

**LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT
COEFFICIENTS:**

An alternative to the method of undetermined coefficients

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ABSTRACT

Here we describe and illustrate an alternative to the method of undetermined coefficients for obtaining a particular solution of a linear differential equation with constant coefficients. The method requires only polynomial differentiation and some elementary algebra. The procedure has also been expressed as a recursive algorithm. Examples have been included to show the usefulness of the recursive analogue. Both the technique and its recursive equivalent can be suitably reformulated for similar difference equations.

1 Introduction

In this paper we are concerned with nonhomogeneous linear differential equations with constant coefficients. The term which makes the equation nonhomogeneous is a linear combination of the terms of the type $e^{\alpha x} p_n(x)$, where α is a (real or complex) constant and p_n is a polynomial of finite degree n in x .

The method of undetermined coefficients is, perhaps, one of the most widely used procedures for obtaining a particular solution of such a differential equation. The procedure is affected by choosing an appropriate trial solution containing unknown constants. Evaluation of these constants so that the trial solution satisfies the given differential equation leads to the required particular solution. The trial solution can be obtained by the annihilator method, but usually, it is obtained by following a set of rules [1], and unknown constants are determined by solving a system of linear equations. The procedure may be quite involved and often leads to tedious algebra.

For instance, consider the problem of finding a particular solution of

$$\begin{aligned} L(D)y &= (D^6 + D^5 + D^4 - D^2 - D - 1)y \\ &= x^3 + 4(11 - 6x^2)e^{-x} + 24x^2 \sin x + 6e^{-x/2} \cos(x\sqrt{3}/2), \quad D \equiv \frac{d}{dx}. \end{aligned} \quad (1.1)$$

Appropriate choice for a particular solution of this differential equation (1.1) will contain 17 constants, to be determined by solving a linear system of 17 equations in 17 unknowns! However, by the superposition principle, its particular solution would be the sum of particular solutions of

$$L(D)y = x^3 \quad (1.2)$$

$$L(D)y = 4(11 - 6x^2)e^{-x} \quad (1.3)$$

$$L(D)y = 24x^2 \sin x \quad (1.4)$$

$$L(D)y = 6e^{-x/2} \cos(x\sqrt{3}/2). \quad (1.5)$$

Computation of particular solution for the above equations would respectively lead to solutions of different systems of linear equations in four, three, six and four unknowns. In the following we present a much simpler alternative which uses only differentiation and some very simple algebra.

2 The procedure

Consider the equation

$$P(D)y = \sum_{i=0}^m b_i D^{m-i} y = e^{\alpha x} p_n(x), \quad (2.1)$$

where $b_i, 0 \leq i \leq m, b_0 \neq 0$ are constants, D denotes d/dx and other symbols have already been described. In general, the integers m and n are not the same. Set $y = e^{\alpha x} u$ in (2.1) to get

$$P(D)(e^{\alpha x} u) = e^{\alpha x} P(D + \alpha)u = e^{\alpha x} p_n(x).$$

This leads to finding a particular solution of

$$P(D + \alpha)u = \sum_{i=0}^m c_i D^{m-i} u = p_n(x), \quad (\text{say}) \quad (2.2)$$

Here $c_i, 0 \leq i \leq m$ are real or complex constants, and we assume $c_m \neq 0$. Now differentiate both sides of (2.2) repeatedly n times, so that the right of the last equation becomes a constant ($= D^n p_n(x)$). Then a particular solution would be given by

$$\begin{aligned} D^n u &= D^n p_n(x) / c_m = \text{constant} \\ D^{n+s} u &= 0, s = 1, 2, \dots \end{aligned} \quad (2.3)$$

Finally by backsolving we obtain a particular solution of (2.2) and since $y = e^{\alpha x} u$, a particular solution of (2.1) is obtained. It may be remarked here that the procedure leads to a particular solution in terms of lowest order derivative of u appearing in (2.2), which on integration leads to a particular solution of (2.2).

This procedure is based on the description given by Love [2] and its generalisation [3].

3 Examples

To illustrate we obtain particular solutions of equations (1.2)-(1.5).

- Differentiate (1.2) three times to get

$$(D^7 + D^6 + D^5 - D^3 - D^2 - D)y = 3x^2 \quad (3.1)$$

$$(D^8 + D^7 + D^6 - D^4 - D^3 - D^2)y = 6x \quad (3.2)$$

$$(D^9 + D^8 + D^7 - D^5 - D^4 - D^3)y = 6. \quad (3.3)$$

A particular solution of this equations is

$$D^3 y = -6, D^{3+r} y = 0, r = 1, 2, \dots$$

Substitute these in (3.2) to get $D^2 y = 6 - 6x$. Combining these with (3.1) we get $Dy = 6x - 3x^2$, and finally (1.2) gives $y = -6 + 3x^2 - x^3$.

- To obtain a particular solution of (1.3) set $y = e^{-x} u(x)$ to arrive at

$$L(D - 1)u = (D^6 - 5D^5 + 11D^4 - 14D^3 + 10D^2 - 4D)u = 4(11 - 6x^2). \quad (3.4)$$

Differentiating this we obtain

$$(D^7 - 5D^6 + 11D^5 - 14D^4 + 10D^3 - 4D^2)u = -48x$$

$$(D^8 - 5D^7 + 11D^6 - 14D^5 + 10D^4 - 4D^3)u = -48$$

Its obvious solution is $D^3 u = 12, D^{3+k} u = 0, k = 1, 2, \dots$. Backsubstituting in the preceding equation we get

$$D^2 u = 30 + 12x,$$

and (3.4) gives

$$Du = 22 + 30x + 6x^2.$$

This is satisfied by $u = x(22 + 15x + 2x^2)$.

Thus Eq. (1.3) has a particular solution $y = xe^{-x}(22 + 15x + 2x^2)$.

••• By the principle of superposition a particular solution of (1.4) would be the imaginary part of the particular solution of

$$L(D)y = 24x^2e^{ix} \quad (3.5)$$

Here we set $y = e^{ix}u$ so that (3.5) becomes

$$(D^6 + (1 + 6i)D^5 + (-14 + 5i)D^4 - (10 + 16i)D^3 + (8 - 10i)D^2 + 4D)u = 24x^2$$

Proceeding as before we get

$$u = 2x^3 - 3(4 - 5i)x^2 + 3(1 - 24i)x,$$

and

$$Im(ue^{ix}) = 3x(5x - 24) \cos x + x(2x^2 - 12x + 3) \sin x$$

is a particular solution of (1.4).

•••• Again by superposition principle a particular solution of (1.5) is real part of the particular solution of

$$L(D)y = 6e^{(-1+i\sqrt{3})x/2}. \quad (3.6)$$

The substitution $y = e^{(-1+i\sqrt{3})x/2}u$ in this equation gives

$$(D^6 + (-2 + 3\sqrt{3}i)D^5 - (9 + 5\sqrt{3}i)D^4 + (13 - 3\sqrt{3}i)D^3 + \frac{3}{2}(-1 + 3\sqrt{3}i)D^2 - \frac{3}{2}(1 + i\sqrt{3})D)u = 6.$$

Obviously this equation has $Du = (-1 + i\sqrt{3})$ as its solution and

$$Re((-1 + i\sqrt{3})xe^{(-1+i\sqrt{3})x/2}) = -xe^{-x/2}(\cos(x\frac{\sqrt{3}}{2}) + \sqrt{3}\sin(x\frac{\sqrt{3}}{2}))$$

is a particular solution of (1.5).

4 The recursive algorithm

We notice [3] that the problems of finding a particular solution of a nonhomogeneous linear differential equation of the form (2.1) is reduced to finding a particular solution of (2.2). In view of (2.3) we rewrite (2.2) as

$$\sum_{i=0}^n \beta_i D^{n-i}u = p_n(x). \quad (4.1)$$

This has been obtained from (2.2) by ignoring all the terms containing $D^{(n+s)}u, s = 1, 2, \dots, m-n$ when $m > n$ and adding terms containing $D^{(m+r)}u, r = 1, 2, \dots, n-m$ with

zero coefficients when $m < n$. For convenience, we take $\beta_n \neq 0$ in (4.1). The process of n -times differentiation and backsubstitution can be expressed as the recursive relation

$$\beta_n D^{n-j} u(x) = D^{n-j} p_n(x) - \sum_{i=n-j}^{n-1} \beta_i D^{2n-i-j} u(x), j = 0, 1, \dots, n \quad (4.2)$$

For $j = 0$ this gives a particular solution of the equation obtained by differentiating (4.1) n times and its recursive use gives $u^{(0)}(x) (= u(x))$ when $j = n$, which is a particular solution of (4.1).

However, if $\beta_n = 0$ and $\beta_{n-1} \neq 0$, one obtain a solution u' which, after one integration, gives the required u . In fact, this recursive scheme gives a particular solution in terms of lowest order derivative in (4.1).

To illustrate the use of this algorithm (4.2) we obtain particular solutions of some differential equations.

• Following the above remarks, a particular solution of (1.2) is the same as that of the equation

$$(0D^3 - D^2 - D - 1)y = x^3$$

Here we have $n = 3, \beta_0 = 0, \beta_1 = -1 = \beta_2 = \beta_3, p_3(x) = x^3$, and the equation (4.2) takes the form

$$-D^{3-j}y = D^{3-j}p_3 - \sum_{i=3-j}^2 \beta_i D^{6-i-j}y, j = 0, 1, 2, 3$$

This gives

$$\begin{aligned} (j = 0) \quad & -D^3y = 6 \\ (j = 1) \quad & -D^2y = 6x - (-1)(D^3y) \\ & = 6x - 6 \\ (j = 2) \quad & -Dy = 3x^2 - (-1)(-6) - (-1)(6 - 6x) \\ & = 3x^2 - 6x \\ (j = 3) \quad & -y = x^3 - (-1)(6 - 6x) - (-1)(6x - 3x^2) \\ & = x^3 - 3x^2 + 6 \end{aligned}$$

which gives the same particular solution of (1.2) as obtained earlier.

•• Particular solution of (3.4) is the same as that of

$$(-7D^2 + 5D - 2)v = 2(11 - 6x^2)$$

with $v = Du$. For this equation $n = 2, \beta_0 = -7, \beta_1 = 5, \beta_2 = -2, p_2(x) = 2(11 - 6x^2)$. The scheme (4.2) becomes

$$-2D^{2-j}v = D^{2-j}p_2 - \sum_{i=2-j}^1 \beta_i D^{4-i-j}u, j = 0, 1, 2.$$

This yields

$$(j = 0) \quad -2D^2v = -24$$

$$\begin{aligned}
(j=1) \quad -2Dv &= -24x - 5(D^2v) \\
&= -24x - 60 \\
(j=2) \quad -2v &= 2(11 - 6x^2) - (-7)(12) - 5(12x + 30) \\
&= -44 - 60x - 12x^2,
\end{aligned}$$

as expected, leading to the same particular solution of (3.4) as computed earlier.

••• The differential equation

$$(D^7 + D^5 - D^4 - D^2)u = 30x^4 \quad (4.3)$$

is the same as

$$(D^5 + D^3 - D^2 - 1)v = 30x^4$$

with $v = D^2u$. Its particular solution is the same as that of the equation

$$(0D^4 + D^3 - D^2 + 0D - 1)v = 30x^4$$

For this equation $n = 4, \beta_0 = 0, \beta_1 = 1, \beta_2 = -1, \beta_3 = 0, \beta_4 = -1, p_4(x) = 30x^4$. The algorithm (7) becomes

$$-D^{4-j}v = D^{4-j}p_4 - \sum_{i=4-j}^3 \beta_i D^{8-i-j}v, j = 0, 1, 2, 3, 4.$$

This gives

$$\begin{aligned}
(j=0) \quad -D^4v &= 720 \\
(j=1) \quad -D^3v &= 720x \\
(j=2) \quad -D^2v &= 360x^2 - (-1)(D^4v) \\
&= 360x^2 - 720 \\
(j=3) \quad -Dv &= 120x^3 - (D^4v) - (-1)(D^3v) \\
&= 120x^3 - 720x + 720 \\
(j=4) \quad -v &= 30x^4 - (-720x) - (-1)(720 - 360x^2) \\
&= 30x^4 - 360x^2 + 720x + 720.
\end{aligned}$$

Since $v = D^2u$, this gives $u = 30x^2(-12 - 4x + x^2) - x^6$ as a particular solution of (4.3).

The procedure used here completely replaces the method of undetermined coefficients for a particular solution of nonhomogeneous linear differential equation with constant coefficients.

5 Epilogue

Here we consider ordinary linear differential equations with constant coefficients in which the nonhomogeneous term is a linear combination of the terms of the type $e^{\alpha x}p_n(x)$, where α is a (real or complex) constant and p_n is a polynomial of finite degree n in x .

The method of undetermined coefficients is commonly used to find a particular solution of such differential equation. Both from teaching and learning point of view,

this is usually quite demanding. It would, therefore, be pedagogically interesting to have a simpler alternative to this method. In this paper we have presented such a method which requires only differentiation and some additions. Several examples have been included to manifest its versatility.

It has been observed that the problem is finally reduced to finding a particular solution of a linear differential equation with constant coefficients with nonhomogeneous term being a polynomial of a finite degree. The procedure in such a situation can be expressed as a recursive algorithm [3]. This has also been featured and illustrated by obtaining particular solutions of several differential equations.

Thus the present paper contains a simpler alternative to the method of undetermined coefficients in its totality. The differential equations which are amenable to the method of undetermined coefficients are taught almost everywhere at the undergraduate level, perhaps, due to the fact that their applications to the real world problems can not be over emphasized. In view of this, the procedure presented herein is didactically relevant and should attract the attention of everyone involved in the teaching of ordinary differential equations. Finally, it is remarked that the procedure can suitably be reformulated for similar difference equations [4, 5].

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