SUB-RIEMANNIAN GEOMETRY: A BRIEF REVIEW

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Abstract. We review the basics of sub-Riemannian geometry. This is the bulk of a talk given for the 90 years of the Department of Mathematics of the Aristotle University of Thessaloniki

1. Introduction

Riemannian geometry is quite familiar to most of us. It is produced out of a model space, i.e., a differentiable manifold endowed with an inner product at its tangent bundle. In the *sub-Riemannian* geometry, we have again a manifold as a model space, but this time we assume that there is a distribution with a fibre inner product. Recall that a distribution is a family of k-planes, i.e., a linear subbundle of the tangent bundle of the manifold. The distribution shall be called the *horizontal tangent space* and objects tangent to it shall be called *horizontal*. In a sub-Riemannian world, the distance traveled between two points is defined as in Riemannian geometry but here, we are allowed to travel along *horizontal curves* which join the two points. These curves are that which their velocity vector is always lying in the horizontal tangent space.

We can trace the awakening of sub-Riemannian geometry in a theorem of C. Carathéodory; this theorem is related to Carnot's Thermodynamic laws. The reasoning behind calling sub-Riemannian geometry as *Carnot-Carathéodory geometry* by Gromov and others, lies exactly in that fact. Carathéodory's theorem is about codimension one distributions. Such a distribution is defined by a single Plaffian equation $\omega = 0$, where ω is a nowhere vanishing 1-form. Recall that this distribution is *integrable* if through each point there passes a hypersurface which is everywhere tangent to the distribution. By the celebrated Frobenius' Theorem, an integrable distribution is *involutive*: for codimension one distributions, this means that locally there exists functions λ and f such that $\omega = \lambda df$. In this case, any horizontal path passing through a point p_0 must lie in $S = f(p_0)$. Consequently, pairs of points p_0 and p_0' that lie in different hypersurfaces cannot be connected by a horizontal path. Carathéodory's theorem is the converse of this statement.

Theorem 1. *(C. Carathéodory) Let M be a connected manifold endowed with a real analytic codimension one distribution. If there exist two points that cannot be connected by a horizontal path then the distribution is integrable.*

Carathéodory was asked to prove this theorem by the German physicist Max Born; Born's problem was to prove the second law of Thermodynamics and the existence of the entropy function S. From the work of Carnot, Joules and others it was known that there exist thermodynamic states $A = p_0$ and $B = p_0'$ that cannot be connected to each other by adiabatic processes; these are slow processes where no heat is exchanged. So to Carathéodory, and thus to sub-Riemannian geometry, an adiabatic process is a horizontal curve and the horizontal constraint is the Plaffian equation $\omega = 0$. The integral of ω over a curve is the net heat exchange undergone by the process represented by the curve. So eventually, Carathéodory's theorem implies the existence of integrating factors $\lambda = T$ and $s = f$ so that $\omega = TdS$ (here, T is the temperature and S is the entropy).

Carathéodory's theorem also can be stated as follows: if a codimension one distribution is not integrable, then any two points can be connected with a horizontal path. In distributions of arbitary codimension, this generalises to what is known as Chow's thorem, see Section 2.3 for details. We shall only make some comments now about Chow's Theorem which is considered as the cornerstone of sub-Riemannian geometry. First, let us review Frobenius' integrability theorem in its full force. Let M be an n-dimensional manifold and $\mathcal{D}$ be a distribution of codimension $n > k \geq 1$. Then $\mathcal{D}$ is called *integrable* if through each point p lying on a plane of $\mathcal{D}$, there is k-dimensional submanifold tangent to that plane. It is called *involutive* if for every X and Y vector fields of $\mathcal{D}$, the Lie bracket $[X, Y] \in \mathcal{D}$.

Theorem 2. *(Frobenius) Let M and $\mathcal{D}$ as above. Then $\mathcal{D}$ is integrable if and only if is involutive.*

In sub-Riemannian geometry we find ourselves in the opposite extreme of integrability In a *bracket generating* or *completely non integrable* distribution any tangent vector field may be written as the sum of iterated Lie Brackets $[X_1, [[X_2, [X_3, ...]]]$ of horizontal vector fields. Chow's theorem simply says that for a completely non integrable distribution on a connected manifold,any two points can be connected by a horizontal path. It follows that on a connected sub-Riemannian manifold whose underlying distribution is non integrable, the distance between any two points is finite, since there exists at least one horizontal curve joining these two points. Summing up, sub-Riemannian geometry is a riemannian geometry together with a constraint on admissible directions of movements. In Riemannian geometry any smoothly embedded curve has locally finite length. In sub-Riemannian geometry, a curve failing to satisfy the obligation of the constraint has necessarily infinite length.

Not very surprisingly eventually, sub-Riemannian geometry is connected to the Isoperimetric Problem (Dido's problem, or Pappu's problem). Dido's problem is formulated in the Aeneid, Virgil's epos glorifying the beginning of Rome:

Given a length, maximise the area of domains whose perimeter is this length.

Dido, a princess of Phoenicia, fled across the Mediterranean sea with a few servants and friends due to her entirely dysfunctional family: Her brother, Pygmalion, murdered her husband and took all her possessions. Arriving penniless in a part of a coast line of Africa ruled by king Jarbas, she persuaded him to give her as much land as she could enclose with an oxide. Dido then smartly enclosed the simicircular city of Carthage. This is the solution to the isoperimetric problem.

We shall now formulate this problem in mathematical terms. In $\mathbb{R}^2$ the volume form is $dvol = dx \wedge dy$ which is the differential da of the one form

$$a = \frac{1}{2}(xdy - ydx).$$

Using Stokes' theorem we get that if a closed smooth positively oriented curve γ in $\mathbb{R}^2$ encloses a domain $D_\gamma = int(\gamma)$, then the area $\mathcal{A}(D_\gamma)$ is given by

$$\mathcal{A}(D_\gamma) = \iint_{D_\gamma} dx \wedge dy = \int_\gamma a.$$

Therefore, Dido's problem is:

Maximize $\int_\gamma a$ under the condition $l(\gamma) = \int_\gamma ds = \int_c^b \|\dot\gamma(t)\|$.

If we start from a curve $\gamma(t) = (x(t), y(t))$ in $\mathbb{R}^2$ such as $\gamma(0) = (0, 0)$, we can lift it into a curve in $\mathbb{R}^3$ where the third coordinate $z(t)$ is the signed area enclosed to $\gamma[0, t]$ and the segment from the origin to $\gamma(t)$. That is,

$$z(t) = \int_{\gamma[0,t]} a = \frac{1}{2}\int_{\gamma[0,t]} xdy - ydx.$$

Differentiating with respect to t we get

$$\dot z(t) = \frac{1}{2}(\dot x(t)y(t) - \dot y(t)x(t)).$$

Set $\omega = dz - \frac{1}{2}(xdy - ydx)$ and consider curves

$$\tilde{\gamma} = (\gamma_1, \gamma_2, \gamma_3) \quad \tilde{\gamma} : [0,1] \to \mathbb{R}^3, \quad \tilde{\gamma}(0) = (0,0,0).$$

Then lifted curves are exactly those which satisfy

$$\dot{\gamma} \in \ker \omega \iff \omega(\dot{\gamma}(t)) = 0, \quad t \in [0,1].$$

The form

$$\omega = dz - \frac{1}{2}(xdy - ydx),$$

is called the *standard contact form*. Recall that a contact form in a $(2n+1)$-dimensional manifold is a 1-form ω satisfying

$$\omega \wedge (d\omega)^n \neq 0.$$

In the case of the standard contact form, $\omega \wedge (d\omega) = dx \wedge dy \wedge dz$ and the distribution $\mathcal{D}$ determined by ω at each point $p = (x,y,z)$ is

$$\mathcal{D}_p = \ker(\omega_p) = \{(v_1, v_2, v_3) \in \mathbb{R}^3 : v_3 = \frac{1}{2}(xv_2 - yv_1)\}.$$

Consider the following linear product in $\mathcal{D}_p$: For $v, w \in \mathcal{D}_p$,

$$< v, w >= v_1 w_1 + v_2 w_2. \tag{1.1}$$

Observe that $< v, v >\equiv 0$ if and only if $v_1 = v_2 = 0$, that is if the z-axis is included in $\mathcal{D}_p$; this can never happen, therefore $< \cdot, \cdot >$ is positively defined. We now fix a frame $\{X, Y, Z\}$ where

$$X = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}, \quad Z = \frac{\partial}{\partial z}, \tag{1.2}$$

and we declare it orthonormal. Since

$$\frac{\partial}{\partial x} = X + \frac{y}{2}Z \ , \ \frac{\partial}{\partial y} = Y - \frac{x}{2}Z,$$

we have on each $\mathcal{D}_p$ that

$$v = v_1 X + v_2 Y + (\frac{v_1}{2}y - \frac{v_2}{2}x + v_3)Z = v_1 X + v_2 Y$$

In this manner, a Riemannian metric is given by the linear product above.

In contact geometry, a curve γ is called *Legendrian* if

$$\omega(\dot{\gamma}) = 0 \iff \dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}$$

for all t in the domain of γ. Given a Legendrian curve γ, we define its length $l(\gamma)$ as the integral of the norm of $\dot{\gamma}$ with respect to the linear product. In other words, $l(\gamma)$ is exactly the Euclidean length of $\mathrm{pr}_{\mathbb{C}}(\gamma)$, the projection of γ into the plane. We may now introduce a new distance in $\mathbb{R}^3$: For $p, q \in \mathbb{R}^3$,

$$d_{cc}(p,q) = \inf\{l(\gamma) : \gamma \text{ Legendrian joining p and q}\}.$$

Do Legendrian joining curves exist? To connect, say $(0,0,0)$ and (x,y,z), take a curve γ in $\mathbb{R}^2$ from $(0,0)$ to (x,y) with the property that the signed area engulfed by γ and the line segment from $(0,0)$ to (x,y) is exactly z. Then, the lifted curve $\tilde{\gamma}$ will connect $(0,0,0)$ and (x,y,z). Now the Riemannian length of $\tilde{\gamma}$ equals the Euclidean length of γ. Thus there is a correspondence between d_{cc} geodesic (i.e.. a curve realising the infimum) and solutions of the dual Dido's problem: Fix a value for the area and minimize the perimetre.

One of the most standard examples of sub-Riemannian objects is the Heisenberg group. Perhaps the most crucial property of its geometry that we are about to define is that it is isometrically homogeneous. We may endow $\mathbb{R}^3$ with a group structure different from the standard Euclidean one in a way that all previous constructions are preserved by the action of the group onto itself. Consider the group law

$$(x,y,z)*(x',y',z')=(x+x',y+y',z+z'+\frac{1}{2}(xy'-yx')). \tag{1.3}$$

It can be shown that left translations $L_{(s,t,u)}$ defined by $L_{(s,t,u)}(x,y,z)=(s,t,u)*(x,y,z)$ preserve the distribution $\mathcal{D}$ and the orthonormal basis $\{X,Y,Z\}$ as in (1.2).

Proposition 1.1. *Heisenberg geometry is isometrically homogeneous. The Heisenberg group has a Lie group structure so that left translations are isometries with respect to the contact distance d_{cc}.*

The Heisenberg group has also a (nilpotent, non Abelian) matrix group model. This is described by the subgroup $G<GL(3,\mathbb{R})$, where

$$G=\left\{\begin{bmatrix}1&a&c\\0&1&b\\0&0&1\end{bmatrix}:a,b,c\in\mathbb{R}\right\}.$$

Its Lie Algebra is

$$\mathfrak{g}=\left\{\begin{bmatrix}0&a&c\\0&0&b\\0&0&0\end{bmatrix}:a,b,c\in\mathbb{R}\right\}$$

and a basis for $\mathfrak{g}$ is

$$X=\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix},Y=\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix},Z=\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}.$$

One parameter subgroups are of the form

$$\gamma_{(a,b,c)}(t)=exp(t\begin{bmatrix}0&a&c\\0&0&b\\0&0&0\end{bmatrix})=\sum_{t=0}^{\infty}t^n\begin{bmatrix}0&a&c\\0&0&b\\0&0&0\end{bmatrix}^n=$$

(Why?)

$$=I+t\begin{bmatrix}0&a&c\\0&0&b\\0&0&0\end{bmatrix}+t^2\begin{bmatrix}0&0&ab\\0&0&0\\0&0&0\end{bmatrix}=\begin{bmatrix}1&at&act+abt^2\\0&1&bt\\0&0&1\end{bmatrix}$$

The map

$$\phi:(x,y,z)\mapsto\begin{bmatrix}1&x&z+\frac{1}{2}xy\\0&1&y\\0&0&1\end{bmatrix}$$

is a Lie group isomorphism from the Lie group $\mathbb{R}^3$ with product (1.3) to the Lie group G with the usual matrix product. Straightforward calculations show that ϕ is a group homomorhpishm and that its differential at the identity is the identity matrix. More than this is true. Heisenberg group is a two-step nilpotent 3 dimensional Lie group and there are only two simply connected nilpotent Lie groups of dimension 3: The Heisenberg group and the Euclidean group.

2. Elements of the general theory of sub-Riemannian geometry

2.1. **Basics.** We shall denote a metric space by (X,d), where $X\neq\emptyset$ is a set and d is a distance. A path (or curve) γ is continuous map $\gamma:I\mapsto X$ where I is an interval $[a,b]$ of $\mathbb{R}$. The length of γ is defined by

$$l(\gamma) = \sup\left\{\sum_{i=1}^{n} d(\gamma(t_i), \gamma(t_{i-1})) : n \in \mathbb{N}, a = t_0 < t_1 < ... < t_n = b\right\}.$$

A rectifiable curve is a curve of finite length. A curve γ is called a *geodesic* if for all $t_1, t_2 \in [a, b]$,

$$l(\gamma[t_1, t_2]) = |t_2 - t_1|$$

The metric space (X, d) is said to be a *path metric space* if for all $x, y \in X$,

$$d(x, y) = \inf\{l(\gamma), \gamma \text{ joins } x, y\}.$$

If the above infimum is attained by a geodesic then (X, d) is called a *geodesic metric space.* A criterion for a path metric space to be geodesic is the following:

Theorem 3. *(Hops-Rinow-Cohn Vossen) A path metric space (X, d) which is complete and locally compact is geodesic.*

A function $f : (X, d_X) \mapsto (Y, d_Y)$ is called *Lipshitz* if

$$\exists K > 0 : \ d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2)$$

for all $x_1, x_2 \in X$. It is *locally Lipschitz* if for every $x \in X$ there exists a neighbourhood U_x such that $f|_{U_x}$ is Lipscitz. If there exists a $K \geq 1$ such that

$$\frac{1}{K} d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2),$$

for all $x_1, x_2 \in X$, then f is called *bi-Lipschitz* (or, more accurately, K-bi-Lipschitz). A K-bi-Lipschitz map is a homeomorphism onto its image. *Isometries* are 1-bi-Lipschitz maps.

Let $S \subset X$ and $m > 0$. Define for $\delta > 0$ the sets

$$H^m_\delta(S) = \inf\left\{\sum_{i=1}^{\infty} \operatorname{diam}(U_i)^m : \bigcup_{i=1}^{\infty} U_i \subset S, \ \operatorname{diam}(U_i)^m < \delta\right\}.$$

Then,

$$\mathcal{H}^m(S) = \sup_{\delta > 0} H^m_\delta(S) = \lim_{\delta \to 0} H^m_\delta(S),$$

is the m-dimensional ıHausdorff measure on S. The *Hausdorff dimension of* S is then

$$dim_{Haus}(S) = \inf\{d \geq 0 : \mathcal{H}^m(S) = 0\} = \sup\{\{d \geq 0 : \mathcal{H}^m(S) = \infty\} \cup \{0\}\}.$$

Let now M be a differentiable manifold of dimension n. For $p \in M$, the fibre $T_p(M)$ of the tangent bundle TM is a derivation of germs of $\mathcal{C}^\infty$ functions at p, i.e. ,an $\mathbb{R}$-linear map from $\mathcal{C}^\infty(p)$ to $\mathbb{R}$ satisfying the Leibniz rule. Suppose that $F : M \mapsto N$ is a smooth mapping between manifolds and $p \in M$. Then the differential $(F_p)_* : T_p(M) \mapsto T_{F(p)}(N)$ is defined as follows: If $X \in T_p(M)$,

$$F_{*,p}(X)(f) = X_p(f \circ F) \text{ , for all } f \in \mathcal{C}^\infty(F(p)).$$

Let $\Gamma(TM)$ be the linear space of smooth vector fields, that is, smooth sections of TM. For $X, Y \in \Gamma(TM)$, their Lie Bracket $[X, Y]$ is defined by

$$[X, Y]f = X(Yf) - Y(Xf), \quad f \in \mathcal{C}^\infty(M).$$

The set $\Gamma(TM)$ together with $[\cdot, \cdot]$ is a Lie algebra. If $F : M \mapsto N$ is a smooth and invertible, then for $X \in \Gamma(TM)$ the push-forward vector field is defined by

$$(F_* X)_{F(p)} = (F_{*,p})(X_p), \quad p \in M.$$

The push forward commutes with the Lie Bracket:

$$[F_*X, F_*Y] = F_*[X, Y], \quad X, Y \in \Gamma(TM).$$

If $F : M \mapsto N$ and γ is a smooth curve, then

$$(F_{*,\gamma(t)})(\dot{\gamma}(t)) = \frac{d(F \circ \gamma)}{dt},$$

where $\dot{\gamma}(t) \in T_{\gamma(t)}M$ and $\frac{d(F\circ\sigma)}{dt} \in T_{F(\sigma(t))}(N)$. If $f \in \mathcal{C}^\infty(M)$, by identifying $T_{f(p)}(\mathbb{R})$ with $\mathbb{R}$, we may write

$$df_p(X) = X_p(f), \quad X \in \Gamma(TM)$$

A *Riemannian metric* on M is a family of positive definite inner products

$$g_p : T_p(M) \times T_p(M) \mapsto \mathbb{R}, \quad p \in M,$$

such that for all $X, Y \in \Gamma(TM)$ the function

$$p \mapsto g_p(X_p, Y_p) \text{ is differentiable.}$$

In a local coordinate system $\{U_p, x_1, ..., x_n\}$, the vector fields

$$\left\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\right\},$$

form a basis for the tangent vectors at U_p. The components of the metric tensor with respect to the coordinate system are

$$(g_{ij})_p = g_p\Big(\Big(\frac{\partial}{\partial x_i}\Big)_p, \Big(\frac{\partial}{\partial x_j}\Big)_p\Big),$$

or, equivalently,

$$g = \sum_{i,j} g_{ij} dx_i \otimes dx_j.$$

The pair (M, g) is called a *Riemanniann manifold*.

A *Finsler structure* on a differentiable manifold M is given by a function

$$\|\cdot\| : TM \mapsto \mathbb{R}$$

which is smooth on the complement of the zero section of TM and its restriction to each fiber $T_p(M)$ is a symmetric norm. A Riemannian manifold has a naturally induced Finsler structure: $\|X\| = g^{1/2}(X, X)$. Connected Riemannian and Finsler manifolds carry the structure of path metric spaces. If $(M, \|\cdot\|)$ is a connected Finsler manifold and $\gamma : [a, b] \mapsto M$ is a parametrised curve in M which is differentiable with velocity vector $\dot{\gamma}$, then the length of γ is defined by

$$l(\gamma) = \int_a^b \|\dot{\gamma}(t)\| dt.$$

Since we may always parametrise γ by its arc length, $l(\gamma)$ does not depend on the parametrisation. The distance function $d : M \times M \mapsto [0, +\infty)$ is given by

$$d(p, q) = \inf\{l(\gamma),\ \gamma \text{ differentiable, joining } p, q\}.$$

The distance d satisfies all the properties of a distance function in a metric space. To prove the property $d(p, q) = 0 \Rightarrow p = q$ on a Riemannian manifold M, we use normal coordinates which also show as that the manifold M and the metric space (M, d) have the same topology. If M is Finsler, one shows that any Finsler structure is locally bi-Lipschitz, equivalently to a Riemannian structure.

2.2. **Carnot-Carathéodory distance.** Let $(M, \|\cdot\|)$ be a Finsler manifold and suppose that $\mathcal{D}$ is a distribution on M. Then the triple $(M, \mathcal{D}, \|\cdot\|)$ is called a *subFinsler manifold*; if the Finsler structure is Riemannian than we are in the case of sub-Riemannian manifold. An absolutely continuous curve γ in M is said to be horizontal with respect to $\mathcal{D}$ if $\dot{\gamma}(t) \in \mathcal{D}$ for almost all t.The length of γ is

$$l_h(\gamma) = \int_0^1 \|\dot{\gamma}(t)\| dt.$$

We consider the metric on M induced by $\mathcal{D}$ and $\|\cdot\|$. For $p, q \in M$,

$$d_{cc}(p, q) = \inf\{l_h(\gamma) : \gamma \text{ horizontal from p to q}\}.$$

This is the *(Finsler) Carnot-Carathéodory distance.*

2.3. **Hörmander's condition-statement of Chow's Theorem.** A distribution $\mathcal{D} \subset TM$ is called *bracket generating* if any local frame $\{X_1, ..., X_k\}$ for $\mathcal{D}$ together with all of its iterated Lie brackets

$$[X_i, X_j],\ [X_i, [X_j, X_k]], \dots,$$

spans TM. If $\mathcal{D}_p^{(j)}$ is the span of all contents of order $\leq j$, then the above is exctly *Hörmander's condition*:

$$T_p(M) = \mathcal{D}_p^{(j)}, \quad j \in \mathbb{N}.$$

The *metric or Hausdorff dimension* is

$$\sum_j j(\dim \mathcal{D}_p^j - \dim \mathcal{D}_p^{j-1}).$$

A bracket generating distribution (that is, a distribution that satisfies Hörmander's condition) lies on the extreme opposite of an integrable distribution. We now state Chow' Theorem:

Theorem 4. *(Chow 1959, Rashevskyi 1938) If $\mathcal{D}$ is a bracket generating distribution on a connected manifold M, then any two points of M can be connected by a horizontal path.*

In the case of Heisenberg group $\mathcal{H}$, equations $[X, Y] = Z, [X, Z] = [Y, Z] = 0$ and Chow's theorem guarantee that we can connect any two points by a horizontal path.

The next two theorems are essentially equivalent versions of Chow's Theorem.

Theorem 5. *If $\mathcal{D}$ is bracket generating on M, than the topology of M induced by the cc-distance is the manifold topology.*

The *endpoint map* associated to $\mathcal{D}$ and which is based at a point $p_0 \in M$ is the map that takes each horizontal curve with starting point p_0 to its endpoint.

Theorem 6. *If $\mathcal{D}$ is bracket generating, then the endpoint map is open.*

For any distribution $\mathcal{D}$ on M and for any point $p_0 \in M$, the *accessible set* $\mathbb{A}(p_0)$ is the image of the endpoint map associated to $\mathcal{D}$ with starting point p_0.

Below we shall present a sketch of the proof of Chow's Theorem; prior to this we remark that its converse fails. There are distributions which are not bracket generating but still are horizontally path connected.

2.4. **Proof of Chow's Theorem-sub-Riemannian Hopf-Rinow.** We fix a point p and let $X \in \mathcal{D}$. Consider the curve γ solving the d.e.

$$\gamma(0) = p \quad \dot{\gamma}(t) = X_{\gamma(t)}.$$

Then γ is a horizontal curve and X_p is tangent to $\mathbb{A}(p)$. Therefore, the whole $\mathcal{D}_p$ is tangent to the accessible set $\mathbb{A}(p)$. We assume for the moment that $\mathbb{A}(p)$ is an embedded submanifold of M. Then its tangent space $T_p\mathbb{A}(p)$ is closed under the Lie bracket. That is, the Lie span of $\mathcal{C}_p(M)$ is tangent to $\mathbb{A}(p)$. Therefore, $\dim(M) = \dim(\mathbb{A}(p))$ and $\mathbb{A}(p)$ is the whole of M.

Note that the crucial step for the proof of Chow's Theorem is the assertion that $A(p)$ is an embedded submanifold. This holds true by a theorem of Sussmann (1973).

Theorem 7. *(Sub-Riemannian Hopf-Rinow) If $\mathcal{D}$ is bracket generating then sufficiently neighbouring points can be joined by a d_{cc} geodesic. Moreover, if M is connected and (M, d_{cc}) is complete, then any two points of M can be joined by a d_{cc} geodesic.*

Proof. By Theorem 5, the topology induced by the d_{cc} metric is the manifold topology. In particular, the space is locally compact. Applying Arzela-Ascoli's Theorem in a compact ball we obtain the existence of geodesics at a small scale. Applying the Hopf-Rinow Theorem for complete, locally compact length spaces, we obtain the existence of global geodesics. □

2.5. **Ball-box theorem and Hausdorff dimension.** Let $\mathcal{D} \subset TM$ a distribution. We shall make the following assumptions:

(1) There exist $X_1, ..., X_n \in \Gamma(TM)$ such that for all $p \in M$,

$$\{X_1, ..., X_\kappa\}_p,$$

is a basis for $\mathcal{D}_p$ and

$$\{X_1, ..., X_n\}_p,$$

is a basis for $T_p(M)$.

(2) For all $j = 1, ..., n$ there exists a $d_j \in \mathbb{N}$, (the *degree* of X_j), such that

$$(X_j)_p \in \Delta_p^{[d_j]} \setminus \Delta_p^{[d_j - 1]}, \ \forall p \in M,$$

where $\Delta^{[d_j]}$ is the space of commutators of $X_1, ..., X_j$ of order d_j.

The latter condition is a regularity assumption for $\mathcal{D}$; endowed with this condition $\mathcal{D}$ is called *equiregular*.

We shall parametrise M using flows of linear sums of vector fields in $\mathcal{D}$. Recall that, for $p \in M$ and $X \in \Gamma(TM)$, the exponential map

$$\exp_p(X) = \gamma(1),$$

the value at time 1 of the integral curve γ of the vector field starting at p, i.e., the solution of

$$\dot{\gamma}(t) = X_{\gamma(t)}, \ \gamma(0) = p.$$

For fixed $p \in M$, exponential coordinates are defined by $\Phi : \mathbb{R}^n \mapsto M$, where

$$\Phi(t_1, ..., t_n) = \exp_p(t_1 X_1 + ... + t_n X_n).$$

This map is in general only local, that is, defined around a neighbourhood of $0 \in \mathbb{R}^n$.

The *box* with respect to $X_1, ..., X_n$ is

$$\text{Box}(r) = \{(t_1, ..., t_n) \in \mathbb{R}^n : |t_j| \leq r^{d_j}\}.$$

The following theorem, which isdue to Mitchell, Gershkovic, Nagel-Stein-Wainger, etal., compares boxes $\text{Box}(r)$ in $\mathbb{R}^n$ with *cc* balls $B_{cc}(p, r)$:

Theorem 8. *(Ball-box theorem) Let $(M, \mathcal{D}, \|\cdot\|)$ be a sub-Finsler manifold with an equiregular distribution $\mathcal{D}$. Let Φ be an exponential coordinate map from a point $p \in M$ constructed with respect to some regular basis $X_1, ..., X_n$. there exist $c > 1$ and $p > 0$ such that*

$$\Phi(\mathrm{Box}(c^{-1}r)) \subseteq B(p, r) \subseteq \Phi(\mathrm{Box}(cr)),\ \forall r \in (0, p).$$

We note the following open question:

Are all (sufficiently-small) Finsler balls and spheres homeomorphic to the usual Euclidean balls and spheres?

An almost direct corollary to the Ball-Box Theorem is that locally, each sub-Finsler manifold is Hölder equivalent to a Riemannian manifold. To see this, let $(M, \mathcal{D}, \|\cdot\|)$ be the manifold in question. Let g be a Riemannian tensor whose norm is smaller than $\|\cdot\|$ and denote by d_R the Riemannian distance. The identity map $id : M \mapsto M$ is 1-Lipschitz with respect to d_{cc}, d_r and thus it is Hölder. Let now $\alpha = \max_j d_j$ be the maximum of the degrees d_j of the vector fields of some equiregular basis $\{X_j\}$. Since for $p \in (0, 1)$, we have

$$\prod_{j=1}^{n} [-r^a, r^a] \subset \mathrm{Box}(r)$$

and since the exponential maps have surjective differentials at the origin, from the second inclusion of the Ball-Box theorem we obtain that $id : (M, d_R) \mapsto (M, d_{cc})$ is a-Hölder.

We shall denote by Q the homogeneous (Hausdorff) dimension

$$Q = \sum_{j=1}^{n} d_j = \sum_{j=1}^{n} j(\dim \Delta^{(j)} - \dim \Delta^{(j-1)}).$$

If a sub-Finsler manifold $(M, \mathcal{D}, \|\cdot\|)$ has equiregular distribution then

$$\dim_{Haus}(M, d_{cc}) = Q.$$

Moreover,the Q-Hausdorff measure of (M, d_{cc}) is locally equivalent (up to multiplication by a function) to the Finsler volume form.

It is natural to ask how to compute Hausdorff dimension and Hausdorff measure of submanifolds of sub-Finsler manifolds with respect to the cc distance. These questions were answered by Gromov and Magnani, the first in full, the second only partially.

Theorem 9. *(Gromov) Let $(M, \mathcal{D}, \|\cdot\|)$ be a sub-Finsler manifold with an equiregular distribution $\mathcal{D}$ and cc distaance d_{cc}. Let $\Sigma \subset M$ be a smooth submanifold. Then*

$$\dim_{Haus}(\Sigma, d_{cc}) = \max \left\{ \sum_{j=1}^{n} j \dim(T_p(M) \cap \Delta^j(p)) \setminus (T_p(M) \cap \Delta^{(j-1)}(p)) : p \in \Sigma \right\}.$$

The question of finding the Hausdorff dimension of smooth submanifolds is yet to be answered in full.

3. Carnot groups

3.1. **Review of Lie groups and Lie algebras.** A Lie group G is a differentiable manifold with a group structure such that the map

$$G \times G \mapsto G$$
$$(x, y) \mapsto x^{-1}y,$$

is smooth. We shall denote by e the identity element. $R_g(h) = hg$ and $L_g(h) = gh$ are right and left translations by g in G, respectively. The set of vector fields $\Gamma(TG)$ from a Lie algebra; the bilinear operation is the Lie bracket: $[\cdot, \cdot] : g \times g \mapsto g$ such that for all $X, Y, Z \in g$

(1) $[X, Y] = -[Y, X]$and
(2) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

There is a special Lie algebra associated to a Lie group G, that is, the tangent space $T_e(G)$. In brief, each element of $T_e(G)$ is extended to an element of $\Gamma(TG)$ by left translations to produce vector fields $X \in \Gamma(TG)$ such that $(L_g)_*X = X$ for all $p \in G$. Then $(L_{g_{*,p}})X = X_{L_g(p)}$ and we have an isomorphism

$$T_e(G) \mapsto \mathfrak{g} \text{ (=left invariant vector fields)}$$
$$V \longmapsto X_g = (L_g)_*V.$$

A Lie group homomorhpism $F : G \mapsto H$ is a $\mathcal{C}^\infty$ group homomorphism. A map $\Phi : g \mapsto h$ is a Lie algebra homomorhism if it is linear and preserves brackets: $\Phi([X, Y]) = [\Phi(X), \Phi(Y)]$ for all $X, Y \in g$. A Lie group homomorphism induces a Lie algebra homomorphism: We have $F(e) = e$ and the differential:

$$(F_*)_e : T_e(G) \mapsto T_e(H)$$

preserves brackets. For the converse we have the following:

Proposition 3.1. *Let G and H be two Lie groups with Lie algebras $\mathfrak{g}$ and$\mathfrak{h}$, respectively. Assume that G is simply connected. If $\emptyset : \mathfrak{g} \mapsto \mathfrak{h}$ is a Lie algebra homomorhpism, then there exists a unique Lie group homomorphism $F : G \mapsto H$ such that $F_* = \emptyset$.*

The above implies that if Lie groups G and H have isomorphic Lie algebras and both are simply connected, then G and H are isomorphic.

By a theorem of Ado, every Lie algebra has a faithful representation in $\mathfrak{gl}(n, \mathbb{R})$ for some $n \in \mathbb{N}$. Hence, if $\mathfrak{g}$ is a Lie algebra, then there exists a simply connected group G with Lie algebra $\mathfrak{g}$. Therefore, isomorphism classes of Lie algebras are into 1–1 correspondence with isomorphism classes of simply connected Lie groups.

Recall the definition of the exponential map is an arbitary manifold M. Let $X \in \Gamma(M)$ be a vector field and fix a point $p \in M$ of the manifold. Then there is a unique curve $\gamma(t)$ such that $\gamma(0) = p$ and $\dot{\gamma}(t) = X_{\gamma(t)}$. Then $\exp_p(X) = \gamma(1)$. In general $\exp_p$ is locally defined: It only takes a small neighbourhood of the zero section of TM to a neighbourhood U_p of M. In Lie groups though, exp is a map $\mathfrak{g} \mapsto G$ and $\mathfrak{g} \subset \Gamma(TG)$ with the definition making sense for $p = e$, and is also defined globally. The following holds:

Theorem 10. *Let $X \in \mathfrak{g}$ be an element of the Lie algebra $\mathfrak{g}$ of a Lie group G. Then*

(1) $\exp((s + t)X) = \exp(sX) \cdot \exp(tX)$, *$s, t \in \mathbb{R}$.*
(2) $\exp(-X) = (\exp(X))^{-1}$.
(3) *$\exp : g \mapsto G$ and $(\exp)_* = id_g : g \mapsto g$. Therefore there exists a diffeomorhpism of a neighbourhood of 0 in $\mathfrak{g}$ onto a neighbourhood of e in G.*
(4) *The curve $\gamma(t) = \exp(tX)$ is the flow of X at time t starting from e. More generally, the curve $g(\exp(tX)) = L_g(\gamma(t))$ is the flow starting at g.*

(5) *The flow of X at time t is the right translation $R_{\exp}(tX)$*

We also have

Theorem 11. *If $F : G \mapsto H$ is a Lie group homomorphism, then*

$$F \circ \exp = \exp \circ F_*.$$

Note that in case where G is compact, it also has a Riemannian metric invariant under left and right translations. Then the Lie group exponential map is the Riemannian exponential map of this Riemannian metric.

3.2. **Nilpotent Lie groups and nilpotent Lie algebras.** Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$. The *central series of* $\mathfrak{g}$ are

$$\mathfrak{g}^{(1)} = \mathfrak{g}, \quad \mathfrak{g}^{(i+1)} = [\mathfrak{g}, \mathfrak{g}^{(i)}].$$

The Lie algebra $\mathfrak{g}$ is called *nilpotent* if there is an integer s such that $\mathfrak{g}^{(s+1)} = 0$. The minimal s for which $\mathfrak{g}^{(s+1)} = \{0\}$ is called the *step* of $\mathfrak{g}$. A nilpotent Lie group G is a Lie group whose Lie algebra is nilpotent. If $\mathfrak{g}$ is s-step nilpotent, then we have the following for the centre $cZ(\mathfrak{g}^s)$:

$$\mathcal{Z}(\mathfrak{g}^s) = \{X \in \mathfrak{g}^{(s)} : [X, Y] = 0, \text{ for all } Y \in \mathfrak{g}^{(s)}\} = \mathfrak{g}^{(s)},$$

that is, $\mathfrak{g}^{(s)}$ (and all $\mathfrak{g}^{(k)}, k \leq s$) are central. It is worth to remark here that a Lie algebra $\mathfrak{g}$ has always non-trivial centre. In fact, the centre

$$\mathcal{Z}(G) = \{g \in G : gh = hg, \forall h \in G\}$$

is a closed subgroup with Lie algebra $\mathcal{Z}(\mathfrak{g})$ if G is connected.

Remark 3.2. The Heisenberg group is a 2-step nilpotent Lie group.

3.3. **Simply connected nilpotent Lie groups.** Recall that if two simply connected Lie groups have isomorphic Lie algebras then they are isomorphic. In the case of nilpotent connected and simply connected Lie groups we have the following:

Theorem 12. *Let G be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Then:*

(1) *The exponential map* $\exp : \mathfrak{g} \mapsto G$ *is an analytic diffeomorphism.*
(2) *The Baker-Campbell-Hausdorff (BCH) formula holds for all $X, Y \in \mathfrak{g}$.*

The BCH formula (which is quite complicated to be written down here) allows us to locally reconstruct any Lie group G with its multiplication law, by only knowing the structure of its Lie algebra $\mathfrak{g}$. It expresses the inverse of the exponential (which quite naturally we shall denote by Log) of the product of two Lie group elements as a lie algebra elements, that is

$$\mathrm{Log}(\mathrm{e}^{\mathrm{X}} \cdot \mathrm{e}^{\mathrm{Y}}) = \text{an element of } \mathfrak{g}.$$

Below we state various consequences of this theorem:

- Every Lie subgroup H of a connected, simply connected nilpotent Lie group G is closed and simply connected.
- Every connected, simply connected Lie group which is nilpotent has a faithful embedding as a closed subgroup of the group N_h whose Lie algebra are the strictly upper triangular matrices.
- With the aid of the exponetial map, we may identify G and $\mathfrak{g}$ when G is a simply connected, connected nilpotent Lie group. In this manner, we may transfer coordinates from $\mathfrak{g}$ to G.

3.4. **Carnot groups.** A *Carnot group with step* $s \geq 1$ is a connected, simply connected nilpotent Lie group whose Lie algebra admits a unique up to isomorphism step s stratification. That is,

$$g = V_1 \oplus ... \oplus V_s \text{ with}$$

$$[V_j, V_1] = V_{j+1}, 1 \leq j \leq s-1, V_s \neq \{0\}$$

We remark that there exist simply connected nilpotent Lie groups which are not Carnot groups: For instance, there exist 6-dimensional nilpotent Lie algebras that cannot be stratified.

The topological dimension of a Carnot group G is $n = \sum_i \dim V_i$ whereas its homogeneous dimension is

$$Q = \sum_{i=1}^{s} i \dim V_i.$$

In fact, each Carnot group may be equipped with a sub-Riemannian structure which is unique up to bi-Lipschitz equivalence and has an additional property which we shall explain later. Fix a stratification for G and let $\mathcal{D}$ be a left invariant subbundle of TG which is such that $\mathcal{D}_e = V_1$. Let $\|\cdot\|$ be any left invariant Finsler norm on G. The triple $(M, \mathcal{D}, \|\cdot\|)$ is a sub-Finsler manifold, since

$$\Delta_e^{(j)} = V_1 \oplus ... \oplus V_j$$

satisfies Hörmander's condition. Thus one may consider the cc distance d_{cc} associated to this sub-Finsler structure. Another choice of the norm does not effect the bi-Lipschitz equivalence class of the sub-Finsler manifold. If $\|\cdot\|_l$ is another left invariant Finsler norm then

$$\text{id} : (G, d_{cc,\|\cdot\|}) \mapsto (G, d_{cc,\|\cdot\|_l})$$

is globally bi-Lipschitz. For that reason, we may assume that $\|\cdot\|$ is coming from the usual scalar product.

It is quite clear that the value of the scalar product in V_1 is important for the definition of the d_{cc} metric. If $m = \dim V_1$, we fix $X_1, ..., X_m$ at V_1. Then,

$$d_{cc}(x, y) = \inf \left\{ \int_0^1 \sqrt{\sum_{i=1}^{n} |\gamma_i(t)|^2 dt} : \gamma(0) = x, \gamma(1) = y \right\},$$

where the infinum is taken over all absolutely continuous curves such that $\gamma : [0,1] \mapsto G$ and

$$\dot{\gamma}(t) = \sum_{i=1}^{m} \gamma_i(t)(X_i)_{\gamma(t)} \; t \in [0,1].$$

We conclude this section by presenting an additional structure of Carnot groups, that is, their *dilation structure*. Let $g = V_1 \oplus ... \oplus V_s$, and $\lambda > 0$. Dilations $\widetilde{\delta_\lambda}$ are defined by the homogeneity conditions

$$\widetilde{\delta_\lambda} X = \lambda^k X, \quad \forall X \in V_k, 1 \leq k \leq s.$$

These are self maps of $\mathfrak{g}$ and we may equivalently write

$$\widetilde{\delta_\lambda}(\sum_{i=1}^{s} V_i) = \sum_{i=1}^{s} \lambda^i V_i,$$

whenever $X = \sum_{i=1}^{s} V_i$ with $v_i \in V_i$, $1 \leq i \leq s$.

Using the fact that $\exp : \mathfrak{g} \mapsto G$ is a diffeomorphism, we may define $\delta_\lambda : G \mapsto G$ by $\exp \circ \widetilde{\delta_\lambda} = \delta_\lambda \circ \exp$. Below we list some properties of dilations:

- $\delta_\lambda(xy) = \delta_\lambda(x) \cdot \delta_\lambda(y)$, for $all x, y \in G$. This follows from BCH formula.
- $\delta_\lambda \circ \delta_\mu = \delta_{\lambda\mu}$.

- $(\delta_\lambda)_* X = \widetilde{\delta_\lambda} X$.
- $\widetilde{\delta_\lambda}([X,Y]) = [\widetilde{\delta_\lambda} X, \widetilde{\delta_\lambda} Y]$.
- $d_{cc}(\delta_\lambda x, \delta_\lambda y) = \lambda d_{cc}(x,y)$, for all$x, y \in G$.

3.4.1. *Nilpotentation.* Nilpotentation is the procedure where a Carnot group appears as tangent to an equiregular distribution. Let $\mathcal{D}$ be a bracket generating and equiregular distribution in a manifold M ,i.e.,

$$\mathcal{D} = \mathcal{D}^{(1)} \subset \mathcal{D}^{(2)} \subset ... \subset \mathcal{D}^{(s)} = TM,$$

is a sequence of subbundles of TM where

$$\mathcal{D}^{(j+1)} = \mathcal{D}^{(j)} + [\mathcal{D}, \mathcal{D}^{(j)}].$$

The sum is not necessarily direct. The crucial fact here is

$$[\mathcal{D}^{(k)}, \mathcal{D}^{(1)}] \subset \mathcal{D}^{(k+1)}.$$

This relation is obvious for $k = 1$. The above relation may be proved by induction using Jacobi's identity.

We now define $H_1 = \mathcal{D}$ and $H_j = \mathcal{D}^{(j)} \setminus \mathcal{D}^{(j-1)}, j = 2, ..., n$. H_j are bundles but not subbundles of TM for $j \geq 1$. It is clear that

$$TM \simeq \bigoplus_{i=1}^{s} H_i.$$

The following holds:

Theorem 13. *For each $p \in M$, T_pM inherits the structure of a Carnot group with respect to the stratification $H_j(p)$ .This Carnot group is the nilpotentation of $T_p(M)$ with respect to $\mathcal{D}$.*

Proof. Let $V_j = H_j(p)$. Then

$$T_p(M) \cong V_1 \oplus ... \oplus V_s.$$

We need to define a Lie algebra product and then show that $[V_j, V_1] = V_{j+1}$. Let $x, y \in T_p(M)$ with $x \in V_j$ and $y \in V_1$.

Since

$$V_j = H_j(p) = \mathcal{D}_p^{(j)} \setminus \mathcal{D}_p^{(j-1)},$$

there exist a $X \in \mathcal{D}^{(j)}$ and a $Y \in \mathcal{D}^{(l)}$ such that

$$x = X_p + \mathcal{D}_p^{(j-1)} \ , \ y = Y_p + \mathcal{D}_p^{(l-1)}.$$

We then define

$$[x, y] = [X, Y]_p + \mathcal{D}_p^{(j+l-1)}.$$

This bracket is well defined: If $u \in \mathcal{D}_p^{(j-1)}$, then $[X + u, Y] = [X, Y] + [u, Y]$, with

$$[u, Y] \in [\mathcal{D}^{(j-1)}, \mathcal{D}^{(l)}] \subset \mathcal{D}^{(j+l-1)}.$$

Therefore, $[X + u, Y]_p = [X, Y]_p \mod \mathcal{D}_p^{(j+l-1)}$.

Now, if $y \in V_1, [x, y] \in \mathcal{D}_p^{(j+1)} \setminus \mathcal{D}_p^{(j)} = V_{j+1}$ and thus $[V_j, V_1] \subseteq V_{j+1}$. To show the reverse inclusion, let $z \in \mathcal{D}^{(j+1)}$ such that $z = Z_p + \mathcal{D}_p^{(j)}$. By definition ,$\mathcal{D}^{(j+1)} = \mathcal{D}^{(j)} + [\mathcal{D}, \mathcal{D}^{(j)}]$ so there exist a $W \in \mathcal{D}^{(j)}$,$X_l \in \mathcal{D}^{(j)}$,$Y_l \in \mathcal{D}$ such that $Z = W + \sum_l [X_l, Y_l]$. Take

$$x_l = (X_l)_p \mod \mathcal{D}^{(j-1)} \ , \ y_l = (Y_l)_p.$$

One then shows that $\sum_l [X_l, Y_l] = Z_p \mod \mathcal{D}_p^{(j)}$ and therefore $V_{j+1} \subseteq [V_j, V_1]$. □

3.5. **Mitchell's theorem.** We start with Gromov's notion of tangent space to a metric space. Given a metric space (X,d), consider the dilated metric space $(X,\lambda d)$, $\lambda>0$. The distance λd is given by

$$(\lambda d)(p,q)=\lambda d(p,q),\quad p,q\in X.$$

A metric space (Z,ρ) is tangent to (X,d) at $p\in X$ if there exists a $\bar{p}\in Z$ and a sequence $\lambda_j\to\infty$ such that

$$\lim_j(X,p,\lambda_j d)=(Z,\bar{p},\rho).$$

We may understand this definition in terms of Gromov-Hausdoff distance. Let B_1,B_2 be compact metric spaces. Then

$$GH(B_1,B_2)=\inf_{\Psi_1,\Psi_2}H(\Psi_1B_1,\Psi_2B_2)$$

over all isometric embeddings Ψ_1,Ψ_2 of B_1,B_2, respectively, into the same metric space C of the Hausdorff distance $H(\Psi_1B_1,\Psi_2B_2)$ of their images as subsets of C. In this way the definition of tangent to a metric space implies that for each $r>0$, there exists a sequence $\epsilon_j\to 0$ such that the ball of radius $r+\epsilon_j$ in $(X,\lambda_j d)$ about the point p converges to a ball of radius r about $\bar{p}$. Namely,the infinum of the GH distance between those compact abstract metric spaces tends to 0 as $\lambda_j\to\infty$.

A distribution $\mathcal{D}$ is called *generic*, if for each j, $\dim\mathcal{D}_p^{(j)}$ is independent of $p\in M$.

Theorem 14. *(Mitchell) For a generic distribution $\mathcal{D}$ on M, the tangent cone of a sub-Riemannian manifold (M,d_{cc}) at $p\in M$ is isometric to (G,d_∞) where G is a Carnot group with a left-invariant cc metric. In fact, G is the nilpotentation of $T_p(M)$ with respect to $\mathcal{D}$.*

We remark the following:

- The tangent(or the tangent cone) to a Carnot group G is G itself. G admits dilations δ_λ which provide isometries between (G,d_{cc}) and $(G,\lambda d_{cc})$.
- In contrast to the Riemannian case where the exponential map is a locally biLipschitz map between the tangent cone and the manifold, Mitchell's map is not in general locally biLipschitz.

Pansu in 1985 and later Margulis and Mostow in 1995 explained why the latter happens, as we shall see in the following section.

3.6. **Pansu's Rademacher theorem.**

Theorem 15. *(Pansu,Margulis-Mostow) For the typical sub-Riemannian manifold there is no bi-Lipschitz map between a neighbourhood of a point of the manifold and its nilpotentiation at this point.*

The classical Rademacher theorem in real analysis asserts that a Lipschitz map between Euclidean spaces is a.e. differentiable. Pansu(1989) extended the theorem to the setting of Carnot groups endowed with their sub-Riemannian distance function. Let $F:G_1\mapsto G_2$ be a map between two Carnot groups with dilations $\delta_t:G_i\mapsto G_i$, $i=1,2$. For $g,h\in G_1$, the *pansu derivative* is defined by

$$D_pF(g)(h)=\lim_{t\to 0}(\delta_t^{-1})(F(g)F(g\delta_t h)).$$

Note that if the G_i's are Abelian Carnot groups (that is, vector spaces with vector addition as the multiplication),the Pansu's derivative D_pF is the usual derivative. In general, if the Pansu derivative exists and is continuous then it is a group homomorphism from G_1 to G_2.

Theorem 16. *(Pansu's Rademacher theorem) At almost all points, the tangent map of a Lipschitz map between sub-Riemannian manifolds exists, it is unique, and is a group homomorphism of the tangent and equivariant with respect to dilations.*

We have seen above that in the Carnot group setting, the tangent map is just Pansu's differential. Let us clarify what we mean by a tangent map between tangent cones. Each map $f : (X, d) \mapsto (X', d')$ induces a map $f_\lambda : (X, \lambda d) \mapsto (X', \lambda d')$ for each $\lambda > 0$. Setwise,this is the map $f_\lambda(x)$. For fixed $x \in X$,assume that (Z, ρ) and (Z', ρ') are tangent cones to (X, d) at x and to (X', d') at $f(x)$,respectively then $Df : (Z, \rho) \mapsto (Z', \rho')$ is a tangent map of f at x if for some sequence $\lambda_j \to \infty$, f_{λ_j} converges to Df uniformly at compact sets.

- With this definition, the tangent map can not be unique or even linear. But for Lipschitz maps between sub-Riemannian manifolds, Pansu's-Rademacher theorem states that not only a tangent map exists at almost every point, but also that outside a small set the limit is a Lie group homomorphism between Carnot groups which commutes with dilations.
- Any sub-Riemannian manifold is a differentiable manifold, therefore we always have the notion of the differential of a smooth map. But this does not coincide with the notion of tangent map which on the other hand takes place on horizontal spaces and on the other one is defined in geometric terms.
- There can be no bi-Lipschitz map between Carnot groups which are not isomorphic.

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