7.3

(3) Εξίσωση εφαπτομένου επιπέδου

$x = u^2, \quad y = u \sin e^v, \quad z = \frac{1}{3} u \cos e^v$ στο $(13, -2, 1)$

$T_u = \left(2u, \sin e^v, \frac{\cos e^v}{3}\right), \quad T_v = \left(0, e^v u \cos e^v, -\frac{u e^v \sin e^v}{3}\right)$

$$T_u \times T_v = \begin{vmatrix} i & j & k \\ 2u & \sin e^v & \cos e^v/3 \\ 0 & e^v u \cos e^v & -u e^v \sin e^v/3 \end{vmatrix} =$$

$$= \left(-\frac{u e^v}{3}\sin^2 e^v - \frac{u e^v}{3}\cos^2 e^v, \; -\left(-\frac{2u^2}{3}e^v \sin e^v\right), \; 2u^2 e^v \cos e^v\right) =$$

$$= \left(-\frac{u e^v}{3}, \; \frac{2u^2}{3} e^v \sin e^v, \; 2u^2 e^v \cos e^v\right) =$$

$$= e^v \left(-\frac{u}{3}, \; \frac{2u^2}{3} \sin e^v, \; 2u^2 \cos e^v\right)$$

για $x_0 = 13$, $u_0^2 = 13 \Rightarrow u_0 = \pm\sqrt{13}$

$y_0 = -2$, $-2 = \pm\sqrt{13} \sin e^v \Rightarrow \sin e^{v_0} = \mp \frac{2\sqrt{13}}{13}$

$z_0 = 1$, $\cos e^{v_0} = \pm \frac{3\sqrt{13}}{13}$

για $u_0 = \sqrt{13}$: $e^{v_0}\left(-\frac{\sqrt{13}}{3}, \; -\frac{2 \cdot \cancel{13} \cdot 2\sqrt{13}}{3 \cdot \cancel{13}}, \; \frac{2 \cdot \cancel{13} \cdot 3\sqrt{13}}{\cancel{13}}\right) =$

$$= e^{v_0}\sqrt{13}\left(-\frac{1}{3}, -\frac{4}{3}, 6\right)$$

άρα $e^{v_0}\sqrt{13}\left(-\frac{1}{3}, -\frac{4}{3}, 6\right)(x-13, y+2, z-1) = 0$

$-x + 13 - 4y - 8 + 18z - 18 = 0 \Rightarrow -(x-13) - 4(y+2) + 18(z-1)$ $\Rightarrow$

ή για $u_0 = -\sqrt{13}$

$-x - 4y + 18z - 13 = 0.$

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(5) Μοναδιαίο κάθετο διαν. της

$x = \cos v \sin u$, $y = \sin v \sin u$, $z = \cos u$

$u : [0, \pi]$, $v : [0, 2\pi]$

$T_u = (\cos u \cos v , \cos u \sin v , -\sin u)$

$T_v = (-\sin u \sin v , \sin u \cos v , 0)$

$$n = \begin{vmatrix} i & j & k \\ \cos u \cos v & \cos u \sin v & -\sin u \\ -\sin u \sin v & \sin u \cos v & 0 \end{vmatrix} =$$

$= (\sin^2 u \cos v , \sin^2 u \sin v , \cos u \sin u)$

μοναδιαία σφαίρα

$x^2 + y^2 + z^2 = 1$

$\|n\| = \sqrt{\sin^4 u \cos^2 v + \sin^4 u \sin^2 v + \cos^2 u \sin^2 u} =$

$= \sqrt{\sin^2 u (\sin^2 u + \cos^2 u)} = \sin u$

άρα μοναδιαίο κάθετο: $(\sin u \cos v , \sin u \sin v , \cos u)$

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9) α) εφαπτόμενο επίπεδο όταν $x=h(y,z)$

$$x = x_0 + (y-y_0)\frac{\partial h}{\partial y}(y_0,z_0) + (z-z_0)\frac{\partial h}{\partial z}(y_0,z_0)$$

β) $y = k(x,z)$

$$y = y_0 + (x-x_0)\frac{\partial k}{\partial x}(x_0,z_0) + (z-z_0)\frac{\partial k}{\partial z}(x_0,z_0)$$

11) $\Phi(r,\theta) = (r\cos\theta, r\sin\theta, \theta)$, $0 \le r \le 1$ και $\theta \in [0,4\pi]$

α) Ελικοειδής. Περιστρέφεται 2 φορές αφού $\theta \in [0,4\pi]$

β) $T_r = (\cos\theta, \sin\theta, 0)$
$T_\theta = (-r\sin\theta, r\cos\theta, 1)$

$$T_r \times T_\theta = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 1 \end{vmatrix} = (\sin\theta, -\cos\theta, r)$$

$$\|T_r \times T_\theta\| = \sqrt{1+r^2}$$

$$n = \frac{1}{\sqrt{1+r^2}} \pm (\sin\theta, -\cos\theta, r)$$

γ) στο (x_0,y_0,z_0) $r_0\cos\theta_0 = x_0$, $r_0\sin\theta_0 = y_0$, $z_0 = \theta_0$
$r^2 = x_0^2 + y_0^2$, $r = \sqrt{x_0^2+y_0^2}$

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Εφαπτόμενο επίπεδο:

$$(x-x_0, y-y_0, z-z_0)\cdot\left(\frac{y_0}{r}, -\frac{x_0}{r}, \sqrt{x_0^2+y_0^2}\right)=0$$

$$(x-x_0)\frac{y_0}{r} + (y-y_0)\frac{(-x_0)}{r} + (z-z_0)\sqrt{x_0^2+y_0^2} = 0$$

$$xy_0 - \cancel{x_0y_0} - yx_0 + \cancel{y_0x_0} + z(x_0^2+y_0^2) - z_0(x_0^2+y_0^2) = 0$$

$$xy_0 - yx_0 + z(x_0^2+y_0^2) = z_0(x_0^2+y_0^2)$$

d) $(x_0, y_0, z_0) = (r_0\cos\theta_0, r_0\sin\theta_0, \theta_0)$

γράφουμε το ευθύγραμμο τμήμα στη μορφή

$$\{(r\cos\theta_0, r\sin\theta_0, \theta_0) \mid 0 \le r \le 1\}$$

η ευθεία αυτή στην επιφάνεια

Γράφουμε την ευθεία στη μορφή

$$\{(tx_0, ty_0, tz_0) \mid 0 \le t \le \frac{1}{x_0^2+y_0^2}\}$$

και αντιστοιχεί στο c.

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(13) $x^2+y^2-z^2=25$

α) $x=5\cosh u \cos v$, $y=5\cosh u \sin v$
$z=5\sinh u$
$(u,v)\in R^2$

$\left(x^2+y^2-z^2=25\left(\cosh^2 u\cos^2 v+\cosh^2 u\sin^2 v-\sinh^2 u\right)= \right.$
$\left. =25 \right)$

b) $T_u=(5\sinh u\cos v, 5\sinh u\sin v, 5\cosh u)$
$T_v=(-5\cosh u\sin v, 5\cosh u\cos v, 0)$

$T_u\times T_v=$
$=(25\cosh^2 u\cos v, -25\cosh^2 u\sin v, 25\cosh u\sinh u)$

$\|T_u\times T_v\|=\sqrt{25^2(\cos^4 hu\cos^2 v+\cos^4 hu\sin^2 v+\cosh^2 u\sinh^2 u}=$
$=25\cosh u\sqrt{\cos^2 hu+\sin^2 hu}=$
$=25\cosh u\sqrt{\cosh(2u)}$

άρα $n=\left(\frac{\cosh u\cos v}{\sqrt{\cosh(2u)}}, -\frac{\cosh u\sin v}{\sqrt{\cosh(2u)}}, \frac{\sinh u}{\sqrt{\cosh(2u)}}\right.$

<u>Άλλη παράσταση</u>

α) $\varphi(z,\theta)=((25+z^2)\cos\theta, (25+z^2)\sin\theta, z)$, $z\in R$
$\theta\in[0,2\pi]$

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b) $\varphi_z = (2z\cos\theta,\ 2z\sin\theta,\ 1)$

$\varphi_\theta = (-(25+z^2)\sin\theta,\ (25+z^2)\cos\theta,\ 0)$

$\varphi_z \times \varphi_\theta = (-(25+z^2)\cos\theta,\ -(25+z^2)\sin\theta,\ 2z(25+z^2)) =$
$= (25+z^2)\ (-\cos\theta,\ -\sin\theta,\ 2z)$

$\|\varphi_z \times \varphi_\theta\| = (25+z^2)\sqrt{1+4z^2}$

άρα $n = \dfrac{1}{\sqrt{1+4z^2}}(\cos\theta,\ \sin\theta,\ -2z)$

c) $x_0 = (25+z_0^2)\cos\theta_0$, $y_0 = (25+z_0^2)\sin\theta_0$
$z_0 = 0$

άρα $-\cos\theta_0(25+z_0^2) = -x_0$ $\quad -\sin\theta_0\cdot(25+z_0^2) = -y_0$

$(\varphi_z \times \varphi_\theta)(x-x_0,\ y-y_0,\ z-z_0) = 0$

$\Rightarrow (-x_0,\ -y_0,\ 0)(x-x_0,\ y-y_0,\ z) = 0$

$-x_0(x-x_0) - y_0(y-y_0) = 0$
$-x_0x + x_0^2 - y_0y + y_0^2 = 0$

$$\boxed{x_0x + y_0y = 25}$$

d) αντικαθιστώ τις συντεταγμένες των τυχαίων σημείων της ευθείας στην εξίσωση της επιφάνειας και των εφαπτόμενων επιπέδων.

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