A STUDY OF CLASSROOM PROCESSES RELATED TO THE PRODUCTION OF MEANING FOR 'FUNCTION': THE CONTEXT OF REAL ANALYSIS VS THE CONTEXT OF DUAL VECTOR SPACES[1]

Teresita NORIEGA
Mathematics Department
University of La Habana, Cuba

Romulo LINS
Mathematics Department/Postgraduate Program in Mathematics Education
UNESP-Rio Claro, Brazil

ABSTRACT

Usually we expect our students to produce meaning for functions in Real Analysis as 'a correspondence between sets of real numbers'. In Algebra we generally start with function as 'a particular subset of a cartesian product', but when working with dual vector spaces we expect them to understand functions as elements of the base set of an algebraic structure. While our teaching experience had already confirmed the results of previous studies showing that students remain attached to the 'analytical' understanding of function, we decided to conduct a study that could further our understanding of this process. This study happened in the context of a regular Algebra course (undergraduate mathematics degree) particularly the section on duality of vector spaces. The data we will present and discuss come from transcriptions of lessons and from tests applied during the course. The theoretical support comes from: (i) EP ('Enseñanza Problémica', in Spanish), a didactical model developed in the former Soviet Union during the second half of the 20th century, based in the historical-cultural theory of Vygotsky, and which provides us with a set of categories that allows us to organise in a dynamical way professor-students interaction; and, (ii) the Theoretical Model of Semantic Fields, developed by R. Lins, an epistemological model that allows us to 'read' the processes of meaning production as they happen, 'on the fly'.

Keywords: function, meaning production, semantic fields, dual vector spaces, problematising teaching (enseñanza problémica)

Introduction

Usually we expect our students to produce meaning for functions in Real Analysis as a correspondence between sets of real numbers. In Algebra we generally start with function as a particular subset of a cartesian product, but when working with dual vector spaces we expect them to understand functions as elements of the base set of an algebraic structure.

What we mean by 'analytical understanding' of function is a function as a correspondence between variables, given or not by an expression or a formula. It's not common in Analysis to consider as different objects a function and the function obtained by a restriction of the domain. Similarly, in Analysis the difference between a function f and the image f (x) of an element of its domain is not generally emphasised.

Differently, in Abstract Algebra one has to consider those aspects carefully. In Algebra, not taking a function and its restriction as different objects can hinder the understanding of important theorems of Linear Algebra. Also, it would not be possible to find an 'inverse' for a not invertible function, by means of its restriction to an appropriate subset of the domain. Considering this 'inverse' is a situation that frequently appears in Algebra and its applications.

Not making distinctions between a function f and the image f (x) of an element is a serious problem when one is working with vector spaces duality, where the student has to deal with functions whose domain is a set of functions. This is the cases of the transpose of a linear map, or the isomorphism (in finite dimension) between a space E and its bidual space E**.

'Algebraic understanding' of functions, in the case of vector spaces duality, will mean for us the acceptance of a function as an element of the base set of a vector space structure.

After trying several approaches to the teaching of duality in Linear Algebra, students´ difficulties with an algebraic understanding of functions persisted.

As our teaching experience had already confirmed the results of previous studies [3], [4], [6] showing that students remain attached to the analytical understanding of functions (which is essentially the one found in school mathematics), we decided to conduct a study that could further our understanding of this process.

We decided that in this study we would change the focus of our analysis from looking to what was missing in the students´ conceptions to eliciting what they really where thinking about functions. This shift in our approach implied not only to consider a didactical model to support the organisation of classroom processes, but also an epistemological model to support the 'reading' of those processes as they happened.

The didactical model chosen was the 'Problematising Teaching' (from the Spanish 'Enseñanza Problémica'; PT on what follows in this paper); the epistemological model chosen was the Theoretical Model of Semantic Fields (TMSF on what follows). They are described in the next two sections.

Based on PT and on TMSF a course on Linear Algebra for undergraduate mathematics students was conducted at the University of Havana (Cuba). The lessons relating to vector space duality were audio taped and analysed; the fourth section of this paper has a discussion of one classroom episode and of the results of a test.

The ‘problematising teaching’ model

PT was developed during the second half of the twentieth century mainly in the former Soviet Union. It integrates principles, categories and methods, which support a coherent didactical strategy. It is based on the Historical-Cultural Theory of Lev Vygotsky, in particular on the psychological thesis that “cognitive activity always grows from conflict between the known and the sought” (see, for instance, [7]).

Conflict is established in a situation in which what the subject knows or believes does not match what is presented to him or is not sufficient to explain it. Such situation is called 'Problemic Situation' (SP) and it is the main category for PT; it 'rules' all the other categories in the model.

From the assimilation of the conflict by the subject results the 'Didactical Problem', which is the form in which the PS is actually going to be approached by the students; it points out the directions in which we are going to conduct the search to solve the SP. The other categories of PT are: 'Didactical Tasks' (which point out the ways in which we are going to conduct the search); 'Didactical Questions' (they help to solve specific conflicts, which remained concealed when the Didactical Tasks were posed); and, 'Problemic Complex' (defined as an structuring of the previous elements that is established from the ways in which they relate to a given concept, or by considering how an element can be derived from the preceding ones).

PT allows the professor to organise classroom processes in a dynamical way, combining the categories provided by PT and taking into account students’ answers, reactions, comments or conclusions. He does not need to remain attached to preconceived ideas about what is going to happen at the classroom.[2]

The theoretical model of semantic fields

This epistemological model was developed to provide a basis for a sufficiently fine reading of the process of meaning production, particularly in classrooms (see [2]).

Its central notions are those of 'knowledge' and 'meaning'. 'Knowledge' is characterised as a statement in which a person believes (a statement-belief), together with a justification he has for making that statement. It departs radically from other characterisations of knowledge by assuming that the justification is a constitutive part of it, not simply a part of the process of that person being said to know something. However, in line with many other authors, it does not work with the notion of implicit knowledge, a quite problematic one; instead, ‘implicit knowledge’ is at best described as third-person knowledge, that is *I* am producing knowledge *about someone else*. 'Meaning' is characterised as what a person actually says about an object, in a given situation. It is not everything that person could eventually say about that object. Meaning production and knowledge production always happen together; at the same time objects are constituted through meaning production.

From those two central notions a third one is characterised, that of 'semantic field', which is the activity (in the sense of Activity Theory) of producing meaning in relation to a given set of local stipulations (statements locally taken as true by the person without requiring any further justification).

---

2 As far as we know, there are not well stablished terms for those categories, so we will use our own translation into English.

A number of other notions related to why and how meaning production occurs, and to explaining why it is necessarily a social process are characterised (interlocutor and legitimacy, for instance) but they will not be presented here.

From the point of view of our interest in this study, two questions guided our reading of what was happening: (i) which are the objects the students are thinking with? and, (ii) what are the meanings being produced for those objects, that is, what are they saying about those objects? The two questions must necessarily be understood as a single one, as there are only objects as long as meaning is produced for them. It is important to notice that according to the TMSF the answers to those questions have to be taken as they come, in the sense that one must avoid 'completing', with his own meanings, what the other has said.

The study

As we have said, the study partially reported in this paper happened in the context of a regular Linear Algebra course (undergraduate mathematics degree).

Dual spaces were introduced inside the study of Inner Product Spaces (finite dimension), as a tool for studying the endomorphisms of such spaces; at the moment of the introduction of dual spaces we would normally expect the students to have mastered the basic theory of finite dimensional vector spaces as well as linear maps and their matrix representations. We reached the introduction of duality following a path traced through the use of the categories of PT and TMSF.

The excerpt we will discuss now happened at a point in which we were engaging in the study of the relation between hyperplanes in a vector space E and straight lines in the dual space E*. This relation involves a possible conflict between the students' previous understanding of vectors and functions, and the fact that vectors in the dual vector space are linear maps. According to the historical-cultural theory, in particular considering the concept of internalisation as developed by Vygotsky, teaching and learning can only be understood as a single process, so teacher intervention is not, in our analysis, a component strange to the process (as it would be seen from other theoretical perspectives).

After identifying the kernel of a linear form as a hyperplane, the fact that there is more than one non-zero linear form sharing the same kernel was established.

At this moment the following Problemic Situation was posed: "Is it possible to give a geometric interpretation of the relation between an hyperplane H in E and the linear forms in E* having H as kernel?"

This Problemic Situation was transformed into the Didactical Problem of comparing two linear forms y*, z* having H as kernel. Then, using the categories of PT, we organised the process so the students could move from comparing the images of y* and z* in a point x (not belonging to H) to comparing y* and z* as maps, as vectors in E*.

In what follows P is the professor and the S are students.

> P. Let us consider two non-zero linear forms y* and z* such that they have the same kernel H, Is it possible to find some relation between them?
>
> S_1. Yes.........(in a low voice)
>
> P. I need to compare y*(x) and z*(x) for all x in E?
>
> S_2. (Whispering) One x

P. There is $x_0 \in$ H such that E= H $\oplus$ $\langle x_0 \rangle$. Isn't it?

S_3. Ah, because x= x_1+k x_0 and this decomposition is unique

P. Then y*(x)= y*(x_1) +k y*(x_0)

S_3. y*(x_1)=0

S_1. For z* is the same thing!

P. And H is a?

SILENCE....

The professor continues the calculation until the following statement is reached:

y*(x)= z*(x) + α x, for some α in K, and she remarks that α does not depend on x.

P. What is the relation between y* and z*?

SILENCE....

P. y*= z* isn't it? Then if we have a hyperplane in E what do we have in E*?

SILENCE....

P. A straight line isn't it?

S_2. A family of straight lines, a vectorial straight line!

Examining the transcription from the point of view of the TMSF and considering the students' answers (silences included), it seems the students did not produce meaning for function as a vector. Silences came out at the moments of shifting from point-wise equality (the analytical understanding of function) to the equality of functions as vectors. The same happened when, after having established the equality **y*= z*** the professor asks for a geometrical interpretation of it. Only after the professor gives the interpretation of the equality **y*= z*** as representing a straight line in E*, one student repeats what the professor had said.

We will now analyse the student's answers to two questions:

I-Let E and F be vector spaces over K and y* and z* non- zero, non-proportional linear forms in e-mail. Prove that dim (Ker y*$\cap$ Ker z*) = n-2

With the exception of two students who wrote y* = z*, the others wrote y*(x) = z*(x) (some of them without specifying that this last equality holds for all x in E) or they went directly to consider kernels as hyperplanes and tried to apply formulas for dimension of subspaces.

II- If f: E$\rightarrow$F is a linear map, with E and F vector spaces over K.

Consider ^{t}f: F*$\rightarrow$E* given by ^{t}f (y*)= y* $\circ$ f

Prove that: (a) ^{t}f is a linear map from F* to E*; (b) Ker ^{t}f = [Im f]^o

The answers to these questions can be categorised into two groups:

Group1: Students who identified the function f to the image f (x).

S_4: y*$\circ$ f defines a map that goes from E$\rightarrow$K. It is, it belongs to E*. As the following is a composition of linear maps, ^{t}f (y*) = y* $\circ$ f is a linear map.

(Identifying ^{t}f to ^{t}f(y*), proving that ^{t}f (y*) is linear, but not that ^{t}f is linear)

S_5: To prove that ^{t}f is linear:

^{t}f(y*) (α x + β y) =α ^{t}f(y*)(x) + β ^{t}f(y*)(y)

(Identifying ^{t}f to ^{t}f(y*), proving that ^{t}f (y*) is linear, but not that ^{t}f is linear)

S_6: [Im f]^o = {y* in F*| y*(y) = 0, $\forall$ y$\in$ Imf}

[Im f]^o = y*(y) = y* $\circ$ y = y* $\circ$ f (x) = 0 = Ker ^{t}f

(Identifying y (=f (x)) to f)

Group2: Students who felt a "need to evaluate"
S_7 : Let $y_1^*, y_2^* \in F^*$
$^t[f(y_1^* + y_2^*)] = {}^t[f(y_1^*) + f(y^*_2)] = {}^t[f(y_1^*)] + {}^t[f(y_2^*)]$
(Interpreting t as a function and evaluating)
S_8: $^tf(\alpha y^* + \beta z^*) = (\alpha y^* + \beta z^*) \circ f = (\alpha y^* + \beta z^*) \circ f(x) = (\alpha y^* + \beta z^*)(f(x))$
(An x appears!)
S_9: Ker $^tf = y^*$: $^tf(y^*) = 0 = y^* \circ f = y^*(f(x))$, $x \in E$, $f(x) \in F$
(Again...)

Using the TMSF we would say that those students were operating in a semantic field in which the analytical understanding of functions was central (the 'evaluating' behaviour).

Conclusions

Being able to reveal that the difficulties faced by the students were not due to something missing (the algebraic understanding of functions), but rather due to something strongly present (the analytical understanding) clearly suggests, we think, that it is not enough to present the new object and its properties; it is also necessary to bring forth the 'old' object and to promote the explicit discussion of how they relate. We also suggest that this is a quite widespread phenomenon in mathematics classrooms; our research group is currently conducting other studies and the findings strongly support this suggestion.

In order to promote such a discussion it is necessary that classroom processes be organised in an open and flexible way, so the students can voice their understandings. That means that the professor must be capable both of handling the didactical task and of dealing with the meanings being produced by the students; in both cases one is dealing with what is emerging, rather than simply guiding the students through a pre-established path and helping them somehow to overcome the hurdles.

We think the association of PT and the TMSF has proven to be a quite useful and effective way to help professors to move towards more efficient approaches in mathematics courses.

REFERENCES

[1] Leontiev, A.N. (1981) "The problem of activity in Psychology" in J. Wertsch, The concept of activity in Soviet Psychology, N.Y.

[2] Lins, R. (2001) "The production of meaning for Algebra a perspective based on a theoretical model of semantic fields" in R Sutherland, T. Rojano, A., R. Lins (eds.), Kluwer Academic Press Publishers, The Netherlands.

[3] Sfard, A (1987) "Two conceptions of mathematical notions: operational and structural". Proceedings of PMEXIII, Montreal, 162-9

[4] Sierpinska A, (1988) "Epistemological remarks on functions" Proceedings of PME XII. Vezprem, Hungary, 508-573

[5] Noriega T and Nuñez R (2001) "La problematización del contenido y la producción de significados en Algebra: Primeras reflexiones, Revista Epsilon de la SAEM Thales, número 50,237-248.

[6] Malik, M.A. (1980) "Historical and pedagogical aspects of the definition of functions, International Journal for Math Education Science and Technology, Vol. 11 no 489-492.

[7] Majmutov, M.I. (1983) La Enseñanza Problémica; Editorial Pueblo y Educación (Cuba)