THE ROLE OF INSTRUMENTAL AND RELATIONAL UNDERSTANDING IN PROOFS ABOUT GROUP ISOMORPHISMS

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ABSTRACT

The ability to construct proofs is a crucial skill in advanced mathematics that most students lack. To investigate the causes of students' difficulty, we observed a small group of undergraduates and doctoral students constructing proofs about group isomorphisms. Undergraduates were able to construct very few proofs, despite having an understanding of mathematical logic and often possessing the instrumental knowledge needed to prove the propositions in our study. Doctoral students proved every proposition in our study. Our analysis reveals that doctoral students regularly used their relational understanding of group isomorphisms to guide their proof attempts, while undergraduates seldom did. We conclude that what one can prove solely using instrumental understanding is often limited, and using a relational understanding may be necessary to be an effective proof constructor.

1. Introduction

The ability to construct proofs about mathematical concepts is a crucial skill for any student of mathematics. Unfortunately, most college have serious difficulties constructing proofs (e.g. Moore, 1994). As students have difficulty with this crucial skill, it is natural to try to locate the cause of their difficulty. There has been considerable research on this topic, most of which has focussed on the logical aspect of proof construction. For instance, Harel and Sowder (1998) observed most students do not have an accurate conception of what constitutes a mathematical proof and Selden and Selden (1987) give examples of common invalid student proofs. While this research has produced rich data that is clearly important, there is a large and significant class of proofs that it cannot explain. Often, students fail to construct proofs because they do not know how to begin, spend all their time pursuing dead-ends, or reach an impasse where they simply cannot decide how to proceed (e.g. Moore, 1994; Schoenfeld, 1985). In these situations, the students' shortcomings are not logical in nature. Why students fail to construct proofs in these situations is poorly understood.

2. Instrumental and relational proofs

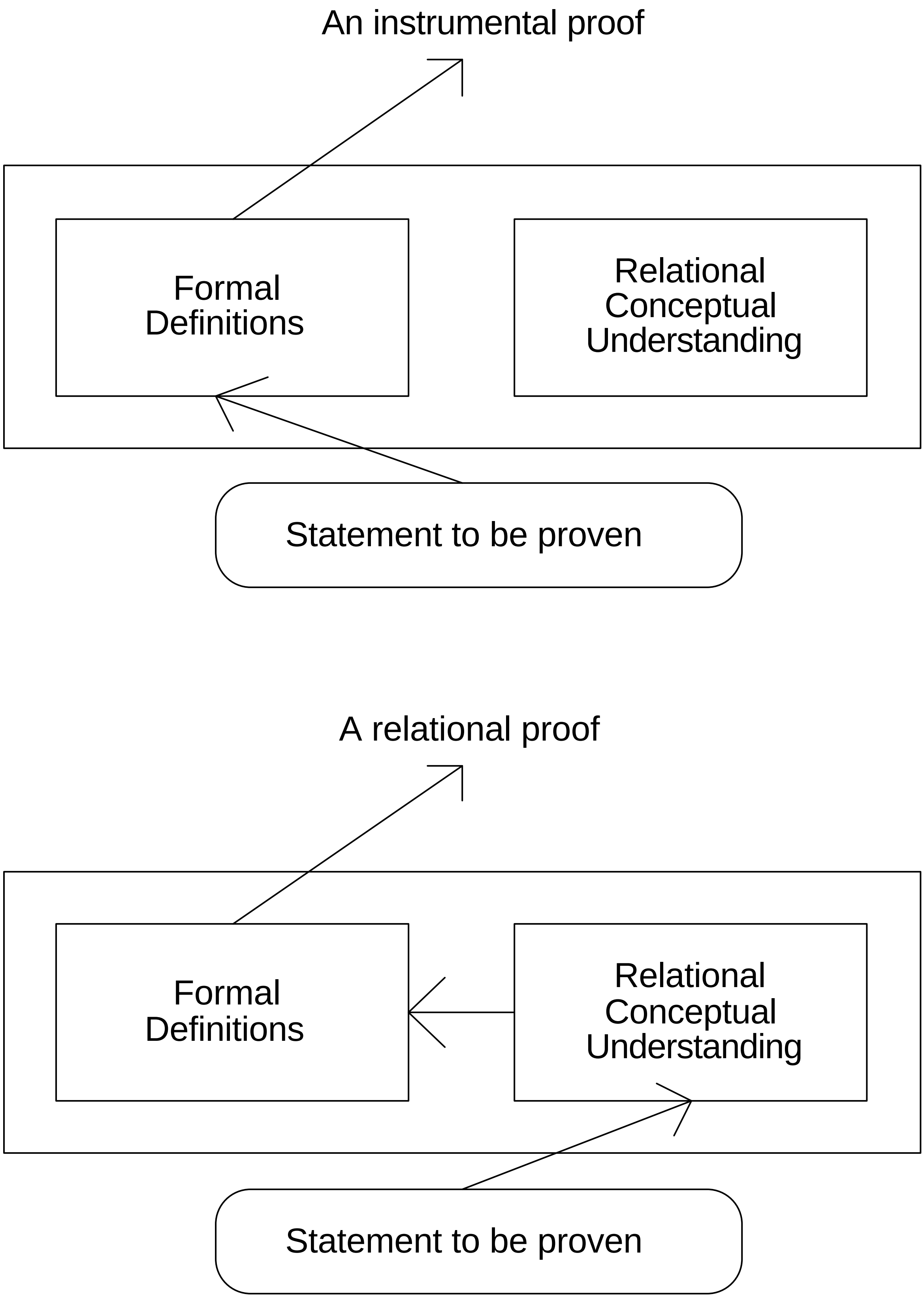
It is often said that there are two ways to understand a mathematical algorithm. An individual has an instrumental understanding of an algorithm if he or she can recall that algorithm and is capable of executing it; the individual has a relational understanding of an algorithm if he or she knows the purpose of the algorithm and why the algorithm works (Skemp, 1987).

We extend these types of understanding to include advanced mathematical concepts. We say an individual has an instrumental understanding of a concept if he or she can state the definition of the concept, is aware of the important theorems associated with that concept, and can apply those theorems in specific instances. We say an individual has a relational understanding of a concept if he or she understands the informal notion this concept was created to exhibit, why the definition is a rigorous demonstration of this intuitive notion, and why the theorems associated with this concept are true. (A relational understanding of a concept is somewhat akin to Tall and Vinner's concept image (Tall and Vinner, 1981)).

We use these types of understanding to describe two different types of proofs, as illustrated in Figure 1. An instrumental proof is a proof in which one primarily uses definitions and logical manipulations without referring to his or her intuitive understanding of a concept. A relational proof is a proof in which one uses his or her intuitive understanding of a concept as a basis for constructing a formal argument. An instrumental and a relational proof are essentially what Vinner (1991) calls a purely formal deduction and a deduction following intuitive thought.

We illustrate our definitions within the context of isomorphic groups, the concept used in our investigation. An individual with an instrumental understanding of isomorphic groups would know that the groups G and H are isomorphic if there exists a bijective homomorphism f from G to H, know basic theorems associated with isomorphic groups (e.g. an abelian group is not isomorphic to a non-abelian group), and be able to apply these theorems (e.g. S_3 is not isomorphic to Z_6). An individual with a relational understanding of isomorphic groups might recognize that isomorphic groups are "essentially the same" and that one is simply a re-labelling of the other. The definition

Figure 1. An instrumental and a relational proof



of isomorphic groups follows as the mapping f serves as the re-labelling (it is obvious then that f should be bijective and respect the groups' operations). The justification of many of the theorems about isomorphic groups, such as isomorphic groups must share all group theoretic properties, become self-evident once one views isomorphic groups as "essentially the same".

In our study, we ask participants to prove or disprove that two given groups are isomorphic. An instrumental proof of these propositions might consist of proving the two groups are isomorphic by constructing a bijective homomorphism between the groups or proving the groups are not isomorphic by demonstrating that no bijective mappings between the groups are homomorphisms. A relational proof would consist of first determining whether or not the groups in questions are essentially the same and then formalizing this intuitive reasoning.

In this paper, we observe undergraduates and doctoral students proving propositions about isomorphisms. We illustrate many examples where undergraduates failed to construct a proof despite possessing the instrumental knowledge required to do so. Further, we analyze both groups' proof attempts to shed light on the roles that instrumental and relational understanding play in proof construction.

3. Methods

Participants

Two groups of participants participated in this study. The first group of participant consisted of four undergraduate students at a university in the northeast United States. These students had recently completed their first abstract algebra course. Each student had also completed two linear algebra courses - the second of which stressed abstract vector spaces and rigorous proofs.

The second group of participants consisted of four doctoral students completing dissertations in an algebraic topic at a university in the mid-west United States. These students had approximately four more years of schooling than the undergraduate students.

Materials

Participants were first asked to prove the following Basic Propositions:

Basic Propositions

B1. Let G and H be groups and f be a homomorphism from G to G. Prove that for all x and y in G, $[f(xy)] = f(y^{-1})f(x^{-1})$.

B2. G is a group and f is a mapping from G to G such that $f(g) = g^{-1}$. Show that f is a homomorphism if and only if G is abelian.

The Basic Propositions were included to determine if the participants possessed an ability to construct rudimentary proofs. Participants were then asked to prove the more difficult Isomorphism Propositions:

I1. Prove or disprove: $\mathbf{Z}_{n!}$ is isomorphic to S_n

I2. Prove or disprove: $\mathbf{Q}$ is isomorphic to $\mathbf{Z}$

I3. Prove or disprove: $\mathbf{Z}_p$ x $\mathbf{Z}_q$ is isomorphic to $\mathbf{Z}_{pq}$ (when p and q are coprime)

I4. Prove or disprove: $\mathbf{Z}_p$ x $\mathbf{Z}_q$ is isomorphic to $\mathbf{Z}_{pq}$ (when p and q are not coprime)

I5. Prove or disprove: S_4 is isomorphic to D_{12}

(where $\mathbf{Z}_p$ represents the integers under addition modulo p, $\mathbf{Z}$ the integers under addition, $\mathbf{Q}$ the rationals under addition, S_n the set of permutations of n elements, and D_{12} the dihedral group with 24 elements).

Procedure

This procedure is similar to the one used in an earlier study reported in Weber (in press).

- Using verbal protocol analysis (Ericcson and Simon, 1993), participants were asked to 'think aloud' as they attempted to prove the propositions listed above. At any point, the participants were allowed to refer to the textbook used in the undergraduate abstract algebra course.

- After attempting to prove the propositions, the participants completed a paper-and-pencil test about the facts needed to prove the propositions in this study. This test contained open-ended questions (e.g. "State the definition of isomorphic groups") as well as yes-or-no questions (e.g. "Can an Abelian group be isomorphic to a non-abelian group?"). After each questions, the participants were asked to indicate how confident they were of their answer with an integer between 0 and 2, where 0 represented "just guessing" and 2 represented "absolutely certain".

- If participants had been previously unable to prove a proposition, they were invited to try again by making use of their work on the paper-and-pencil test.

Each proof attempt was coded using the following scheme:

Correct- The participant produced a valid proof

Failure to apply instrumental knowledge- The participant failed to construct a proof. However, the participant indicated that he or she had the instrumental knowledge to construct the proof by answering the relevant questions on the paper-and-pencil test correctly with some degree of confidence (1 or 2). When told to use his or her work on the paper-and-pencil test, the participant produced a valid proof. Therefore, the participant could construct a proof if specifically told which facts to use, but failed to construct a proof without this prompting.

Lack of instrumental knowledge- The participant failed to construct a valid proof and either indicated that he or she was not aware of a fact required to prove the theorem (or indicated that he or she was aware of the fact, but was just guessing), or the participant could not prove the theorem when told to use the facts on the paper-and-pencil test.

Invalid proof- The participant produced an invalid proof.

4. Results

All participants in this study could prove the Basic Propositions. Although these proofs were not difficult, the participants' success indicates that they all had some basic notion of proof, familiarity with group theoretic concepts, and an ability to logically manipulate symbols.

Each doctoral student was able to prove or disprove every Isomorphism Proposition in this study. The undergraduates' performance on each of the Isomorphism Propositions is presented in Table 1. Collectively the undergraduates were only able to prove two of the Isomorphism Propositions. However, there were nine instances where the undergraduates failed to construct a proof because they did not apply their instrumental knowledge. To be specific, when the undergraduates were specifically told to use the facts needed to prove the propositions, they were able to construct a proof. When they had previously attempted to construct proofs without this prompting, they failed to construct a proof. Hence, the data indicate that even if one has an accurate conception of proof, possessing an instrumental understanding of a mathematical concept does not imply that one can effectively prove statements about that concept. There were eleven instances in which the undergraduates demonstrated an instrumental understanding of isomorphisms and the groups in question; in only two of those instances did they produce a valid proof.

Table 1. Undergraduates' performance on proving the Isomorphism Propositions

Proposition Number	Valid proof	Failure to apply instrumental knowledge	Lack of instrumental knowledge	Invalid proof
I1	1	2	1	0
I2	0	4	0	0
I3	0	1	3	0
I4	1	2	1	0
I5	0	0	4	0
Total	2	9	9	0

To investigate the role that instrumental and relational understanding plays in constructing proofs, we analyzed the behavior of the participants as they attempted to construct their proofs. Below, we present a brief description of the undergraduates' and the doctoral students' behavior for each of the propositions. We conclude by offering a summary of both groups' performance in this study.

Prove or disprove S_n is isomorphic to $\mathbf{Z}_{n!}$

Each doctoral student proved these two groups were not isomorphic (when n was greater than two) within forty seconds. Three doctoral students did so by realizing that $\mathbf{Z}_{n!}$ was abelian and S_n was not. The other student pointed out that S_n had no element of order n!.

After attempting to inappropriately apply Cayley's theorem, one undergraduate was able to disprove the proposition (by noting that $\mathbf{Z}_{n!}$ was cyclic and S_n was not). Another undergraduate made no meaningful progress on this problem. The other two undergraduates tried unsuccessfully to construct a bijection between the two groups.

Prove or disprove **Q** *is isomorphic to* **Z**

The protocol of one doctoral student's proof is given below:

"**Z** is isomorphic to **Q**? That's false. Let's see… why? Well **Q** is dense and **Z** is not. No wait, denseness isn't a group property. Well then **Z** is cyclic and **Q** is not. So they can't be isomorphic".

Two other doctoral students proved that the groups could not be isomorphic because **Z** was cyclic and **Q** was not, with one adding, "I was tempted to add something about **Q** having a field structure, but that's not really the point". The final doctoral student proved the proposition by demonstrating that no homomorphism from **Z** to **Q** could be bijective.

The following excerpt of one undergraduate's protocol is given below:

"Um I think that **Q** and **Z** have different cardinalities so… no wait, **R** has a different cardinality, **Q** doesn't. Well, I guess we'll just use that as a proof. Yeah so I remember like seeing this proof on the board. I just don't remember what it is. There's something about being able to form a uh homomorphism by just counting diagonally [the student proceeds to create a complicated bijection between **Z** and **Q** by using a Cantorian diagonalization argument] Yeah I don't think we're on the right track here. Um… what you are describing is… it's um a bijection, but not a homomorphism"

This excerpt was representative of all four undergraduates' proof attempts. Upon realizing that **Z** and **Q** were equinumerous, all undergraduates constructed or attempted to construct a bijection between the groups. They seemingly showed little regard as to whether their bijections would respect the groups' operations. None successfully proved the groups were not isomorphic.

Prove or disprove $\mathbf{Z}_p$ x $\mathbf{Z}_q$ is isomorphic to $\mathbf{Z}_{pq}$ (assuming p and q are coprime)

An excerpt from one doctoral student's proof attempt is given below:

"OK, sufficient to find an element (g, h) in $\mathbf{Z}_p$ times $\mathbf{Z}_q$ that has order pq, because $\mathbf{Z}_p$ times $\mathbf{Z}_q$ has order pq and so if there's an element with the same order as the group, the group is cyclic and must be the same group as $\mathbf{Z}_{pq}$. OK um the element we're looking for is going to be (1, 1)."

The student then proceeded to show (1, 1) had order pq. The other doctoral students all proceeded to prove these groups were isomorphic by first observing that equinumerous cyclic groups were isomorphic and then showing that $\mathbf{Z}_p$ x $\mathbf{Z}_q$ was cyclic. No doctoral student constructed an explicit isomorphism between the two groups.

The two undergraduates that made progress on this problem attempted to construct a bijection between the two groups, one of which was a somewhat absurd mapping that mapped (a, b) in $\mathbf{Z}_p$ x $\mathbf{Z}_q$ to ab (mod pq) in $\mathbf{Z}_{pq}$. This mapping was neither bijective nor a homomorphism. Neither of these undergraduates used the fact that p and q were coprime. The other two undergraduates did not know how to begin their proof attempts.

Prove or disprove $\mathbf{Z}_p$ x $\mathbf{Z}_q$ is isomorphic to $\mathbf{Z}_{pq}$ (assuming p and q are coprime)

Three doctoral students proved that $\mathbf{Z}_p$ x $\mathbf{Z}_q$ was not cyclic and therefore could not be isomorphic to the cyclic group $\mathbf{Z}_{pq}$. The other doctoral student disproved this proposition by noting that $\mathbf{Z}_2$ x $\mathbf{Z}_2$ was not isomorphic to $\mathbf{Z}_4$.

One undergraduate disproved the proposition by offering the same counterexample. The other three undergraduates made no attempt to prove or disprove this proposition, explicitly reasoning that they made no useful progress on the last proposition, and there was nothing indicating their techniques would be more successful on this proposition.

Prove or disprove S_4 is isomorphic to D_{12}

The undergraduates had little familiarity with the dihedral groups so none were able to make much progress on this problem. Upon noting that the both groups were equinumerous, non-abelian groups, the doctoral students attempted to identify a distinct property of one group and demonstrate that the other group did not share this property. Some of the doctoral students' efforts were ineffective, as these groups do share some surprising properties. However, eventually all doctoral students were able to determine the groups were not isomorphic by finding a structural property possessed by one group that the other group did not share.

Summary

In most of the cases where the undergraduates seriously attempted to prove an Isomorphism Proposition, their proof attempt was of the following form: Upon realizing that the groups in question were equinumerous, they attempted to construct an arbitrary bijection between the groups. If this construction was successful, they were dismayed to find that the bijection did not respect the groups' operations and abandoned their proof attempts. If the construction was unsuccessful, they also gave up as they did not know how to proceed. Rarely did the undergraduates employ structural information about the groups in question. In our view, these types of proof attempts would be classified as instrumental, or purely deductive. Given the definition of isomorphic groups, the approach the undergraduates took was a logically viable option, perhaps the most viable option. However, we should note that this approach is unlikely to be successful. If one constructs an arbitrary bijection between two isomorphic groups, rarely will this mapping happen to be one of the few bijections that preserves the groups' operations. For this to occur, the bijection that one constructs must be based upon one's knowledge of the two groups. Likewise, it is a nearly impossible task to demonstrate that every bijective mapping between two groups is not a homomorphism without using structural information about the groups.

On the other hand, we would classify many of the doctoral students' proofs as relational proofs. The doctoral students seldom employed the definition of isomorphic groups; in fact there was only one instance where a doctoral student made any mention of an explicit mapping between the groups with which he was working. The doctoral students seemed quite consistent with their proof attempts: Prove that the two groups were the same or find a way that they were different. To show the groups in proposition three were isomorphic, the doctoral students did not attempt to construct an isomorphism between the groups, rather they tried to show the groups had the same essence- that they were both equinumerous cyclic groups. When the groups were not isomorphic, the doctoral students almost always attempted to find a property that one group possessed and the other did not. This was illustrated most sharply in their proofs of the last proposition. In the second proposition, one doctoral student recalled that **Q** was dense and another recalled **Q** formed a field. These observations were irrelevant from a group theoretic point of view, but they were indicative of the doctoral students' strategy.

5. Conclusions

There are three limitations of this study preventing broad conclusions. First, this study employed a small number of participants proving theorems within a narrow mathematical domain. More research is necessary to determine how general the effects observed in this study are. Second, it is unclear whether the undergraduates lacked a relational understanding of isomorphisms or simply declined to use it during their proof attempts (our paper-and-pencil tests are too crude to measure something as complex as relational understanding). Third, one reason that the doctoral students performed better than the undergraduates was that they had more mathematical experience. It seems unreasonable to hope that we can design short-term pedagogy lead undergraduates to achieve the doctoral students' level of performance, as the undergraduates will always lack the doctoral students' experience.

Leron, Hazzan, and Zazkis (1995) suggest students be taught a “naï ve” conception of isomorphisms long before learning their formal definition. Vinner (1991) offers similar advice in a more general setting; he advocates building an intuitive understanding of a mathematical concept before giving a precise definition. Skemp (1987) also endorses this view, recommending that students learn the essence of a concept through the judicious use of examples before learning the rule that defines the concept. We concur with these suggestions. We believe that students will best build a relational understanding of isomorphic groups if we present them with carefully selected examples of isomorphic and non-isomorphic groups. After students understand the essence of this concept, a formal definition can be given to them. Perhaps the students can generate this definition themselves. Whether this suggested pedagogy would improve students' ability to prove statements about isomorhpisms is a testable hypothesis and would be an interesting topic of future research.

Formal definitions play a crucial role in advanced mathematics. However, relying exclusively on definitions has severe weaknesses. Vinner (1991) notes that except for students well-versed in technical mathematics, students will use their intuitive understanding of a concept far more than the definition of the concept in their work. Therefore, a definition that is not consistent with a student's intuitive understanding of a concept will seldom be used. Our results indicate that students with a strong logical background can prove very little with definitions, facts, and theorems, if they do not also use relational understanding.

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