense reasoning to be sophisticated enough mathematically, and therefore didn't believe she should work with a picture to explain the solution. If it was the latter, then this is similar to what Raman (2001) found in her study of students and teachers in collegiate calculus: students viewed thinking mathematically as involving algebraic tricks and formal language. Raman found that students were less willing than instructors to accept a pictorial proof that the derivative of an even function is odd.

Thus students need time and experience to develop the idea that *reasoning* is a cornerstone of mathematics, and that this reasoning can not only involve equations and formulas, but can also refer to pictures and experiences.

4 Will it Transfer to the School Classroom?

A major challenge in mathematics teacher education is to help teachers carry explaining and sense-making into their own classrooms.

The National Council of Teachers of Mathematics (NCTM) has promoted a vision of reasoning and sense-making in school mathematics (NCTM, 1989, 2000). Yet studies in which teachers are trained in accordance with this vision show mixed results when the teachers enter the classroom (Wilcox, Lanier, Schram, and Lappan, 1992; Frykholm, 1999). Frykholm (1999), in his study of secondary mathematics student teachers, found that most student teachers were not able to put the NCTM Standard's vision of reform into practice:

> ... the student teachers reported that, although the Standards were valuable inasmuch as they articulated a compelling vision for what mathematics instruction could be, there had been little offered in the way of practical advice and examples of innovative pedagogy that could be used as a model for implementing such instructional strategies. (p. 94)

Mathematics classes that focus on reasoning and sense-making often seem diffuse and inefficient as described in the literature. One wonders whether students will be able to pull the ideas together in the end; one wonders if class time has been used effectively. For example, Simon and Blume (1996, pp. 10–17) describe a class for prospective elementary teachers in which the students and the instructor discuss why the number of cardboard rectangles covering a table can be determined by multiplying. There is a lot of fumbling and searching; there is a lot of confusion. In the end, some of the students were able to explain clearly why it is valid to multiply, but excerpts from journals of other students show that several students left the class still uncertain and confused. Learning mathematics is necessarily messy and imperfect; it inevitably involves some fumbling and false starts. But I can't help wondering if the important class discussion described in the article couldn't have helped the students learn more effectively and efficiently if it had taken place in a narrower context. What if the instructor had given the class a definition of multiplication, and had asked the class to use the definition to explain why it is valid to multiply? In my own experiments with teaching in different ways, I have found that being too much of a "guide on the side" leaves too many students confused and unable to pull the ideas together in a coherent way.

Could it be that in our desire to help students make sense of mathematics for themselves, and in our desire not to lecture, that we sometimes give students *too little structure* in which to learn *efficiently*? And if prospective teachers view sense-making as too inefficient and unstructured, will they feel that they do not have the luxury of engaging their own students in sense-making? After all, as teachers, they will be responsible that their students achieve specific learning objectives on specific topics, which may be tested on high-stakes state or national tests.

5 Recommended Features of Explanations

In light of the discussion above, I offer the following recommendations for choosing explanations for prospective elementary teachers to engage in.

1. **Choose many explanations that are fairly closely linked to the actual practice of teaching mathematics in elementary school.**

For example:

> Jim thinks that because $30 \times 40 = 1200$, and $1 \times 1 = 1$, therefore
>
> $$31 \times 41 = 1200 + 1 = 1201.$$
>
> Draw a picture and use your picture to help you explain to Jim how 30×40 and 31×41 are actually related. (Beckmann, 2003)

2. **Choose explanations that will help teachers organize their thinking around key principles and concepts.** In some cases, state the principle or definition to be used in order to provide structure and context.

In her study of American and Chinese elementary teachers, Ma (1999) found that some of the Chinese teachers developed what she called *Profound Understanding of Fundamental Mathematics* (PUFM). One key component of PUFM is a focus on basic ideas. As Ma explains:

> Teachers with PUFM display mathematical attitudes and are particularly aware of the "simple but powerful basic concepts and principles of mathematics" (e.g., the idea of an equation). They tend to revisit and reinforce these basic ideas. By focusing on these basic ideas, students are not merely *encouraged* to approach problems, but are *guided* to conduct real mathematical activity. (p. 122, emphasis in original.)

These key concepts include fundamental definitions, such as the definition of multiplication and the definition of fraction. In some situations, the principle or definition can be referred to explicitly in asking for an explanation. For example:

> John, Trey, and Miles want to know how many two-letter acronyms there are that don't have a repeated letter. For example, they want to count acronyms such as BA and AT, but they don't want to count acronyms such as ZZ or XX.
>
> John says there are $26 + 25$ because you don't want to use the same letter twice, that's why the second number is 25.
>
> Trey says he thinks it should be *times*, not *plus*: 26×25.
>
> Miles says the number is $26 \times 26 - 26$ because you need to take away the double letters.
>
> Discuss the boys' ideas. Which answers are correct, which are not, and *why*? Explain your answers clearly and thoroughly, drawing on the meaning of multiplication. (Beckmann, 2003)

Or:

> The grid lines below are 1 cm apart. Use the *moving* and *combining* principles about area to help you determine the *exact* area of each of the triangles below. Explain why your answers are correct. *Do not* use a formula for areas of triangles. [The problem includes a picture of triangles on a grid.]

Or:

> Use the *meaning of fractions* to explain why
>
> $$\frac{2}{3} = \frac{2 \cdot 57}{3 \cdot 57}.$$
>
> (In other words, explain why $\frac{2}{3} = \frac{114}{171}$.) *Do not* use multiplying by 1 to explain this.

We can ask not only for explanations of why things are the way they are, but also for explanations of why things aren't the way they aren't. In these cases, the underlying principles are not given, but must be uncovered in order to give a full explanation. For example:

> Frank thinks that it would be easier to add fractions by "adding the tops and adding the bottoms." So for example, Frank wants to add $\frac{1}{2}$ and $\frac{3}{4}$ this way:
>
> $$\frac{1}{2}+\frac{3}{4}=\frac{1+3}{2+4}=\frac{4}{6}.$$
>
> Frank uses the picture below to explain why his method makes sense. Why is Frank's method not a valid way to add fractions, and why does Frank's picture not prove that fractions can be added in his way? Do not just state the proper way to add fractions, explain what is wrong with Frank's reasoning.

$$\boxed{\mathrm{X}}\boxed{\mathrm{O}} + \boxed{\mathrm{X}}\boxed{\mathrm{X}}\boxed{\mathrm{X}}\boxed{\mathrm{O}} = \boxed{\mathrm{X}}\boxed{\mathrm{X}}\boxed{\mathrm{X}}\boxed{\mathrm{X}}\boxed{\mathrm{O}}\boxed{\mathrm{O}}$$

In order to explain what is wrong with Frank's method, prospective teachers must focus on the crucial role of the whole associated to each fraction, as in the following explanation given by a student.

> Although Frank's reasoning looks good at first, he is not using the same wholes to get the fractions $\frac{1}{2}$ and $\frac{3}{4}$. When adding fractions, it is important to consider the wholes. He starts with 2 blocks, 1 shaded [X] and 1 white [O] which is equal to $\frac{1}{2}$, but then he adds two more blocks to show $\frac{3}{4}$. The wholes (2 blocks) and (4 blocks) are not equal and therefore we cannot add these fractions [yet].

3. Give students specific guidelines for writing mathematical explanations.

I give my students the following guidelines characterizing good explanations in mathematics:

A. The explanation is factually correct, or nearly so, with only minor flaws (for example, a minor mistake in a calculation).

B. The explanation addresses the specific question or problem that was posed. It is focused, detailed, and precise. There are no irrelevant or distracting points.

C. The explanation is clear, convincing, and logical. A clear and convincing explanation is characterized by the following:

   (a) The explanation could be used to teach another (college) student, possibly even one who is not in the class.

   (b) The explanation could be used to convince a skeptic.

   (c) The explanation does not require the reader to make a leap of faith.

   (d) Key points are emphasized.

(e) If applicable, supporting pictures, diagrams, and/or equations are used appropriately and as needed.

(f) The explanation is coherent.

(g) Clear, complete sentences are used.

For example, we could respond to the problem "use the meaning of fractions to explain why $\frac{2}{3} = \frac{2 \cdot 57}{3 \cdot 57}$" as follows.

> According to the meaning of fractions, $\frac{2}{3}$ of a pie is the amount formed by 2 parts when the pie is divided into 3 equal parts. This amount is shown shaded in the picture below. [Show the relevant picture of a pie.] If I divide each of those 3 equal parts into 57 small equal parts, the pie will now be divided into $3 \cdot 57 = 171$ small parts. Because the 2 original shaded parts representing $\frac{2}{3}$ of the pie have each been subdivided into 57 small parts, these 2 original shaded parts become $2 \cdot 57 = 114$ small parts, as indicated in the picture. [Show another picture of the same pie, indicating that each piece is now subdivided into many smaller pieces of equal size.] It's still the same amount of pie that is shaded either way you look at it. So 2 of the original 3 parts of pie is the same amount of pie as $2 \cdot 57$ small parts of the total $3 \cdot 57$ small parts. This is why $\frac{2}{3}$ of a pie is the same amount of pie as $\frac{2 \cdot 57}{3 \cdot 57} = \frac{114}{171}$ of the pie.

Notice that even though we can also use multiplication by 1, in the form $\frac{57}{57}$, to explain why $\frac{2}{3} = \frac{2 \cdot 57}{3 \cdot 57}$, the explanation above addresses the specific problem that was posed, namely to use the meaning of fractions. The explanation is written to explain in a natural and convincing way, and not just to establish truth.

With attention to the matters that I have described in this paper, it is possible to teach an efficient course in which prospective teachers learn to "explain why" and make sense of mathematics.

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