STUDENT' ALGORITHMIC, FORMAL AND INTUITIVE KNOWLEDGE: THE CASE OF INEQUALITIES

Pessia TSAMIR
School of Education, Tel Aviv University, Tel Aviv. ISRAEL
pessia@post.tau.ac.il

Luciana BAZZINI
Dept. of Mathematics, University of Torino, ITALY
luciana.bazzini@unito.it

ABSTRACT

This paper presents a study that aimed at investigating Italian and Israeli high school students' solutions to algebraic inequalities. The findings of this study are analysed with reference to Fischbein' notions of intuitive and algorithmic knowledge. Students' intuitive ideas and their algorithmic models when solving algebraic inequalities are identified and discussed. Our claim is that students intuitively used the solutions of equations as a prototype for solving inequalities. They developed an equation-algorithmic model for solving inequalities. Students were intuitively drawing analogies to either correct or incorrect solutions of related equations, either by excluding only zero values when dividing both sides of an inequality by a not-necessarily positive value, or when dividing by an expression without the exclusion of zero values as well. The latter tended to use the balance model when solving both equations and inequalities, explaining that it is always permitted to "do the same thing on both sides" of either an equation or an inequality.

Student' algorithmic, formal and intuitive knowledge: The case of inequalities

In his analysis of students' mathematical performance, Fischbein (1993) related to the algorithmic, the formal and the intuitive components. Algorithmic knowledge is the ability to activate procedures in solving given tasks, and understand why these procedures "work". Formal knowledge refers to a wider perspective of the mathematical realm, what is accepted as valid and how to validate statements in mathematical context. Intuitive knowledge is described as an immediate self-evident cognition of which students are sure, feeling no need of validation.

The three components are usually inseparable, and Fischbein explained that "sometimes, the intuitive background manipulates and hinders the formal interpretation or the use of algorithmic procedures" (Fischbein, 1993. p. 14). He presented a number of algorithmic procedures, which he called algorithmic models, when referring, for instance, to methods of reduction in processes of simplifying algebraic or trigonometric expressions. For example, students' tendencies to treat $(a+b)^5$ as a^5+b^5 or log(x+t) as logx + logt were interpreted by Fischbein as evolving from the application of the distributive law, which he identified as a prototype for simplifying algebraic and trigonometric expressions (Fischbein, 1993; Fischbein & Barash, 1993). In this paper we use Fischbein's theory to analyze Italian and Israeli secondary-school students' knowledge, their intuitive ideas and their algorithmic models when solving algebraic inequalities.

We chose algebraic inequalities for several reasons. First, inequalities provide a complementary perspective to equations and are part of many mathematical topics including algebra, trigonometry, linear programming and the investigation of functions (e.g, Mahmood & Edwards, 1999). Accordingly, various document, such as the NCTM standards(2000) recommend that all students in Grades 9-12 should learn to represent situations that involve equations and inequalities, and that students should "understand the meaning of equivalent forms of expressions, equations, [and] inequalities .. and solve them with fluency" (ibid., p. 296). However, in both Italy and Israel, algebraic inequalities usually receive relatively little attention and are commonly presented in an algorithmic way, merely by discussing various algebraic manipulations. Moreover, in both countries, the researchers witnessed students' and teachers' frustration with the difficulties encountered when dealing with inequalities.

We believe that teaching algebraic inequalities, like any other mathematical topic, should take into consideration students' correct and incorrect ideas when solving related tasks. Therefore, an understanding of students' ways of thinking about inequalities is a necessary condition for making didactical decisions, and for implementing the recommendations made by the NCTM. As a first step for investigating students' common solutions to algebraic inequalities we decided to study the publications in the professional literature. We found that many related articles deal with suggestions for instructional approaches, usually with no research support, and considerably less attention has been paid to students' conceptions of inequalities (e.g., Bazzini, 2000; Linchevski & Sfard, 1991; Tsamir & Bazzini, 2001). The latter studies pointed, for instance, to students' difficulties in grasping the role of the sign, to their tendency to reject the R or Φ solutions, and to difficulties they encounter when using logical connectives.

The present study was designed in order to extend the existing body of knowledge regarding students' ways of thinking and their difficulties when solving various types of algebraic inequalities. In this paper we focus on the question: What intuitive ideas and what algorithmic models can be identified in Italian and Israeli secondary school students' solutions to algebraic inequalities?

Methodology

Participants

One-hundred-and-ninety two Italian and 210 Israeli high school students participated in this study. All participants were 16-17 year old who planned to take final mathematics examinations in high school. Success in these examinations is a condition for acceptance to academic institutions, such as universities.

Tools

Italian and Hebrew versions of a 15-task questionnaire were administered to the students. Here we focus on five tasks. Of these, three deal with dividing an inequality by a not-necessarily-positive factor, and the other two dealing with quadratic inequalities. The first three tasks are:

Task I: Examine the following claim: for any a in R, $a \cdot x < 5 \Longrightarrow x < 5/a$

Task II: Examine the following statement: for any $a \neq 0$ in R, $a \cdot x < 5 \Longrightarrow x < 5/a$

Task III: Solve the inequality: $(a-5) \cdot x > 2a-1$, x being the variable and 'a' being a parameter.

Research findings indicate that when solving rational inequalities, students frequently multiply both sides of the inequality by a negative number without changing the direction of the inequality (e.g. Tsamir & Almog, 2001). It was also reported that students encounter difficulties when solving mathematical tasks, presented in a way different from the way they are used to. For example, when having to deal with parametric equations and inequalities that are commonly not discussed in class (e.g., Furinghetti & Paola, 1994; Ilani, 1998). We took theses data into account when constructing tasks I, II, and III.

The second type of tasks included two "solve" tasks.

Task IV: Indicate which of the following is the truth set (the solution) of $5x^4 \leq 0$,

(a) $\{x: x>0\}$ (b) R (c) $\{x: x<-5\}$ (d) $\{x: 0<x<1/5\}$ (e) ϕ (f) $x=0$ (g) $\{x: x\leq 0\}$

Explain your choice

Task V: Indicate which of the following is the truth set (the solution) of $\frac{1}{4} \cdot x^2 \geq 0$,

(a) $\{x: x>0\}$ (b) R (c) $\{x: x \geq 4\}$ (d) $\{x: x \geq 2\}$ (e) ϕ (f) $\{x: x \geq 0\}$ (g) $\{x: x\leq 0\}$

Explain your choice

Tasks IV results in a single value and Task IV or in any real number. They and were presented in a manner similar to other tasks presented in Israeli and Italian classes. As such, we assumed that students would feel they can solve the tasks, but that a substantial number of them will reach incorrect solutions in accordance with findings reported in the literature regarding such tasks (e.g., Tsamir & Almog, 2001).

Procedure

The students were given approximately one hour, during mathematics lessons, to complete their written solutions. In order to get a better insight into the students' ways of thinking, forty-five students were individually interviewed. In the interviews we asked students to elaborate on their written solutions. Each interview lasted 30 to 45 minutes.

Results

No significant differences between Italian and Israeli students' solutions, thus, we present the data of all students.

Students' Answers to Task I

In their responses to Task I, about three-quarters of the students correctly judged the claim: "for any a in R, a·x < 5 ==> x < 5/a" as being false, and accompanied their correct judgement with an acceptable justification. While about 20% elaborated on the role of the sign of the value substituted for a in determining the direction of the ">", about 55% mentioned only the "zero case" as a counterexample to the given statement.

In their explanations, students who *related to the sign of a* did it either in a general, verbal manner, or by means of a specific counterexample. Those who provided a verbal explanation, wrote, for instance, "there are three cases: when a>0 the statement is correct, for a = 0 it is impossible to divide by zero, and when a is negative the conclusion is that x<5/a". Some students just mentioned that, "for a>0 this is a correct statement, but for a≤0 it is not", and others explained more briefly that, "this statement is correct only for positive 'a's"; or "when a is negative the direction of the sign changes". A number of students provided specific *counterexamples*. They wrote, for instance, "if a=(-1) the statement is not correct"; or "if –5x<5 the conclusion is that x>(-1), instead of x<(-1)". A few students exhibited a good understanding of the role of their single counterexample in refuting the given statement, by adding, "I gave one example, but a single counterexample is sufficient for proving that the statement is false."

Most prevalent was the students' tendency to use *only the 'a=0' case*, as a *counterexample* to refute the statement. They wrote, for instance, "this statement is false when a equals zero"; or "the statement is false, because of the case of a=0". Many added "division by zero is undefined, therefore the statement is not *always* correct."

In their oral interviews these students' typically commented,

Sophia: the statement here refers to any number. BUT, since it is false for a=0, the statement is not true for *any* number. It is, therefore, false.

The few students, who incorrectly judged the statement as "true" either explained "we divided both sides by the same thing" or provided an example, "5x<5, for example, means that x<1". In their oral interviews these students typically added a confirmation like,

Jonathan: It's OK to do the same thing on both sides. When doing the same operation on both sides, the equivalency is preserved.

Jonathan went on to talk about equations and when the interviewer commented on his shift to equations he said: "It's the same…"

Students' Answers to Task II

Only about 30% of the participants correctly responded that the claim "for any a≠0 in R, a·x < 5 ==> x<5/a" is false and accompanied their response with a valid justification. All of them related to the role of the sign in their decision. They explained, for instance, "if a is negative then x>5/a"; or "the claim is correct only for positive a". Some students added specific examples, "It is false, because it holds only when a is positive. For example, if a=(-2), then –2·x<5 ==> x>(-2.5)". Students were usually satisfied with a single counterexample, occasionally explaining, "one counterexample is sufficient in order to show that the proposition is false".

Most prevalent (over 50%) was the incorrect response that the statement is true, accompanied by a comment explicitly based on the given that a≠0. Students wrote, for instance, "It is correct because of the given condition that a≠0." In the interviews, these students pointed to connections they made between equations and inequalities. For example, Daniel had used the example of $2x=6$ to explain his solution to the given inequality in the questionnaire. When the interviewer related to it, he said,

Daniel: It's the same [pause]. In a way inequalities are a certain type of equations. Just that equations are easier, so I simply use examples of equations when I have a difficult inequality.

Several students gave explanations, similar to the following one,

John: In equations and inequalities, dividing by zero is problematic. But if we solve an inequality by operating with identical numbers on both sides, it is not only permitted, it is actually *the way* to solve the given tasks.

Students' Answers to Task III

Only a little more than 10% of the participants provided a comprehensive analysis of the various (positive, zero and negative) options for 'a'. About 45% of the participants either wrote that $x>(2a-1)/(a-5)$ for $a\neq5$ (about 30%), or were satisfied with writing $x>(2a-1)/(a-5)$ without any limiting condition. Surprisingly, in their interviews, a substantial number of the students clearly mentioned drawing analogies to equations. Bettina, for instance, wrote in her solution that $x>(2a-1)/(a-5)$ for $a\neq5$, and she explained in her interview,

Bettina: I divided both sides by the same expression, but I had to make sure that it is a non-zero expression. So, I wrote that a cannot be 5, because then a-5 equals zero…

Interviewer: Are you sure that five is the only problematic value here?

Bettina: [confidently] sure. I have done that a million times when solving equations.

On the other hand, Anna who gave the $x>(2a-1)/(a-5)$ solution (without mentioning any limitation), also mentioned the use of equation-ideas in the oral interviews. Like others who provided this solution, she explained that it is allowed to "do the same thing on both sides". She related interchangeably to equations and to inequalities,

Interviewer: Is it OK to divide both sides by a-5?

Anna: Yes. I have done the same thing on both sides. If you do the same thing on both sides of an equation [pause], I mean an inequality [pause], actually both, you reach an equation or an inequality that has the same solution as the given one.

Interviewer: Always?

Anna: It is not only allowed, it is necessary to do that in order to solve the problem.

Students' Amswers to Task IV

About 60% of the participants correctly wrote that the solution for $\frac{1}{4}x^2 \geq 0$, is R. The common error made by the other participants was that the set of solutions is $\{x: x\geq0\}$. This conclusion was usually reached in the following algorithmic manner:

$$1/4x^2 \geq 0 / \bullet 4$$
$$x^2 \geq 0 / \sqrt{\ }$$
$$x \geq 0$$

In their interviews, these students usually elaborated on the way they had solved the inequality, and mentioned having in mind the way they usually solve equations. Kim, for instance, wrote a solution like the one written above, and in her interview she explained,

Kim: Here [pointing to the / $\bullet$ 4 that she wrote in the first line] I showed that I multiplied both sides by four, and here [pointing to the / $\sqrt{\ }$ written in the second line] I showed that I calculated the square root of both sides. I used on both sides the same operation with the same number, until I isolated x on the right side and thus reached $x \geq 0$, [pause] the solution.

Interviewer: Why is this the solution?

Kim: I reached it by means of permitted actions in each stage [pause].

Interviewer: [Questioning look] ???

Kim: Like in the case of $3x = 6$, I divide both sides by 3 so I get $x = 2$. I do the same thing on both sides. The same operation, I divide by the same number, three [pause] and therefore 2 is the solution.

Interviewer: You used an equation, while here we have an inequality…

Kim: It is the same thing.

Interviewer: [Questioning look] ???

Kim: There is no difference between the ways of solving equations and inequalities.

The connections that Kim made between equations and inequalities could be identified both in her intuitive choice of an equation to exemplify the solution of an inequality, and in her explicit saying that "it is the same thing".

Students' Reactions to Task V

Only about 50% of the participants correctly responded to this task, marking $x = 0$ as the solution of the inequality. A substantial number of students in both Israel (about 20%) and Italy (about 15%) incorrectly wrote that the set of solutions of $5x^4 \leq 0$ is $\{x{:}x\leq 0\}$, which was usually reached in the following algorithmic manner:

$$5x^4 \leq 0 \;/ :5$$
$$x^4 \leq 0 \;/ \sqrt[4]{}$$
$$x \leq 0$$

In their interviews of these students, most students related to connections they made between the solutions of equations and those of inequalities. Betty, for instance, said,

Betty: I divided both sides of the inequality [$5x^4 \leq 0$] by five and reached $x^4 \leq 0$ [pause]

…

Betty: I calculated the fourth root of both sides, and got $x \leq 0$.

Interviewer: Is it OK to calculate the fourth root of both sides of an inequality?

Betty: Sure. The fourth root is a root of an even order, so we can calculate it when the given expressions are not negative. This is exactly the case here. Neither $5x^4$ nor zero is negative, so it's OK to perform this calculation for which I got $x \leq 0$.

Interviewer: Are you sure?

Betty: Sure. These are all procedures I know very well from solving equations.

Betty performed a valid manipulation of "dividing both sides by 5", but incorrectly defended the conclusions derived from her "calculation of the fourth root" of both sides. She explained that her certainty in the correctness of her solution was rooted in her experience with such procedures for solving equations.

Final Comments

The aim of this study was to deepen the understanding of students' performance with inequalities, by identifying intuitive ideas and algorithmic models in Israeli and Italian secondary school students' solutions to algebraic inequalities. As mentioned before, according to Fischbein algorithmic models evolve when students' intuitive ideas manipulate their formal reasoning and/or their use of algorithmic procedures. The latter usually express formal overgeneralizations and/or rigid algorithms (e.g., Fischbein, 1993). These models are usually coercive, used with confidence and grasped as being self-evident, even though they frequently lead to erroneous solutions.

We found that equations serve as a prototype in the algorithmic model of solving inequalities. This algorithmic model had mainly the appearance of "doing the same operation with the same numbers on both sides is valid for any operation with any number", in solving equations and inequalities. Students tended to correctly multiply both sides by 4 in Task IV, and divide both sides by 5 in Task V. However, they also frequently applied this algorithmic model when incorrectly calculating the square root of both sides in Task IV, the 4th root in Task V, and when dividing both sides by a not necessarily positive number in Tasks I, II and III.

Participants who applied this algorithmic model actually overgeneralized the balance model when incorrectly solving both equations and inequalities. They assumed that "doing the same thing on both sides of an equation *always* leads to an equivalent equation, and consequently to the solution". This assumption which is not even always true for equations, is much more problematic in the case of inequalities. By drawing the equation-analogy to cases of inequalities, these students reached incorrect solutions. In Tasks I and II they judged the statements to be true, in Task III the erroneously wrote that $x>(2a-1)/(a-5)$ is the solution to the inequality $(a-5)\cdot x>(2a-1)$, in Task IV the concluded that $x \geq 0$and in Task V they wrote that $x \leq 0$. Consequently, most prevalent errors were rooted in this algorithmic model, which was clearly and explicitly referred to in the students' oral interviews.

A substantial number of participants applied a version of this algorithmic model. That is to say, they knew that when solving equations they should be careful not to divide by zero, and since they held the equation-model for solving inequalities, they imposed the same condition in the case of inequalities. This assumption which is true for equations, is problematic in the case of inequalities. Again, the equation-analogy derived from inequalities, led to incorrect solutions for inequalities. These students usually answered Task I correctly, providing the '$a=0$' case as a counterexample to refute the given proposition. However, they frequently, incorrectly regarded the proposition of Task II as valid or suggested $x>(2a-1)/(a-5)$ $a\neq 5$ as the solution for Task III.

We have seen that students tend to apply the equation algorithmic model when solving inequalities. This was done by students who correctly solved and by those who incorrectly solved the related equations. This understanding of students' solutions should be considered when planning instruction. Fischbein recommended that when teaching, students' be made aware of their erroneous ways of thinking (e.g., Fischbein, 1987). How to promote students' awareness is another issue for research. One way to go about it is by presenting students with parametric inequalities, similar to the ones given here. These inequalities were found helpful in triggering students, who hold different equation-based models, to answer differently, and occasionally also incorrectly. The various solutions should be discussed in class, while shedding light on the mathematical similarities and differences between equations and inequalities, and on students' intuitive ideas and the resulting algorithmic model that they intuitively use. How to implement such instruction and their impact on students' performance should further be investigated.

REFERENCES

Bazzini, L. (2000). Cognitive processes in algebraic thinking and implications for teaching, Paper presented at ICME 9, Tokyo, Japan.

Fischbein, E. (1987). *Intuition in science and mathematics – An educational approach*, Reidel, The Netherlands.

Fischbein, E. (1993). The interaction between the formal, the algorithmic and the intuitive components in a mathematical activity. In R. Biehler, R. W. Scholz, R. Straser, & B. Winkelmann (Eds.), *Didactics of Mathematics as a Scientific Discipline*, 231-245. Netherlands, Dordrecht: Kluwer.

Fischbein, E., & Barash, A. (1993). Algorithmic models and their misuse in solving algebraic problems. *Proceedings of PME 17*, Tsukuba, Japan, Vol. I, pp. 162-172.

Furinghetti, F., & Paola, D. (1994). Parameters, unknowns and variables: A little difference? *Proceedings of PME 18*, Lisbon, Portugal, Vol. II, pp. 368-375.

Ilani, B. (1998). The elusive parameter. *Proceedings of PME 22*, Stelenbosch, South Africa, Vol. IV, pp. 265.

Linchevski, L., & Sfard, A. (1991). Rules without reasons as processes without objects – The case of equations and inequalities. *Proceedings of PME 15*, Assisi, Italy, Vol. II, pp. 317-324.

Mahmood, M., Edwards, P. (1999). Some inequalities and sequences converging to e. *International Journal of Mathematical Education in Science and Technology*, 30(3), 430-434.

NCTM [National Council of Teachers of Mathematics] (1989). *Curriculum and Evaluation Standards for School Mathematics*, Reston, Virginia.

NCTM [National Council of Teachers of Mathematics]. (2000). *Principles and Standards for School Mathematics*, Reston, Virginia.

Tsamir, P. & Almog, N. (2001) Students' Strategies and difficulties: The case of algebraic inequalities. *International Journal of Mathematical Education in Science and Technology*. 32(4), 513-524

Tsamir, P., & Bazzini, L., (2001). Can x=3 be the solution of an inequality? A study of Italian and Israeli students, *Proceedings of PME 25*, Utrecht, The Netherlands, Vol. IV, pp. 303-310.