GEOMETRICAL AND FIGURAL MODELS IN LINEAR ALGEBRA

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ABSTRACT

According to many university teachers, “geometrical intuition” can help students in their learning and understanding of linear algebra. Fischbein’s theory about intuition and intuitive models provided us with a framework that confers a precise meaning to “geometrical intuition”, and permits to examine its possible effects on students practices in linear algebra. We study especially geometrical models, stemming from a geometry, and figural models, whose elements are drawings. We describe here some aspects of the use of these models by teachers and students in linear algebra.

1 Introduction

University teachers in France often declare that "geometrical intuition" might help students in their learning and understanding of linear algebra. Several educational researches mention possible interactions between geometry and linear algebra (Dorier 2000) but none of them tries to clarify the specific problem of intuition. In order to design a theoretical frame allowing us to tell and answer research questions about geometrical intuition in linear algebra, we used Fischbein's theory about intuition in mathematics, and models as intuition factors. We will first briefly present that theory, and the questions it raises in the case we study. Then we will expose elements of our work about the use of geometrical and figural models by teachers and students in linear algebra.

2 Using models in mathematics : the theory of Fischbein

We will set out here the elements of Fischbein's work which are relevant for the present study.

Intuition and the use of models

According to Fischbein, every human being needs to act in accordance with a credible reality. Even within a conceptual structure, the reasoning endeavor needs a form of certitude. The role of intuition is to provide that kind of certitude. Intuition is for Fischbein a type of cognition characterized by self evidence, immediacy and certitude ; it always exceeds the given facts. Models are a central factor of intuition in mathematics ; Fischbein defines a model as follows :

> "A system B represents a model of system A if, on the basis of a certain isomorphism, a description or a solution produced in terms of A may be reflected consistently in terms of B and vice versa"(Fischbein 1987 p.121)

Fischbein distinguishes several kinds of models. The ones we use in our study are intuitive models. An intuitive model can be perceived like a concrete reality ; it can stemm from a mathematical theory, if it stays connected with a certain reality (the opposite is a theoretical model, i.e. a mathematical modelisation of a physical reality). There are also several kinds of intuitive models, in particular :

- *Analogical and paradigmatic models*

  An analogical model must be independent of the original ; in that case, the model and the original belong to two distinct conceptual systems. On the opposite, a paradigmatic model is a subclass of objects, used as a model. It is not a mere example, but a particular examplar, representative for the whole class.

- *Intramathematical and extramathematical analogies*

Fischbein also distinguishes different sorts of analogical models in mathematics. The main distinction is beetween intra and extramathematical models. In the case of an intramathematical analogy, the original and the model are both mathematical theories. On the opposite, extramathematical analogies occur with extramathematical models. In our study, it will be the case when the model is a material representation (we use in that case the term "drawing", or "picture", referring to (Laborde and Capponi 1992)). We will refer to such models as "figural models".

We define geometrical intuition in linear algebra as the use of geometrical or figural models.

Geometrical and figural models in linear algebra

We define here a geometry as a mathematical theory whose main objective is to provide a theoretical model for physical space (it is notably restricted to dimension 3). A geometrical model is a model stemming from a geometry ; it is an intuitive model, because the geometry is connected with physical space. It is an intramathematical model ; it can be either paradigmatic, or analogical, depending on the corresponding geometry (that geometry can be indeed a subclass of linear algebra, or can be independent of it). It is always associated with a figural model. The geometrical model can thus smuggle uncontrolled elements in the reasoning process. For example, when studying the general notion of quadratic form, students encounter in some cases vectors orthogonal to themselves. That property cannot be associated with a result in two-dimensional Euclidean geometry ; it is opposed to the drawing usually used to represent two orthogonal vectors in the plane. Thus in that case, the reference to a geometrical model stemming from Euclidean geometry might prevent the understanding of the general theory.
We also study the use of figural models in linear algebra for themselves, independantly of any geometry.

In the following study, we will rather refer to the use of models than to the general expression "geometrical intuition". The questions we study can then be formulated as follows :

- What are the possibilities and the limits of the use of geometrical and figural models in linear algebra ?
- What are the effective uses of models, by teachers and students ?

The results we present in the two following sections are partial answers to the second question.

3 Teacher's choices

We addressed a questionnaire to university teachers, who were used to teach linear algebra (in France). It included several parts, related to various aspects of the use of geometrical and figural models in linear algebra. We will give here details about their use of figural models, and the conclusions of the whole questionnaire.

3.1 Teacher's use of figural models in linear algebra

In our questionnaire, two tables were proposed for the teachers to fill in : one with three drawings, that are sometimes used in linear algebra (according to a previous textbooks study) ; the teachers were asked to say if they use them, and what they illustrate with them ; and an empty table (with five lines), where the teachers were asked to present other drawings they use.
The drawings of the first table, and examples of drawings and interpretations proposed by the teachers, are presented in Annex 1.
Analysis of the answers led to the following conclusions.

Little use of a figural model

A first global statement is that teachers do not use many drawings in their linear algebra courses. Only 16 of the 28 teachers who answered that question proposed drawings in the second part of the question, i.e. other drawings they might use in their courses. And the average number of drawings proposed by these 16 teachers is 2.25 ; this is very low, considering the fact that there were five lines to be filled in the table figuring in the questionnaire. The average number of drawing per teacher, for both parts of the question, is only 3.2.

No specific figural model ?

Moreover, most of the drawings are used to illustrate situations in dimension ≤ 3 , in fact situations occuring in $\mathbb{R}^2$ and $\mathbb{R}^3$. Only 43% of the teachers propose more interpretations referring to an abstract vector space than to $\mathbb{R}^2$ and $\mathbb{R}^3$. For example, for the first drawing (see Annex 1), nine teachers propose the interpretation : "Basis of the space[1]", and three "Orthogonal basis of the space", while only three of them quote the general notion of "Orthogonal basis", and only one the general notion of basis.
For the second drawing, eight teachers mention an intersection of planes, and only five an intersection of subspaces.
The drawings proposed by the teachers are not very different from what we proposed in the questionnaire : except for two quadric surfaces, they are mostly combinations of parallelograms, lines (plain or dotted) and arrows. Only five drawings represent a 2-space ; the thirty-one others are perspective drawings, evoking the 3-space, even if they are used to illustrate situations in a general vector space ; 3-space seems probably more representative than the plane, a better candidate for a paradigmatic model.
The notions illustrated by at least two teachers are projections, orthogonal projections, symmetries, rotations, supplementary subspaces, coordinates of a vector.

In fact, most of the notions and properties quoted by the teachers have already been encountered at secondary school in France, in the geometry course : lines, planes, symmetries, projections (it is not the case for supplementary subspaces and rotations around an axis).

[1]The term "space" refers here directly to geometry. In French indeed, "space" used on his own means "geometrical 3-space".

So the teachers do not seem to develop a specific figurative model, independant of a geometrical model, in linear algebra. For some teachers, drawings intervene only when they mention an affine geometry in their linear algebra course. Some others (a minority) use drawings in linear algebra, but only for $\mathbb{R}^2$ and $\mathbb{R}^3$. In that case, linear algebra in $\mathbb{R}^2$ and $\mathbb{R}^3$ could then be used as an intuitive, paradigmatic model for the whole theory. But there is is no evidence that the students will be able to use that model, especially if the teachers use no drawings in general vector spaces (we will not study that question here).

3.2 Conclusions of the teacher's questionnaire

Considering the answers to the whole questionnaire leads to distinguish two main tendencies among the teachers.
Some of them praise a structural approach to linear algebra, with almost no figural model associated. Geometry will then be presented as a mere application of the general theory.
On the opposite, the others choose to present an affine geometry, with an associated figural model, before introducing linear algebra.
This is a clear symptom of the influence, still very strong, of the discussions held before and during the reform of modern mathematics in France (1960-1970).
Only a minority of teachers propose a figural model especially elaborated for linear algebra. It might have negative consequences on the students practices : if some students need a figural model to help their reasoning in linear algebra, they will probably use a model associated with affine geometry, unsuitable in a vectorial space (they can for example mention "parallel" subspaces, when asked for their possible relative positions).

4 Use of models by students

4.1 Presentation of the test

Description of the activity

We have chosen to submit to first year university students an unusual linear algebra task : for two given sets of vectors of the plane, represented by two drawings, they were asked to say if there exists a linear application sending the first onto the second. Six couples of drawings were proposed ; the first was a parallelogram (except in the sixth case, where it was a segment) ; two basis vectors were drawn on the sides of the parallelogram. The second was either a parallelogram, or a circle, or a triangle, or a segment (see Annex 2).
The students were also asked to provide a justification for their positive or negative answer, but no proof, because we only wanted to observe the elements used to base their reasoning process.

Possible uses of models, and related difficulties

Several models can be used by students in that context ; we will briefly describe them here.

- *Geometrical models*

  - *Usual geometrical transformations*
    Students can use the model of the "usual" applications of the plane : rotations, projections, symmetries, dilations. That model can stemm from linear algebra in a two-space, but also from secondary school geometry. The main problem here is that students may answer negatively if they do not identify a usual geometrical transformation sending the first set of vectors onto the second. That problem has been pointed out by Sierpinska (Sierpinska 2000), in a research work about the learning of linear applications. She calls that kind of phenomenon "thinking of mathematical concepts in terms of prototypical examples".

  - *Preservation of spatial properties*
    Students can associate with linearity, or at least with linear applications of $\mathbb{R}^2$, some preservation properties. For example : "A linear application preserves alignment", or "A linear application preserves parallelograms". It can lead to wrong answers if only alignment is taken into account ; in that case, some students can declare that a parallelogram can be transformed into a triangle.

- *Linear algebra properties associated with a figural model*

  Students can use figural models, associated with different aspects of linearity, and different properties of linear applications.

  - *Stability properties*
    The stability properties, for the sum and the scalar multiplication, can be associated with drawings. For example, the drawing of a parallelogram can illustrate the sum of two vectors, and the coresponding stability.

  - *Transformation of the basis vectors*
    The students we asked know that a linear application of the plane is characterized by the images of two basis vectors. So they can draw on the second picture two arrows representing these images, as a justification for the existence of a convenient linear application. The problem that can arise here is that students only care for the two vectors, forgetting the rest of the figure. In that case they can even answer that a parallelogram can be transformed into a circle.

  - *Other properties*
    Figural models associated with various other properties of linear applications can intervene.
    "A linear application sends a subspace on another subspace" ; "A linear application sends the nul vector on itself"... Some of these properties involve the notion of dimension : "the dimension of the image of a subspace F is less or equal than the dimension of F", for example. There is in that case a special difficulty, stemming from a figural model associated with the notion of dimension. The given drawings can be misinterpreted ; in particular, a confusion between "dimension" and "direction" can occur.

4.2 Answers analysis

The test was proposed during the second semester of the first university year ; 43 students answered it. They already had linear algebra during the first semester, with different teachers (the tutorial groups are reorganized between the two semesters).
Uses of the models mentioned above clearly appear in their answers, with the associated difficulties. Several models can intervene in the same answer. In fact, three main types emerge, corresponding to the following use of models :

- *Usual transformations and dimensional properties (13 students, labelled "U")*
  These students use the two models together ; they propose for example a usual transformation to justify their positive answers, and use a dimensional argument in a negative case.
  The association of these two models is surprising at first sight, because they are of different natures. But they both correspond to an attempt of students to elaborate a figural model that can help them in their task. For that purpose, they use familiar objects ; but these objects are unsufficient to provide here an appropriate model.

- *Transformations of the basis vectors (14 students, labelled "B")*
  Only one model intervenes in these answers : the characterization of a linear application of the plane by the images of two basis vectors. These students reduce to a minimum their use of a figural model. Their reasoning is based on a theoretical property ; they draw two vectors on the second picture, because they are asked to do so. But most of them neglict to condider the whole drawing ; they claim, for example, that a parallelogram can be transformed into a triangle, because they can represent two "image vectors" on the sides of the triangle.

- *Preservation of spatial properties and stability properties (10 students, labelled "P")*
  The two models used in these answers are in fact closely related. The properties : "A linear application preserves parallelograms", and : "A linear application preserves sum and scalar multiplication" can indeed be associated with the same drawing ; the first can be used as an intuitive model for the second. The associated figural model is well adaptated for the task we proposed here.

(37 answers are gathered in these types ; for the 6 remaining answers, there is no evidence of the used models, but all of them are wrong).

The following crosstable shows the distribution of the students answers in the three types, together with their success or failure to the test.

	Correct answer	Incomplete answer	Wrong answer	Total
U	1	1	11	13
B	2	2	10	14
P	4	2	4	10

A correct answer means here that the choice of a positive or negative answer was right in the six cases ; in a wrong answer, there is at least one mistake. Only seven

students are right in the six cases ; it is indeed a difficult task, where drawings play a major part, on the opposite of the students habits.
Despite the low number of students in each box, it appears clearly that the ones belonging to the type labelled "P" are more likely to succeed than the others.

5 Conclusion

We presented here very local results ; but they point out general phenomena, confirmed by the rest of our work (Gueudet-Chartier 2000).
Some students need a figural model to help their reasoning in linear algebra (in the experiment we presented, they were obliged to deal with drawings ; that statement comes from other parts of our work).
But most teachers do not propose in their linear algebra courses a suitable, specific figural model. What are the consequences for the students practices ? According to our observations (the test presented above provides an example of it), three main types stand out :

- Some students do not seem to use any figural model. A minority of these students proves nevertheless a good understanding of linear algebra.
- Others try to construct by themselves such a model, using for example secondary school geometry ; but it turns out to be inadaptated for linear algebra.
- Some students elaborate a suitable figural model ; moreover, they are quite successfull in various linear algebra tasks. However, it is difficult to decide if that model is a factor, or on the contrary an evidence, of their understanding of linear algebra.

Studying further the students uses of figural models in linear algebra would require the organization of a teaching experiment, allowing us to know exactly wich models have been proposed, and to observe their influence on the students practices.

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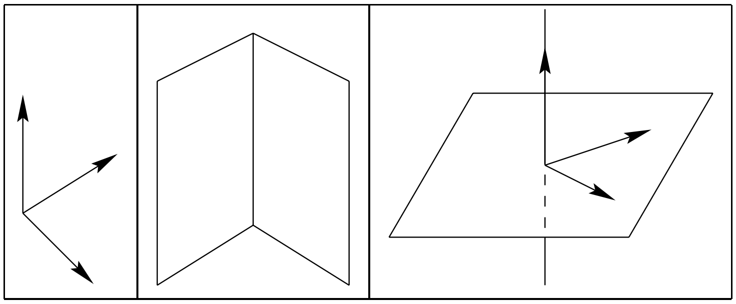
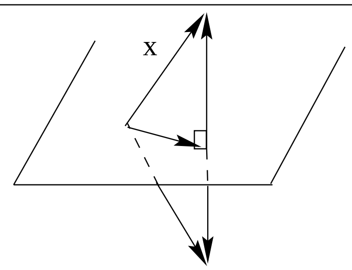
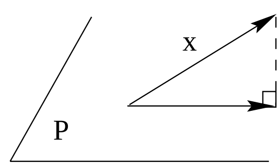
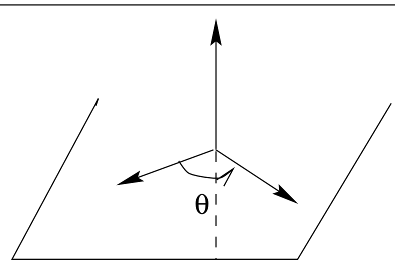
ANNEX 1

Drawings proposed to the teachers :



Examples of drawings proposed by the teachers :

Drawing	That drawing illustrates
	Orthogonal projection on a plane Orthogonal symmetry
	x and its orthogonal projection on P
	Rotation about an axis

ANNEX 2

Drawings proposed in the students test :