

USING MATLAB TO SOLVE A CLASSIFICATION PROBLEM IN FINITE RINGS

Chiteng'a John CHIKUNJI

Department of Mathematics and Statistics,
University of Zambia, P.O. Box 32379, Lusaka, ZAMBIA
email: jchikunji@natsci.unza.zm

ABSTRACT

Although most linear algebra problems can be solved using a number of software packages, in our judgment MATLAB (MATrix LABoratory) is the most suitable package. MATLAB is a versatile and powerful, yet user - friendly software package designed to handle wide - ranging problems involving matrix computations and linear algebra concepts. MATLAB incorporates professionally developed quality computer routines for linear algebra computations.

In this paper, we make use of elements from MATLAB to devise a programme that helps in determining the structure and classification, up to isomorphism, of a naturally arising class of finite associative local rings.

We demonstrate this in the case where the finite local ring has a finite residue field K of characteristic p , although our results apply in fact over any field K .

1 Introduction

The use of computer technology has been widely discussed as having the potential to radically change higher education.

The ways in which information is put to work today seem almost countless, and computers are continually assuming a larger role in preparing this information suitably for the needs of teachers and students. Problem solving is an important example.

For computers to play a part in problem solving, it is necessary that communication be established between them and their users. In presenting a problem for a solution, the user still has a substantial role to play. It is still not possible to address the machine in one's own language, but it is fairly easy to learn and use programming language that resembles it and the effort to improve that resemblance is relentless. Computers are gradually being taught to accept more and more of the communications burden.

MATLAB (MATrix LABoratory) is a high-performance interactive software package for scientific and engineering numeric computations. MATLAB integrates numerical analysis, matrix computation, signal processing, and graphics in an easy-to-use environment where problems and solutions are expressed just as they are written mathematically - *without traditional programming*.

In this paper, we make use of elements from MATLAB to devise a programme that helps in determining the structure and classification, up to isomorphism, of a naturally arising class of finite associative local rings. In particular, we consider rings of the form

$$R = K \oplus J$$

in which $K = F_q$, a finite field of $q = p^r$ elements, (with p a prime, r a positive integer) and the Jacobson radical J is such that $J^3 = (0)$ and $J^2 \neq (0)$.

2 A Problem in Finite Rings

In investigating the structure of finite associative local rings, one is led to consider such a ring of the form

$$R = K \oplus J$$

in which $K = F_q$, a finite field of $q = p^r$ elements, and the Jacobson radical J is such that $J^3 = (0)$ and J/J^2 is two-dimensional and J^2 is three-dimensional over $R/J = K$.

Rings with $J^3 = (0)$ and $J^2 \neq (0)$ form an object of study (e.g. Chikunji 1999), the case $J^2 = (0)$ having long been settled (e.g. Corbas 1969, 1970).

If

$$J = Kx_1 \oplus Kx_2 \oplus J^2$$

and

$$J^2 = Ky_1 \oplus Ky_2 \oplus Ky_3,$$

then we may write

$$x_i x_j = \alpha_{ij} y_1 + \beta_{ij} y_2 + \gamma_{ij} y_3,$$

with $\alpha_{ij}, \beta_{ij}, \gamma_{ij} \in K$, and these four products span J^2 . The ring structure is now determined by the triple of 2×2 matrices $A = (\alpha_{ij})$, $B = (\beta_{ij})$, and $C = (\gamma_{ij})$, which

are linearly independent over K and any triple of linearly independent matrices defines such a ring.

In Chikunji 2002, on the basis of computational calculations, we conjectured that there are 5 isomorphism classes for $p = 2$ and when p is odd, the number of isomorphism classes of such rings is $p + 4$. We further conjectured that exactly one of these rings is commutative, for every prime p .

In this paper, we extend the above results to all finite fields F_q , where $q = p^r$.

If $(x'_1, x'_2, y'_1, y'_2, y'_3)$ is a new basis of J with corresponding matrices A', B', C' , then x'_1, x'_2 are linear combinations of x_1, x_2, y_1, y_2, y_3 . Since $J^3 = (0)$, we may assume that the coefficients of y_1, y_2, y_3 are zero and write

$$x'_i = p_{1i}x_1 + p_{2i}x_2,$$

so that $P = (p_{ij})$ is the transition matrix from the basis (\bar{x}_1, \bar{x}_2) of J/J^2 to the basis (\bar{x}'_1, \bar{x}'_2) .

Equally, let $Q = (q_{ij})$ be the transition matrix from the basis (y_1, y_2, y_3) to (y'_1, y'_2, y'_3) .

If we now calculate $x'_i x'_j$ and compare coefficients of y_i , we obtain equations which, in matrix form, are

$$\begin{aligned} P^t A P &= q_{11} A' + q_{12} B' + q_{13} C' \\ P^t B P &= q_{21} A' + q_{22} B' + q_{23} C' \\ P^t C P &= q_{31} A' + q_{32} B' + q_{33} C', \end{aligned}$$

where P^t is the transpose of the matrix P .

Evidently, the problem of classifying our rings up to isomorphism amounts to that of classifying triples of linearly independent matrices under the above relation of *equivalence*, P and Q being arbitrary invertible matrices, and it is this problem of linear algebra that the paper is devoted to illustrate using elements of MATLAB.

If $\langle A, B, C \rangle$ is a subspace of $M_2(K)$ spanned by A, B and C , we may equally speak of $\langle A, B, C \rangle$ and $\langle A', B', C' \rangle$ being "congruent" via P . Also, if \mathcal{X} is the set of all triples (A, B, C) , then $GL_2(K)$ acts on the right of \mathcal{X} by

$$(A, B, C) \cdot P = (P^t A P, P^t B P, P^t C P)$$

and on the left by

$$Q \cdot (A, B, C) = (q_{11}A + q_{12}B + q_{13}C, q_{21}A + q_{22}B + q_{23}C, q_{31}A + q_{32}B + q_{33}C),$$

where $Q = (q_{ij})$.

These two actions are permutable and define a (left) action of $G = GL_2 \times GL_3$ on \mathcal{X} :

$$(P, Q) \cdot (A, B, C) = Q \cdot (A, B, C) \cdot P^{-1}.$$

By restriction, G acts on the subset Y consisting of triples with A, B, C linearly independent. This amounts to studying the congruence action (via P) of GL_2 on the subset \mathcal{Y} of 3-dimensional subspaces of $M_2(K)$, Q just representing a change of basis in a given subspace. In the same way, the whole action of G on \mathcal{X} may be reinterpreted as an action of GL_3 on the subset X of subspaces of dimension ≤ 3 . The two triples in the same G -orbit will be called *equivalent*.

The complex nature of this problem prompts us to look for ways of finding the number of non-isomorphic classes.

3 Problem Analysis

With all this superlative hardware and software in place, how much is left for the user to do? As already noted, there is a programming language to be learned, which in this case is MATLAB, to complete the closing of the communications gap. But before programming can begin, the problem to be solved needs preparation. In spite of their impressive capabilities, computers still have to be told exactly what to do, in a step-by-step fashion. This process of satisfactorily achieving the required level of detail is called *Problem Analysis*, and it is the user's responsibility. It is by no means an easy assignment.

4 A MATLAB Programme With Elements From The Field $K = F_3$

In this section, we devise a programme that illustrates the use of MATLAB to solve the problem of §2. We illustrate this for the case where the ring R is of characteristic $p = 3$ and the residue field R/J is isomorphic to F_3 .

In our programme, the invertible matrices P and Q given in §2 are denoted by the matrices M and N , respectively.

```
function jo(a)
global A
global B
global C
T=[ ];
for i=1:12
    if a >= 2*3^(12 - i) T(i) = 2; a = a - 2*3^(12 - i);
        elseif a >= 3^(12 - i) T(i) = 1; a = a - 3^(12 - i);
            else T(i) = 0;
        end
    end
end
A = [T(1:2); T(3:4)];
B = [T(5:6); T(7:8)];
C = [T(9:10); T(11:12)];
```

```
function joh(a)
global M
T = [ ];
for i = 1:4
    if a >= 2*3^(4 - i) T(i) = 2; a = a - 2*3^(4 - i);
        elseif a >= 3^(4 - i) T(i) = 1; a = a - 3^(4 - i);
            else T(i) = 0;
        end
    end
end
M = [T(1:2); T(3:4)];
```

```
function john(a)
global N
T = [ ];
for i = 1:9
    if a >= 2*3^(9 - i) T(i) = 2; a = a - 2*3^(9 - i);
        elseif a >= 3^(9 - i) T(i) = 1; a = a - 3^(9 - i);
```


$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix};$$

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This programme may be modified several times to obtain results over other finite fields F_q , where $q = p^r$, p a prime and r a positive integer.

We may now state the following result based on our computational calculations using MATLAB programmes.

4.1 Theorem *For the rings of §2, there are 5 isomorphism classes for $p = 2$ and when p is odd, the number of isomorphism classes of such rings is $p^r + 4$. Furthermore, exactly one of these rings is commutative, for every prime p .*

5 Conclusion

MATLAB is an interactive system whose basic element is a matrix that does not require dimensioning. This allows one to solve many numerical problems in a fraction of the time it would take to write a programme in a language such as Fortran, Basic or C. Furthermore, as may be seen from the above problem, solutions are expressed in MATLAB almost exactly as they are written mathematically.

In university environments, it has become the standard instructional tool for introductory courses in applied linear algebra, as well as advanced courses in other areas. Just like in trying to find a solution to the above classification problem in finite rings, MATLAB can be used for research and to solve practical engineering and mathematical problems.

REFERENCES

- Chikunji C. J., 1999, "On a class of Finite Rings", *Comm. Algebra*, **27** (10), 5049-5081.
- Chikunji C. J., 2002, "Enumeration of Finite Rings with Jacobson Radical of Cube Zero", (to appear in *SAMSA Journal*, **1**(2)).
- Corbas B., 1969, "Rings with few zero divisors", *Math. Ann.*, **181**, 1-7.
- Corbas B., 1970, "Finite rings in which the product of any two zero divisors is zero", *Archiv der Math.*, **XXI**, 466-469.
- PC-MATLAB, 1989, "User's Guide".