AN APPROACH OF LINEAR ALGEBRA THROUGH EXAMPLES AND APPLICATIONS

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ABSTRACT

Linear algebra is a *language* which is used in all sciences (and beyond). For a class consisting of students in mathematics, computer science, physics, engineering, microtechnics, chemistry, we use a multidisciplinary approach to this field by example and application. Starting with linear systems, we extract the general features from three motivating examples.

In the first one, we show that it is impossible to cover a sphere with (curvilinear) hexagons only. In any subdivision using hexagons and pentagons, a fixed number of twelve pentagons is needed. This is shown by row operations on a system of 4 equations in 5 variables. Here, the *surprise* is that although the system is under-determined, one variable has a fixed value. Several natural examples may illustrate this necessity: Football ball, buckminsterfullerene C_{60}, architecture, protozoa... From the dodecahedron we get a special solution having no hexagons. All others are derived from this one by addition of a solution of the associated homogeneous system.

In the second example, we consider a chemical reaction (composition of the atmosphere, according to Lord Rayleigh), in which the coefficients have to be determined. The superposition principle for homogeneous systems appears quite naturally in this context.

Finally, to exhibit the power of the general principles, we consider a huge system obtained by digitalization of a potential on a grid. If the values are given on the boundary, then there is one and only one solution for which the value at each interior point is the mean value of the four neighboring points. It is indeed easy to show that the associated homogeneous system has only the trivial solution.

In our opinion, these motivating examples are accessible to undergraduate students. Linear equations may be amplified and added; thus linear combinations appear. They can be dependent, whence the interest in giving a maximal number of independent ones; here is the rank. Linear equations thus furnish an ideal approach for the language of vector spaces and their dimension.

Introduction

Linear algebra is a cornerstone in undergraduate mathematical education. It develops a general language used by all scientists and is interdisciplinary in essence. It hence evolves naturally towards abstraction. For most students, it is a first contact with modern mathematics. I propose to approach it by concrete examples. In this way, its power and relevance is immediately realized.

Let me only sketch here a possible start with linear systems, already furnishing a meaningful and valuable part of linear algebra. Two by two (and three by three?) systems may have been solved in high school. But now it is important to consider more general ones, and choose examples creating surprise, leading to questions, general methods. . . , with cultural relevance, aesthetic sense, or having as many of these qualities as possible!

1 First Example: Covering a Sphere with Hexagons and Pentagons

Question: Is it possible to cover the surface of a sphere with (curved) hexagons only?

Answer: From a bee "it is difficult!"; from Euler: "*It is impossible*!"

To prove the impossibility, we consider a generalization. Let us try to cover a sphere with hexagons and pentagons only. We know that this is possible.[1] The dodecahedron yields such a covering with 12 pentagons (and no hexagon). By convention, we juxtapose two polygons along a common edge, three polygons having a common vertex. It is easy to find a few equations, linking the unknown numbers of such polygons. More precisely, let us introduce

$$x : \text{number of pentagons}, \quad y : \text{number of hexagons},$$
$$e : \text{number of edges}, \quad f : \text{number of faces}, \quad v : \text{number of vertices}.$$

The number of faces is equal to the sum of the numbers of pentagons and hexagons, hence a first obvious relation: $f = x + y$. Since each pentagon has five edges, and each hexagon has six, the expression $5x+6y$ counts twice the number of edges (edges belong to exactly two polygons). Hence a second relation $5x + 6y = 2e$. Our convention shows that the sum $5x + 6y$ also counts vertices three times and we get $5x + 6y = 3v$. From this follows $2e = 3v$, but this relation adds nothing new since it is a consequence of the previous ones. Another, more subtle relation was discovered by Euler, namely[2] $f + v = e + 2$. We have obtained a system consisting of four equations linking the five variables x, y, e, f, and v:

$$\begin{cases} x+y &= f, \\ 5x+6y &= 2e, \\ 5x+6y &= 3v, \\ f+v &= e+2. \end{cases}$$

[1] Such configurations occur in architecture, sport, chemistry. . .

[2] It is valid for any decomposition of the sphere into polygons, with no restriction on the number of incidences at the vertices.

Grouping the variables in the left-hand side in the order $e,\ f,\ v,\ x,\ y$, these equations are

$$\begin{cases} f-x-y &= & 0, \\ 2e-5x-6y &= & 0, \\ 3s-5x-6y &= & 0, \\ e-f-v &= & -2. \end{cases}$$

To save space—this savings has enormous benefits—we replace an equation by the sequence of its coefficients, not forgetting to include a 0 in the place of a variable that does not appear explicitly. For example, the equation $f-x-y=0$ stands for

$$0e+1f+0v-1x-1y=0 \quad \text{abbreviated by the row } (0\ 1\ 0\ -1\ -1 \vdots 0),$$

separating the left- and right-hand sides by vertical dots. The whole system is thus

$$\begin{pmatrix} 0 & 1 & 0 & -1 & -1 & \vdots & 0 \\ 2 & 0 & 0 & -5 & -6 & \vdots & 0 \\ 0 & 0 & 3 & -5 & -6 & \vdots & 0 \\ 1 & -1 & -1 & 0 & 0 & \vdots & -2 \end{pmatrix}.$$

The big parentheses have the sole purpose of isolating the system from the context(!). It is advisable to start the enumeration by an equation containing the first variable, so we permute the first and last equations and obtain an equivalent system... As is explained in any linear algebra textbook, *row operations* may be used to bring the system into a staircase form

$$\begin{pmatrix} 1 & -1 & -1 & 0 & 0 & \vdots & -2 \\ 0 & 1 & 0 & -1 & -1 & \vdots & 0 \\ 0 & 0 & 1 & -3/2 & -2 & \vdots & 2 \\ 0 & 0 & 0 & -1/2 & 0 & \vdots & -6 \end{pmatrix}.$$

The last equation of this equivalent system is $-x/2=-6$ implying $x=12$.

Here comes a *surprise*: Although the system is *under-determined* (only four equations linking five variables), the number of pentagons in any subdivision of the sphere (into hexagons and pentagons only) is fixed and equal to 12. Isn't this remarkable! On the other hand, the number of hexagons is not fixed. Several natural examples illustrate this. (Recall that the audience is not necessarily interested in pure mathematics, so why not spend a few minutes to show the importance and ubiquity of the result found; a few slides may help.)

(a) We already mentioned that a partition of the sphere is easily obtained with twelve pentagons and no hexagon: $x=12$ and $y=0$ (simply project a regular dodecahedron onto the surface of a sphere).

(b) Another solution with $y=20$ (and $x=12$) is obtained as follows. Start with a regular icosahedron (12 vertices and 20 faces formed by equilateral triangles). Cut the vertices, replacing them by pentagonal faces (thus replacing the triangular faces by hexagonal ones). The polyhedron thus obtained has 60 vertices representing the positions of the carbon atoms in the buckminsterfullerene C_{60}.

(c) One can construct a geometrical solution with $y = 2$. Start with six pentagons attached to one hexagon. This roughly covers a hemisphere. Two such hemispheres—placed symmetrically—will cover the sphere.

General solution. Mathematically speaking, one can take for y any value—say $y = t$—and then

$$x = 12, \quad y = t, \quad e = 3t + 30, \quad f = t + 12, \quad v = 2t + 20.$$

provides the algebraic solution of the proposed linear system. [3]

General Principle. The general solution is the sum of the particular solution coming from the dodecahedron and the *general solution of the associated homogeneous system*, here depending on the choice of a parameter t (there is one free variable).

Further themes. (1) Construct infinitely many geometrical solutions with two groups of 6 pentagons (Hint: Consider two types of tubes). (2) What happens if the sphere is replaced by the surface of a torus? (The associated homogeneous system appears.)

2 Second Example: A Chemical Reaction

The first example has shown that homogeneous systems are both important and simpler to study. Let us turn to one of them. When Lord Rayleigh started his investigations on the composition of the atmosphere around 1894, he blew ammoniac and air on a red-hot copper wire and analysed the result. Let us imitate him, and consider a typical reaction of the form[4]

$$x\, NH_3 + y\, O_2 + z\, H_2 \to u\, H_2O + v\, N_2,$$

where the proportions $x, \ldots, v$ have to be found. Equilibrium of N-atoms requires $x = 2v$. Similarly, equilibrium of hydrogen atoms requires $3x + 2z = 2u$ and finally, for oxygen, we get $2y = u$. Proceeding systematically, we have to choose an order for the variables. We adopt their order of occurrence in the chemical reaction: x, y, z, u, and v, hence write the system in the form

$$\begin{cases} x & & & & -2v & = 0, \\ 3x & & +2z & -2u & & = 0, \\ & 2y & & -u & & = 0. \end{cases}$$

Now, observing that the right-hand sides are all zero, it is superfluous to include the last coefficient 0 common to all equations. Thus we simply replace the first equation by the row $(1\ 0\ 0\ 0\ -2)$, so that the system is represented by the array

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -2 \\ 3 & 0 & 2 & -2 & 0 \\ 0 & 2 & 0 & -1 & 0 \end{pmatrix}.$$

[3] Notice that many algebraic solutions have no geometric realization. For example, one may take $y = \frac{1}{2}$ ($x = 12$) and adapt correspondingly $e = 31.5$, $f = 12.5$, $v = 21$. Similarly, one can take $y = -1$ together with $e = 27$, $f = 11$, $v = 18$. A necessary condition is that y should be a nonnegative integer! But this condition is not sufficient. There is no covering of the sphere consisting of twelve pentagons and just *one* hexagon.

[4] We add hydrogen for mathematical interest, but be careful of the explosive character!

From the second row (or equation), subtract three times the first one, and then, permute the second and third equations. This leads to the staircase shape system

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & 2 & -2 & 6 \end{pmatrix}.$$

The last equation is

$$2z - 2u + 6v = 0 \quad \text{or simply} \quad z - u + 3v = 0.$$

If we choose arbitrarily u and v—say $u = a$ and $v = b$—we have to take

$$z = a - 3b.$$

The second equation then leads to $2y = a$, and the first one furnishes $x = 2b$. Thus, for each choice of a pair of values for u and v, there is one and only one solution set[5]

$$\begin{cases} x = & 2b \\ y = & \frac{1}{2}a \\ z = & a - 3b \\ u = & a \\ v = & b \end{cases} \quad \text{or equivalently} \quad \begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix} = \begin{pmatrix} 2b \\ \frac{1}{2}a \\ a - 3b \\ a \\ b \end{pmatrix}.$$

Observations. This problem concerns proportions. We can deal with numbers of atoms, or numbers of moles.[6] If a solution is found, any *multiple* will also be one. We may also *add* or *combine* multiples of solutions to obtain new ones. A first case is given by the choice $u = 2$, $v = 0$, hence $x = 0$ (no ammoniac); it corresponds to the elementary reaction

$$O_2 + 2H_2 \to 2H_2O,$$

namely the synthesis of water. Another one—in which Lord Rayleigh was interested—is given by $u = 6$, $v = 2$, hence $z = 0$ (no danger of explosion!) which corresponds to the elementary reaction

$$4\,NH_3 + 3\,O_2 \to 6\,H_2O + 2\,N_2.$$

Any solution is a combination of these two basic solutions. The general solution of the system depends on *two arbitrary parameters*. It is easy to generalize.

Results. Any homogeneous system having more variables than relations has a nonzero solution. The solutions of a homogeneous system exhibit the following structure

- ⋄ Any multiple of a solution is again a solution,
- ⋄ The sum of two solutions is also a solution.

Linear equations (rows of a certain type) may be amplified and added; solutions (vertical lists) may similarly be combined. The language of vector spaces emerges in a relatively general context.

[5] A solution set is a list of solutions, written *vertically*.

[6] Each mole contains approximately $0.60221367 \times 10^{24}$ atoms. This is the *Avogadro number*, namely the number of atoms in 12g. of carbon, or the number of oxygen molecules O_2 in 32g. of oxygen, etc.

3 Third Example: Potentials on a Grid

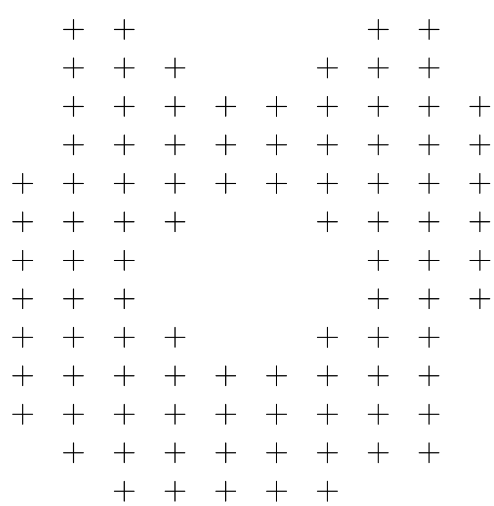
It is important to realize that systems containing several hundred or even thousands of equations and variables occur frequently. These systems are often incompatible, or under-determined, and it is highly desirable to have efficient algorithms to discuss them. In particular, it is impossible to use tricks or guess work to solve them! This is why a systematic discussion has to be carried out. The first problem is that the alphabet is too poor to code so many variables and we have to number them, thereby ordering them:

$$x_1, x_2, x_3, \ldots, x_n.$$

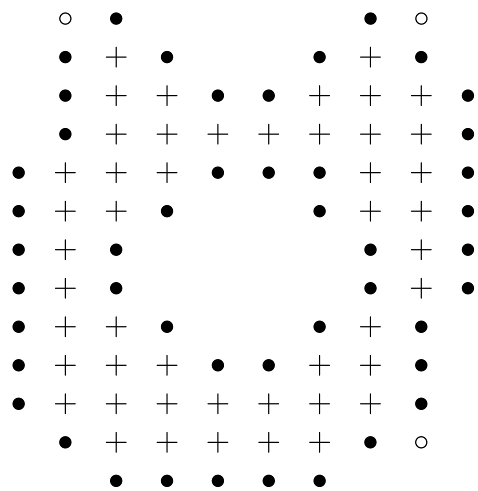
As before, instead of the equation $a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$, we simply write the row of its coefficients: $(a_1\ a_2\ a_3\ \ \ldots\ \ a_n \vdots b)$. At this point, one should explain the following

Basic Principle. A linear system having as many equations as variables can always be solved in a unique way if the rank of the associated homogeneous system is maximal. (Indeed, the reduced staircase system exhibits no compatibility condition, and there are no free variables.)

Consider now in the plane $\mathbf{R}^2$, a certain bounded domain D (e.g. a disc, the interior of an ellipse, or a rectangle, etc.). We are looking for a potential inside D, taking prescribed values on the boundary. To approach this physical problem, we introduce a square mesh in the plane, and only keep the squares having a nonempty intersection with D. We are left with a certain set of vertices P_i, edges and square faces. Here is an example



Replacing the boundary points by a bullet, we get



The vertices which are not boundary ones have four neighbors, conveniently called North, East, South, and West. We are looking for a function (potential) defined at all interior points having the mean value property. Starting from known values at the boundary points,[7] we introduce variables x_i for the unknown values at the interior points P_i. If the four neighbors of an interior point P_i are P_p, P_q, P_r, and P_s, there is a corresponding equation

$$x_p + x_q + x_r + x_s = 4x_i.$$

Here, $p = N(i)$ is the index of the northern neighbor of P_i, etc. It may happen that all x_j are unknown, in which case we get a homogeneous equation

$$x_p + x_q + x_r + x_s - 4x_i = 0.$$

Or it may happen that certain values are prescribed, because the corresponding point lies on the boundary. For instance, we may encounter an equation of the form

$$x_p + x_q + x_r - 4x_i = -b_s$$

where b_s is the given value for the potential at the boundary point P_s. In any case, we group the unknown variables in the left-hand sides, while the known ones are gathered in the right-hand sides. Thus we get a linear system (S) for the variables x_i. *We are going to show that this linear system is compatible, and has a unique solution for each data on the boundary.*

If there are N interior points P_i, the system contains N variables x_i and also N equations: To prove that (S) has maximal rank $r = N$, we consider the associated homogeneous system (HS), simply obtained by requiring zero values on the boundary. In this case, it is enough to show that there is only one solution to the problem, namely the trivial one $x_i = 0$ for all indices i (corresponding to interior points P_i). Here is the crucial observation. For any solution set (x_i), select a variable x_j taking the maximal value (in a finite list, there is always a maximum!). Since this value x_j is the average of the four values at neighboring points, the only possibility is that these four values

[7] Certain boundary values may be irrelevant: Here, they are denoted by a $\circ$ instead of a $\bullet$.

are equal, and equal to the maximal value. Iterating this observation on neighboring points, we eventually reach a boundary point, where the value is 0. Hence the maximal value is itself 0. By symmetry, the minimal value is 0. Finally, we see that all $x_i = 0$, which proves the claim.[8]

4 Notes on a Teaching Approach

From my experience, the last example is much more difficult to grasp than the two preceding ones. But even if it is not possible to convey its significance, it will serve a purpose, namely to show that linear algebra is not a trivial matter. Linear equations in a large number of variables are used in extremely sophisticated situations, like weather forecasting, devising profiles for wings of supersonic planes, etc. This is not apparent on 4×5 examples, and is only suggested by the last example.

The preceding examples lead to the systematic elimination theory based on row operations. Each of them can easily fit in a one hour (or 45 min.) presentation, possibly followed by a discussion. In parallel exercise sessions (is it necessary to repeat that exercises constitute a must in the learning procedure?), one may try to lead the students to the *question* of the invariance of the rank. As soon as the vocabulary of independence, generation, and dimension is acquired, it is possible to give a positive answer.

It is widely recognized now that a *first part* of linear algebra should be devoted to linear systems, rank/dimension theory, linear maps and their kernels, eigenvectors (geometrical theory, incl. diagonalization). A *second part* should introduce inner product spaces with metric relations, orthogonality (Pythagoras theorem), best approximation (mean squares method). This is the "bilinear" part of linear algebra. Symmetric operators can be treated in this part (with their diagonalization). Finally, in a *third part*, the determinant is presented as a generalized volume, or volume amplification factor. Having some experience from bilinear algebra, the students may now grasp multilinearity. Applications abound with the characteristic polynomial. Spectral values for orthogonal, antisymmetric (and more generally normal) operators can be discussed.

It is important to me that a student able to follow only a first section of the course, can already apply it in his field. I hope that this type of introduction yields a valuable primer in linear algebra, complementing the classical approach by vectors in the usual 2- and 3-dimensional spaces.

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[8]The same reasoning shows more generally that any solution will take its values between the minimum and maximum on the boundary. Any solution attains both a maximum and a minimum at a boundary point.