DEVELOPING COLLEGE STUDENTS' VIEWS ON MATHEMATICAL THINKING IN A HISTORICAL APPROACH, PROBLEM-BASED CALCULUS COURSE

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ABSTRACT

It has been held that heuristic training alone is not enough for developing one's mathematical thinking. One missing component is a mathematical point of view. Many educational researchers propose problem-based curricula to improve students' views of mathematical thinking. Meanwhile, scholars in different areas advocate using historical problems to attain this end. This paper reports findings regarding effects of a historical approach, problem-based curriculum to foster Taiwanese college students' views of mathematical thinking.

The present study consisted of three stages. During the initial phase, 44 engineering majors' views on mathematical thinking were tabulated by an open-ended questionnaire and follow-up interviews. Students then received an 18-week historical approach, problem-based calculus course in which mathematical concepts were problematizing to challenge their intuition-based empirical beliefs in doing mathematics. Several historical problems and handouts served to reach the goal.

Near the end of the semester, participants answered the identical questionnaire and were interviewed to pinpoint what shift their views on mathematical thinking had undergone. It was found that participants were more likely to value logical sense, creativity, and imagination in doing mathematics. Further, students were leaning toward a conservative attitude toward certainty of mathematical knowledge. Participants' focus seemingly shifted from mathematics as a product to mathematics as a process.

1. Introduction

Polya's four-phase theory sketches a blueprint for mathematical problem solving and initiates study of heuristics during the 1970s and 1980s. Contemporary studies suggest that teaching heuristics could significantly improve students' ability to employ heuristics in solving non-routine mathematical problems, yet research in this phase has long been questioned by many scholars for its limited capacity for preparing students to extrapolate the ability (Lester, 1994; Owen & Sweller, 1989; Sweller, 1990). Relevant researchers thus revisited the ultimate goals of mathematics instruction and how problem solving fits within the goals. National Council of Teachers of Mathematics (1991) defines the aims of teaching mathematics as "to help all students develop mathematical power" and "all students can learn to think mathematically" (p. 21). Learning to think mathematically means developing a mathematical point of view (Schoenfeld, 1994), a missing part in traditional training of problem solving (Schoenfeld, 1992).

On the other hand, scholars in different areas have evoked the use of historical problems in developing students' mathematical thinking (Barbin, 1996; Rickey, 1995; Siu, 1995a, 1995b; Swetz, 1995a, 1995b). The gist of this argument is that using historical problems in a classroom can benefit students in not only the affective domain but also the cognitive domain. Ernest (1998) interprets the rationale for using historical problems as indicating mathematicians in history struggled to create mathematical processes and strategies that are still valuable in learning and doing mathematics to this day.

Note that the relationship between students' views of or beliefs about doing mathematics and their learning behaviors has attracted considerable attention in recent years (Carlson, 1999; Franke & Carey, 1997; Higgins, 1997; Kloosterman and Stage, 1991; Schoenfeld, 1989). Empirical investigations suggest students who view doing mathematics as a rigid process may be more reluctant to engage in creative mathematical activities. Conversely, an active view would potentially promote an individual's desire to undertake challenging tasks (Carlson, 1999; Franke & Carey, 1997; Henningsen, & Stein, 1997; Higgins, 1997, Schoenfeld, 1989, 1992). A basic understanding of the intrinsic essence of mathematical knowledge is requisite for mathematical literacy. To reach the goal, learners need to comprehend the nature of mathematical thinking (American Association for the Advancement of Science, 1990). On the basis of empirical evidence, investigating and developing problem solvers' views of mathematical thinking are noteworthy issues to receive further attention.

2. Purpose of The Study

Though scholars in various fields have addressed the critical role that history of mathematics plays in mathematics education for years, empirical studies designed to explore the issue are rare. This research aims to investigate interrelationships between a historical approach, problem-based calculus course and Taiwanese technological college students' views of mathematical thinking, particularly regarding in what aspects and to what extent participants' views on mathematical thinking evolve during such a course.

3. Procedure

Data collection proceeded in three stages of instruction: initial, intermediate, and late. The instructor, meanwhile, was the researcher of the present study. A six-item questionnaire (developed in four stages of pilot studies) examining participants' pre-instruction views of mathematical thinking (Appendix A) was administered to 44 Taiwanese engineering-major college students and collected at the first class meeting. Students were also requested to hand in their math biography at the next class meeting, serving as auxiliary data for interpreting pre-instruction views. The questionnaire and mathematics biography were followed by several semi-structured individual interviews to validate written data and elicit more information from nine randomly selected students.

The course was scheduled generally in accordance with historical order, handouts relevant to historical knowledge assigned as supplemental materials. In class, mathematical concepts were problematizing to challenge students' intuition-based empirical beliefs in doing mathematics, comprehend the necessity of rigorizing mathematical ideas, appreciate alternative strategies for attacking identical problems. Historical problems (Appendix B), differing from ordinary exercises in nature, served as demanding tasks to motivate intrinsic thinking. All problems assigned were related to curriculum taught at the time. As answers were collected, students demonstrating elaborative thinking were invited to share their ideas on the board, followed by a whole-class discussion.

In the late instruction stage, the identical questionnaire was again conducted on all participants, followed by several one-to-one interviews validating written responses and comparing interviewees' views before and after instruction. To minimize potential bias, respondents were never informed about the purpose of study.

4. Pre-instruction Views

Data analysis began the first day of data collection. Participants' initial views on mathematical thinking were analyzed on the basis of written responses on six-item, open-ended questionnaires and transcriptions of follow-up interviews conducted with nine randomly selected interviewees. Moreover, students' past learning experiences, as told in their mathematics biographies, served as auxiliary data for interpreting initial views.

In the first item, all respondents defined mathematical thinking, aiming to profile the essence of the construct in their minds. Twenty (45%) associated mathematical thinking with ways of solving problems or deriving answers. Further, participants tended to relate solving problems to derive answers by following predetermined routes and perceived pondering on mathematics more as recalling and applying formulas. On the other hand, 12 participants (27%) referred to mathematical thinking as a process of logical thinking or reasoning; several interviewees expressing this view but confessed they had never experienced the merit.

How good a problem solver in some sense is subject to how well one copes with untried and demanding tasks. The second questionnaire item aimed at exploring how students reacted to predicaments; 15 (34%) reported that the first thing they would do is seeking outer assistance or skip it entirely. Others adopted conservative strategies to evade difficult positions by recalling formulas or similar problems, eight (18%) claiming they would think on their own before asking for help. One of the interviewees, Ming, reported he was usually persistent. When asked about his motivation, he responded:

> There is little to do with confidence. *This is what mathematics is all about* [italics added]; you have to think. …You would feel it easy when you achieve a breakthrough in your thinking. (Ming, pre-instruction interview)

It appears that Ming demonstrated a thoughtful belief about mathematics as well as an active view on mathematical thinking, a mathematician-like disposition.

The mathematician is typically regarded as the perfect mathematical problem solver and laypersons usually conceptualize mathematicians' ways of thinking as an archetype. On the basis of this notion, participants were asked to propose how the mathematician thinks of a mathematical problem. Ten respondents (23%) considered mathematicians as generally being able to attack problems from diverse angles or apply alternative approaches. Many attributed mathematicians' ability to owning solid knowledge background, as evidenced by the following quote:

> Mathematicians' brains *must be filled with various kinds of definitions and solutions for solving problems* [italics added]. . . they are able to solve problems *by using very simple, quick and precise approaches* [italics added]. (Mong, pre-instruction questionnaire)

In contrast, four respondents cited hard thinking as critical to mathematicians' vocation. Chang plotted a vibrant mode involving activities like survey, making/testing conjecture, and verifying results, revealing the empirical aspect of mathematics.

Mathematics is typically seen as requiring creativity, yet memorization is usually viewed as the best way to learn it (Schoenfeld, 1989). It is noteworthy to scrutinize participating Taiwanese college freshmen's views on this concern. Twenty-six respondents (59%) thought problem solving in mathematics is much like a creative activity. Among them, 12 claimed that solving problems involves personal creativity because there are always various ways to do mathematics. Eleven respondents (25%) took a neutral position (both creativity and preset procedure are required for doing mathematics), and some perceived the issue as doer-dependent—creativity for experts, preset procedure for novices.

It is presumed that students must own certain impressions, adequate or inadequate, regarding mathematics after years of learning the discipline. Surprisingly, when asked to define mathematics, eight (18%) were mute on this concern. Among those responding to the item, nine (20%) associated mathematics with numbers; seven (16%) interpreted mathematics as a practical tool in daily life; five (11%) professed that mathematical results must be infallible through the ages. Contrarily, some saw mathematics from alternative windows, viewing it as fundamental to science and inextricably related to the study of reality. Moreover, participants were asked to address, at their best understanding, how mathematical knowledge developed. Thirteen (30%) considered growth of mathematics progressive and subject to human demand. On the other hand, interviewees were further asked whether mathematics could exist parallel or unrelated to human demand. They in general showed poor understanding of this issue; an appreciation of abstract thinking was seemingly lacking.

5. Post-Instruction Views

Analysis of students' post-instruction views was mainly based on written responses to post-instruction questionnaires and selected interviewees' transcriptions. Initial and late views were compared and contrasted to identify any commonality or distinction. Several short essays regarding classroom activity, written by participants, served as auxiliary data sources for interpreting professed statements.

Similarly, while responding to what mathematical thinking is, participants were more likely to associate it with the process of solving mathematical problems; 18 of them (41%) claimed mathematical thinking means figuring out a way to reach answers. Their wording, however, differed in some way. They tended to conceptualize mathematical thinking as solving problems in one's own way, multiple approaches, or peculiar ideas. In addition, participants were more likely to value logical sense in doing mathematics this time. For instance, Liu, who considered mathematical thinking merely as a route leading to answer at the outset, professed:

> Mathematical thinking could mean that attaining reasonable answers through logic of making sense and reasonable generalization. In sum, it is a process of solving problems by means of reasonable ideas and procedure. (Liu, post-instruction questionnaire)

By reasonable procedure, Liu meant evidential and meaningful facts. Several respondents also cited mathematical thinking as a way of exploring rationale of formulas and intuition alone as unreliable, suggesting justification began to loom larger in their minds.

Participants' strategies reacting to predicaments generally showed wide diversity. In addition to looking for relevant material and asking for outer assistance, 11 (compared to two at the beginning) emphasized they would try to understand a problem, identify all knowns and unknowns, then make a plan. Moreover, several participants exhibited more willingness to discuss with others, yet neither written nor oral responses manifested any significant improvement of individual persistence while doing mathematics.

During their instruction, participants witnessed several ancient mathematicians' approaches to specific problems. It is therefore noteworthy to investigate again their thought about how the mathematician thinks. Contrast of answers yielded an unchanged point of view: mathematicians are good at attacking a problem from multiple facets and diverse angles. Nonetheless, they stressed more a mathematician's imagination and creativity, less one's approach as most convenient and quickest. Shern initially proposed mathematicians tends to think by reasoning, later turned to highlight their capability of association and imagination. In interview, he took Newton and Archimedes as instances:

> Just like capability of association, many figures had discovered calculus but not specific until Newton. I consider imagination is more important is because of Archimedes. I feel he is so strange. He derived the volume of a sphere by means of lever... How did he think of it? Plus, he transferred a circle into a triangle. I feel his imagination is quite strange. (Shern, post-instruction interview)

He further labeled Archimedes' approach inaccessible when merely relying on reasoning, the cause for changing his mind. Moreover, following recognition of mathematicians' imagination, the majority of participants held that doing mathematics involves more individual creativity as opposed to following preset procedure.

An important issue in the present study is, in such a historical approach course, whether participants' epistemological belief regarding mathematics had been affected in some way. By contrasting responses, several distinctions emerged. While a majority still viewed mathematics as a fundamental subject (involving numbers, operations and logic) for exploring other disciplines, one chief difference was that no participant claimed mathematical knowledge is absolute truth. During the semester, several inaccurate mathematical conceptions in history, such as Euler's mistake on infinite series, were presented to students to demonstrate the fallible aspect of mathematical thinking. It appears students were impressed by these examples and leaning toward a conservative attitude toward certainty of mathematical knowledge. Asked about the possibility of new mathematical facts superseding old ones, no interviewees showed doubt; all defended by citing examples given in class. According to them, mathematical criteria evolve over the course of time, and validity of mathematical knowledge is constantly examined.

Calculus taught at school today is entirely credited to European mathematicians, but several concepts of integral calculus had occurred in the oriental world as well. This historical approach course also covered issues of different approaches to deriving area of a circle and volume of a sphere between ancient Chinese mathematicians (Liu Hui and Zu Chongzhi) and Archimedes. Participants were then asked to compare and contrast the different fashions between these types of mathematical thought. Most held that Chinese mathematicians tended to think in intuition, operated mathematical ideas via concrete figures, and usually demonstrated results without justification, whereas the Greek was more likely to approach a problem from unusual angles by integrating physical concepts and verify answers in a meticulous manner. In short, Chinese method is direct and intuitive rather than theory-laden; Archimedes' thinking is indirect and skillful with rigorous confirmation.

6. Summary and Discussion

The aforementioned findings suggest that, as a rule, participants initially viewed doing mathematics as a solution-oriented activity, in which mathematical thinking is degraded as fixed processes leading to final answers. Thinking of mathematics as such was interpreted as a way of recalling content; mathematicians therefore were seen as figures possessing more solid knowledge background and experiences in solving problems. The phenomenon can be explained by the mathematics biographies, revealing exam-oriented mathematics teaching in Taiwan had intensely, but distortedly, shaped recognition of mathematical thinking. Conscious reflection was lacking while engaging in mathematical activity, resulting in superficial understanding of the essence of mathematics.

After an 18-week historical approach, problem-based calculus course, students' views of mathematical thinking in particular, mathematics in general, had shifted in some ways. Though still referring to mathematical thinking as a procedure for deriving answers, post-instruction responses showed an inclination to stress the role of creativity in solving problems and necessity of involving relevant concepts of other disciplines. Such leaning, on the basis of interview transcripts, may be attributed to ancient mathematicians' imaginative approaches learned in class, demonstrating a wide range of possibilities in attacking a problem. Meanwhile, after exposure to historical mistakes, they were less likely to believe mathematical knowledge is time-independent truth and more likely to value necessity of justification. Participants' focus seemingly shifted from mathematics as a product to mathematics as a process.

Despite these above inspiring outcomes, several emergent issues merit further attention. Firstly, many participants showed more eagerness to try, whereas individual persistence in thinking on mathematics did not significantly improve. Strategy most often adopted by them was discussing with others, mostly because of the difficulty of assigned problems. Selecting moderate tasks from history, challenging but accessible, thus is a critical factor in success of a study of this type. Secondly, most participants were impressed by Archimedes' fashion of thinking, but his ideas were not viewed as applicable by most interviewees. In their minds, a good method ought to be simple, precise, and intuitive. Asked to compare Zu Chongzhi's and Archimedes' approaches to deriving volume of a sphere, eight interviewees preferred Zu's thinking; Archimedes' peculiar thought was more like models in the shop window, drawing gaze but not approach. In some sense, one chief purpose of the effort made in the present research is to foster students' appreciation of ingenuity and beauty of mathematical thinking. This finding nevertheless reveals a restriction of this study. Is this an educational challenge or a cultural issue? Cross-cultural study may help to resolve these doubts. Thirdly, several respondents showing not much difference in their professed views not only expressed a rigid view about the concerns, but also demonstrated conservative performance on the challenging tasks. They tended to approach problems in a fixed and traditional fashion. The interplay between an individual's pre- and post-instruction views and degree of consistency between one's views and behavior make noteworthy issues for further study.

Integrating history into mathematics curricula has been promulgated for decades, yet cannot be accepted without question. The present study is not an experimental design, so no cause-effect inference can be made; this is an exploratory attempt laying groundwork for further research in this respect. Fried (2001) raises several critical concerns regarding possibility of combining mathematics education and history. Best strategy for revealing the doubt is probing what history can and cannot do for mathematics education through empirical investigations. History of mathematics is by no means the prescription of mathematics education, but definitely can be a guide to it.

REFERENCES

-American Association for the Advancement of Science (1990). Science for All Americans. New York: Oxford University Press.

-Barbin, E. (1996). The role of problems in the history and teaching of mathematics. In R. Calinger (Ed.), *Vita mathematica* (pp.17-25). Washington, D.C.: Mathematics Association of America.

-Carlson, M. P. (1999). The mathematical behavior of six successful mathematics graduate students: Influences leading to mathematical success. *Educational Studies in Mathematics, 40*, 237-258.

-Ernest, P. (1998). The history of mathematics in the classroom. *Mathematics in School, 27(4)*, 25-32.

-Franke, M. L., & Carey, D.A. (1997). Young children's perceptions of mathematics in problem-solving environments. *Journal for Research in Mathematics Education, 28(1)*, 8-25.

-Fried, M. N. (2001). Can mathematics education and history of mathematics coexist? *Science and Education*, 10, 391-408.

-Henningsen, M. & Stein, M. K. (1997). Mathematical tasks and student cognition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education*, *28*(5), 524-549.

-Higgins, K. H. (1997). The effect of year-long instruction in mathematical problem solving on middle school students' attitudes, beliefs, and abilities. *The Journal of Experimental Education, 66(1)*, 5-28.

-Kloosterman, P. & Stage, F. K. (1991). Relationships between ability, belief and achievement in remedial college mathematics classrooms. *Research and Teaching in Developmental Education, 8*(1), 27-36.

-Lester, F. K. (1994). Musing about mathematical problem solving research: 1970-1994. *Journal for Research in Mathematics Education*, *25*, 660-675.

-National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: Author.

-Owen, E., & Sweller, J. (1989). Should problem-solving be used as a learning device in mathematics? *Journal for Research in Mathematics Education, 20*, 322-328.

-Rickey, V. F. (1995). My favorite ways of using history in teaching calculus. In F. Swetz, J. Fauvel, O. Bekken, B. Johansson, & V. Katz (Eds.), *Learn from the masters* (pp.123-134). Washington, D.C.: Mathematics Association of America.

-Schoenfeld, A. H. (1989). Explorations of students' mathematical beliefs and behavior. *Journal for Research in Mathematics Education, 20(4)*, 338-355.

-Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 340-370). New York: Macmillan.

-Schoenfeld, A. H. (1994). Reflections on doing and teaching mathematics. In A. H. Schoenfeld (Ed.), *Mathematical thinking and problem solving* (pp. 53-75). Hillsdale, NJ: Lawrence Erlbaum Associates.

-Siu, M. (1995a). Euler and heuristic reasoning. In F. Swetz, J. Fauvel, O. Bekken, B. Johansson, & V. Katz (Eds.), *Learn from the masters* (pp.145-160). Washington, D.C.: Mathematics Association of America.

-Siu, M. (1995b). Mathematical thinking and history of mathematics. In F. Swetz, J. Fauvel, O. Bekken, B. Johansson, & V. Katz (Eds.), *Learn from the masters* (pp.279-282). Washington, D.C.: Mathematics Association of America.

-Sweller, J. (1990). On the limited evidence for the effectiveness of teaching general problem-solving strategies. *Journal for Research in Mathematics Education, 21(5),* 411-415.
-Swetz, F. J. (1995a). To know and to teach: Mathematical pedagogy from a historical context. *Educational Studies in Mathematics. 29,* 73-88.
-Swetz, F. J. (1995b). Using problems from the history of mathematics in classroom instruction. In F. Swetz, J. Fauvel, O. Bekken, B. Johansson, & V. Katz (Eds.), *Learn from the masters* (pp.25-38). Washington, D.C.: Mathematics Association of America.

Appendix A (open-ended questionnaire)

1. In your understanding, what is mathematical thinking? Please explain your answer with examples.
2. When you are stuck on an unfamiliar mathematics problem, what is your instant reaction to and strategy for this?
3. In your understanding and imagination, how do mathematicians think while solving a problem? Is there any difference between a mathematician's way of thinking and a layperson's?
4. Some hold that solving mathematical problems is a thinking activity involving personal creativity; others argue that getting correct answers requires following predetermined, known procedures. What is your opinion about this? Why? Please defend your answer with examples.
5. In your opinion, what is mathematics? What makes mathematics differ from other disciplines?
6. In your opinion, how does mathematical knowledge develop? Does the development of mathematical knowledge follow any rule? Please defend your answer with examples.

Appendix B Historical problems

1. Finding the area of a circle (Archimedes, Liu Hui, Seki Kowa)
2. The method for finding the area of a circle on Rhind Papyrus
3. Archimedes' quadrature of the parabola
4. Fibonacci sequence
5. Computing the sum of 1–1+1–1+1–1+…
6. Approaches to finding the tangent line to a curve (Descartes, Fermat, and Barrow)
7. Napier's logarithm
8. Fermat's approach to find extreme values
9. The curve of witch of Agnesi
10. The Tractrix problem
11. Finding the volume of a sphere
12. Finding the volume of a sphere inscribed in a cylinder