ELEMENTS BOOK 11

Elementary Stereometry

Ὅροι.

αʹ. Στερεόν ἐστι τὸ μῆκος καὶ πλάτος καὶ βάθος ἔχον.

βʹ. Στερεοῦ δὲ πέρας ἐπιφάνεια.

γʹ. Εὐθεῖα πρὸς ἐπίπεδον ὀρθή ἐστιν, ὅταν πρὸς πάσας τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ [ὑποκειμένῳ] ἐπιπέδῳ ὀρθὰς ποιῇ γωνίας.

δʹ. Ἐπίπεδον πρὸς ἐπίπεδον ὀρθόν ἐστιν, ὅταν αἱ τῇ κοινῇ τομῇ τῶν ἐπιπέδων πρὸς ὀρθὰς ἀγόμεναι εὐθεῖαι ἐν ἑνὶ τῶν ἐπιπέδων τῷ λοιπῷ ἐπιπέδῳ πρὸς ὀρθὰς ὦσιν.

εʹ. Εὐθείας πρὸς ἐπίπεδον κλίσις ἐστίν, ὅταν ἀπὸ τοῦ μετεώρου πέρατος τῆς εὐθείας ἐπὶ τὸ ἐπίπεδον κάθετος ἀχθῇ, καὶ ἀπὸ τοῦ γενομένου σημείου ἐπὶ τὸ ἐν τῷ ἐπιπέδῳ πέρας τῆς εὐθείας εὐθεῖα ἐπιζευχθῇ, ἡ περιεχομένη γωνία ὑπὸ τῆς ἀχθείσης καὶ τῆς ἐφεστώσης.

ϛʹ. Ἐπιπέδου πρὸς ἐπίπεδον κλίσις ἐστὶν ἡ περιεχομένη ὀξεῖα γωνία ὑπὸ τῶν πρὸς ὀρθὰς τῇ κοινῇ τομῇ ἀγομένων πρὸς τῷ αὐτῷ σημείῳ ἐν ἑκατέρῳ τῶν ἐπιπέδων.

ζʹ. Ἐπίπεδον πρὸς ἐπίπεδον ὁμοίως κεκλίσθαι λέγεται καὶ ἕτερον πρὸς ἕτερον, ὅταν αἱ εἰρημέναι τῶν κλίσεων γωνίαι ἴσαι ἀλλήλαις ὦσιν.

ηʹ. Παράλληλα ἐπίπεδά ἐστι τὰ ἀσύμπτωτα.

θʹ. Ὅμοια στερεὰ σχήματά ἐστι τὰ ὑπὸ ὁμοίων ἐπιπέδων περιεχόμενα ἴσων τὸ πλῆθος.

ιʹ. Ἴσα δὲ καὶ ὅμοια στερεὰ σχήματά ἐστι τὰ ὑπὸ ὁμοίων ἐπιπέδων περιεχόμενα ἴσων τῷ πλήθει καὶ τῷ μεγέθει.

ιαʹ. Στερεὰ γωνία ἐστὶν ἡ ὑπὸ πλειόνων ἢ δύο γραμμῶν ἁπτομένων ἀλλήλων καὶ μὴ ἐν τῇ αὐτῇ ἐπιφανείᾳ οὐσῶν πρὸς πάσαις ταῖς γραμμαῖς κλίσις. ἄλλως· στερεὰ γωνία ἐστὶν ἡ ὑπὸ πλειόνων ἢ δύο γωνιῶν ἐπιπέδων περιεχομένη μὴ οὐσῶν ἐν τῷ αὐτῷ ἐπιπέδῳ πρὸς ἑνὶ σημείῳ συνισταμένων.

ιβʹ. Πυραμίς ἐστι σχῆμα στερεὸν ἐπιπέδοις περιχόμενον ἀπὸ ἑνὸς ἐπιπέδου πρὸς ἑνὶ σημείῳ συνεστώς.

ιγʹ. Πρίσμα ἐστὶ σχῆμα στερεὸν ἐπιπέδοις περιεχόμενον, ὧν δύο τὰ ἀπεναντίον ἴσα τε καὶ ὅμοιά ἐστι καὶ παράλληλα, τὰ δὲ λοιπὰ παραλληλόγραμμα.

ιδʹ. Σφαῖρά ἐστιν, ὅταν ἡμικυκλίου μενούσης τῆς διαμέτρου περιενεχθὲν τὸ ἡμικύκλιον εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, τὸ περιληφθὲν σχῆμα.

ιεʹ. Ἄξων δὲ τῆς σφαίρας ἐστὶν ἡ μένουσα εὐθεῖα, περὶ ἣν τὸ ἡμικύκλιον στρέφεται.

ιϛʹ. Κέντρον δὲ τῆς σφαίρας ἐστὶ τὸ αὐτό, ὃ καὶ τοῦ ἡμικυκλίου.

ιζʹ. Διάμετρος δὲ τῆς σφαίρας ἐστὶν εὐθεῖά τις διὰ τοῦ κέντρου ἠγμένη καὶ περατουμένη ἐφ᾽ ἑκάτερα τὰ μέρη ὑπὸ τῆς ἐπιφανείας τῆς σφαίρας.

ιηʹ. Κῶνός ἐστιν, ὅταν ὀρθογωνίου τριγώνου μενούσης μιᾶς πλευρᾶς τῶν περὶ τὴν ὀρθὴν γωνίαν περιενεχθὲν τὸ τρίγωνον εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῇ, ὅθεν ἤρξατο

Definitions

1. A solid is a (figure) having length and breadth and depth.

2. The extremity of a solid (is) a surface.

3. A straight-line is at right-angles to a plane when it makes right-angles with all of the straight-lines joined to it which are also in the plane.

4. A plane is at right-angles to a(nother) plane when (all of) the straight-lines drawn in one of the planes, at right-angles to the common section of the planes, are at right-angles to the remaining plane.

5. The inclination of a straight-line to a plane is the angle contained by the drawn and standing (straight-lines), when a perpendicular is lead to the plane from the end of the (standing) straight-line raised (out of the plane), and a straight-line is (then) joined from the point (so) generated to the end of the (standing) straight-line (lying) in the plane.

6. The inclination of a plane to a(nother) plane is the acute angle contained by the (straight-lines), (one) in each of the planes, drawn at right-angles to the common segment (of the planes), at the same point.

7. A plane is said to have been similarly inclined to a plane, as another to another, when the aforementioned angles of inclination are equal to one another.

8. Parallel planes are those which do not meet (one another).

9. Similar solid figures are those contained by equal numbers of similar planes (which are similarly arranged).

10. But equal and similar solid figures are those contained by similar planes equal in number and in magnitude (which are similarly arranged).

11. A solid angle is the inclination (constituted) by more than two lines joining one another (at the same point), and not being in the same surface, to all of the lines. Otherwise, a solid angle is that contained by more than two plane angles, not being in the same plane, and constructed at one point.

12. A pyramid is a solid figure, contained by planes, (which is) constructed from one plane to one point.

13. A prism is a solid figure, contained by planes, of which the two opposite (planes) are equal, similar, and parallel, and the remaining (planes are) parallelograms.

14. A sphere is the figure enclosed when, the diameter of a semicircle remaining (fixed), the semicircle is carried around, and again established at the same (position) from which it began to be moved.

15. And the axis of the sphere is the fixed straight-line about which the semicircle is turned.

φέρεσθαι, τὸ περιληφθὲν σχῆμα. κἂν μὲν ἡ μένουσα εὐθεῖα ἴση ᾖ τῇ λοιπῇ [τῇ] περὶ τὴν ὀρθὴν περιφερομένῃ, ὀρθογώνιος ἔσται ὁ κῶνος, ἐὰν δὲ ἐλάττων, ἀμβλυγώνιος, ἐὰν δὲ μείζων, ὀξυγώνιος.

ιθ΄. Ἄξων δὲ τοῦ κώνου ἐστὶν ἡ μένουσα εὐθεῖα, περὶ ἣν τὸ τρίγωνον στρέφεται.

κ΄. Βάσις δὲ ὁ κύκλος ὁ ὑπὸ τῆς περιφερομένης εὐθείας γραφόμενος.

κα΄. Κύλινδρός ἐστιν, ὅταν ὀρθογωνίου παραλληλογράμμου μενούσης μιᾶς πλευρᾶς τῶν περὶ τὴν ὀρθὴν γωνίαν περιενεχθὲν τὸ παραλληλόγραμμον εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, τὸ περιληφθὲν σχῆμα.

κβ΄. Ἄξων δὲ τοῦ κυλίνδρου ἐστὶν ἡ μένουσα εὐθεῖα, περὶ ἣν τὸ παραλληλόγραμμον στρέφεται.

κγ΄. Βάσεις δὲ οἱ κύκλοι οἱ ὑπὸ τῶν ἀπεναντίον περιαγομένων δύο πλευρῶν γραφόμενοι.

κδ΄. Ὅμοιοι κῶνοι καὶ κύλινδροί εἰσιν, ὧν οἵ τε ἄξονες καὶ αἱ διάμετροι τῶν βάσεων ἀνάλογόν εἰσιν.

κε΄. Κύβος ἐστὶ σχῆμα στερεὸν ὑπὸ ἓξ τετραγώνων ἴσων περιεχόμενον.

κϛ΄. Ὀκτάεδρόν ἐστὶ σχῆμα στερεὸν ὑπὸ ὀκτὼ τριγώνων ἴσων καὶ ἰσοπλεύρων περιεχόμενον.

κζ΄. Εἰκοσάεδρόν ἐστι σχῆμα στερεὸν ὑπὸ εἴκοσι τριγώνων ἴσων καὶ ἰσοπλεύρων περιεχόμενον.

κη΄. Δωδεκάεδρόν ἐστι σχῆμα στερεὸν ὑπὸ δώδεκα πενταγώνων ἴσων καὶ ἰσοπλεύρων καὶ ἰσογωνίων περιεχόμενον.

16. And the center of the sphere is the same as that of the semicircle.

17. And the diameter of the sphere is any straight-line which is drawn through the center and terminated in both directions by the surface of the sphere.

18. A cone is the figure enclosed when, one of the sides of a right-angled triangle about the right-angle remaining (fixed), the triangle is carried around, and again established at the same (position) from which it began to be moved. And if the fixed straight-line is equal to the remaining (straight-line) about the right-angle, (which is) carried around, then the cone will be right-angled, and if less, obtuse-angled, and if greater, acute-angled.

19. And the axis of the cone is the fixed straight-line about which the triangle is turned.

20. And the base (of the cone is) the circle described by the (remaining) straight-line (about the right-angle which is) carried around (the axis).

21. A cylinder is the figure enclosed when, one of the sides of a right-angled parallelogram about the right-angle remaining (fixed), the parallelogram is carried around, and again established at the same (position) from which it began to be moved.

22. And the axis of the cylinder is the stationary straight-line about which the parallelogram is turned.

23. And the bases (of the cylinder are) the circles described by the two opposite sides (which are) carried around.

24. Similar cones and cylinders are those for which the axes and the diameters of the bases are proportional.

25. A cube is a solid figure contained by six equal squares.

26. An octahedron is a solid figure contained by eight equal and equilateral triangles.

27. An icosahedron is a solid figure contained by twenty equal and equilateral triangles.

28. A dodecahedron is a solid figure contained by twelve equal, equilateral, and equiangular pentagons.

α΄.

Εὐθείας γραμμῆς μέρος μέν τι οὐκ ἔστιν ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ, μέρος δέ τι ἐν μετεωροτέρῳ.

Εἰ γὰρ δυνατόν, εὐθείας γραμμῆς τῆς ΑΒΓ μέρος μέν τι τὸ ΑΒ ἔστω ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ, μέρος δέ τι τὸ ΒΓ ἐν μετεωροτέρῳ.

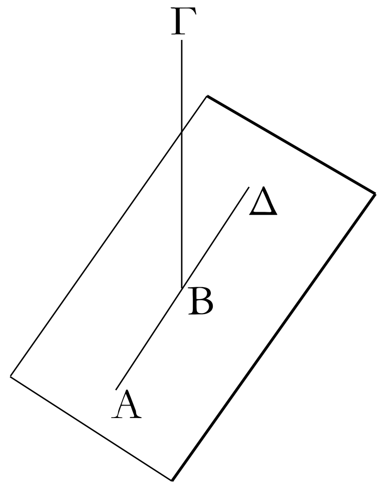
Ἔσται δή τις τῇ ΑΒ συνεχὴς εὐθεῖα ἐπ᾽ εὐθείας ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ. ἔστω ἡ ΒΔ· δύο ἄρα εὐθειῶν τῶν ΑΒΓ, ΑΒΔ κοινὸν τμῆμά ἐστιν ἡ ΑΒ· ὅπερ ἐστὶν ἀδύνατον, ἐπειδήπερ ἐὰν κέντρῳ τῷ Β καὶ διαστήματι τῷ ΑΒ κύκλον γράψωμεν, αἱ διάμετροι ἀνίσους ἀπολήψονται τοῦ κύκλου

Proposition 1†

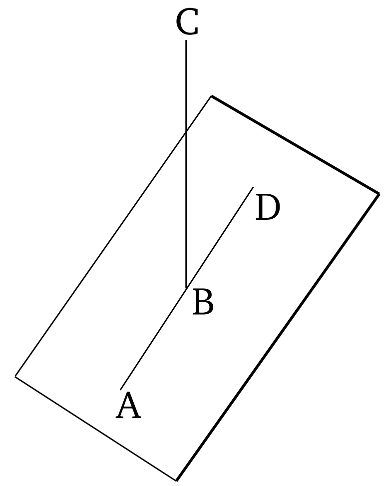
Some part of a straight-line cannot be in a reference plane, and some part in a more elevated (plane).

For, if possible, let some part, AB, of the straight-line ABC be in a reference plane, and some part, BC, in a more elevated (plane).

In the reference plane, there will be some straight-line continuous with, and straight-on to, AB.‡ Let it be BD. Thus, AB is a common segment of the two (different) straight-lines ABC and ABD. The very thing is impossible, inasmuch as if we draw a circle with center B and περιφερείας.



radius AB then the diameters (ABD and ABC) will cut off unequal circumferences of the circle.



Εὐθείας ἄρα γραμμῆς μέρος μέν τι οὐκ ἔστιν ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ, τὸ δὲ ἐν μετεωροτέρῳ· ὅπερ ἔδει δεῖξαι.

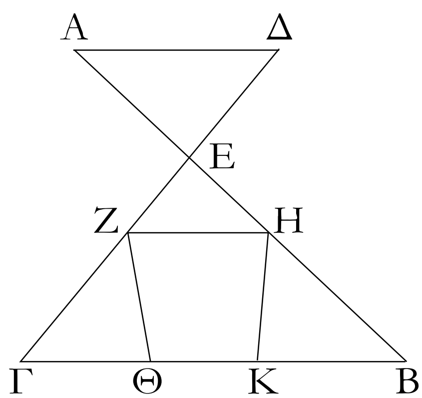
Thus, some part of a straight-line cannot be in a reference plane, and (some part) in a more elevated (plane). (Which is) the very thing it was required to show.

† The proofs of the first three propositions in this book are not at all rigorous. Hence, these three propositions should properly be regarded as additional axioms.

‡ This assumption essentially presupposes the validity of the proposition under discussion.

β´.

Ἐὰν δύο εὐθεῖαι τέμνωσιν ἀλλήλας, ἐν ἑνί εἰσιν ἐπιπέδῳ, καὶ πᾶν τρίγωνον ἐν ἑνί ἐστιν ἐπιπέδῳ.

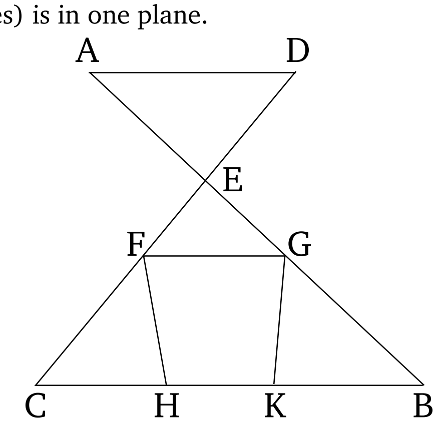


Δύο γὰρ εὐθεῖαι αἱ ΑΒ, ΓΔ τεμνέτωσαν ἀλλήλας κατὰ τὸ Ε σημεῖον. λέγω, ὅτι αἱ ΑΒ, ΓΔ ἐν ἑνί εἰσιν ἐπιπέδῳ, καὶ πᾶν τρίγωνον ἐν ἑνί ἐστιν ἐπιπέδῳ.

Εἰλήφθω γὰρ ἐπὶ τῶν ΕΓ, ΕΒ τυχόντα σημεῖα τὰ Ζ, Η, καὶ ἐπεζεύχθωσαν αἱ ΓΒ, ΖΗ, καὶ διήχθωσαν αἱ ΖΘ, ΗΚ· λέγω πρῶτον, ὅτι τὸ ΕΓΒ τρίγωνον ἐν ἑνί ἐστιν ἐπιπέδῳ. εἰ γάρ ἐστι τοῦ ΕΓΒ τριγώνου μέρος ἤτοι τὸ ΖΘΓ ἢ τὸ ΗΒΚ ἐν τῷ ὑποκειμένῳ [ἐπιπέδῳ], τὸ δὲ λοιπὸν ἐν ἄλλῳ, ἔσται καὶ μιᾶς τῶν ΕΓ, ΕΒ εὐθειῶν μέρος μέν τι ἐν τῷ ὑποκειμένῳ

Proposition 2

If two straight-lines cut one another then they are in one plane, and every triangle (formed using segments of both lines) is in one plane.



For let the two straight-lines AB and CD have cut one another at point E. I say that AB and CD are in one plane, and that every triangle (formed using segments of both lines) is in one plane.

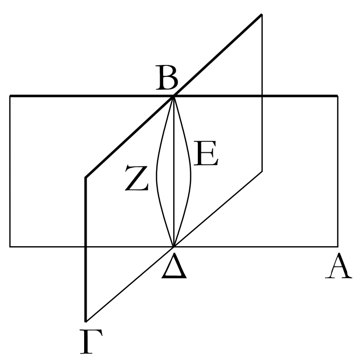
For let the random points F and G have been taken on EC and EB (respectively). And let CB and FG have been joined, and let FH and GK have been drawn across. I say, first of all, that triangle ECB is in one (reference) plane. For if part of triangle ECB, either FHC

ἐπιπέδῳ, τὸ δὲ ἐν αλλῳ. εἰ δὲ τοῦ ΕΓΒ τριγώνου τὸ ΖΓΒΗ μέρος ᾖ ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ, τὸ δὲ λοιπὸν ἐν ἄλλῳ, ἔσται καὶ ἀμφοτέρων τῶν ΕΓ, ΕΒ εὐθειῶν μέρος μέν τι ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ, τὸ δὲ ἐν ἄλλῳ· ὅπερ ἄτοπον ἐδείχθη. τὸ ἄρα ΕΓΒ τρίγωνον ἐν ἑνί ἐστιν ἐπιπέδῳ. ἐν ᾧ δέ ἐστι τὸ ΕΓΒ τρίγωνον, ἐν τούτῳ καὶ ἑκατέρα τῶν ΕΓ, ΕΒ, ἐν ᾧ δὲ ἑκατέρα τῶν ΕΓ, ΕΒ, ἐν τούτῳ καὶ αἱ ΑΒ, ΓΔ. αἱ ΑΒ, ΓΔ ἄρα εὐθεῖαι ἐν ἑνί εἰσιν ἐπιπέδῳ, καὶ πᾶν τρίγωνον ἐν ἑνί ἐστιν ἐπιπέδῳ· ὅπερ ἔδει δεῖξαι.

or GBK, is in the reference [plane], and the remainder in a different (plane) then a part of one the straight-lines EC and EB will also be in the reference plane, and (a part) in a different (plane). And if the part $FCBG$ of triangle ECB is in the reference plane, and the remainder in a different (plane) then parts of both of the straight-lines EC and EB will also be in the reference plane, and (parts) in a different (plane). The very thing was shown to be absurb [Prop. 11.1]. Thus, triangle ECB is in one plane. And in whichever (plane) triangle ECB is (found), in that (plane) EC and EB (will) each also (be found). And in whichever (plane) EC and EB (are) each (found), in that (plane) AB and CD (will) also (be found) [Prop. 11.1]. Thus, the straight-lines AB and CD are in one plane, and every triangle (formed using segments of both lines) is in one plane. (Which is) the very thing it was required to show.

γʹ.

Ἐὰν δύο ἐπίπεδα τεμνῇ ἄλληλα, ἡ κοινὴ αὐτῶν τομὴ εὐθεῖά ἐστιν.



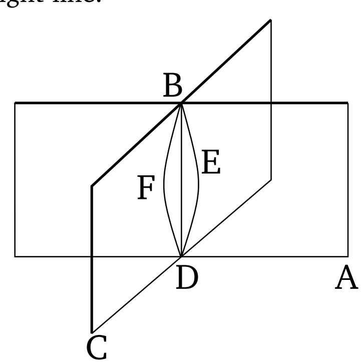
Δύο γὰρ ἐπίπεδα τὰ ΑΒ, ΒΓ τεμνέτω ἄλληλα, κοινὴ δὲ αὐτῶν τομὴ ἔστω ἡ ΔΒ γραμμή· λέγω, ὅτι ἡ ΔΒ γραμμὴ εὐθεῖά ἐστιν.

Εἰ γὰρ μή, ἐπεζεύχθω ἀπὸ τοῦ Δ ἐπὶ τὸ Β ἐν μὲν τῷ ΑΒ ἐπιπέδῳ εὐθεῖα ἡ ΔΕΒ, ἐν δὲ τῷ ΒΓ ἐπιπέδῳ εὐθεῖα ἡ ΔΖΒ. ἔσται δὴ δύο εὐθειῶν τῶν ΔΕΒ, ΔΖΒ τὰ αὐτὰ πέρατα, καὶ περιέξουσι δηλαδὴ χωρίον· ὅπερ ἄτοπον. οὐκ ἄρα αἱ ΔΕΒ, ΔΖΒ εὐθεῖαί εἰσιν. ὁμοίως δὴ δείξομεν, ὅτι οὐδὲ ἄλλη τις ἀπὸ τοῦ Δ ἐπὶ τὸ Β ἐπιζευγνυμένη εὐθεῖα ἔσται πλὴν τῆς ΔΒ κοινῆς τομῆς τῶν ΑΒ, ΒΓ ἐπιπέδων.

Ἐὰν ἄρα δύο ἐπίπεδα τέμνῃ ἄλληλα, ἡ κοινὴ αὐτῶν τομὴ εὐθεῖά ἐστιν· ὅπερ ἔδει δεῖξαι.

Proposition 3

If two planes cut one another then their common section is a straight-line.



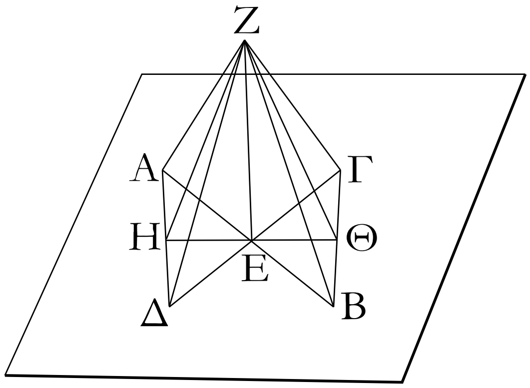
For let the two planes AB and BC cut one another, and let their common section be the line DB. I say that the line DB is straight.

For, if not, let the straight-line DEB have been joined from D to B in the plane AB, and the straight-line DFB in the plane BC. So two straight-lines, DEB and DFB, will have the same ends, and they will clearly enclose an area. The very thing (is) absurd. Thus, DEB and DFB are not straight-lines. So, similarly, we can show than no other straight-line can be joined from D to B except DB, the common section of the planes AB and BC.

Thus, if two planes cut one another then their common section is a straight-line. (Which is) the very thing it was required to show.

δ΄.

Ἐὰν εὐθεῖα δύο εὐθείαις τεμνούσαις ἀλλήλας πρὸς ὀρθὰς ἐπὶ τῆς κοινῆς τομῆς ἐπισταθῇ, καὶ τῷ δι᾽ αὐτῶν ἐπιπέδῳ πρὸς ὀρθὰς ἔσται.



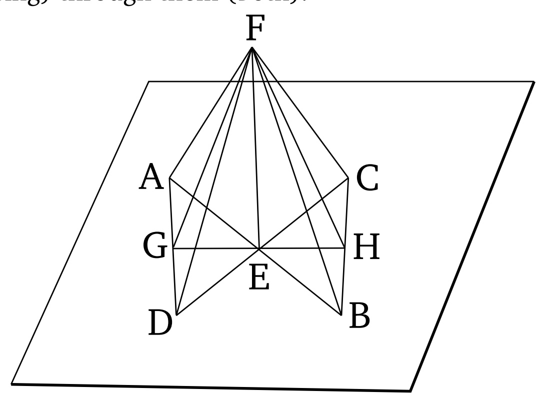
Εὐθεῖα γάρ τις ἡ ΕΖ δύο εὐθείαις ταῖς ΑΒ, ΓΔ τεμνούσαις ἀλλήλας κατὰ τὸ Ε σημεῖον ἀπὸ τοῦ Ε πρὸς ὀρθὰς ἐφεστάτω· λέγω, ὅτι ἡ ΕΖ καὶ τῷ διὰ τῶν ΑΒ, ΓΔ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν.

Ἀπειλήφθωσαν γὰρ αἱ ΑΕ, ΕΒ, ΓΕ, ΕΔ ἴσαι ἀλλήλαις, καὶ διήχθω τις διὰ τοῦ Ε, ὡς ἔτυχεν, ἡ ΗΕΘ, καὶ ἐπεζεύχθωσαν αἱ ΑΔ, ΓΒ, καὶ ἔτι ἀπὸ τυχόντος τοῦ Ζ ἐπεζεύχθωσαν αἱ ΖΑ, ΖΗ, ΖΔ, ΖΓ, ΖΘ, ΖΒ.

Καὶ ἐπεὶ δύο αἱ ΑΕ, ΕΔ δυσὶ ταῖς ΓΕ, ΕΒ ἴσαι εἰσὶ καὶ γωνίας ἴσας περιέχουσιν, βάσις ἄρα ἡ ΑΔ βάσει τῇ ΓΒ ἴση ἐστίν, καὶ τὸ ΑΕΔ τρίγωνον τῷ ΓΕΒ τριγώνῳ ἴσον ἔσται· ὥστε καὶ γωνία ἡ ὑπὸ ΔΑΕ γωνίᾳ τῇ ὑπὸ ΕΒΓ ἴση [ἐστίν]. ἔστι δὲ καὶ ἡ ὑπὸ ΑΕΗ γωνία τῇ ὑπὸ ΒΕΘ ἴση. δύο δὴ τρίγωνά ἐστι τὰ ΑΗΕ, ΒΕΘ τὰς δύο γωνίας δυσὶ γωνίαις ἴσας ἔχοντα ἑκατέραν ἑκατέρᾳ καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην τὴν πρὸς ταῖς ἴσαις γωνίαις τὴν ΑΕ τῇ ΕΒ· καὶ τὰς λοιπὰς ἄρα πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξουσιν. ἴση ἄρα ἡ μὲν ΗΕ τῇ ΕΘ, ἡ δὲ ΑΗ τῇ ΒΘ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΕ τῇ ΕΒ, κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ ΖΕ, βάσις ἄρα ἡ ΖΑ βάσει τῇ ΖΒ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΖΓ τῇ ΖΔ ἐστιν ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΔ τῇ ΓΒ, ἔστι δὲ καὶ ἡ ΖΑ τῇ ΖΒ ἴση, δύο δὴ αἱ ΖΑ, ΑΔ δυσὶ ταῖς ΖΒ, ΒΓ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ βάσις ἡ ΖΔ βάσει τῇ ΖΓ ἐδείχθη ἴση· καὶ γωνία ἄρα ἡ ὑπὸ ΖΑΔ γωνίᾳ τῇ ὑπὸ ΖΒΓ ἴση ἐστίν. καὶ ἐπεὶ πάλιν ἐδείχθη ἡ ΑΗ τῇ ΒΘ ἴση, ἀλλὰ μὴν καὶ ἡ ΖΑ τῇ ΖΒ ἴση, δύο δὴ αἱ ΖΑ, ΑΗ δυσὶ ταῖς ΖΒ, ΒΘ ἴσαι εἰσίν. καὶ γωνία ἡ ὑπὸ ΖΑΗ ἐδείχθη ἴση τῇ ὑπὸ ΖΒΘ· βάσις ἄρα ἡ ΖΗ βάσει τῇ ΖΘ ἐστιν ἴση. καὶ ἐπεὶ πάλιν ἴση ἐδείχθη ἡ ΗΕ τῇ ΕΘ, κοινὴ δὲ ἡ ΕΖ, δύο δὴ αἱ ΗΕ, ΕΖ δυσὶ ταῖς ΘΕ, ΕΖ ἴσαι εἰσίν· καὶ βάσις ἡ ΖΗ βάσει τῇ ΖΘ ἴση· γωνία ἄρα ἡ ὑπὸ ΗΕΖ γωνίᾳ τῇ ὑπὸ ΘΕΖ ἴση ἐστίν. ὀρθὴ ἄρα ἑκατέρα τῶν ὑπὸ ΗΕΖ, ΘΕΖ γωνιῶν. ἡ ΖΕ ἄρα πρὸς τὴν ΗΘ τυχόντως διὰ τοῦ Ε ἀχθεῖσαν ὀρθή ἐστιν. ὁμοίως δὴ δείξομεν, ὅτι ἡ ΖΕ καὶ

Proposition 4

If a straight-line is set up at right-angles to two straight-lines cutting one another, at the common point of section, then it will also be at right-angles to the plane (passing) through them (both).



For let some straight-line EF have (been) set up at right-angles to two straight-lines, AB and CD, cutting one another at point E, at E. I say that EF is also at right-angles to the plane (passing) through AB and CD.

For let AE, EB, CE and ED have been cut off from (the two straight-lines so as to be) equal to one another. And let GEH have been drawn, at random, through E (in the plane passing through AB and CD). And let AD and CB have been joined. And, furthermore, let FA, FG, FD, FC, FH, and FB have been joined from the random (point) F (on EF).

For since the two (straight-lines) AE and ED are equal to the two (straight-lines) CE and EB, and they enclose equal angles [Prop. 1.15], the base AD is thus equal to the base CB, and triangle AED will be equal to triangle CEB [Prop. 1.4]. Hence, the angle DAE [is] equal to the angle EBC. And the angle AEG (is) also equal to the angle BEH [Prop. 1.15]. So AGE and BEH are two triangles having two angles equal to two angles, respectively, and one side equal to one side—(namely), those by the equal angles, AE and EB. Thus, they will also have the remaining sides equal to the remaining sides [Prop. 1.26]. Thus, GE (is) equal to EH, and AG to BH. And since AE is equal to EB, and FE is common and at right-angles, the base FA is thus equal to the base FB [Prop. 1.4]. So, for the same (reasons), FC is also equal to FD. And since AD is equal to CB, and FA is also equal to FB, the two (straight-lines) FA and AD are equal to the two (straight-lines) FB and BC, respectively. And the base FD was shown (to be) equal to the base FC. Thus, the angle FAD is also equal to the angle FBC [Prop. 1.8]. And, again, since AG was shown (to be) equal to BH, but FA (is) also equal to πρὸς πάσας τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ ὀρθὰς ποιήσει γωνίας. εὐθεῖα δὲ πρὸς ἐπίπεδον ὀρθή ἐστιν, ὅταν πρὸς πάσας τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ αὐτῷ ἐπιπέδῳ ὀρθὰς ποιῇ γωνίας· ἡ ΖΕ ἄρα τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. τὸ δὲ ὑποκείμενον ἐπίπεδόν ἐστι τὸ διὰ τῶν ΑΒ, ΓΔ εὐθειῶν. ἡ ΖΕ ἄρα πρὸς ὀρθάς ἐστι τῷ διὰ τῶν ΑΒ, ΓΔ ἐπιπέδῳ.

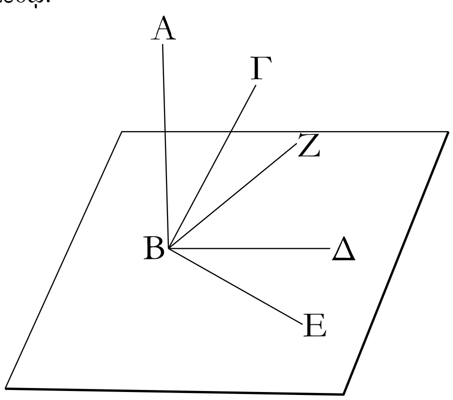
Ἐὰν ἄρα εὐθεῖα δύο εὐθείαις τεμνούσαις ἀλλήλας πρὸς ὀρθὰς ἐπὶ τῆς κοινῆς τομῆς ἐπισταθῇ, καὶ τῷ δι᾽ αὐτῶν ἐπιπέδῳ πρὸς ὀρθὰς ἔσται· ὅπερ ἔδει δεῖξαι.

FB, the two (straight-lines) FA and AG are equal to the two (straight-lines) FB and BH (respectively). And the angle FAG was shown (to be) equal to the angle FBH. Thus, the base FG is equal to the base FH [Prop. 1.4]. And, again, since GE was shown (to be) equal to EH, and EF (is) common, the two (straight-lines) GE and EF are equal to the two (straight-lines) HE and EF (respectively). And the base FG (is) equal to the base FH. Thus, the angle GEF is equal to the angle HEF [Prop. 1.8]. Each of the angles GEF and HEF (are) thus right-angles [Def. 1.10]. Thus, FE is at right-angles to GH, which was drawn at random through E (in the reference plane passing though AB and AC). So, similarly, we can show that FE will make right-angles with all straight-lines joined to it which are in the reference plane. And a straight-line is at right-angles to a plane when it makes right-angles with all straight-lines joined to it which are in the plane [Def. 11.3]. Thus, FE is at right-angles to the reference plane. And the reference plane is that (passing) through the straight-lines AB and CD. Thus, FE is at right-angles to the plane (passing) through AB and CD.

Thus, if a straight-line is set up at right-angles to two straight-lines cutting one another, at the common point of section, then it will also be at right-angles to the plane (passing) through them (both). (Which is) the very thing it was required to show.

εʹ.

Ἐὰν εὐθεῖα τρισὶν εὐθείαις ἁπτομέναις ἀλλήλων πρὸς ὀρθὰς ἐπὶ τῆς κοινῆς τομῆς ἐπισταθῇ, αἱ τρεῖς εὐθεῖαι ἐν ἑνί εἰσιν ἐπιπέδῳ.

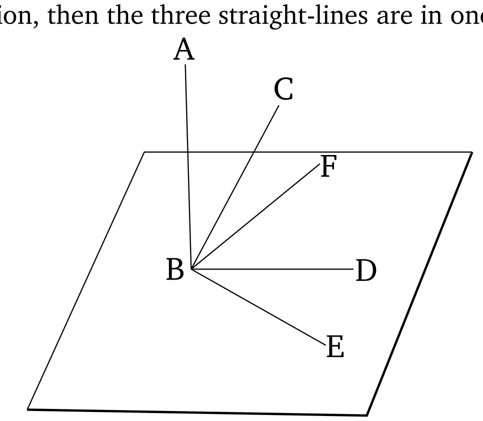


Εὐθεῖα γάρ τις ἡ ΑΒ τρισὶν εὐθείαις ταῖς ΒΓ, ΒΔ, ΒΕ πρὸς ὀρθὰς ἐπὶ τῆς κατὰ τὸ Β ἁφῆς ἐφεστάτω· λέγω, ὅτι αἱ ΒΓ, ΒΔ, ΒΕ ἐν ἑνί εἰσιν ἐπιπέδῳ.

Μὴ γάρ, ἀλλ᾽ εἰ δυνατόν, ἔστωσαν αἱ μὲν ΒΔ, ΒΕ ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ, ἡ δὲ ΒΓ ἐν μετεωροτέρῳ, καὶ ἐκβεβλήσθω τὸ δῖα τῶν ΑΒ, ΒΓ ἐπίπεδον· κοινὴν δὴ τομὴν

Proposition 5

If a straight-line is set up at right-angles to three straight-lines cutting one another, at the common point of section, then the three straight-lines are in one plane.



For let some straight-line AB have been set up at right-angles to three straight-lines BC, BD, and BE, at the (common) point of section B. I say that BC, BD, and BE are in one plane.

For (if) not, and if possible, let BD and BE be in the reference plane, and BC in a more elevated (plane).

ποιήσει ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ εὐθεῖαν. ποιείτω τὴν ΒΖ. ἐν ἑνὶ ἄρα εἰσὶν ἐπιπέδῳ τῷ διηγμένῳ διὰ τῶν ΑΒ, ΒΓ αἱ τρεῖς εὐθεῖαι αἱ ΑΒ, ΒΓ, ΒΖ. καὶ ἐπεὶ ἡ ΑΒ ὀρθή ἐστι πρὸς ἑκατέραν τῶν ΒΔ, ΒΕ, καὶ τῷ διὰ τῶν ΒΔ, ΒΕ ἄρα ἐπιπέδῳ ὀρθή ἐστιν ἡ ΑΒ. τὸ δὲ διὰ τῶν ΒΔ, ΒΕ ἐπίπεδον τὸ ὑποκείμενόν ἐστιν· ἡ ΑΒ ἄρα ὀρθή ἐστι πρὸς τὸ ὑποκείμενον ἐπίπεδον. ὥστε καὶ πρὸς πάσας τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ ὀρθὰς ποιήσει γωνίας ἡ ΑΒ. ἅπτεται δὲ αὐτῆς ἡ ΒΖ οὖσα ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ· ἡ ἄρα ὑπὸ ΑΒΖ γωνία ὀρθή ἐστιν. ὑπόκειται δὲ καὶ ἡ ὑπὸ ΑΒΓ ὀρθή· ἴση ἄρα ἡ ὑπὸ ΑΒΖ γωνία τῇ ὑπὸ ΑΒΓ. καί εἰσιν ἐν ἑνὶ ἐπιπέδῳ· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἡ ΒΓ εὐθεῖα ἐν μετεωροτέρῳ ἐστὶν ἐπιπέδῳ· αἱ τρεῖς ἄρα εὐθεῖαι αἱ ΒΓ, ΒΔ, ΒΕ ἐν ἑνί εἰσιν ἐπιπέδῳ.

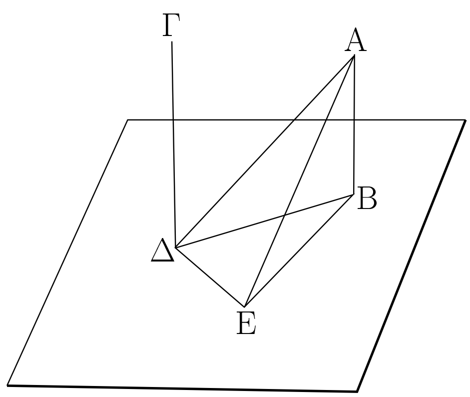
Ἐὰν ἄρα εὐθεῖα τρισὶν εὐθείαις ἁπτομέναις ἀλλήλων ἐπὶ τῆς ἁφῆς πρὸς ὀρθὰς ἐπισταθῇ, αἱ τρεῖς εὐθεῖαι ἐν ἑνί εἰσιν ἐπιπέδῳ· ὅπερ ἔδει δεῖξαι.

And let the plane through AB and BC have been produced. So it will make a straight-line as a common section with the reference plane [Def. 11.3]. Let it make BF. Thus, the three straight-lines AB, BC, and BF are in one plane—(namely), that drawn through AB and BC. And since AB is at right-angles to each of BD and BE, AB is thus also at right-angles to the plane (passing) through BD and BE [Prop. 11.4]. And the plane (passing) through BD and BE is the reference plane. Thus, AB is at right-angles to the reference plane. Hence, AB will also make right-angles with all straight-lines joined to it which are also in the reference plane [Def. 11.3]. And BF, which is in the reference plane, is joined to it. Thus, the angle ABF is a right-angle. And ABC was also assumed to be a right-angle. Thus, angle ABF (is) equal to ABC. And they are in one plane. The very thing is impossible. Thus, BC is not in a more elevated plane. Thus, the three straight-lines BC, BD, and BE are in one plane.

Thus, if a straight-line is set up at right-angles to three straight-lines cutting one another, at the (common) point of section, then the three straight-lines are in one plane. (Which is) the very thing it was required to show.

ϛʹ.

Ἐὰν δύο εὐθεῖαι τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ὦσιν, παράλληλοι ἔσονται αἱ εὐθεῖαι.



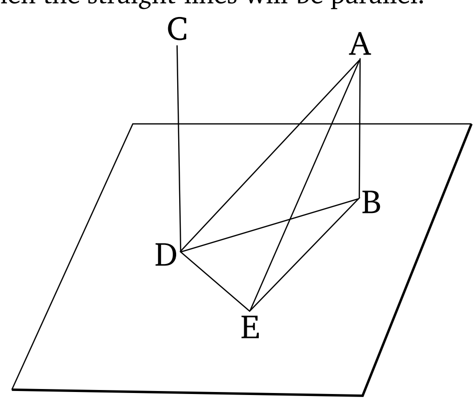
Δύο γὰρ εὐθεῖαι αἱ ΑΒ, ΓΔ τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθὰς ἔστωσαν· λέγω, ὅτι παράλληλός ἐστιν ἡ ΑΒ τῇ ΓΔ.

Συμβαλλέτωσαν γὰρ τῷ ὑποκειμένῳ ἐπιπέδῳ κατὰ τὰ Β, Δ σημεῖα, καὶ ἐπεζεύχθω ἡ ΒΔ εὐθεῖα, καὶ ἤχθω τῇ ΒΔ πρὸς ὀρθὰς ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ ἡ ΔΕ, καὶ κείσθω τῇ ΑΒ ἴση ἡ ΔΕ, καὶ ἐπεζεύχθωσαν αἱ ΒΕ, ΑΕ, ΑΔ.

Καὶ ἐπεὶ ἡ ΑΒ ὀρθή ἐστι πρὸς τὸ ὑποκείμενον ἐπίπεδον, καὶ πρὸς πάσας [ἄρα] τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ ὀρθὰς ποιήσει γωνίας. ἅπτεται δὲ τῆς ΑΒ ἑκατέρα τῶν ΒΔ, ΒΕ οὖσα ἐν τῷ ὑπο-

Proposition 6

If two straight-lines are at right-angles to the same plane then the straight-lines will be parallel.†



For let the two straight-lines AB and CD be at right-angles to a reference plane. I say that AB is parallel to CD.

For let them meet the reference plane at points B and D (respectively). And let the straight-line BD have been joined. And let DE have been drawn at right-angles to BD in the reference plane. And let DE be made equal to AB. And let BE, AE, and AD have been joined.

And since AB is at right-angles to the reference plane, it will [thus] also make right-angles with all straight-lines joined to it which are in the reference plane [Def. 11.3].

κειμένῳ ἐπιπέδῳ· ὀρθὴ ἄρα ἐστὶν ἑκατέρα τῶν ὑπὸ ΑΒΔ, ΑΒΕ γωνιῶν. διὰ τὰ αὐτὰ δὴ καὶ ἑκατέρα τῶν ὑπὸ ΓΔΒ, ΓΔΕ ὀρθή ἐστιν. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΒ τῇ ΔΕ, κοινὴ δὲ ἡ ΒΔ, δύο δὴ αἱ ΑΒ, ΒΔ δυσὶ ταῖς ΕΔ, ΔΒ ἴσαι εἰσίν· καὶ γωνίας ὀρθὰς περιέχουσιν· βάσις ἄρα ἡ ΑΔ βάσει τῇ ΒΕ ἐστιν ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΒ τῇ ΔΕ, ἀλλὰ καὶ ἡ ΑΔ τῇ ΒΕ, δύο δὴ αἱ ΑΒ, ΒΕ δυσὶ ταῖς ΕΔ, ΔΑ ἴσαι εἰσίν· καὶ βάσις αὐτῶν κοινὴ ἡ ΑΕ· γωνία ἄρα ἡ ὑπὸ ΑΒΕ γωνίᾳ τῇ ὑπὸ ΕΔΑ ἐστιν ἴση. ὀρθὴ δὲ ἡ ὑπὸ ΑΒΕ· ὀρθὴ ἄρα καὶ ἡ ὑπὸ ΕΔΑ· ἡ ΕΔ ἄρα πρὸς τὴν ΔΑ ὀρθή ἐστιν. ἔστι δὲ καὶ πρὸς ἑκατέραν τῶν ΒΔ, ΔΓ ὀρθή. ἡ ΕΔ ἄρα τρισὶν εὐθείαις ταῖς ΒΔ, ΔΑ, ΔΓ πρὸς ὀρθὰς ἐπὶ τῆς ἁφῆς ἐφέστηκεν· αἱ τρεῖς ἄρα εὐθεῖαι αἱ ΒΔ, ΔΑ, ΔΓ ἐν ἑνί εἰσιν ἐπιπέδῳ. ἐν ᾧ δὲ αἱ ΔΒ, ΔΑ, ἐν τούτῳ καὶ ἡ ΑΒ· πᾶν γὰρ τρίγωνον ἐν ἑνί ἐστιν ἐπιπέδῳ· αἱ ἄρα ΑΒ, ΒΔ, ΔΓ εὐθεῖαι ἐν ἑνί εἰσιν ἐπιπέδῳ. καί ἐστιν ὀρθὴ ἑκατέρα τῶν ὑπὸ ΑΒΔ, ΒΔΓ γωνιῶν· παράλληλος ἄρα ἐστὶν ἡ ΑΒ τῇ ΓΔ.

Ἐὰν ἄρα δύο εὐθεῖαι τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ὦσιν, παράλληλοι ἔσονται αἱ εὐθεῖαι· ὅπερ ἔδει δεῖξαι.

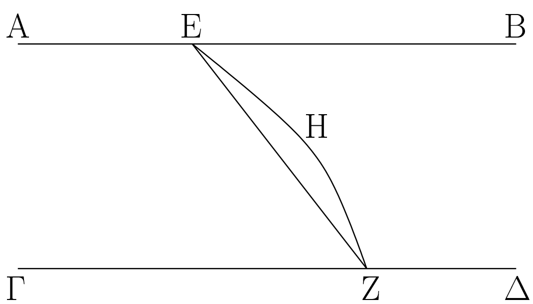
And BD and BE, which are in the reference plane, are each joined to AB. Thus, each of the angles ABD and ABE are right-angles. So, for the same (reasons), each of the angles CDB and CDE are also right-angles. And since AB is equal to DE, and BD (is) common, the two (straight-lines) AB and BD are equal to the two (straight-lines) ED and DB (respectively). And they contain right-angles. Thus, the base AD is equal to the base BE [Prop. 1.4]. And since AB is equal to DE, and AD (is) also (equal) to BE, the two (straight-lines) AB and BE are thus equal to the two (straight-lines) ED and DA (respectively). And their base AE (is) common. Thus, angle ABE is equal to angle EDA [Prop. 1.8]. And ABE (is) a right-angle. Thus, EDA (is) also a right-angle. ED is thus at right-angles to DA. And it is also at right-angles to each of BD and DC. Thus, ED is standing at right-angles to the three straight-lines BD, DA, and DC at the (common) point of section. Thus, the three straight-lines BD, DA, and DC are in one plane [Prop. 11.5]. And in which(ever) plane DB and DA (are found), in that (plane) AB (will) also (be found). For every triangle is in one plane [Prop. 11.2]. And each of the angles ABD and BDC is a right-angle. Thus, AB is parallel to CD [Prop. 1.28].

Thus, if two straight-lines are at right-angles to the same plane then the straight-lines will be parallel. (Which is) the very thing it was required to show.

† In other words, the two straight-lines lie in the same plane, and never meet when produced in either direction.

ζ´.

Ἐὰν ὦσι δύο εὐθεῖαι παράλληλοι, ληφθῇ δὲ ἐφ᾽ ἑκατέρας αὐτῶν τυχόντα σημεῖα, ἡ ἐπὶ τὰ σημεῖα ἐπιζευγνυμένη εὐθεῖα ἐν τῷ αὐτῷ ἐπιπέδῳ ἐστὶ ταῖς παραλλήλοις.

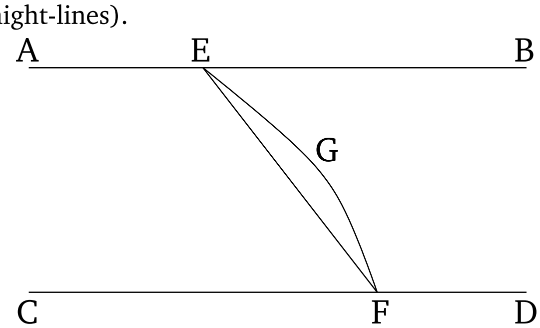


Ἔστωσαν δύο εὐθεῖαι παράλληλοι αἱ ΑΒ, ΓΔ, καὶ εἰλήφθω ἐφ᾽ ἑκατέρας αὐτῶν τυνχόντα σημεῖα τὰ Ε, Ζ· λέγω, ὅτι ἡ ἐπὶ τὰ Ε, Ζ σημεῖα ἐπιζευγνυμένη εὐθεῖα ἐν τῷ αὐτῷ ἐπιπέδῳ ἐστὶ ταῖς παραλλήλοις.

Μὴ γάρ, ἀλλ᾽ εἰ δυνατόν, ἔστω ἐν μετεωροτέρῳ ὡς ἡ ΕΗΖ, καὶ διήχθω διὰ τῆς ΕΗΖ ἐπίπεδον· τομὴν δὴ ποιήσει

Proposition 7

If there are two parallel straight-lines, and random points are taken on each of them, then the straight-line joining the two points is in the same plane as the parallel (straight-lines).



Let AB and CD be two parallel straight-lines, and let the random points E and F have been taken on each of them (respectively). I say that the straight-line joining points E and F is in the same (reference) plane as the parallel (straight-lines).

For (if) not, and if possible, let it be in a more elevated ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ εὐθεῖαν. ποιείτω ὡς τὴν ΕΖ· δύο ἄρα εὐθεῖαι αἱ ΕΗΖ, ΕΖ χωρίον περιέξουσιν· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἡ ἀπὸ τοῦ Ε ἐπὶ τὸ Ζ ἐπιζευγνυμένη εὐθεῖαι ἐν μετεωροτέρῳ ἐστὶν ἐπιπέδῳ· ἐν τῷ διὰ τῶν ΑΒ, ΓΒ ἄρα παραλλήλων ἐστὶν ἐπιπέδῳ ἡ ἀπὸ τοῦ Ε ἐπὶ τὸ Ζ ἐπιζευγνυμένη εὐθεῖα.

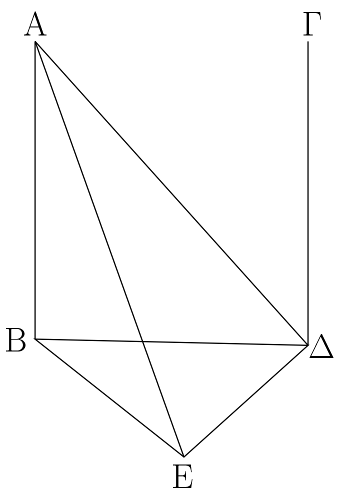
Ἐὰν ἄρα ὦσι δύο εὐθεῖαι παράλληλοι, ληφθῇ δὲ ἐφ᾽ ἑκατέρας αὐτῶν τυχόντα σημεῖα, ἡ ἐπὶ τὰ σημεῖα ἐπιζευγνυμένη εὐθεῖα ἐν τῷ αὐτῷ ἐπιπέδῳ ἐστὶ ταῖς παραλλήλοις· ὅπερ ἔδει δεῖξαι.

(plane), such as EGF. And let a plane have been drawn through EGF. So it will make a straight cutting in the reference plane [Prop. 11.3]. Let it make EF. Thus, two straight-lines (with the same end-points), EGF and EF, will enclose an area. The very thing is impossible. Thus, the straight-line joining E to F is not in a more elevated plane. The straight-line joining E to F is thus in the plane through the parallel (straight-lines) AB and CD.

Thus, if there are two parallel straight-lines, and random points are taken on each of them, then the straight-line joining the two points is in the same plane as the parallel (straight-lines). (Which is) the very thing it was required to show.

η΄.

Ἐὰν ὦσι δύο εὐθεῖαι παράλληλοι, ἡ δὲ ἑτέρα αὐτῶν ἐπιπέδῳ τινὶ πρὸς ὀρθὰς ᾖ, καὶ ἡ λοιπὴ τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται.



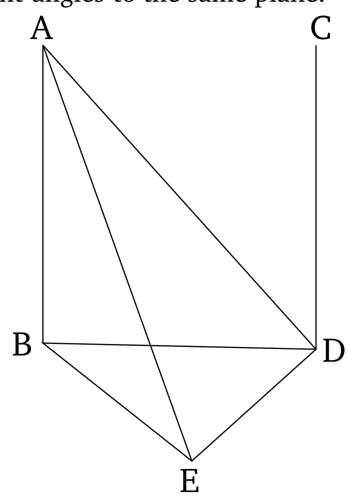
Ἔστωσαν δύο εὐθεῖαι παράλληλοι αἱ ΑΒ, ΓΔ, ἡ δὲ ἑτέρα αὐτῶν ἡ ΑΒ τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθὰς ἔστω· λέγω, ὅτι καὶ ἡ λοιπὴ ἡ ΓΔ τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται.

Συμβαλλέτωσαν γὰρ αἱ ΑΒ, ΓΔ τῷ ὑποκειμένῳ ἐπιπέδῳ κατὰ τὰ Β, Δ σημεῖα, καὶ ἐπεζεύχθω ἡ ΒΔ· αἱ ΑΒ, ΓΔ, ΒΔ ἄρα ἐν ἑνί εἰσιν ἐπιπέδῳ. ἤχθω τῇ ΒΑ πρὸς ὀρθὰς ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ ἡ ΔΕ, καὶ κείσθω τῇ ΑΒ ἴση ἡ ΔΕ, καὶ ἐπεζεύχθωσαν αἱ ΒΕ, ΑΕ, ΑΔ.

Καὶ ἐπεὶ ἡ ΑΒ ὀρθή ἐστι πρὸς τὸ ὑποκείμενον ἐπίπεδον, καὶ πρὸς πάσας ἄρα τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν ἡ ΑΒ· ὀρθὴ ἄρα [ἐστὶν] ἑκατέρα τῶν ὑπὸ ΑΒΔ, ΑΒΕ γωνιῶν. καὶ ἐπεὶ εἰς παραλλήλους τὰς ΑΒ, ΓΔ εὐθεῖα ἐμπέπτωκεν ἡ ΒΔ, αἱ ἄρα ὑπὸ ΑΒΔ, ΓΔΒ γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν. ὀρθὴ δὲ ἡ ὑπὸ ΑΒΔ· ὀρθὴ ἄρα καὶ ἡ ὑπὸ ΓΔΒ· ἡ ΓΔ ἄρα πρὸς τὴν ΒΔ ὀρθή ἐστιν. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΒ τῇ ΔΕ, κοινὴ δὲ ἡ ΒΔ,

Proposition 8

If two straight-lines are parallel, and one of them is at right-angles to some plane, then the remaining (one) will also be at right-angles to the same plane.



Let AB and CD be two parallel straight-lines, and let one of them, AB, be at right-angles to a reference plane. I say that the remaining (one), CD, will also be at right-angles to the same plane.

For let AB and CD meet the reference plane at points B and D (respectively). And let BD have been joined. AB, CD, and BD are thus in one plane [Prop. 11.7]. Let DE have been drawn at right-angles to BD in the reference plane, and let DE be made equal to AB, and let BE, AE, and AD have been joined.

And since AB is at right-angles to the reference plane, AB is thus also at right-angles to all of the straight-lines joined to it which are in the reference plane [Def. 11.3]. Thus, the angles ABD and ABE [are] each right-angles. And since the straight-line BD has met the parallel (straight-lines) AB and CD, the (sum of the) angles ABD and CDB is thus equal to two right-angles δύο δὴ αἱ ΑΒ, ΒΔ δυσὶ ταῖς ΕΔ, ΔΒ ἴσαι εἰσίν· καὶ γωνία ἡ ὑπὸ ΑΒΔ γωνίᾳ τῇ ὑπὸ ΕΔΒ ἴση· ὀρθὴ γὰρ ἑκατέρα· βάσις ἄρα ἡ ΑΔ βάσει τῇ ΒΕ ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν ΑΒ τῇ ΔΕ, ἡ δὲ ΒΕ τῇ ΑΔ, δύο δὴ αἱ ΑΒ, ΒΕ δυσὶ ταῖς ΕΔ, ΔΑ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ. καὶ βάσις αὐτῶν κοινὴ ἡ ΑΕ· γωνία ἄρα ἡ ὑπὸ ΑΒΕ γωνίᾳ τῇ ὑπὸ ΕΔΑ ἐστιν ἴση. ὀρθὴ δὲ ἡ ὑπὸ ΑΒΕ· ὀρθὴ ἄρα καὶ ἡ ὑπὸ ΕΔΑ· ἡ ΕΔ ἄρα πρὸς τὴν ΑΔ ὀρθή ἐστιν. ἔστι δὲ καὶ πρὸς τὴν ΔΒ ὀρθή· ἡ ΕΔ ἄρα καὶ τῷ διὰ τῶν ΒΔ, ΔΑ ἐπιπέδῳ ὀρθή ἐστιν. καὶ πρὸς πάσας ἄρα τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ διὰ τῶν ΒΔΑ ἐπιπέδῳ ὀρθὰς ποιήσει γωνίας ἡ ΕΔ. ἐν δὲ τῷ διὰ τῶν ΒΔΑ ἐπιπέδῳ ἐστὶν ἡ ΔΓ, ἐπειδήπερ ἐν τῷ διὰ τῶν ΒΔΑ ἐπιπέδῳ ἐστὶν αἱ ΑΒ, ΒΔ, ἐν ᾧ δὲ αἱ ΑΒ, ΒΔ, ἐν τούτῳ ἐστὶ καὶ ἡ ΔΓ. ἡ ΕΔ ἄρα τῇ ΔΓ πρὸς ὀρθάς ἐστιν· ὥστε καὶ ἡ ΓΔ τῇ ΔΕ πρὸς ὀρθάς ἐστιν. ἔστι δὲ καὶ ἡ ΓΔ τῇ ΒΔ πρὸς ὀρθάς. ἡ ΓΔ ἄρα δύο εὐθείαις τεμνούσαις ἀλλήλας ταῖς ΔΕ, ΔΒ ἀπὸ τῆς κατὰ τὸ Δ τομῆς πρὸς ὀρθὰς ἐφέστηκεν· ὥστε ἡ ΓΔ καὶ τῷ διὰ τῶν ΔΕ, ΔΒ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. τὸ δὲ διὰ τῶν ΔΕ, ΔΒ ἐπίπεδον τὸ ὑποκείμενόν ἐστιν· ἡ ΓΔ ἄρα τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν.

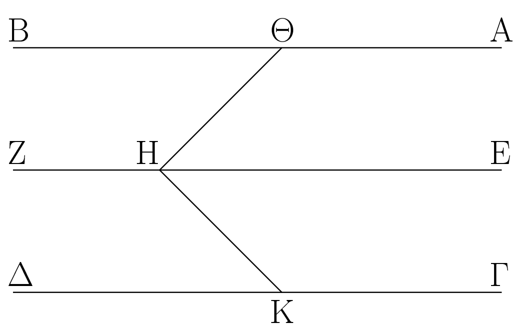
Ἐὰν ἄρα ὦσι δύο εὐθεῖαι παράλληλοι, ἡ δὲ μία αὐτῶν ἐπιπέδῳ τινὶ πρὸς ὀρθὰς ᾖ, καὶ ἡ λοιπὴ τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται· ὅπερ ἔδει δεῖξαι.

[Prop. 1.29]. And ABD (is) a right-angle. Thus, CDB (is) also a right-angle. CD is thus at right-angles to BD. And since AB is equal to DE, and BD (is) common, the two (straight-lines) AB and BD are equal to the two (straight-lines) ED and DB (respectively). And angle ABD (is) equal to angle EDB. For each (is) a right-angle. Thus, the base AD (is) equal to the base BE [Prop. 1.4]. And since AB is equal to DE, and BE to AD, the two (sides) AB, BE are equal to the two (sides) ED, DA, respectively. And their base AE is common. Thus, angle ABE is equal to angle EDA [Prop. 1.8]. And ABE (is) a right-angle. EDA (is) thus also a right-angle. Thus, ED is at right-angles to AD. And it is also at right-angles to DB. Thus, ED is also at right-angles to the plane through BD and DA [Prop. 11.4]. And ED will thus make right-angles with all of the straight-lines joined to it which are also in the plane through BDA. And DC is in the plane through BDA, inasmuch as AB and BD are in the plane through BDA [Prop. 11.2], and in which(ever plane) AB and BD (are found), DC is also (found). Thus, ED is at right-angles to DC. Hence, CD is also at right-angles to DE. And CD is also at right-angles to BD. Thus, CD is standing at right-angles to two straight-lines, DE and DB, which meet one another, at the (point) of section, D. Hence, CD is also at right-angles to the plane through DE and DB [Prop. 11.4]. And the plane through DE and DB is the reference (plane). CD is thus at right-angles to the reference plane.

Thus, if two straight-lines are parallel, and one of them is at right-angles to some plane, then the remaining (one) will also be at right-angles to the same plane. (Which is) the very thing it was required to show.

ϑʹ.

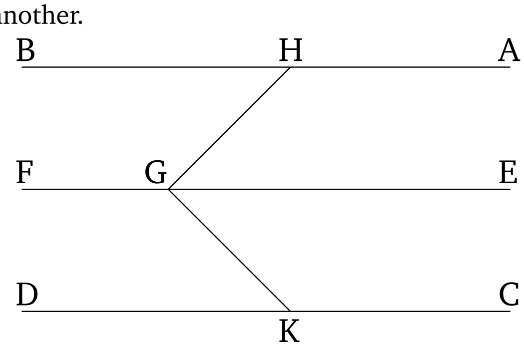
Αἱ τῇ αὐτῇ εὐθείᾳ παράλληλοι καὶ μὴ οὖσαι αὐτῇ ἐν τῷ αὐτῷ ἐπιπέδῳ καὶ ἀλλήλαις εἰσὶ παράλληλοι.



Ἔστω γὰρ ἑκατέρα τῶν ΑΒ, ΓΔ τῇ ΕΖ παράλληλος μὴ οὖσαι αὐτῇ ἐν τῷ αὐτῷ ἐπιπέδῳ· λέγω, ὅτι παράλληλός

Proposition 9

(Straight-lines) parallel to the same straight-line, and which are not in the same plane as it, are also parallel to one another.



For let AB and CD each be parallel to EF, not being in the same plane as it. I say that AB is parallel to CD.

ἐστιν ἡ ΑΒ τῇ ΓΔ.

Εἰλήφθω γὰρ ἐπὶ τῆς ΕΖ τυχὸν σημεῖον τὸ Η, καὶ ἀπ᾿ αὐτοῦ τῇ ΕΖ ἐν μὲν τῷ διὰ τῶν ΕΖ, ΑΒ ἐπιπέδῳ πρὸς ὀρθὰς ἤχθω ἡ ΗΘ, ἐν δὲ τῷ διὰ τῶν ΖΕ, ΓΔ τῇ ΕΖ πάλιν πρὸς ὀρθὰς ἤχθω ἡ ΗΚ.

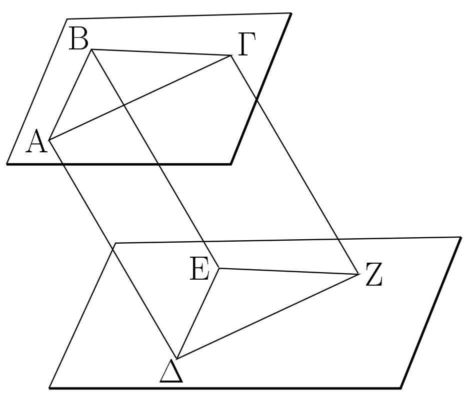
Καὶ ἐπεὶ ἡ ΕΖ πρὸς ἑκατέραν τῶν ΗΘ, ΗΚ ὀρθή ἐστιν, ἡ ΕΖ ἄρα καὶ τῷ διὰ τῶν ΗΘ, ΗΚ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. καί ἐστιν ἡ ΕΖ τῇ ΑΒ παράλληλος· καὶ ἡ ΑΒ ἄρα τῷ διὰ τῶν ΘΗΚ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΓΔ τῷ διὰ τῶν ΘΗΚ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν· ἑκατέρα ἄρα τῶν ΑΒ, ΓΔ τῷ διὰ τῶν ΘΗΚ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. ἐὰν δὲ δύο εὐθεῖαι τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ὦσιν, παράλληλοί εἰσιν αἱ εὐθεῖαι· παράλληλος ἄρα ἐστὶν ἡ ΑΒ τῇ ΓΔ· ὅπερ ἔδει δεῖξαι.

For let some point G have been taken at random on EF. And from it let GH have been drawn at right-angles to EF in the plane through EF and AB. And let GK have been drawn, again at right-angles to EF, in the plane through FE and CD.

And since EF is at right-angles to each of GH and GK, EF is thus also at right-angles to the plane through GH and GK [Prop. 11.4]. And EF is parallel to AB. Thus, AB is also at right-angles to the plane through HGK [Prop. 11.8]. So, for the same (reasons), CD is also at right-angles to the plane through HGK. Thus, AB and CD are each at right-angles to the plane through HGK. And if two straight-lines are at right–angles to the same plane then the straight-lines are parallel [Prop. 11.6]. Thus, AB is parallel to CD. (Which is) the very thing it was required to show.

ιʹ.

Ἐὰν δύο εὐθεῖαι ἁπτόμεναι ἀλλήλων παρὰ δύο εὐθείας ἁπτομένας ἀλλήλων ὦσι μὴ ἐν τῷ αὐτῷ ἐπιπέδῳ, ἴσας γωνίας περιέξουσιν.



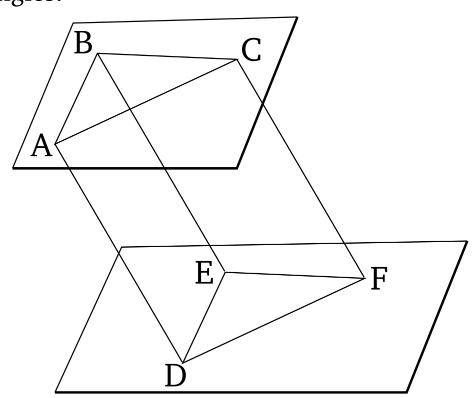
Δύο γὰρ εὐθεῖαι αἱ ΑΒ, ΒΓ ἁπτόμεναι ἀλλήλων παρὰ δύο εὐθείας τὰς ΔΕ, ΕΖ ἁπτομένας ἀλλήλων ἔστωσαν μὴ ἐν τῷ αὐτῷ ἐπιπέδῳ· λέγω, ὅτι ἴση ἐστὶν ἡ ὑπὸ ΑΒΓ γωνία τῇ ὑπὸ ΔΕΖ.

Ἀπειλήφθωσαν γὰρ αἱ ΒΑ, ΒΓ, ΕΔ, ΕΖ ἴσαι ἀλλήλαις, καὶ ἐπεζεύχθωσαν αἱ ΑΔ, ΓΖ, ΒΕ, ΑΓ, ΔΖ.

Καὶ ἐπεὶ ἡ ΒΑ τῇ ΕΔ ἴση ἐστὶ καὶ παράλληλος, καὶ ἡ ΑΔ ἄρα τῇ ΒΕ ἴση ἐστὶ καὶ παράλληλος. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΓΖ τῇ ΒΕ ἴση ἐστὶ καὶ παράλληλος· ἑκατέρα ἄρα τῶν ΑΔ, ΓΖ τῇ ΒΕ ἴση ἐστὶ καὶ παράλληλος. αἱ δὲ τῇ αὐτῇ εὐθείᾳ παράλληλοι καὶ μὴ οὖσαι αὐτῇ ἐν τῷ αὐτῷ ἐπιπέδῳ καὶ ἀλλήλαις εἰσὶ παράλληλοι· παράλληλος ἄρα ἐστὶν ἡ ΑΔ τῇ ΓΖ καὶ ἴση. καὶ ἐπιζευγνύουσιν αὐτὰς αἱ ΑΓ, ΔΖ· καὶ ἡ ΑΓ ἄρα τῇ ΔΖ ἴση ἐστὶ καὶ παράλληλος. καὶ ἐπεὶ δύο αἱ ΑΒ, ΒΓ δυσὶ ταῖς ΔΕ, ΕΖ ἴσαι εἰσίν, καὶ βάσις ἡ ΑΓ βάσει τῇ ΔΖ ἴση, γωνία ἄρα ἡ ὑπὸ ΑΒΓ γωνίᾳ τῇ ὑπὸ ΔΕΖ ἐστιν

Proposition 10

If two straight-lines joined to one another are (respectively) parallel to two straight-lines joined to one another, (but are) not in the same plane, then they will contain equal angles.



For let the two straight-lines joined to one another, AB and BC, be (respectively) parallel to the two straight-lines joined to one another, DE and EF, (but) not in the same plane. I say that angle ABC is equal to (angle) DEF.

For let BA, BC, ED, and EF have been cut off (so as to be, respectively) equal to one another. And let AD, CF, BE, AC, and DF have been joined.

And since BA is equal and parallel to ED, AD is thus also equal and parallel to BE [Prop. 1.33]. So, for the same reasons, CF is also equal and parallel to BE. Thus, AD and CF are each equal and parallel to BE. And straight-lines parallel to the same straight-line, and which are not in the same plane as it, are also parallel to one another [Prop. 11.9]. Thus, AD is parallel and equal to CF. And AC and DF join them. Thus, AC is also equal and ἴση.

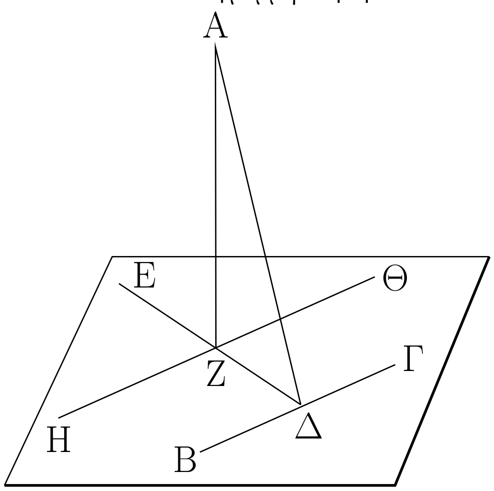
Ἐὰν ἄρα δύο εὐθεῖαι ἁπτόμεναι ἀλλήλων παρὰ δύο εὐθείας ἁπτομένας ἀλλήλων ὦσι μὴ ἐν τῷ αὐτῷ ἐπιπέδῳ, ἴσας γωνίας περιέξουσιν· ὅπερ ἔδει δεῖξαι.

parallel to DF [Prop. 1.33]. And since the two (straight-lines) AB and BC are equal to the two (straight-lines) DE and EF (respectvely), and the base AC (is) equal to the base DF, the angle ABC is thus equal to the (angle) DEF [Prop. 1.8].

Thus, if two straight-lines joined to one another are (respectively) parallel to two straight-lines joined to one another, (but are) not in the same plane, then they will contain equal angles. (Which is) the very thing it was required to show.

ιαʹ.

Ἀπὸ τοῦ δοθέντος σημείου μετεώρου ἐπὶ τὸ δοθὲν ἐπίπεδον κάθετον εὐθεῖαν γραμμὴν ἀγαγεῖν.



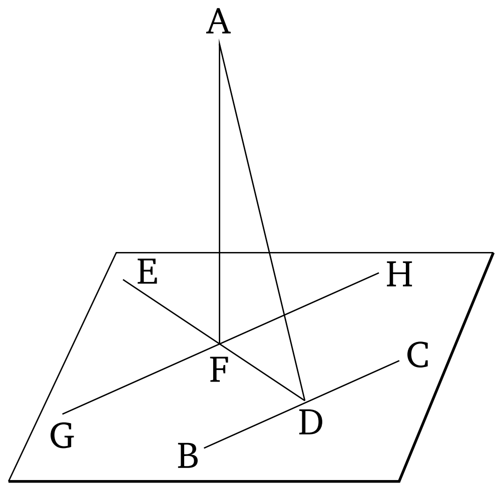
Ἔστω τὸ μὲν δοθὲν σημεῖον μετέωρον τὸ Α, τὸ δὲ δοθὲν ἐπίπεδον τὸ ὑποκείμενον· δεῖ δὴ ἀπὸ τοῦ Α σημείου ἐπὶ τὸ ὑποκείμενον ἐπίπεδον κάθετον εὐθεῖαν γραμμὴν ἀγαγεῖν.

Διήχθω γάρ τις ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ εὐθεῖα, ὡς ἔτυχεν, ἡ ΒΓ, καὶ ἤχθω ἀπὸ τοῦ Α σημείου ἐπὶ τὴν ΒΓ κάθετος ἡ ΑΔ. εἰ μὲν οὖν ἡ ΑΔ κάθετός ἐστι καὶ ἐπὶ τὸ ὑποκείμενον ἐπίπεδον, γεγονὸς ἂν εἴη τὸ ἐπιταχθέν. εἰ δὲ οὔ, ἤχθω ἀπὸ τοῦ Δ σημείου τῇ ΒΓ ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθὰς ἡ ΔΕ, καὶ ἤχθω ἀπὸ τοῦ Α ἐπὶ τὴν ΔΕ κάθετος ἡ ΑΖ, καὶ διὰ τοῦ Ζ σημείου τῇ ΒΓ παράλληλος ἤχθω ἡ ΗΘ.

Καὶ ἐπεὶ ἡ ΒΓ ἑκατέρᾳ τῶν ΔΑ, ΔΕ πρὸς ὀρθάς ἐστιν, ἡ ΒΓ ἄρα καὶ τῷ διὰ τῶν ΕΔΑ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. καί ἐστιν αὐτῇ παράλληλος ἡ ΗΘ· ἐὰν δὲ ὦσι δύο εὐθεῖαι παράλληλοι, ἡ δὲ μία αὐτῶν ἐπιπέδῳ τινὶ πρὸς ὀρθὰς ᾖ, καὶ ἡ λοιπὴ τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται· καὶ ἡ ΗΘ ἄρα τῷ διὰ τῶν ΕΔ, ΔΑ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. καὶ πρὸς πάσας ἄρα τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ διὰ τῶν ΕΔ, ΔΑ ἐπιπέδῳ ὀρθή ἐστιν ἡ ΗΘ. ἅπτεται δὲ αὐτῆς ἡ ΑΖ οὖσα ἐν τῷ διὰ τῶν ΕΔ, ΔΑ ἐπιπέδῳ· ἡ ΗΘ ἄρα ὀρθή ἐστι πρὸς τὴν ΖΑ· ὥστε καὶ ἡ ΖΑ ὀρθή ἐστι πρὸς τὴν ΘΗ. ἔστι

Proposition 11

To draw a perpendicular straight-line from a given raised point to a given plane.



Let A be the given raised point, and the given plane the reference (plane). So, it is required to draw a perpendicular straight-line from point A to the reference plane.

Let some random straight-line BC have been drawn across in the reference plane, and let the (straight-line) AD have been drawn from point A perpendicular to BC [Prop. 1.12]. If, therefore, AD is also perpendicular to the reference plane then that which was prescribed will have occurred. And, if not, let DE have been drawn in the reference plane from point D at right-angles to BC [Prop. 1.11], and let the (straight-line) AF have been drawn from A perpendicular to DE [Prop. 1.12], and let GH have been drawn through point F, parallel to BC [Prop. 1.31].

And since BC is at right-angles to each of DA and DE, BC is thus also at right-angles to the plane through EDA [Prop. 11.4]. And GH is parallel to it. And if two straight-lines are parallel, and one of them is at right-angles to some plane, then the remaining (straight-line) will also be at right-angles to the same plane [Prop. 11.8]. Thus, GH is also at right-angles to the plane through δὲ ἡ ΑΖ καὶ πρὸς τὴν ΔΕ ὀρθή· ἡ ΑΖ ἄρα πρὸς ἑκατέραν τῶν ΗΘ, ΔΕ ὀρθή ἐστιν. ἐὰν δὲ εὐθεῖα δυσὶν εὐθείαις τεμνούσαις ἀλλήλας ἐπὶ τῆς τομῆς πρὸς ὀρθὰς ἐπισταθῇ, καὶ τῷ δι᾽ αὐτῶν ἐπιπέδῳ πρὸς ὀρθὰς ἔσται· ἡ ΖΑ ἄρα τῷ διὰ τῶν ΕΔ, ΗΘ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. τὸ δὲ διὰ τῶν ΕΔ, ΗΘ ἐπίπεδόν ἐστι τὸ ὑποκείμενον· ἡ ΑΖ ἄρα τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν.

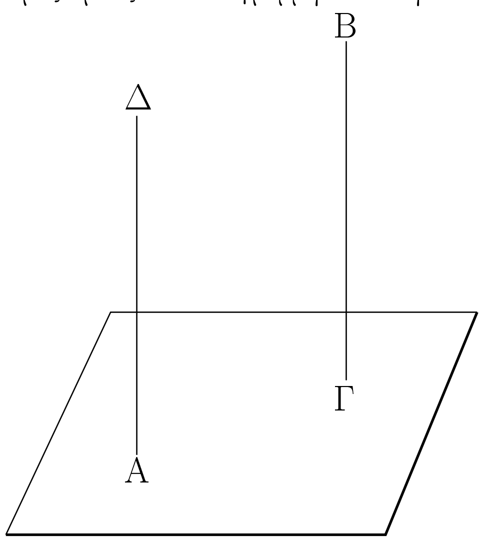
Ἀπὸ τοῦ ἄρα δοθέντος σημείου μετεώρου τοῦ Α ἐπὶ τὸ ὑποκείμενον ἐπίπεδον κάθετος εὐθεῖα γραμμὴ ἦκται ἡ ΑΖ· ὅπερ ἔδει ποιῆσαι.

ED and DA. And GH is thus at right-angles to all of the straight-lines joined to it which are also in the plane through ED and AD [Def. 11.3]. And AF, which is in the plane through ED and DA, is joined to it. Thus, GH is at right-angles to FA. Hence, FA is also at right-angles to HG. And AF is also at right-angles to DE. Thus, AF is at right-angles to each of GH and DE. And if a straight-line is set up at right-angles to two straight-lines cutting one another, at the point of section, then it will also be at right-angles to the plane through them [Prop. 11.4]. Thus, FA is at right-angles to the plane through ED and GH. And the plane through ED and GH is the reference (plane). Thus, AF is at right-angles to the reference plane.

Thus, the straight-line AF has been drawn from the given raised point A perpendicular to the reference plane. (Which is) the very thing it was required to do.

ιβʹ.

Τῷ δοθέντι ἐπιπέδῳ ἀπὸ τοῦ πρὸς αὐτῷ δοθέντος σημείου πρὸς ὀρθὰς εὐθεῖαν γραμμὴν ἀναστῆσαι.



Ἔστω τὸ μὲν δοθὲν ἐπίπεδον τὸ ὑποκείμενον, τὸ δὲ πρὸς αὐτῷ σημεῖον τὸ Α· δεῖ δὴ ἀπὸ τοῦ Α σημείου τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθὰς εὐθεῖαν γραμμὴν ἀναστῆσαι.

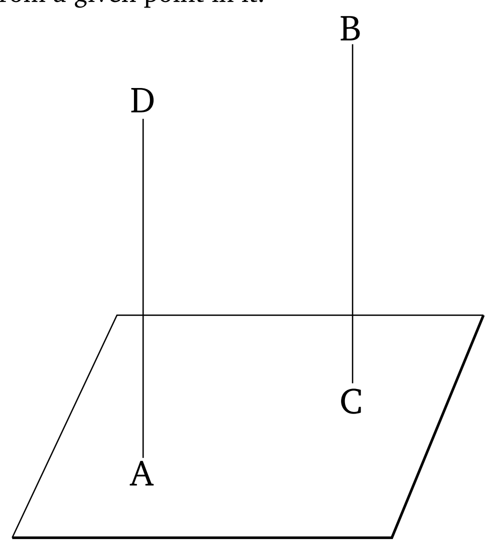
Νενοήσθω τι σημεῖον μετέωρον τὸ Β, καὶ ἀπὸ τοῦ Β ἐπὶ τὸ ὑποκείμενον ἐπίπεδον κάθετος ἤχθω ἡ ΒΓ, καὶ διὰ τοῦ Α σημείου τῇ ΒΓ παράλληλος ἤχθω ἡ ΑΔ.

Ἐπεὶ οὖν δύο εὐθεῖαι παράλληλοί εἰσιν αἱ ΑΔ, ΓΒ, ἡ δὲ μία αὐτῶν ἡ ΒΓ τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν, καὶ ἡ λοιπὴ ἄρα ἡ ΑΔ τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν.

Τῷ ἄρα δοθέντι ἐπιπέδῳ ἀπὸ τοῦ πρὸς αὐτῷ σημείου τοῦ Α πρὸς ὀρθὰς ἀνέσταται ἡ ΑΔ· ὅπερ ἔδει ποιῆσαι.

Proposition 12

To set up a straight-line at right-angles to a given plane from a given point in it.



Let the given plane be the reference (plane), and A a point in it. So, it is required to set up a straight-line at right-angles to the reference plane at point A.

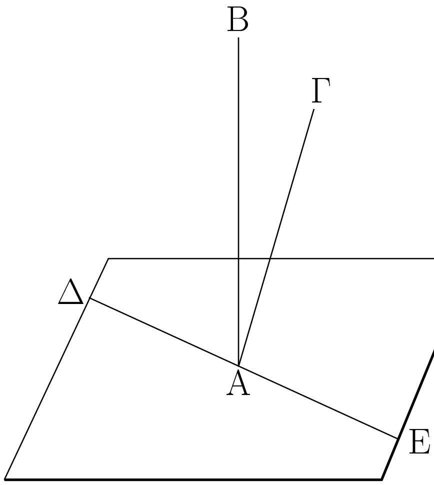
Let some raised point B have been assumed, and let the perpendicular (straight-line) BC have been drawn from B to the reference plane [Prop. 11.11]. And let AD have been drawn from point A parallel to BC [Prop. 1.31].

Therefore, since AD and CB are two parallel straight-lines, and one of them, BC, is at right-angles to the reference plane, the remaining (one) AD is thus also at right-angles to the reference plane [Prop. 11.8].

Thus, AD has been set up at right-angles to the given plane, from the point in it, A. (Which is) the very thing it was required to do.

ιγ΄.

Ἀπὸ τοῦ αὐτοῦ σημείου τῷ αὐτῷ ἐπιπέδῳ δύο εὐθεῖαι πρὸς ὀρθὰς οὐκ ἀναστήσονται ἐπὶ τὰ αὐτὰ μέρη.

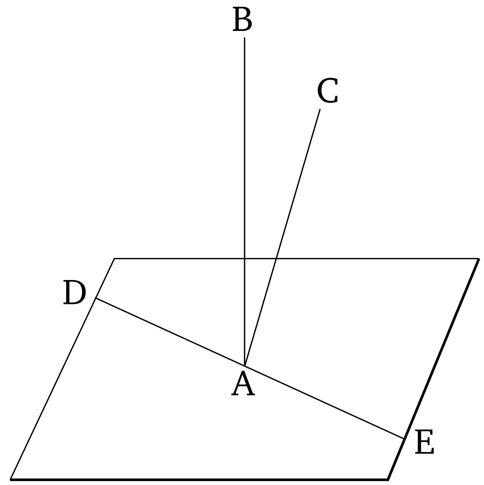


Εἰ γὰρ δυνατόν, ἀπὸ τοῦ αὐτοῦ σημείου τοῦ Α τῷ ὑποκειμένῳ ἐπιπέδῳ δύο εὐθεῖαι αἱ ΑΒ, ΒΓ πρὸς ὀρθὰς ἀνεστάτωσαν ἐπὶ τὰ αὐτὰ μέρη, καὶ διήχθω τὸ διὰ τῶν ΒΑ, ΑΓ ἐπίπεδον· τομὴν δὴ ποιήσει διὰ τοῦ Α ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ εὐθεῖαν. ποιείτω τὴν ΔΑΕ· αἱ ἄρα ΑΒ, ΑΓ, ΔΑΕ εὐθεῖαι ἐν ἑνι εἰσιν ἐπιπέδῳ. καὶ ἐπεὶ ἡ ΓΑ τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν, καὶ πρὸς πάσας ἄρα τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ ὀρθὰς ποιήσει γωνίας. ἅπτεται δὲ αὐτῆς ἡ ΔΑΕ οὖσα ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ· ἡ ἄρα ὑπὸ ΓΑΕ γωνία ὀρθή ἐστιν. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΒΑΕ ὀρθή ἐστιν· ἴση ἄρα ἡ ὑπὸ ΓΑΕ τῇ ὑπὸ ΒΑΕ καί εἰσιν ἐν ἑνὶ ἐπιπέδῳ· ὅπερ ἐστὶν ἀδύνατον.

Οὐκ ἄρα ἀπὸ τοῦ αὐτοῦ σημείου τῷ αὐτῷ ἐπιπέδῳ δύο εὐθεῖαι πρὸς ὀρθὰς ἀνασταθήσονται ἐπὶ τὰ αὐτὰ μέρη· ὅπερ ἔδει δεῖξαι.

Proposition 13

Two (different) straight-lines cannot be set up at the same point at right-angles to the same plane, on the same side.



For, if possible, let the two straight-lines AB and AC have been set up at the same point A at right-angles to the reference plane, on the same side. And let the plane through BA and AC have been drawn. So it will make a straight cutting (passing) through (point) A in the reference plane [Prop. 11.3]. Let it have made DAE. Thus, AB, AC, and DAE are straight-lines in one plane. And since CA is at right-angles to the reference plane, it will thus also make right-angles with all of the straight-lines joined to it which are also in the reference plane [Def. 11.3]. And DAE, which is in the reference plane, is joined to it. Thus, angle CAE is a right-angle. So, for the same (reasons), BAE is also a right-angle. Thus, CAE (is) equal to BAE. And they are in one plane. The very thing is impossible.

Thus, two (different) straight-lines cannot be set up at the same point at right-angles to the same plane, on the same side. (Which is) the very thing it was required to show.

ιδ΄.

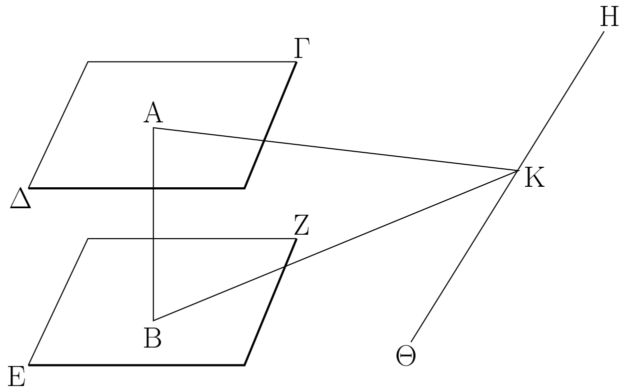
Πρὸς ἃ ἐπίπεδα ἡ αὐτὴ εὐθεῖα ὀρθή ἐστιν, παράλληλα ἔσται τὰ ἐπίπεδα.

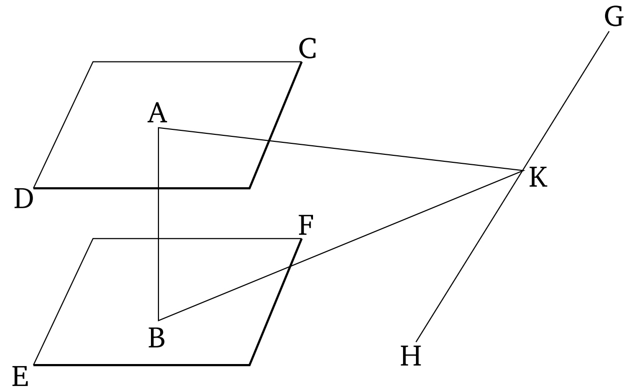
Εὐθεῖα γάρ τις ἡ ΑΒ πρὸς ἑκάτερον τῶν ΓΔ, ΕΖ ἐπιπέδων πρὸς ὀρθὰς ἔστω· λέγω, ὅτι παράλληλά ἐστι τὰ ἐπίπεδα.

Proposition 14

Planes to which the same straight-line is at right-angles will be parallel planes.

For let some straight-line AB be at right-angles to each of the planes CD and EF. I say that the planes are parallel.





Εἰ γὰρ μή, ἐκβαλλόμενα συμπεσοῦνται. συμπιπτέτωσαν· ποιήσουσι δὴ κοινὴν τομὴν εὐθεῖαν. ποιείτωσαν τὴν ΗΘ, καὶ εἰλήφθω ἐπὶ τῆς ΗΘ τυχὸν σημεῖον τὸ Κ, καὶ ἐπεζεύχθωσαν αἱ ΑΚ, ΒΚ.

Καὶ ἐπεὶ ἡ ΑΒ ὀρθή ἐστι πρὸς τὸ ΕΖ ἐπίπεδον, καὶ πρὸς τὴν ΒΚ ἄρα εὐθεῖαν οὖσαν ἐν τῷ ΕΖ ἐκβληθέντι ἐπιπέδῳ ὀρθή ἐστιν ἡ ΑΒ· ἡ ἄρα ὑπὸ ΑΒΚ γωνία ὀρθή ἐστιν. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΒΑΚ ὀρθή ἐστιν. τριγώνου δὴ τοῦ ΑΒΚ αἱ δύο γωνίαι αἱ ὑπὸ ΑΒΚ, ΒΑΚ δυσὶν ὀρθαῖς εἰσιν ἴσαι· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τὰ ΓΔ, ΕΖ ἐπίπεδα ἐκβαλλόμενα συμπεσοῦνται· παράλληλα ἄρα ἐστὶ τὰ ΓΔ, ΕΖ ἐπίπεδα.

Πρὸς ἃ ἐπίπεδα ἄρα ἡ αὐτὴ εὐθεῖα ὀρθή ἐστιν, παράλληλά ἐστι τὰ ἐπίπεδα· ὅπερ ἔδει δεῖξαι.

For, if not, being produced, they will meet. Let them have met. So they will make a straight-line as a common section [Prop. 11.3]. Let them have made GH. And let some random point K have been taken on GH. And let AK and BK have been joined.

And since AB is at right-angles to the plane EF, AB is thus also at right-angles to BK, which is a straight-line in the produced plane EF [Def. 11.3]. Thus, angle ABK is a right-angle. So, for the same (reasons), BAK is also a right-angle. So the (sum of the) two angles ABK and BAK in the triangle ABK is equal to two right-angles. The very thing is impossible [Prop. 1.17]. Thus, planes CD and EF, being produced, will not meet. Planes CD and EF are thus parallel [Def. 11.8].

Thus, planes to which the same straight-line is at right-angles are parallel planes. (Which is) the very thing it was required to show.

ιε΄.

Ἐὰν δύο εὐθεῖαι ἁπτόμεναι ἀλλήλων παρὰ δύο εὐθείας ἁπτομένας ἀλλήλων ὦσι μὴ ἐν τῷ αὐτῷ ἐπιπέδῳ οὖσαι, παράλληλά ἐστι τὰ δι᾽ αὐτῶν ἐπίπεδα.

Δύο γὰρ εὐθεῖαι ἁπτόμεναι ἀλλήλων αἱ ΑΒ, ΒΓ παρὰ δύο εὐθείας ἁπτομένας ἀλλήλων τὰς ΔΕ, ΕΖ ἔστωσαν μὴ ἐν τῷ αὐτῷ ἐπιπέδῳ οὖσαι· λέγω, ὅτι ἐκβαλλόμενα τὰ διὰ τῶν ΑΒ, ΒΓ, ΔΕ, ΕΖ ἐπίπεδα οὐ συμπεσεῖται ἀλλήλοις.

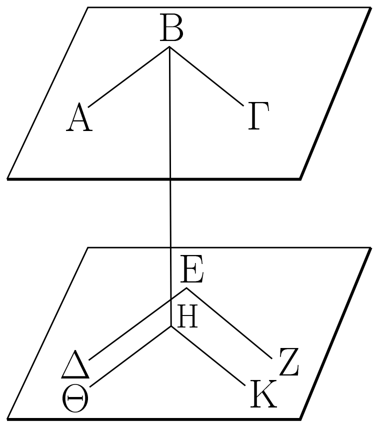
Ἤχθω γὰρ ἀπὸ τοῦ Β σημείου ἐπὶ τὸ διὰ τῶν ΔΕ, ΕΖ ἐπίπεδον κάθετος ἡ ΒΗ καὶ συμβαλλέτω τῷ ἐπιπέδῳ κατὰ τὸ Η σημεῖον, καὶ διὰ τοῦ Η τῇ μὲν ΕΔ παράλληλος ἤχθω ἡ ΗΘ, τῇ δὲ ΕΖ ἡ ΗΚ.

Proposition 15

If two straight-lines joined to one another are parallel (respectively) to two straight-lines joined to one another, which are not in the same plane, then the planes through them are parallel (to one another).

For let the two straight-lines joined to one another, AB and BC, be parallel to the two straight-lines joined to one another, DE and EF (respectively), not being in the same plane. I say that the planes through AB, BC and DE, EF will not meet one another (when) produced.

For let BG have been drawn from point B perpendicular to the plane through DE and EF [Prop. 11.11], and let it meet the plane at point G. And let GH have been drawn through G parallel to ED, and GK (parallel) to EF [Prop. 1.31].



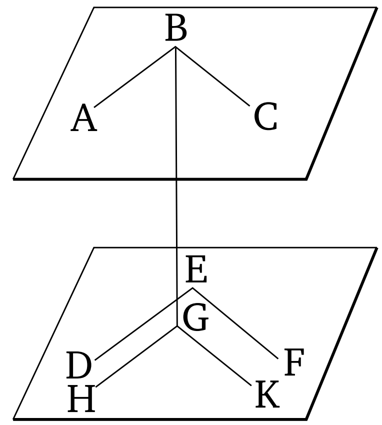
Καὶ ἐπεὶ ἡ ΒΗ ὀρθή ἐστι πρὸς τὸ διὰ τῶν ΔΕ, ΕΖ ἐπίπεδον, καὶ πρὸς πάσας ἄρα τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ διὰ τῶν ΔΕ, ΕΖ ἐπιπέδῳ ὀρθὰς ποιήσει γωνίας. ἅπτεται δὲ αὐτῆς ἑκατέρα τῶν ΗΘ, ΗΚ οὖσα ἐν τῷ διὰ τῶν ΔΕ, ΕΖ ἐπιπέδῳ· ὀρθὴ ἄρα ἐστὶν ἑκατέρα τῶν ὑπὸ ΒΗΘ, ΒΗΚ γωνιῶν. καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΒΑ τῇ ΗΘ, αἱ ἄρα ὑπὸ ΗΒΑ, ΒΗΘ γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν. ὀρθὴ δὲ ἡ ὑπὸ ΒΗΘ· ὀρθὴ ἄρα καὶ ἡ ὑπὸ ΗΒΑ· ἡ ΗΒ ἄρα τῇ ΒΑ πρὸς ὀρθάς ἐστιν. διὰ τὰ αὐτὰ δὴ ἡ ΗΒ καὶ τῇ ΒΓ ἐστι πρὸς ὀρθάς. ἐπεὶ οὖν εὐθεῖα ἡ ΗΒ δυσὶν εὐθείαις ταῖς ΒΑ, ΒΓ τεμνούσαις ἀλλήλας πρὸς ὀρθὰς ἐφέστηκεν, ἡ ΗΒ ἄρα καὶ τῷ διὰ τῶν ΒΑ, ΒΓ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. [διὰ τὰ αὐτὰ δὴ ἡ ΒΗ καὶ τῷ διὰ τῶν ΗΘ, ΗΚ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. τὸ δὲ διὰ τῶν ΗΘ, ΗΚ ἐπίπεδόν ἐστι τὸ διὰ τῶν ΔΕ, ΕΖ· ἡ ΒΗ ἄρα τῷ διὰ τῶν ΔΕ, ΕΖ ἐπιπέδῳ ἐστὶ πρὸς ὀρθάς. ἐδείχθη δὲ ἡ ΗΒ καὶ τῷ διὰ τῶν ΑΒ, ΒΓ ἐπιπέδῳ πρὸς ὀρθάς]. πρὸς ἃ δὲ ἐπίπεδα ἡ αὐτὴ εὐθεῖα ὀρθή ἐστιν, παράλληλά ἐστι τὰ ἐπίπεδα· παράλληλον ἄρα ἐστὶ τὸ διὰ τῶν ΑΒ, ΒΓ ἐπίπεδον τῷ διὰ τῶν ΔΕ, ΕΖ.

Ἐὰν ἄρα δύο εὐθεῖαι ἁπτόμεναι ἀλλήλων παρὰ δύο εὐθείας ἁπτομένας ἀλλήλων ὦσι μὴ ἐν τῷ αὐτῷ ἐπιπέδῳ, παράλληλά ἐστι τὰ δι᾽ αὐτῶν ἐπίπεδα· ὅπερ ἔδει δεῖξαι.

ιϛʹ.

Ἐὰν δύο ἐπίπεδα παράλληλα ὑπὸ ἐπιπέδου τινὸς τέμνηται, αἱ κοιναὶ αὐτῶν τομαὶ παράλληλοί εἰσιν.

Δύο γὰρ ἐπίπεδα παράλληλα τὰ ΑΒ, ΓΔ ὑπὸ ἐπιπέδου τοῦ ΕΖΗΘ τεμνέσθω, κοιναὶ δὲ αὐτῶν τομαὶ ἔστωσαν αἱ ΕΖ, ΗΘ· λέγω, ὅτι παράλληλός ἐστιν ἡ ΕΖ τῇ ΗΘ.



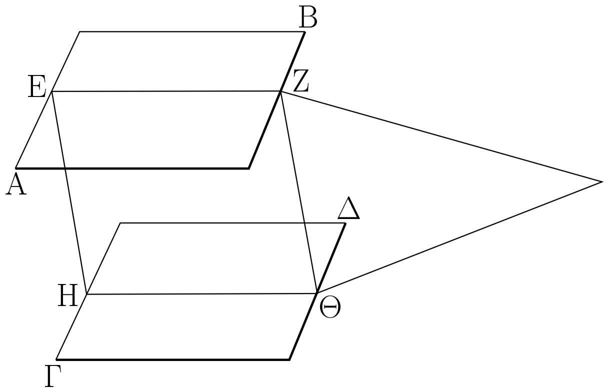
And since BG is at right-angles to the plane through DE and EF, it will thus also make right-angles with all of the straight-lines joined to it, which are also in the plane through DE and EF [Def. 11.3]. And each of GH and GK, which are in the plane through DE and EF, are joined to it. Thus, each of the angles BGH and BGK are right-angles. And since BA is parallel to GH [Prop. 11.9], the (sum of the) angles GBA and BGH is equal to two right-angles [Prop. 1.29]. And BGH (is) a right-angle. GBA (is) thus also a right-angle. Thus, GB is at right-angles to BA. So, for the same (reasons), GB is also at right-angles to BC. Therefore, since the straight-line GB has been set up at right-angles to two straight-lines, BA and BC, cutting one another, GB is thus at right-angles to the plane through BA and BC [Prop. 11.4]. [So, for the same (reasons), BG is also at right-angles to the plane through GH and GK. And the plane through GH and GK is the (plane) through DE and EF. And it was also shown that GB is at right-angles to the plane through AB and BC.] And planes to which the same straight-line is at right-angles are parallel planes [Prop. 11.14]. Thus, the plane through AB and BC is parallel to the (plane) through DE and EF.

Thus, if two straight-lines joined to one another are parallel (respectively) to two straight-lines joined to one another, which are not in the same plane, then the planes through them are parallel (to one another). (Which is) the very thing it was required to show.

Proposition 16

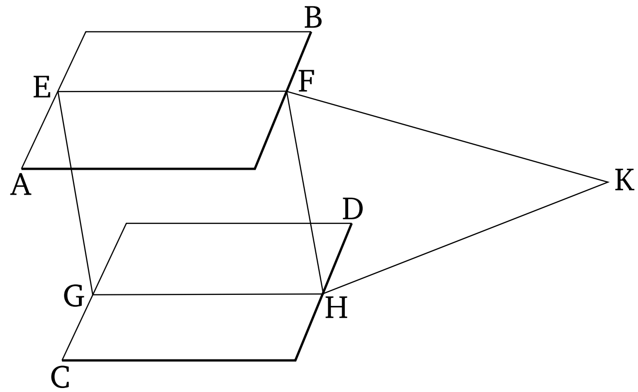
If two parallel planes are cut by some plane then their common sections are parallel.

For let the two parallel planes AB and CD have been cut by the plane $EFGH$. And let EF and GH be their common sections. I say that EF is parallel to GH.



Εἰ γὰρ μή, ἐκβαλλόμεναι αἱ ΕΖ, ΗΘ ἤτοι ἐπὶ τὰ Ζ, Θ μέρη ἢ ἐπὶ τὰ Ε, Η συμπεσοῦνται. ἐκβεβλήσθωσαν ὡς ἐπὶ τὰ Ζ, Θ μέρη καὶ συμπιπτέτωσαν πρότερον κατὰ τὸ Κ. καὶ ἐπεὶ ἡ ΕΖΚ ἐν τῷ ΑΒ ἐστιν ἐπιπέδῳ, καὶ πάντα ἄρα τὰ ἐπὶ τῆς ΕΖΚ σημεῖα ἐν τῷ ΑΒ ἐστιν ἐπιπέδῳ. ἓν δὲ τῶν ἐπὶ τῆς ΕΖΚ εὐθείας σημείων ἐστὶ τὸ Κ· τὸ Κ ἄρα ἐν τῷ ΑΒ ἐστιν ἐπιπέδῳ. διὰ τὰ αὐτὰ δὴ τὸ Κ καὶ ἐν τῷ ΓΔ ἐστιν ἐπιπέδῳ· τὰ ΑΒ, ΓΔ ἄρα ἐπίπεδα ἐκβαλλόμενα συμπεσοῦνται. οὐ συμπίπτουσι δὲ διὰ τὸ παράλληλα ὑποκεῖσθαι· οὐκ ἄρα αἱ ΕΖ, ΗΘ εὐθεῖαι ἐκβαλλόμεναι ἐπὶ τὰ Ζ, Θ μέρη συμπεσοῦνται. ὁμοίως δὴ δείξομεν, ὅτι αἱ ΕΖ, ΗΘ εὐθεῖαι οὐδὲ ἐπὶ τὰ Ε, Η μέρη ἐκβαλλόμεναι συμπεσοῦνται. αἱ δὲ ἐπὶ μηδέτερα τὰ μέρη συμπίπτουσαι παράλληλοί εἰσιν. παράλληλος ἄρα ἐστὶν ἡ ΕΖ τῇ ΗΘ.

Ἐὰν ἄρα δύο ἐπίπεδα παράλληλα ὑπὸ ἐπιπέδου τινὸς τέμνηται, αἱ κοιναὶ αὐτῶν τομαὶ παράλληλοί εἰσιν· ὅπερ ἔδει δεῖξαι.



For, if not, being produced, EF and GH will meet either in the direction of F, H, or of E, G. Let them be produced, as in the direction of F, H, and let them, first of all, have met at K. And since EFK is in the plane AB, all of the points on EFK are thus also in the plane AB [Prop. 11.1]. And K is one of the points on EFK. Thus, K is in the plane AB. So, for the same (reasons), K is also in the plane CD. Thus, the planes AB and CD, being produced, will meet. But they do not meet, on account of being (initially) assumed (to be mutually) parallel. Thus, the straight-lines EF and GH, being produced in the direction of F, H, will not meet. So, similarly, we can show that the straight-lines EF and GH, being produced in the direction of E, G, will not meet either. And (straight-lines in one plane which), being produced, do not meet in either direction are parallel [Def. 1.23]. EF is thus parallel to GH.

Thus, if two parallel planes are cut by some plane then their common sections are parallel. (Which is) the very thing it was required to show.

ιζ′.

Ἐὰν δύο εὐθεῖαι ὑπὸ παραλλήλων ἐπιπεδων τέμνωνται, εἰς τοὺς αὐτοὺς λόγους τμηθήσονται.

Δύο γὰρ εὐθεῖαι αἱ ΑΒ, ΓΔ ὑπὸ παραλλήλων ἐπιπέδων τῶν ΗΘ, ΚΛ, ΜΝ τεμνέσθωσαν κατὰ τὰ Α, Ε, Β, Γ, Ζ, Δ σημεῖα· λέγω, ὅτι ἐστὶν ὡς ἡ ΑΕ εὐθεῖα πρὸς τὴν ΕΒ, οὕτως ἡ ΓΖ πρὸς τὴν ΖΔ.

Ἐπεζεύχθωσαν γὰρ αἱ ΑΓ, ΒΔ, ΑΔ, καὶ συμβαλλέτω ἡ ΑΔ τῷ ΚΛ ἐπιπέδῳ κατὰ τὸ Ξ σημεῖον, καὶ ἐπεζεύχθωσαν αἱ ΕΞ, ΞΖ.

Καὶ ἐπεὶ δύο ἐπίπεδα παράλληλα τὰ ΚΛ, ΜΝ ὑπὸ ἐπιπέδου τοῦ ΕΒΔΞ τέμνεται, αἱ κοιναὶ αὐτῶν τομαὶ αἱ ΕΞ, ΒΔ παράλληλοί εἰσιν. διὰ τὰ αὐτὰ δὴ ἐπεὶ δύο ἐπίπεδα παράλληλα τὰ ΗΘ, ΚΛ ὑπὸ ἐπιπέδου τοῦ ΑΞΖΓ τέμνεται, αἱ κοιναὶ αὐτῶν τομαὶ αἱ ΑΓ, ΞΖ παράλληλοί εἰσιν. καὶ ἐπεὶ τριγώνου τοῦ ΑΒΔ παρὰ μίαν τῶν πλευρῶν τὴν ΒΔ εὐθεῖα ἦκται ἡ ΕΞ, ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΑΕ πρὸς ΕΒ, οὕτως

Proposition 17

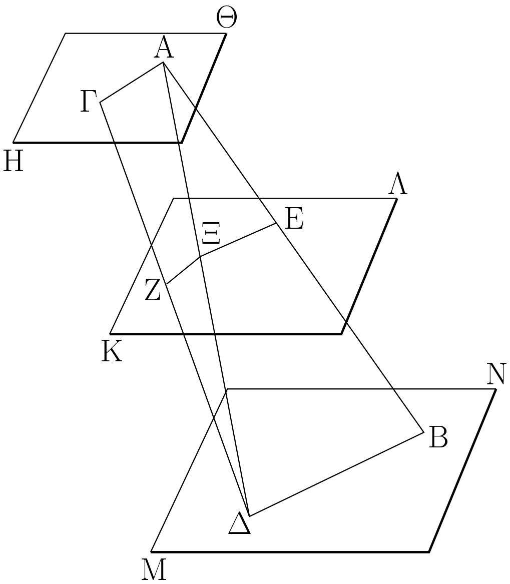
If two straight-lines are cut by parallel planes then they will be cut in the same ratios.

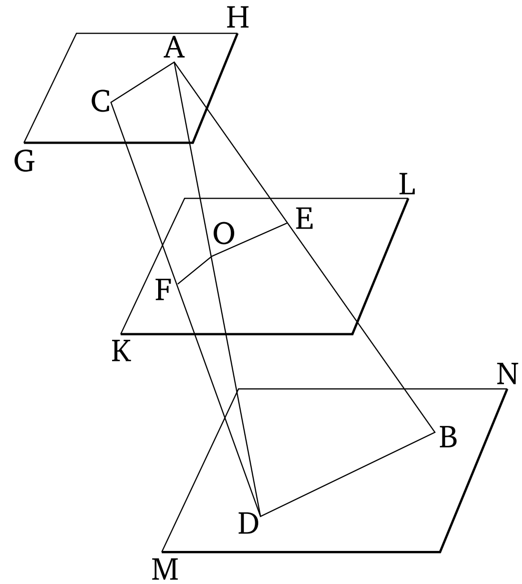
For let the two straight-lines AB and CD be cut by the parallel planes GH, KL, and MN at the points A, E, B, and C, F, D (respectively). I say that as the straight-line AE is to EB, so CF (is) to FD.

For let AC, BD, and AD have been joined, and let AD meet the plane KL at point O, and let EO and OF have been joined.

And since two parallel planes KL and MN are cut by the plane $EBDO$, their common sections EO and BD are parallel [Prop. 11.16]. So, for the same (reasons), since two parallel planes GH and KL are cut by the plane $AOFC$, their common sections AC and OF are parallel [Prop. 11.16]. And since the straight-line EO has been drawn parallel to one of the sides BD of trianἡ ΑΞ πρὸς ΞΔ. πάλιν ἐπεὶ τριγώνου τοῦ ΑΔΓ παρὰ μίαν τῶν πλευρῶν τὴν ΑΓ εὐθεῖα ἦκται ἡ ΞΖ, ἀνάλογόν ἐστιν ὡς ἡ ΑΞ πρὸς ΞΔ, οὕτως ἡ ΓΖ πρὸς ΖΔ. ἐδείχθη δὲ καὶ ὡς ἡ ΑΞ πρὸς ΞΔ, οὕτως ἡ ΑΕ πρὸς ΕΒ· καὶ ὡς ἄρα ἡ ΑΕ πρὸς ΕΒ, οὕτως ἡ ΓΖ πρὸς ΖΔ.

gle ABD, thus, proportionally, as AE is to EB, so AO (is) to OD [Prop. 6.2]. Again, since the straight-line OF has been drawn parallel to one of the sides AC of triangle ADC, proportionally, as AO is to OD, so CF (is) to FD [Prop. 6.2]. And it was also shown that as AO (is) to OD, so AE (is) to EB. And thus as AE (is) to EB, so CF (is) to FD [Prop. 5.11].





Ἐὰν ἄρα δύο εὐθεῖαι ὑπὸ παραλλήλων ἐπιπέδων τέμνωνται, εἰς τοὺς αὐτοὺς λόγους τμηθήσονται· ὅπερ ἔδει δεῖξαι.

Thus, if two straight-lines are cut by parallel planes then they will be cut in the same ratios. (Which is) the very thing it was required to show.

ιη΄.

Ἐὰν εὐθεῖα ἐπιπέδῳ τινὶ πρὸς ὀρθὰς ᾖ, καὶ πάντα τὰ δι᾿ αὐτῆς ἐπίπεδα τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται.

Εὐθεῖα γάρ τις ἡ ΑΒ τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθὰς ἔστω· λέγω, ὅτι καὶ πάντα τὰ διὰ τῆς ΑΒ ἐπίπεδα τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν.

Ἐκβεβλήσθω γὰρ διὰ τῆς ΑΒ ἐπίπεδον τὸ ΔΕ, καὶ ἔστω κοινὴ τομὴ τοῦ ΔΕ ἐπιπέδου καὶ τοῦ ὑποκειμένου ἡ ΓΕ, καὶ εἰλήφθω ἐπὶ τῆς ΓΕ τυχὸν σημεῖον τὸ Ζ, καὶ ἀπὸ τοῦ Ζ τῇ ΓΕ πρὸς ὀρθὰς ἤχθω ἐν τῷ ΔΕ ἐπιπέδῳ ἡ ΖΗ.

Καὶ ἐπεὶ ἡ ΑΒ πρὸς τὸ ὑποκείμενον ἐπίπεδον ὀρθή ἐστιν, καὶ πρὸς πάσας ἄρα τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ ὀρθή ἐστιν ἡ ΑΒ· ὥστε καὶ πρὸς τὴν ΓΕ ὀρθή ἐστιν· ἡ ἄρα ὑπὸ ΑΒΖ γωνία ὀρθή ἐστιν. ἔστι δὲ καὶ ἡ ὑπὸ ΗΖΒ ὀρθή· παράλληλος ἄρα ἐστὶν ἡ ΑΒ τῇ ΖΗ. ἡ δὲ ΑΒ τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν· καὶ ἡ ΖΗ ἄρα τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. καὶ ἐπίπεδον πρὸς ἐπίπεδον ὀρθόν ἐστιν, ὅταν αἱ τῇ κοινῇ τομῇ τῶν ἐπιπέδων πρὸς ὀρθὰς ἀγόμεναι εὐθεῖαι ἐν ἑνὶ τῶν ἐπιπέδων τῷ λοιπῷ ἐπιπέδῳ πρὸς ὀρθὰς ὦσιν. καὶ τῇ κοινῇ τομῇ τῶν ἐπιπέδων τῇ ΓΕ ἐν ἑνὶ τῶν ἐπιπέδων

Proposition 18

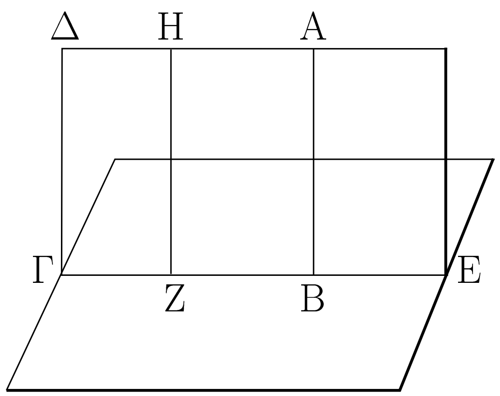
If a straight-line is at right-angles to some plane then all of the planes (passing) through it will also be at right-angles to the same plane.

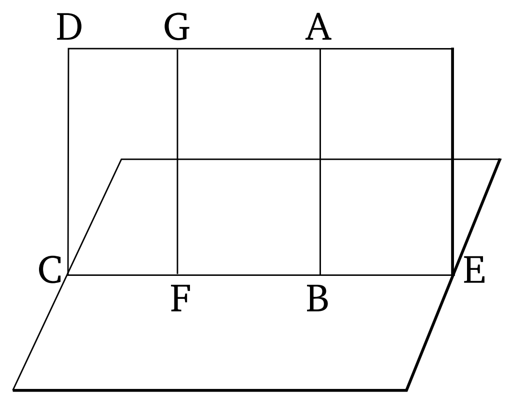
For let some straight-line AB be at right-angles to a reference plane. I say that all of the planes (passing) through AB are also at right-angles to the reference plane.

For let the plane DE have been produced through AB. And let CE be the common section of the plane DE and the reference (plane). And let some random point F have been taken on CE. And let FG have been drawn from F, at right-angles to CE, in the plane DE [Prop. 1.11].

And since AB is at right-angles to the reference plane, AB is thus also at right-angles to all of the straight-lines joined to it which are also in the reference plane [Def. 11.3]. Hence, it is also at right-angles to CE. Thus, angle ABF is a right-angle. And GFB is also a right-angle. Thus, AB is parallel to FG [Prop. 1.28]. And AB is at right-angles to the reference plane. Thus, FG is also τῷ ΔΕ πρὸς ὀρθὰς ἀχθεῖσα ἡ ΖΗ ἐδείχθη τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς· τὸ ἄρα ΔΕ ἐπίπεδον ὀρθόν ἐστι πρὸς τὸ ὑποκείμενον. ὁμοίως δὴ δειχθήσεται καὶ πάντα τὰ διὰ τῆς ΑΒ ἐπίπεδα ὀρθὰ τυγχανοντα πρὸς τὸ ὑποκείμενον ἐπίπεδον.

at right-angles to the reference plane [Prop. 11.8]. And a plane is at right-angles to a(nother) plane when the straight-lines drawn at right-angles to the common section of the planes, (and lying) in one of the planes, are at right-angles to the remaining plane [Def. 11.4]. And FG, (which was) drawn at right-angles to the common section of the planes, CE, in one of the planes, DE, was shown to be at right-angles to the reference plane. Thus, plane DE is at right-angles to the reference (plane). So, similarly, it can be shown that all of the planes (passing) at random through AB (are) at right-angles to the reference plane.





Ἐὰν ἄρα εὐθεῖα ἐπιπέδῳ τινὶ πρὸς ὀρθὰς ᾖ, καὶ πάντα τὰ δι᾿ αὐτῆς ἐπίπεδα τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται· ὅπερ ἔδει δεῖξαι.

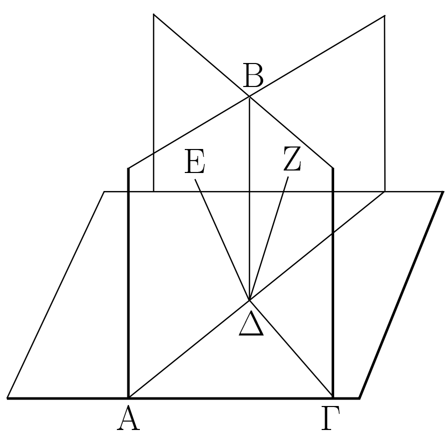
Thus, if a straight-line is at right-angles to some plane then all of the planes (passing) through it will also be at right-angles to the same plane. (Which is) the very thing it was required to show.

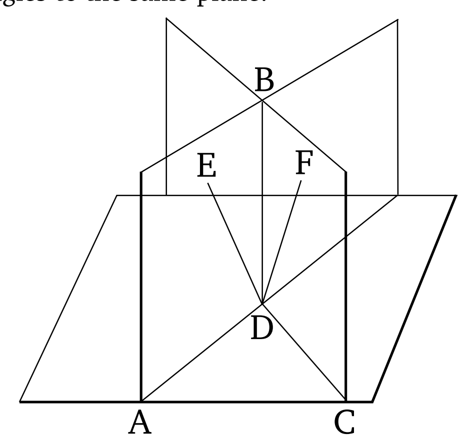
ιθ΄.

Ἐὰν δύο ἐπίπεδα τέμνοντα ἄλληλα ἐπιπέδῳ τινὶ πρὸς ὀρθὰς ᾖ, καὶ ἡ κοινὴ αὐτῶν τομὴ τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται.

Proposition 19

If two planes cutting one another are at right-angles to some plane then their common section will also be at right-angles to the same plane.





Δύο γὰρ ἐπίπεδα τὰ ΑΒ, ΒΓ τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθὰς ἔστω, κοινὴ δὲ αὐτῶν τομὴ ἔστω ἡ ΒΔ· λέγω, ὅτι ἡ ΒΔ τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν.

For let the two planes AB and BC be at right-angles to a reference plane, and let their common section be BD. I say that BD is at right-angles to the reference

Μὴ γάρ, καὶ ἤχθωσαν ἀπὸ τοῦ Δ σημείου ἐν μὲν τῷ ΑΒ ἐπιπέδῳ τῇ ΑΔ εὐθείᾳ πρὸς ὀρθὰς ἡ ΔΕ, ἐν δὲ τῷ ΒΓ ἐπιπέδῳ τῇ ΓΔ πρὸς ὀρθὰς ἡ ΔΖ.

Καὶ ἐπεὶ τὸ ΑΒ ἐπίπεδον ὀρθόν ἐστι πρὸς τὸ ὑποκείμενον, καὶ τῇ κοινῇ αὐτῶν τομῇ τῇ ΑΔ πρὸς ὀρθὰς ἐν τῷ ΑΒ ἐπιπέδῳ ἦκται ἡ ΔΕ, ἡ ΔΕ ἄρα ὀρθή ἐστι πρὸς τὸ ὑποκείμενον ἐπίπεδον. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἡ ΔΖ ὀρθή ἐστι πρὸς τὸ ὑποκείμενον ἐπίπεδον. ἀπὸ τοῦ αὐτοῦ ἄρα σημείου τοῦ Δ τῷ ὑποκειμένῳ ἐπιπέδῳ δύο εὐθεῖαι πρὸς ὀρθὰς ἀνεσταμέναι εἰσὶν ἐπὶ τὰ αὐτὰ μέρη· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τῷ ὑποκειμένῳ ἐπιπέδῳ ἀπὸ τοῦ Δ σημείου ἀνασταθήσεται πρὸς ὀρθὰς πλὴν τῆς ΔΒ κοινῆς τομῆς τῶν ΑΒ, ΒΓ ἐπιπέδων.

Ἐὰν ἄρα δύο ἐπίπεδα τέμνοντα ἄλληλα ἐπιπέδῳ τινὶ πρὸς ὀρθὰς ᾖ, καὶ ἡ κοινὴ αὐτῶν τομὴ τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται· ὅπερ ἔδει δεῖξαι.

plane.

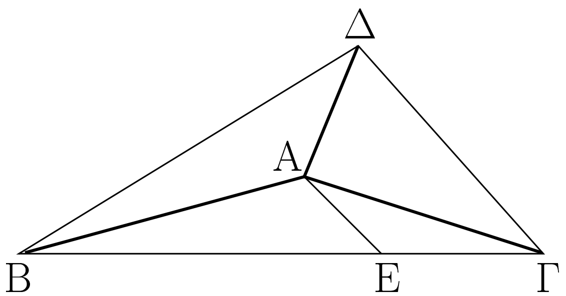
For (if) not, let DE also have been drawn from point D, in the plane AB, at right-angles to the straight-line AD, and DF, in the plane BC, at right-angles to CD.

And since the plane AB is at right-angles to the reference (plane), and DE has been drawn at right-angles to their common section AD, in the plane AB, DE is thus at right-angles to the reference plane [Def. 11.4]. So, similarly, we can show that DF is also at right-angles to the reference plane. Thus, two (different) straight-lines are set up, at the same point D, at right-angles to the reference plane, on the same side. The very thing is impossible [Prop. 11.13]. Thus, no (other straight-line) except the common section DB of the planes AB and BC can be set up at point D, at right-angles to the reference plane.

Thus, if two planes cutting one another are at right-angles to some plane then their common section will also be at right-angles to the same plane. (Which is) the very thing it was required to show.

κ΄.

Ἐὰν στερεὰ γωνία ὑπὸ τριῶν γωνιῶν ἐπιπέδων περιέχηται, δύο ὁποιαιοῦν τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι.



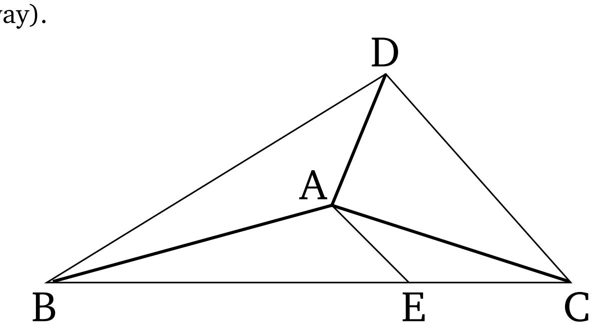
Στερεὰ γὰρ γωνία ἡ πρὸς τῷ Α ὑπὸ τριῶν γωνιῶν ἐπιπέδων τῶν ὑπὸ ΒΑΓ, ΓΑΔ, ΔΑΒ περιεχέσθω· λέγω, ὅτι τῶν ὑπὸ ΒΑΓ, ΓΑΔ, ΔΑΒ γωνιῶν δύο ὁποιαιοῦν τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι.

Εἰ μὲν οὖν αἱ ὑπὸ ΒΑΓ, ΓΑΔ, ΔΑΒ γωνίαι ἴσαι ἀλλήλαις εἰσίν, φανερόν, ὅτι δύο ὁποιαιοῦν τῆς λοιπῆς μείζονές εἰσιν. εἰ δὲ οὔ, ἔστω μείζων ἡ ὑπὸ ΒΑΓ, καὶ συνεστάτω πρὸς τῇ ΑΒ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α τῇ ὑπὸ ΔΑΒ γωνίᾳ ἐν τῷ διὰ τῶν ΒΑΓ ἐπιπέδῳ ἴση ἡ ὑπὸ ΒΑΕ, καὶ κείσθω τῇ ΑΔ ἴση ἡ ΑΕ, καὶ διὰ τοῦ Ε σημείου διαχθεῖσα ἡ ΒΕΓ τεμνέτω τὰς ΑΒ, ΑΓ εὐθείας κατὰ τὰ Β, Γ σημεῖα, καὶ ἐπεζεύχθωσαν αἱ ΔΒ, ΔΓ.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΔΑ τῇ ΑΕ, κοινὴ δὲ ἡ ΑΒ, δύο δυσὶν ἴσαι· καὶ γωνία ἡ ὑπὸ ΔΑΒ γωνίᾳ τῇ ὑπὸ ΒΑΕ ἴση· βάσις ἄρα ἡ ΔΒ βάσει τῇ ΒΕ ἐστιν ἴση. καὶ ἐπεὶ δύο αἱ ΒΔ, ΔΓ τῆς ΒΓ μείζονές εἰσιν, ὧν ἡ ΔΒ τῇ ΒΕ ἐδείχθη ἴση,

Proposition 20

If a solid angle is contained by three plane angles then (the sum of) any two (angles) is greater than the remaining (one), (the angles) being taken up in any (possible way).



For let the solid angle A have been contained by the three plane angles BAC, CAD, and DAB. I say that (the sum of) any two of the angles BAC, CAD, and DAB is greater than the remaining (one), (the angles) being taken up in any (possible way).

For if the angles BAC, CAD, and DAB are equal to one another then (it is) clear that (the sum of) any two is greater than the remaining (one). But, if not, let BAC be greater (than CAD or DAB). And let (angle) BAE, equal to the angle DAB, have been constructed in the plane through BAC, on the straight-line AB, at the point A on it. And let AE be made equal to AD. And BEC being drawn across through point E, let it cut the straight-lines AB and AC at points B and C (respectively). And let DB and DC have been joined.

And since DA is equal to AE, and AB (is) common, λοιπὴ ἄρα ἡ ΔΓ λοιπῆς τῆς ΕΓ μείζων ἐστίν. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΔΑ τῇ ΑΕ, κοινὴ δὲ ἡ ΑΓ, καὶ βάσις ἡ ΔΓ βάσεως τῆς ΕΓ μείζων ἐστίν, γωνία ἄρα ἡ ὑπὸ ΔΑΓ γωνάις τῆς ὑπὸ ΕΑΓ μείζων ἐστίν. ἐδείχθη δὲ καὶ ἡ ὑπὸ ΔΑΒ τῇ ὑπὸ ΒΑΕ ἴση· αἱ ἄρα ὑπὸ ΔΑΒ, ΔΑΓ τῆς ὑπὸ ΒΑΓ μείζονές εἰσιν. ὁμοίως δὴ δείξομεν, ὅτι καὶ αἱ λοιπαὶ σύνδυο λαμβανόμεναι τῆς λοιπῆς μείζονές εἰσιν.

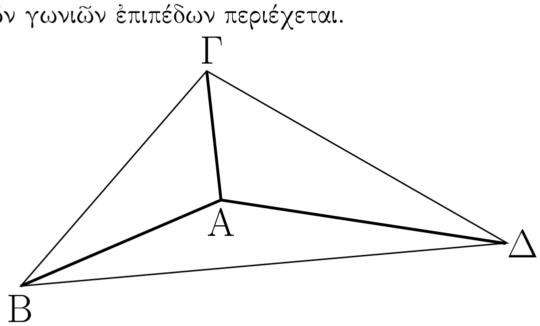
Ἐὰν ἄρα στερεὰ γωνία ὑπὸ τριῶν γωνιῶν ἐπιπέδων περιέχηται, δύο ὁποιαιοῦν τῆς λοιπῆς μείζονές εἰσι πάντη μεταλαμβανόμεναι· ὅπερ ἔδει δεῖξαι.

the two (straight-lines AD and AB are) equal to the two (straight-lines EA and AB, respectively). And angle DAB (is) equal to angle BAE. Thus, the base DB is equal to the base BE [Prop. 1.4]. And since the (sum of the) two (straight-lines) BD and DC is greater than BC [Prop. 1.20], of which DB was shown (to be) equal to BE, the remainder DC is thus greater than the remainder EC. And since DA is equal to AE, but AC (is) common, and the base DC is greater than the base EC, the angle DAC is thus greater than the angle EAC [Prop. 1.25]. And DAB was also shown (to be) equal to BAE. Thus, (the sum of) DAB and DAC is greater than BAC. So, similarly, we can also show that the remaining (angles), being taken in pairs, are greater than the remaining (one).

Thus, if a solid angle is contained by three plane angles then (the sum of) any two (angles) is greater than the remaining (one), (the angles) being taken up in any (possible way). (Which is) the very thing it was required to show.

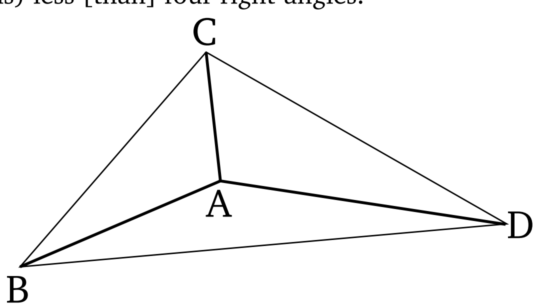
κα΄.

Ἅπασα στερεὰ γωνία ὑπὸ ἐλασσόνων [ἢ] τεσσάρων ὀρθῶν γωνιῶν ἐπιπέδων περιέχεται.



Proposition 21

Any solid angle is contained by plane angles (whose sum is) less [than] four right-angles.†



Ἔστω στερεὰ γωνία ἡ πρὸς τῷ Α περιεχομένη ὑπὸ ἐπιπέδων γωνιῶν τῶν ὑπὸ ΒΑΓ, ΓΑΔ, ΔΑΒ· λέγω, ὅτι αἱ ὑπὸ ΒΑΓ, ΓΑΔ, ΔΑΒ τεσσάρων ὀρθῶν ἐλάσσονές εἰσιν.

Εἰλήφθω γὰρ ἐφ᾿ ἑκάστης τῶν ΑΒ, ΑΓ, ΑΔ τυχόντα σημεῖα τὰ Β, Γ, Δ, καὶ ἐπεζεύχθωσαν αἱ ΒΓ, ΓΔ, ΔΒ. καὶ ἐπεὶ στερεὰ γωνία ἡ πρὸς τῷ Β ὑπὸ τριῶν γωνιῶν ἐπιπέδων περιέχεται τῶν ὑπὸ ΓΒΑ, ΑΒΔ, ΓΒΔ, δύο ὁποιαιοῦν τῆς λοιπῆς μείζονές εἰσιν· αἱ ἄρα ὑπὸ ΓΒΑ, ΑΒΔ τῆς ὑπὸ ΓΒΔ μείζονές εἰσιν. διὰ τὰ αὐτὰ δὴ καὶ αἱ μὲν ὑπὸ ΒΓΑ, ΑΓΔ τῆς ὑπὸ ΒΓΔ μείζονές εἰσιν, αἱ δὲ ὑπὸ ΓΔΑ, ΑΔΒ τῆς ὑπὸ ΓΔΒ μείζονές εἰσιν· αἱ ἓξ ἄρα γωνίαι αἱ ὑπὸ ΓΒΑ, ΑΒΔ, ΒΓΑ, ΑΓΔ, ΓΔΑ, ΑΔΒ τριῶν τῶν ὑπὸ ΓΒΔ, ΒΓΑ, ΓΔΒ μείζονές εἰσιν. ἀλλὰ αἱ τρεῖς αἱ ὑπὸ ΓΒΔ, ΒΔΓ, ΒΓΔ δυσὶν ὀρθαῖς ἴσαι εἰσίν· αἱ ἓξ ἄρα αἱ ὑπὸ ΓΒΑ, ΑΒΔ, ΒΓΑ, ΑΓΔ, ΓΔΑ, ΑΔΒ δύο ὀρθῶν μείζονές εἰσιν. καὶ ἐπεὶ ἑκάστου τῶν ΑΒΓ, ΑΓΔ, ΑΔΒ τριγώνων αἱ τρεῖς γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν, αἱ ἄρα τῶν τριῶν τριγώνων ἐννέα γωνίαι αἱ ὑπὸ

Let the solid angle A be contained by the plane angles BAC, CAD, and DAB. I say that (the sum of) BAC, CAD, and DAB is less than four right-angles.

For let the random points B, C, and D have been taken on each of (the straight-lines) AB, AC, and AD (respectively). And let BC, CD, and DB have been joined. And since the solid angle at B is contained by the three plane angles CBA, ABD, and CBD, (the sum of) any two is greater than the remaining (one) [Prop. 11.20]. Thus, (the sum of) CBA and ABD is greater than CBD. So, for the same (reasons), (the sum of) BCA and ACD is also greater than BCD, and (the sum of) CDA and ADB is greater than CDB. Thus, the (sum of the) six angles CBA, ABD, BCA, ACD, CDA, and ADB is greater than the (sum of the) three (angles) CBD, BCD, and CDB. But, the (sum of the) three (angles) CBD, BDC, and BCD is equal to two

ΓΒΑ, ΑΓΒ, ΒΑΓ, ΑΓΔ, ΓΔΑ, ΓΑΔ, ΑΔΒ, ΔΒΑ, ΒΑΔ ἓξ ὀρθαῖς ἴσαι εἰσίν, ὧν αἱ ὑπὸ ΑΒΓ, ΒΓΑ, ΑΓΔ, ΓΔΑ, ΑΔΒ, ΔΒΑ ἓξ γωνίαι δύο ὀρθῶν εἰσι μείζονες· λοιπαὶ ἄρα αἱ ὑπὸ ΒΑΓ, ΓΑΔ, ΔΑΒ τρεῖς [γωνίαι] περιέχουσαι τὴν στερεὰν γωνίαν τεσσάρων ὀρθῶν ἐλάσσονές εἰσιν.

Ἅπασα ἄρα στερεὰ γωνία ὑπὸ ἐλασσόνων [ἢ] τεσσάρων ὀρθῶν γωνιῶν ἐπιπέδων περιέχεται· ὅπερ ἔδει δεῖξαι.

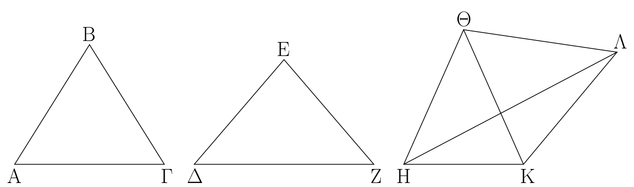
right-angles [Prop. 1.32]. Thus, the (sum of the) six angles CBA, ABD, BCA, ACD, CDA, and ADB is greater than two right-angles. And since the (sum of the) three angles of each of the triangles ABC, ACD, and ADB is equal to two right-angles, the (sum of the) nine angles CBA, ACB, BAC, ACD, CDA, CAD, ADB, DBA, and BAD of the three triangles is equal to six right-angles, of which the (sum of the) six angles ABC, BCA, ACD, CDA, ADB, and DBA is greater than two right-angles. Thus, the (sum of the) remaining three [angles] BAC, CAD, and DAB, containing the solid angle, is less than four right-angles.

Thus, any solid angle is contained by plane angles (whose sum is) less [than] four right-angles. (Which is) the very thing it was required to show.

† This proposition is only proved for the case of a solid angle contained by three plane angles. However, the generalization to a solid angle contained by more than three plane angles is straightforward.

κβʹ.

Ἐὰν ὦσι τρεῖς γωνίαι ἐπίπεδοι, ὧν αἱ δύο τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι, περιέχωσι δὲ αὐτὰς ἴσαι εὐθεῖαι, δυνατόν ἐστιν ἐκ τῶν ἐπιζευγνυουσῶν τὰς ἴσας εὐθείας τρίγωνον συστήσασθαι.

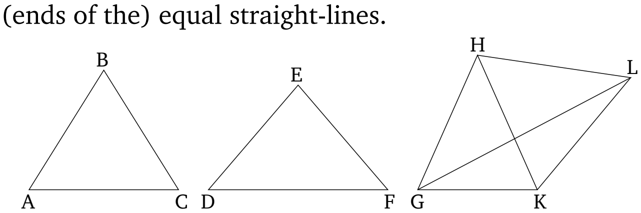


Ἔστωσαν τρεῖς γωνίαι ἐπίπεδοι αἱ ὑπὸ ΑΒΓ, ΔΕΖ, ΗΘΚ, ὧν αἱ δύο τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι, αἱ μὲν ὑπὸ ΑΒΓ, ΔΕΖ τῆς ὑπὸ ΗΘΚ, αἱ δὲ ὑπὸ ΔΕΖ, ΗΘΚ τῆς ὑπὸ ΑΒΓ, καὶ ἔτι αἱ ὑπὸ ΗΘΚ, ΑΒΓ τῆς ὑπὸ ΔΕΖ, καὶ ἔστωσαν ἴσαι αἱ ΑΒ, ΒΓ, ΔΕ, ΕΖ, ΗΘ, ΘΚ εὐθεῖαι, καὶ ἐπεζεύχθωσαν αἱ ΑΓ, ΔΖ, ΗΚ· λέγω, ὅτι δυνατόν ἐστιν ἐκ τῶν ἴσων ταῖς ΑΓ, ΔΖ, ΗΚ τρίγωνον συστήσασθαι, τουτέστιν ὅτι τῶν ΑΓ, ΔΖ, ΗΚ δύο ὁποιαιοῦν τῆς λοιπῆς μείζονές εἰσιν.

Εἰ μὲν οὖν αἱ ὑπὸ ΑΒΓ, ΔΕΖ, ΗΘΚ γωνίαι ἴσαι ἀλλήλαις εἰσίν, φανερόν, ὅτι καὶ τῶν ΑΓ, ΔΖ, ΗΚ ἴσων γινομένων δυνατόν ἐστιν ἐκ τῶν ἴσων ταῖς ΑΓ, ΔΖ, ΗΚ τρίγωνον συστήσασθαι. εἰ δὲ οὔ, ἔστωσαν ἄνισοι, καὶ συνεστάτω πρὸς τῇ ΘΚ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Θ τῇ ὑπὸ ΑΒΓ γωνίᾳ ἴση ἡ ὑπὸ ΚΘΛ· καὶ κείσθω μιᾷ τῶν ΑΒ, ΒΓ, ΔΕ, ΕΖ, ΗΘ, ΘΚ ἴση ἡ ΘΛ, καὶ ἐπεζεύχθωσαν αἱ ΚΛ, ΗΛ. καὶ ἐπεὶ δύο αἱ ΑΒ, ΒΓ δυσὶ ταῖς ΚΘ, ΘΛ ἴσαι εἰσίν, καὶ γωνία ἡ πρὸς τῷ Β γωνίᾳ τῇ ὑπὸ ΚΘΛ ἴση, βάσις ἄρα ἡ ΑΓ βάσει τῇ ΚΛ ἴση. καὶ ἐπεὶ αἱ ὑπὸ ΑΒΓ, ΗΘΚ τῆς

Proposition 22

If there are three plane angles, of which (the sum of any) two is greater than the remaining (one), (the angles) being taken up in any (possible way), and if equal straight-lines contain them, then it is possible to construct a triangle from (the straight-lines created by) joining the (ends of the) equal straight-lines.



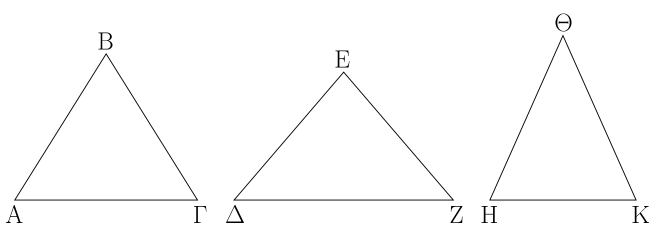
Let ABC, DEF, and GHK be three plane angles, of which the sum of any) two is greater than the remaining (one), (the angles) being taken up in any (possible way)—(that is), ABC and DEF (greater) than GHK, DEF and GHK (greater) than ABC, and, further, GHK and ABC (greater) than DEF. And let AB, BC, DE, EF, GH, and HK be equal straight-lines. And let AC, DF, and GK have been joined. I say that that it is possible to construct a triangle out of (straight-lines) equal to AC, DF, and GK—that is to say, that (the sum of) any two of AC, DF, and GK is greater than the remaining (one).

Now, if the angles ABC, DEF, and GHK are equal to one another then (it is) clear that, (with) AC, DF, and GK also becoming equal, it is possible to construct a triangle from (straight-lines) equal to AC, DF, and GK. And if not, let them be unequal, and let KHL, equal to angle ABC, have been constructed on the straight-line HK, at the point H on it. And let HL be made equal to ὑπὸ ΔΕΖ μείζονές εἰσιν, ἴση δὲ ἡ ὑπὸ ΑΒΓ τῇ ὑπὸ ΚΘΛ, ἡ ἄρα ὑπὸ ΗΘΛ τῆς ὑπὸ ΔΕΖ μείζων ἐστίν. καὶ ἐπεὶ δύο αἱ ΗΘ, ΘΛ δύο ταῖς ΔΕ, ΕΖ ἴσαι εἰσίν, καὶ γωνία ἡ ὑπὸ ΗΘΛ γωνίας τῆς ὑπὸ ΔΕΖ μείζων, βάσις ἄρα ἡ ΗΛ βάσεως τῆς ΔΖ μείζων ἐστίν. ἀλλὰ αἱ ΗΚ, ΚΛ τῆς ΗΛ μείζονές εἰσιν. πολλῷ ἄρα αἱ ΗΚ, ΚΛ τῆς ΔΖ μείζονές εἰσιν. ἴση δὲ ἡ ΚΛ τῇ ΑΓ· αἱ ΑΓ, ΗΚ ἄρα τῆς λοιπῆς τῆς ΔΖ μείζονές εἰσιν. ὁμοίως δὴ δείξομεν, ὅτι καὶ αἱ μὲν ΑΓ, ΔΖ τῆς ΗΚ μείζονές εἰσιν, καὶ ἔτι αἱ ΔΖ, ΗΚ τῆς ΑΓ μείζονές εἰσιν. δυνατὸν ἄρα ἐστὶν ἐκ τῶν ἴσων ταῖς ΑΓ, ΔΖ, ΗΚ τρίγωνον συστήσασθαι· ὅπερ ἔδει δεῖξαι.

one of AB, BC, DE, EF, GH, and HK. And let KL and GL have been joined. And since the two (straight-lines) AB and BC are equal to the two (straight-lines) KH and HL (respectively), and the angle at B (is) equal to KHL, the base AC is thus equal to the base KL [Prop. 1.4]. And since (the sum of) ABC and GHK is greater than DEF, and ABC equal to KHL, GHL is thus greater than DEF. And since the two (straight-lines) GH and HL are equal to the two (straight-lines) DE and EF (respectively), and angle GHL (is) greater than DEF, the base GL is thus greater than the base DF [Prop. 1.24]. But, (the sum of) GK and KL is greater than GL [Prop. 1.20]. Thus, (the sum of) GK and KL is much greater than DF. And KL (is) equal to AC. Thus, (the sum of) AC and GK is greater than the remaining (straight-line) DF. So, similarly, we can show that (the sum of) AC and DF is greater than GK, and, further, that (the sum of) DF and GK is greater than AC. Thus, it is possible to construct a triangle from (straight-lines) equal to AC, DF, and GK. (Which is) the very thing it was required to show.

κγ΄.

Ἐκ τριῶν γωνιῶν ἐπιπέδων, ὧν αἱ δύο τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι, στερεὰν γωνίαν συστήσασθαι· δεῖ δὴ τὰς τρεῖς τεσσάρων ὀρθῶν ἐλάσσονας εἶναι.

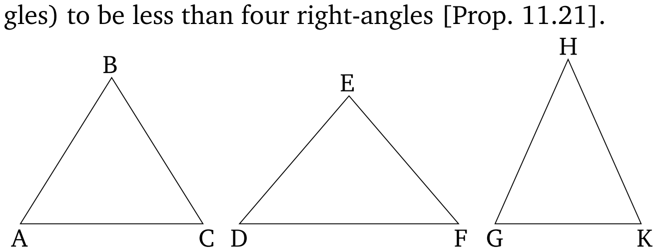


Ἔστωσαν αἱ δοθεῖσαι τρεῖς γωνίαι ἐπίπεδοι αἱ ὑπὸ ΑΒΓ, ΔΕΖ, ΗΘΚ, ὧν αἱ δύο τῆς λοιπῆς μείζονες ἔστωσαν πάντῃ μεταλαμβανόμεναι, ἔτι δὲ αἱ τρεῖς τεσσάρων ὀρθῶν ἐλάσσονες· δεῖ δὴ ἐκ τῶν ἴσων ταῖς ὑπὸ ΑΒΓ, ΔΕΖ, ΗΘΚ στερεὰν γωνίαν συστήσασθαι.

Ἀπειλήφθωσαν ἴσαι αἱ ΑΒ, ΒΓ, ΔΕ, ΕΖ, ΗΘ, ΘΚ, καὶ ἐπεζεύχθωσαν αἱ ΑΓ, ΔΖ, ΗΚ· δυνατὸν ἄρα ἐστὶν ἐκ τῶν ἴσων ταῖς ΑΓ, ΔΖ, ΗΚ τρίγωνον συστήσασθαι. συνεστάτω τὸ ΛΜΝ, ὥστε ἴσην εἶναι τὴν μὲν ΑΓ τῇ ΛΜ, τὴν δὲ ΔΖ τῇ ΜΝ, καὶ ἔτι τὴν ΗΚ τῇ ΝΛ, καὶ περιγεγράφθω περὶ τὸ ΛΜΝ τρίγωνον κύκλος ὁ ΛΜΝ, καὶ εἰλήφθω αὐτοῦ τὸ κέντρον καὶ ἔστω τὸ Ξ, καὶ ἐπεζεύχθωσαν αἱ ΛΞ, ΜΞ, ΝΞ·

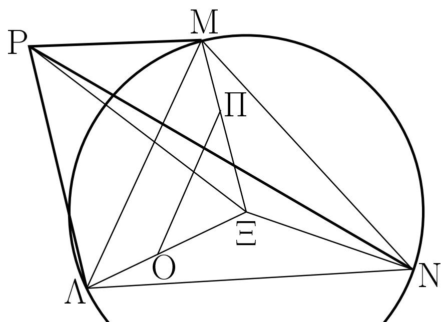
Proposition 23

To construct a solid angle from three (given) plane angles, (the sum of) two of which is greater than the remaining (one, the angles) being taken up in any (possible way). So, it is necessary for the (sum of the) three (angles) to be less than four right-angles [Prop. 11.21].



Let ABC, DEF, and GHK be the three given plane angles, of which let (the sum of) two be greater than the remaining (one, the angles) being taken up in any (possible way), and, further, (let) the (sum of the) three (be) less than four right-angles. So, it is necessary to construct a solid angle from (plane angles) equal to ABC, DEF, and GHK.

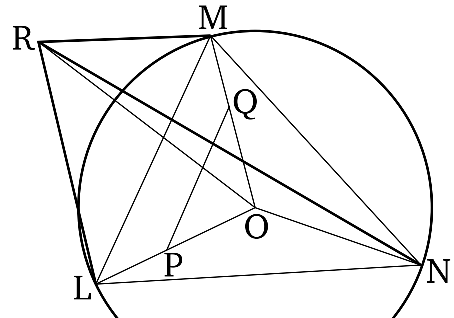
Let AB, BC, DE, EF, GH, and HK be cut off (so as to be) equal (to one another). And let AC, DF, and GK have been joined. It is, thus, possible to construct a triangle from (straight-lines) equal to AC, DF, and GK [Prop. 11.22]. Let (such a triangle), LMN, have be constructed, such that AC is equal to LM, DF to MN, and, further, GK to NL. And let the circle LMN have been circumscribed about triangle LMN [Prop. 4.5]. And let



Λέγω, ὅτι ἡ ΑΒ μείζων ἐστὶ τῆς ΛΞ. εἰ γὰρ μή, ἤτοι ἴση ἐστὶν ἡ ΑΒ τῇ ΛΞ ἢ ἐλάττων. ἔστω πρότερον ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΒ τῇ ΛΞ, ἀλλὰ ἡ μὲν ΑΒ τῇ ΒΓ ἐστιν ἴση, ἡ δὲ ΞΛ τῇ ΞΜ, δύο δὴ αἱ ΑΒ, ΒΓ δύο ταῖς ΛΞ, ΞΜ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ βάσις ἡ ΑΓ βάσει τῇ ΛΜ ὑπόκειται ἴση· γωνία ἄρα ἡ ὑπὸ ΑΒΓ γωνίᾳ τῇ ὑπὸ ΛΞΜ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ μὲν ὑπὸ ΔΕΖ τῇ ὑπὸ ΜΞΝ ἐστιν ἴση, καὶ ἔτι ἡ ὑπὸ ΗΘΚ τῇ ὑπὸ ΝΞΛ· αἱ ἄρα τρεῖς αἱ ὑπὸ ΑΒΓ, ΔΕΖ, ΗΘΚ γωνίαι τρισὶ ταῖς ὑπὸ ΛΞΜ, ΜΞΝ, ΝΞΛ εἰσιν ἴσαι. ἀλλὰ αἱ τρεῖς αἱ ὑπὸ ΛΞΜ, ΜΞΝ, ΝΞΛ τέτταρσιν ὀρθαῖς εἰσιν ἴσαι· καὶ αἱ τρεῖς ἄρα αἱ ὑπὸ ΑΒΓ, ΔΕΖ, ΗΘΚ τέτταρσιν ὀρθαῖς ἴσαι εἰσίν. ὑπόκεινται δὲ καὶ τεσσάρων ὀρθῶν ἐλάσσονες· ὅπερ ἄτοπον. οὐκ ἄρα ἡ ΑΒ τῇ ΛΞ ἴση ἐστίν. λέγω δή, ὅτι οὐδὲ ἐλάττων ἐστὶν ἡ ΑΒ τῆς ΛΞ. εἰ γὰρ δυνατόν, ἔστω· καὶ κείσθω τῇ μὲν ΑΒ ἴση ἡ ΞΟ, τῇ δὲ ΒΓ ἴση ἡ ΞΠ, καὶ ἐπεζεύχθω ἡ ΟΠ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΒ τῇ ΒΓ, ἴση ἐστὶ καὶ ἡ ΞΟ τῇ ΞΠ· ὥστε καὶ λοιπὴ ἡ ΛΟ τῇ ΠΜ ἐστιν ἴση. παράλληλος ἄρα ἐστὶν ἡ ΛΜ τῇ ΟΠ, καὶ ἰσογώνιον τὸ ΛΜΞ τῷ ΟΠΞ· ἔστιν ἄρα ὡς ἡ ΞΛ πρὸς ΛΜ, οὕτως ἡ ΞΟ πρὸς ΟΠ· ἐναλλὰξ ὡς ἡ ΛΞ πρὸς ΞΟ, οὕτως ἡ ΛΜ πρὸς ΟΠ. μείζων δὲ ἡ ΛΞ τῆς ΞΟ· μείζων ἄρα καὶ ἡ ΛΜ τῆς ΟΠ. ἀλλὰ ἡ ΛΜ κεῖται τῇ ΑΓ ἴση· καὶ ἡ ΑΓ ἄρα τῆς ΟΠ μείζων ἐστίν. ἐπεὶ οὖν δύο αἱ ΑΒ, ΒΓ δυσὶ ταῖς ΟΞ, ΞΠ ἴσαι εἰσίν, καὶ βάσις ἡ ΑΓ βάσεως τῆς ΟΠ μείζων ἐστίν, γωνία ἄρα ἡ ὑπὸ ΑΒΓ γωνίας τῆς ὑπὸ ΟΞΠ μείζων ἐστίν. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἡ μὲν ὑπὸ ΔΕΖ τῆς ὑπὸ ΜΞΝ μείζων ἐστίν, ἡ δὲ ὑπὸ ΗΘΚ τῆς ὑπὸ ΝΞΛ. αἱ ἄρα τρεῖς γωνίαι αἱ ὑπὸ ΑΒΓ, ΔΕΖ, ΗΘΚ τριῶν τῶν ὑπὸ ΛΞΜ, ΜΞΝ, ΝΞΛ μείζονές εἰσιν. ἀλλὰ αἱ ὑπὸ ΑΒΓ, ΔΕΖ, ΗΘΚ τεσσάρων ὀρθῶν ἐλάσσονες ὑπόκεινται· πολλῷ ἄρα αἱ ὑπὸ ΛΞΜ, ΜΞΝ, ΝΞΛ τεσσάρων ὀρθῶν ἐλάσσονές εἰσιν. ἀλλὰ καὶ ἴσαι· ὅπερ ἐστὶν ἄτοπον. οὐκ ἄρα ἡ ΑΒ ἐλάσσων ἐστὶ τῆς ΛΞ. ἐδείχθη δέ, ὅτι οὐδὲ ἴση· μείζων ἄρα ἡ ΑΒ τῆς ΛΞ.

Ἀνεστάτω δὴ ἀπὸ τοῦ Ξ σημείου τῷ τοῦ ΛΜΝ κύκλου ἐπιπέδῳ πρὸς ὀρθὰς ἡ ΞΡ, καὶ ᾧ μεῖζόν ἐστι τὸ ἀπὸ τῆς ΑΒ τετράγωνον τοῦ ἀπὸ τῆς ΛΞ, ἐκείνῳ ἴσον ἔστω τὸ ἀπὸ

its center have been found, and let it be (at) O. And let LO, MO, and NO have been joined.



I say that AB is greater than LO. For, if not, AB is either equal to, or less than, LO. Let it, first of all, be equal. And since AB is equal to LO, but AB is equal to BC, and OL to OM, so the two (straight-lines) AB and BC are equal to the two (straight-lines) LO and OM, respectively. And the base AC was assumed (to be) equal to the base LM. Thus, angle ABC is equal to angle LOM [Prop. 1.8]. So, for the same (reasons), DEF is also equal to MON, and, further, GHK to NOL. Thus, the three angles ABC, DEF, and GHK are equal to the three angles LOM, MON, and NOL, respectively. But, the (sum of the) three angles LOM, MON, and NOL is equal to four right-angles. Thus, the (sum of the) three angles ABC, DEF, and GHK is also equal to four right-angles. And it was also assumed (to be) less than four right-angles. The very thing (is) absurd. Thus, AB is not equal to LO. So, I say that AB is not less than LO either. For, if possible, let it be (less). And let OP be made equal to AB, and OQ equal to BC, and let PQ have been joined. And since AB is equal to BC, OP is also equal to OQ. Hence, the remainder LP is also equal to (the remainder) QM. LM is thus parallel to PQ [Prop. 6.2], and (triangle) LMO (is) equiangular with (triangle) PQO [Prop. 1.29]. Thus, as OL is to LM, so OP (is) to PQ [Prop. 6.4]. Alternately, as LO (is) to OP, so LM (is) to PQ [Prop. 5.16]. And LO (is) greater than OP. Thus, LM (is) also greater than PQ [Prop. 5.14]. But LM was made equal to AC. Thus, AC is also greater than PQ. Therefore, since the two (straight-lines) AB and BC are equal to the two (straight-lines) PO and OQ (respectively), and the base AC is greater than the base PQ, the angle ABC is thus greater than the angle POQ [Prop. 1.25]. So, similarly, we can show that DEF is also greater than MON, and GHK than NOL. Thus, the (sum of the) three angles ABC, DEF, and GHK is greater than the (sum of the) three angles LOM, MON, τῆς ΞΡ, καὶ ἐπεζεύχθωσαν αἱ ΡΛ, ΡΜ, ΡΝ.

Καὶ ἐπεὶ ἡ ΡΞ ὀρθή ἐστι πρὸς τὸ τοῦ ΛΜΝ κύκλου ἐπίπεδον, καὶ πρὸς ἑκάστην ἄρα τῶν ΛΞ, ΜΞ, ΝΞ ὀρθή ἐστιν ἡ ΡΞ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΛΞ τῇ ΞΜ, κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ ΞΡ, βάσις ἄρα ἡ ΡΛ βάσει τὴ ΡΜ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΡΝ ἑκατέρᾳ τῶν ΡΛ, ΡΜ ἐστιν ἴση· αἱ τρεῖς ἄρα αἱ ΡΛ, ΡΜ, ΡΝ ἴσαι ἀλλήλαις εἰσίν. καὶ ἐπεὶ ᾧ μεῖζόν ἐστι τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΛΞ, ἐκείνῳ ἴσον ὑπόκειται τὸ ἀπὸ τῆς ΞΡ, τὸ ἄρα ἀπὸ τῆς ΑΒ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΛΞ, ΞΡ. τοῖς δὲ ἀπὸ τῶν ΛΞ, ΞΡ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΛΡ· ὀρθὴ γὰρ ἡ ὑπὸ ΛΞΡ· τὸ ἄρα ἀπὸ τῆς ΑΒ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΡΛ· ἴση ἄρα ἡ ΑΒ τῇ ΡΛ. ἀλλὰ τῇ μὲν ΑΒ ἴση ἐστὶν ἑκάστη τῶν ΒΓ, ΔΕ, ΕΖ, ΗΘ, ΘΚ, τῇ δὲ ΡΛ ἴση ἑκατέρα τῶν ΡΜ, ΡΝ· ἑκάστη ἄρα τῶν ΑΒ, ΒΓ, ΔΕ, ΕΖ, ΗΘ, ΘΚ ἑκάστῃ τῶν ΡΛ, ΡΜ, ΡΝ ἴση ἐστίν. καὶ ἐπεὶ δύο αἱ ΛΡ, ΡΜ δυσὶ ταῖς ΑΒ, ΒΓ ἴσαι εἰσίν, καὶ βάσις ἡ ΛΜ βάσει τῇ ΑΓ ὑπόκειται ἴση, γωνία ἄρα ἡ ὑπὸ ΛΡΜ γωνίᾳ τῇ ὑπὸ ΑΒΓ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ μὲν ὑπὸ ΜΡΝ τῇ ὑπὸ ΔΕΖ ἐστιν ἴση, ἡ δὲ ὑπὸ ΛΡΝ τῇ ὑπὸ ΗΘΚ.

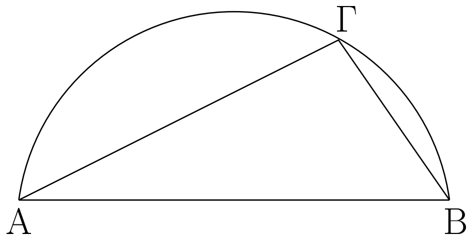
Ἐκ τριῶν ἄρα γωνιῶν ἐπιπέδων τῶν ὑπὸ ΛΡΜ, ΜΡΝ, ΛΡΝ, αἵ εἰσιν ἴσαι τρισὶ ταῖς δοθείσαις ταῖς ὑπὸ ΑΒΓ, ΔΕΖ, ΗΘΚ, στερεὰ γωνία συνέσταται ἡ πρὸς τῷ Ρ περιεχομένη ὑπὸ τῶν ΛΡΜ, ΜΡΝ, ΛΡΝ γωνιῶν· ὅπερ ἔδει ποιῆσαι.

and NOL. But, (the sum of) ABC, DEF, and GHK was assumed (to be) less than four right-angles. Thus, (the sum of) LOM, MON, and NOL is much less than four right-angles. But, (it is) also equal (to four right-angles). The very thing is absurd. Thus, AB is not less than LO. And it was shown (to be) not equal either. Thus, AB (is) greater than LO.

So let OR have been set up at point O at right-angles to the plane of circle LMN [Prop. 11.12]. And let the (square) on OR be equal to that (area) by which the square on AB is greater than the (square) on LO [Prop. 11.23 lem.]. And let RL, RM, and RN have been joined.

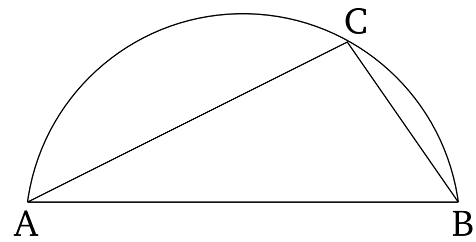
And since RO is at right-angles to the plane of circle LMN, RO is thus also at right-angles to each of LO, MO, and NO. And since LO is equal to OM, and OR is common and at right-angles, the base RL is thus equal to the base RM [Prop. 1.4]. So, for the same (reasons), RN is also equal to each of RL and RM. Thus, the three (straight-lines) RL, RM, and RN are equal to one another. And since the (square) on OR was assumed to be equal to that (area) by which the (square) on AB is greater than the (square) on LO, the (square) on AB is thus equal to the (sum of the squares) on LO and OR. And the (square) on LR is equal to the (sum of the squares) on LO and OR. For LOR (is) a right-angle [Prop. 1.47]. Thus, the (square) on AB is equal to the (square) on RL. Thus, AB (is) equal to RL. But, each of BC, DE, EF, GH, and HK is equal to AB, and each of RM and RN equal to RL. Thus, each of AB, BC, DE, EF, GH, and HK is equal to each of RL, RM, and RN. And since the two (straight-lines) LR and RM are equal to the two (straight-lines) AB and BC (respectively), and the base LM was assumed (to be) equal to the base AC, the angle LRM is thus equal to the angle ABC [Prop. 1.8]. So, for the same (reasons), MRN is also equal to DEF, and LRN to GHK.

Thus, the solid angle R, contained by the angles LRM, MRN, and LRN, has been constructed out of the three plane angles LRM, MRN, and LRN, which are equal to the three given (plane angles) ABC, DEF, and GHK (respectively). (Which is) the very thing it was required to do.



Λῆμμα.

Ὃν δὲ τρόπον, ᾧ μεῖζόν ἐστι τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΛΞ, ἐκείνῳ ἴσον λαβεῖν ἔστι τὸ ἀπὸ τῆς ΞΡ, δείξομεν οὕτως. ἐκκείσθωσαν αἱ ΑΒ, ΛΞ εὐθεῖαι, καὶ ἔστω μείζων ἡ ΑΒ, καὶ γεγράφθω ἐπ᾽ αὐτῆς ἡμικύκλιον τὸ ΑΒΓ, καὶ εἰς τὸ ΑΒΓ ἡμικύκλιον ἐνηρμόσθω τῇ ΛΞ εὐθείᾳ μὴ μείζονι οὔσῃ τῆς ΑΒ διαμέτρου ἴση ἡ ΑΓ, καὶ ἐπεζεύχθω ἡ ΓΒ. ἐπεὶ οὖν ἐν ἡμικυκλίῳ τῷ ΑΓΒ γωνία ἐστὶν ἡ ὑπὸ ΑΓΒ, ὀρθὴ ἄρα ἐστὶν ἡ ὑπὸ ΑΓΒ. τὸ ἄρα ἀπὸ τῆς ΑΒ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΓ, ΓΒ. ὥστε τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΑΓ μεῖζόν ἐστι τῷ ἀπὸ τῆς ΓΒ. ἴση δὲ ἡ ΑΓ τῇ ΛΞ. τὸ ἄρα ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΛΞ μεῖζόν ἐστι τῷ ἀπὸ τῆς ΓΒ. ἐὰν οὖν τῇ ΒΓ ἴσην τὴν ΞΡ ἀπολάβωμεν, ἔσται τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΛΞ μεῖζον τῷ ἀπὸ τῆς ΞΡ· ὅπερ προέκειτο ποιῆσαι.

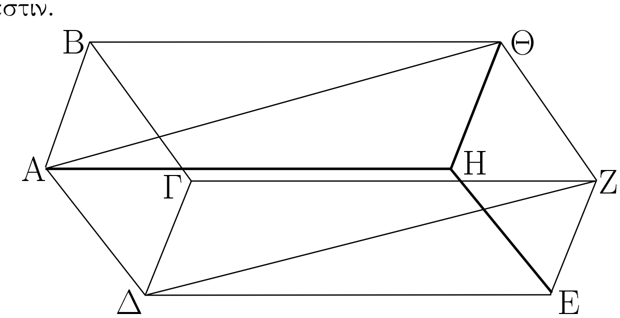


Lemma

And we can demonstrate, thusly, in which manner to take the (square) on OR equal to that (area) by which the (square) on AB is greater than the (square) on LO. Let the straight-lines AB and LO be set out, and let AB be greater, and let the semicircle ABC have been drawn around it. And let AC, equal to the straight-line LO, which is not greater than the diameter AB, have been inserted into the semicircle ABC [Prop. 4.1]. And let CB have been joined. Therefore, since the angle ACB is in the semicircle ACB, ACB is thus a right-angle [Prop. 3.31]. Thus, the (square) on AB is equal to the (sum of the) squares on AC and CB [Prop. 1.47]. Hence, the (square) on AB is greater than the (square) on AC by the (square) on CB. And AC (is) equal to LO. Thus, the (square) on AB is greater than the (square) on LO by the (square) on CB. Therefore, if we take OR equal to BC then the (square) on AB will be greater than the (square) on LO by the (square) on OR. (Which is) the very thing it was prescribed to do.

κδ΄.

Ἐὰν στερεὸν ὑπὸ παραλλήλων ἐπιπέδων περιέχηται, τὰ ἀπεναντίον αὐτοῦ ἐπίπεδα ἴσα τε καὶ παραλληλόγραμμά ἐστιν.

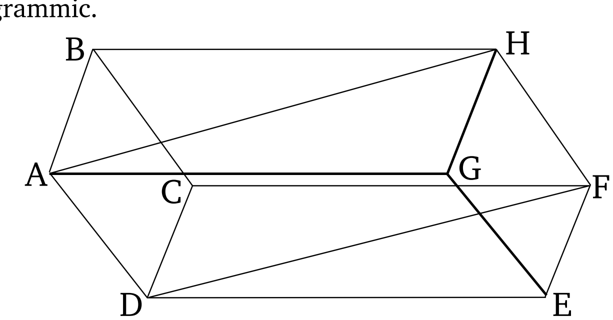


Στερεὸν γὰρ τὸ ΓΔΘΗ ὑπὸ παραλλήλων ἐπιπέδων περιεχέσθω τῶν ΑΓ, ΗΖ, ΑΘ, ΔΖ, ΒΖ, ΑΕ· λέγω, ὅτι τὰ ἀπεναντίον αὐτοῦ ἐπίπεδα ἴσα τε καὶ παραλληλόγραμμά ἐστιν.

Ἐπεὶ γὰρ δύο ἐπίπεδα παράλληλα τὰ ΒΗ, ΓΕ ὑπὸ ἐπιπέδου τοῦ ΑΓ τέμνεται, αἱ κοιναὶ αὐτῶν τομαὶ παράλληλοί εἰσιν. παράλληλος ἄρα ἐστὶν ἡ ΑΒ τῇ ΔΓ. πάλιν, ἐπεὶ δύο ἐπίπεδα παράλληλα τὰ ΒΖ, ΑΕ ὑπὸ ἐπιπέδου τοῦ ΑΓ τέμνεται, αἱ κοιναὶ αὐτῶν τομαὶ παράλληλοί εἰσιν.

Proposition 24

If a solid (figure) is contained by (six) parallel planes then its opposite planes are both equal and parallelogrammic.



For let the solid (figure) $CDHG$ have been contained by the parallel planes AC, GF, and AH, DF, and BF, AE. I say that its opposite planes are both equal and parallelogrammic.

For since the two parallel planes BG and CE are cut by the plane AC, their common sections are parallel [Prop. 11.16]. Thus, AB is parallel to DC. Again, since the two parallel planes BF and AE are cut by the plane παράλληλος ἄρα ἐστὶν ἡ ΒΓ τῇ ΑΔ. ἐδείχθη δὲ καὶ ἡ ΑΒ τῇ ΔΓ παράλληλος· παραλληλόγραμμον ἄρα ἐστὶ τὸ ΑΓ. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἕκαστον τῶν ΔΖ, ΖΗ, ΗΒ, ΒΖ, ΑΕ παραλληλόγραμμόν ἐστιν.

Ἐπεζεύχθωσαν αἱ ΑΘ, ΔΖ. καὶ ἐπεὶ παράλληλός ἐστιν ἡ μὲν ΑΒ τῇ ΔΓ, ἡ δὲ ΒΘ τῇ ΓΖ, δύο δὴ αἱ ΑΒ, ΒΘ ἁπτόμεναι ἀλλήλων παρὰ δύο εὐθείας τὰς ΔΓ, ΓΖ ἁπτομένας ἀλλήλων εἰσὶν οὐκ ἐν τῷ αὐτῷ ἐπιπέδῳ· ἴσας ἄρα γωνίας περιέξουσιν· ἴση ἄρα ἡ ὑπὸ ΑΒΘ γωνία τῇ ὑπὸ ΔΓΖ. καὶ ἐπεὶ δύο αἱ ΑΒ, ΒΘ δυσὶ ταῖς ΔΓ, ΓΖ ἴσαι εἰσίν, καὶ γωνία ἡ ὑπὸ ΑΒΘ γωνίᾳ τῇ ὑπὸ ΔΓΖ ἐστιν ἴση, βάσις ἄρα ἡ ΑΘ βάσει τῇ ΔΖ ἐστιν ἴση, καὶ τὸ ΑΒΘ τρίγωνον τῷ ΔΓΖ τριγώνῳ ἴσον ἐστίν. καί ἐστι τοῦ μὲν ΑΒΘ διπλάσιον τὸ ΒΗ παραλληλόγραμμον, τοῦ δὲ ΔΓΖ διπλάσιον τὸ ΓΕ παραλληλόγραμμον· ἴσον ἄρα τὸ ΒΗ παραλληλόγραμμον τῷ ΓΕ παραλληλογράμμῳ· ὁμοίως δὴ δείξομεν, ὅτι καὶ τὸ μὲν ΑΓ τῷ ΗΖ ἐστιν ἴσον, τὸ δὲ ΑΕ τῷ ΒΖ.

Ἐὰν ἄρα στερεὸν ὑπὸ παραλλήλων ἐπιπέδων περιέχηται, τὰ ἀπεναντίον αὐτοῦ ἐπίπεδα ἴσα τε καὶ παραλληλόγραμμά ἐστιν· ὅπερ ἔδει δεῖξαι.

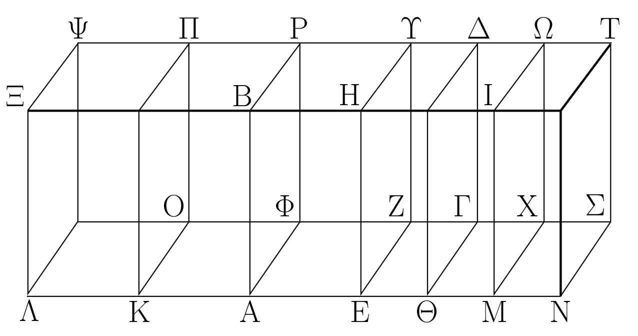
AC, their common sections are parallel [Prop. 11.16]. Thus, BC is parallel to AD. And AB was also shown (to be) parallel to DC. Thus, AC is a parallelogram. So, similarly, we can also show that DF, FG, GB, BF, and AE are each parallelograms.

Let AH and DF have been joined. And since AB is parallel to DC, and BH to CF, so the two (straight-lines) joining one another, AB and BH, are parallel to the two straight-lines joining one another, DC and CF (respectively), not (being) in the same plane. Thus, they will contain equal angles [Prop. 11.10]. Thus, angle ABH (is) equal to (angle) DCF. And since the two (straight-lines) AB and BH are equal to the two (straight-lines) DC and CF (respectively) [Prop. 1.34], and angle ABH is equal to angle DCF, the base AH is thus equal to the base DF, and triangle ABH is equal to triangle DCF [Prop. 1.4]. And parallelogram BG is double (triangle) ABH, and parallelogram CE double (triangle) DCF [Prop. 1.34]. Thus, parallelogram BG (is) equal to parallelogram CE. So, similarly, we can show that AC is also equal to GF, and AE to BF.

Thus, if a solid (figure) is contained by (six) parallel planes then its opposite planes are both equal and parallelogrammic. (Which is) the very thing it was required to show.

κε΄.

Ἐὰν στερεὸν παραλληλεπίπεδον ἐπιπέδῳ τμηθῇ παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις, ἔσται ὡς ἡ βάσις πρὸς τὴν βάσιν, οὕτως τὸ στερεὸν πρὸς τὸ στερεόν.

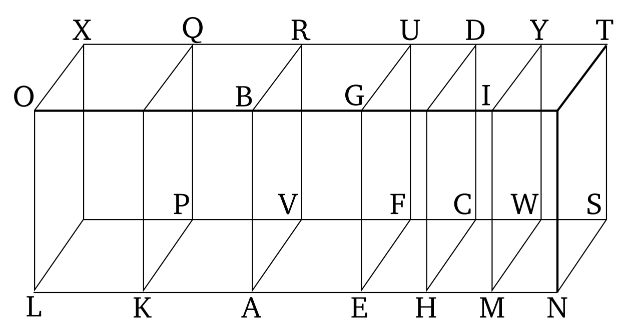


Στερεὸν γὰρ παραλληλεπίπεδον τὸ ΑΒΓΔ ἐπιπέδῳ τῷ ΖΗ τετμήσθω παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις τοῖς ΡΑ, ΔΘ· λέγω, ὅτι ἐστὶν ὡς ἡ ΑΕΖΦ βάσις πρὸς τὴν ΕΘΓΖ βάσιν, οὕτως τὸ ΑΒΖΥ στερεὸν πρὸς τὸ ΕΗΓΔ στερεόν.

Ἐκβεβλήσθω γὰρ ἡ ΑΘ ἐφ᾽ ἑκάτερα τὰ μέρη, καὶ κείσθωσαν τῇ μὲν ΑΕ ἴσαι ὁσαιδηποτοῦν αἱ ΑΚ, ΚΛ, τῇ δὲ ΕΘ ἴσαι ὁσαιδηποτοῦν αἱ ΘΜ, ΜΝ, καὶ συμπεπληρώσθω τὰ ΛΟ, ΚΦ, ΘΧ, ΜΣ παραλληλόγραμμα καὶ τὰ ΛΠ, ΚΡ,

Proposition 25

If a parallelipiped solid is cut by a plane which is parallel to the opposite planes (of the parallelipiped) then as the base (is) to the base, so the solid will be to the solid.



For let the parallelipiped solid $ABCD$ have been cut by the plane FG which is parallel to the opposite planes RA and DH. I say that as the base $AEFV$ (is) to the base $EHCF$, so the solid $ABFU$ (is) to the solid $EGCD$.

For let AH have been produced in each direction. And let any number whatsoever (of lengths), AK and KL, be made equal to AE, and any number whatsoever (of lengths), HM and MN, equal to EH. And let the parallelograms LP, KV, HW, and MS have been completed, ΔΜ, ΜΤ στερεά.

Καὶ ἐπεὶ ἴσαι εἰσὶν αἱ ΛΚ, ΚΑ, ΑΕ εὐθεῖαι ἀλλήλαις, ἴσα ἐστὶ καὶ τὰ μὲν ΛΟ, ΚΦ, ΑΖ παραλληλόγραμμα ἀλλήλοις, τὰ δὲ ΚΞ, ΚΒ, ΑΗ ἀλλήλοις καὶ ἔτι τὰ ΛΨ, ΚΠ, ΑΡ ἀλλήλοις· ἀπεναντίον γάρ. διὰ τὰ αὐτὰ δὴ καὶ τὰ μὲν ΕΓ, ΘΧ, ΜΣ παραλληλόγραμμα ἴσα εἰσὶν ἀλλήλοις, τὰ δὲ ΘΗ, ΘΙ, ΙΝ ἴσα εἰσὶν ἀλλήλοις, καὶ ἔτι τὰ ΔΘ, ΜΩ, ΝΤ· τρία ἄρα ἐπίπεδα τῶν ΛΠ, ΚΡ, ΑΥ στερεῶν τρισὶν ἐπιπέδοις ἐστὶν ἴσα. ἀλλὰ τὰ τρία τρισὶ τοῖς ἀπεναντίον ἐστὶν ἴσα· τὰ ἄρα τρία στερεὰ τὰ ΛΠ, ΚΡ, ΑΥ ἴσα ἀλλήλοις ἐστίν. διὰ τὰ αὐτὰ δὴ καὶ τὰ τρία στερεὰ τὰ ΕΔ, ΔΜ, ΜΤ ἴσα ἀλλήλοις ἐστίν· ὁσαπλασίων ἄρα ἐστὶν ἡ ΛΖ βάσις τῆς ΑΖ βάσεως, τοσαυταπλάσιόν ἐστι καὶ τὸ ΛΥ στερεὸν τοῦ ΑΥ στερεοῦ. διὰ τὰ αὐτὰ δὴ ὁσαπλασίων ἐστὶν ἡ ΝΖ βάσις τῆς ΖΘ βάσεως, τοσαυταπλάσιόν ἐστι καὶ τὸ ΝΥ στερεὸν τοῦ ΘΥ στερεοῦ. καὶ εἰ ἴση ἐστὶν ἡ ΛΖ βάσις τῇ ΝΖ βάσει, ἴσον ἐστὶ καὶ τὸ ΛΥ στερεὸν τῷ ΝΥ στερεῷ, καὶ εἰ ὑπερέχει ἡ ΛΖ βάσις τῆς ΝΖ βάσεως, ὑπερέχει καὶ τὸ ΛΥ στερεὸν τοῦ ΝΥ στερεοῦ, καὶ εἰ ἐλλείπει, ἐλλείπει. τεσσάρων δὴ ὄντων μεγεθῶν, δύο μὲν βάσεων τῶν ΑΖ, ΖΘ, δύο δὲ στερεῶν τῶν ΑΥ, ΥΘ, εἴληπται ἰσάκις πολλαπλάσια τῆς μὲν ΑΖ βάσεως καὶ τοῦ ΑΥ στερεοῦ ἥ τε ΛΖ βάσις καὶ τὸ ΛΥ στερεόν, τῆς δὲ ΘΖ βάσεως καὶ τοῦ ΘΥ στερεοῦ ἥ τε ΝΖ βάσις καὶ τὸ ΝΥ στερεόν, καὶ δέδεικται, ὅτι εἰ ὑπερέχει ἡ ΛΖ βάσις τῆς ΖΝ βάσεως, ὑπερέχει καὶ τὸ ΛΥ στερεὸν τοῦ ΝΥ [στερεοῦ], καὶ εἰ ἴση, ἴσον, καὶ εἰ ἐλλείπει, ἐλλείπει. ἔστιν ἄρα ὡς ἡ ΑΖ βάσις πρὸς τὴν ΖΘ βάσιν, οὕτως τὸ ΑΥ στερεὸν πρὸς τὸ ΥΘ στερεόν· ὅπερ ἔδει δεῖξαι.

and the solids LQ, KR, DM, and MT.

And since the straight-lines LK, KA, and AE are equal to one another, the parallelograms LP, KV, and AF are also equal to one another, and KO, KB, and AG (are equal) to one another, and, further, LX, KQ, and AR (are equal) to one another. For (they are) opposite [Prop. 11.24]. So, for the same (reasons), the parallelograms EC, HW, and MS are also equal to one another, and HG, HI, and IN are equal to one another, and, further, DH, MY, and NT (are equal to one another). Thus, three planes of (one of) the solids LQ, KR, and AU are equal to the (corresponding) three planes (of the others). But, the three planes (in one of the soilds) are equal to the three opposite planes [Prop. 11.24]. Thus, the three solids LQ, KR, and AU are equal to one another [Def. 11.10]. So, for the same (reasons), the three solids ED, DM, and MT are also equal to one another. Thus, as many multiples as the base LF is of the base AF, so many multiples is the solid LU also of the the solid AU. So, for the same (reasons), as many multiples as the base NF is of the base FH, so many multiples is the solid NU also of the solid HU. And if the base LF is equal to the base NF then the solid LU is also equal to the solid NU.† And if the base LF exceeds the base NF then the solid LU also exceeds the solid NU. And if (LF) is less than (NF) then (LU) is (also) less than (NU). So, there are four magnitudes, the two bases AF and FH, and the two solids AU and UH, and equal multiples have been taken of the base AF and the solid AU— (namely), the base LF and the solid LU—and of the base HF and the solid HU—(namely), the base NF and the solid NU. And it has been shown that if the base LF exceeds the base FN then the solid LU also exceeds the [solid] NU, and if (LF is) equal (to FN) then (LU is) equal (to NU), and if (LF is) less than (FN) then (LU is) less than (NU). Thus, as the base AF is to the base FH, so the solid AU (is) to the solid UH [Def. 5.5]. (Which is) the very thing it was required to show.

† Here, Euclid assumes that $LF \gtreqless NF$ implies $LU \gtreqless NU$. This is easily demonstrated.

κϛʹ.

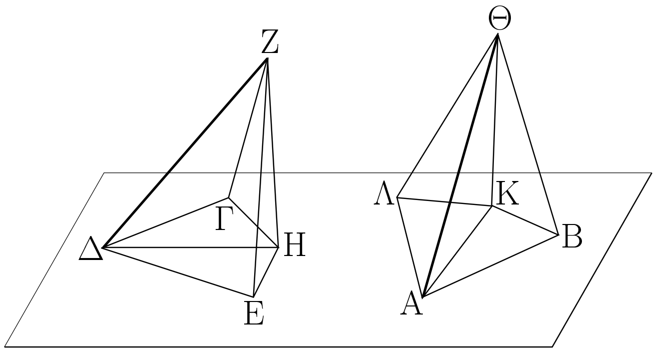
Πρὸς τῇ δοθείσῃ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῇ δοθείσῃ στερεᾷ γωνίᾳ ἴσην στερεὰν γωνίαν συστήσασθαι.

Ἔστω ἡ μὲν δοθεῖσα εὐθεῖα ἡ ΑΒ, τὸ δὲ πρὸς αὐτῇ δοθὲν σημεῖον τὸ Α, ἡ δὲ δοθεῖσα στερεὰ γωνία ἡ πρὸς τῷ Δ περιεχομένη ὑπὸ τῶν ὑπὸ ΕΔΓ, ΕΔΖ, ΖΔΓ γωνιῶν ἐπιπέδων· δεῖ δὴ πρὸς τῇ ΑΒ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α τῇ πρὸς τῷ Δ στερεᾷ γωνίᾳ ἴσην στερεὰν γωνίαν συστήσασθαι.

Proposition 26

To construct a solid angle equal to a given solid angle on a given straight-line, and at a given point on it.

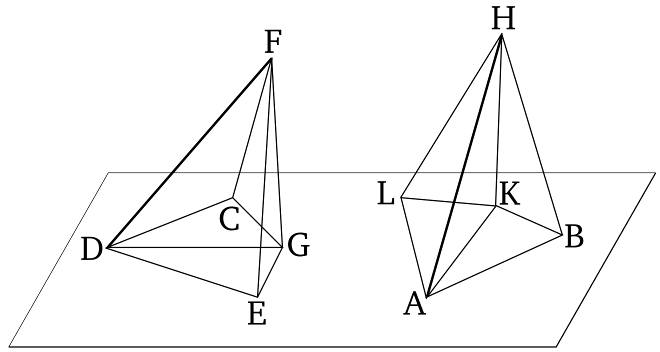
Let AB be the given straight-line, and A the given point on it, and D the given solid angle, contained by the plane angles EDC, EDF, and FDC. So, it is necessary to construct a solid angle equal to the solid angle D on the straight-line AB, and at the point A on it.



Εἰλήφθω γὰρ ἐπὶ τῆς ΔΖ τυχὸν σημεῖον τὸ Ζ, καὶ ἤχθω ἀπὸ τοῦ Ζ ἐπὶ τὸ διὰ τῶν ΕΔ, ΔΓ ἐπίπεδον κάθετος ἡ ΖΗ, καὶ συμβαλλέτω τῷ ἐπιπέδῳ κατὰ τὸ Η, καὶ ἐπεζεύχθω ἡ ΔΗ, καὶ συνεστάτω πρὸς τῇ ΑΒ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α τῇ μὲν ὑπὸ ΕΔΓ γωνίᾳ ἴση ἡ ὑπὸ ΒΑΛ, τῇ δὲ ὑπὸ ΕΔΗ ἴση ἡ ὑπὸ ΒΑΚ, καὶ κείσθω τῇ ΔΗ ἴση ἡ ΑΚ, καὶ ἀνεστάτω ἀπὸ τοῦ Κ σημείου τῷ διὰ τῶν ΒΑΛ ἐπιπέδῳ πρὸς ὀρθὰς ἡ ΚΘ, καὶ κείσθω ἴση τῇ ΗΖ ἡ ΚΘ, καὶ ἐπεζεύχθω ἡ ΘΑ· λέγω, ὅτι ἡ πρὸς τῷ Α στερεὰ γωνία περιεχομένη ὑπὸ τῶν ΒΑΛ, ΒΑΘ, ΘΑΛ γωνιῶν ἴση ἐστὶ τῇ πρὸς τῷ Δ στερεᾷ γωνίᾳ τῇ περιεχομένῃ ὑπὸ τῶν ΕΔΓ, ΕΔΖ, ΖΔΓ γωνιῶν.

Ἀπειλήφθωσαν γὰρ ἴσαι αἱ ΑΒ, ΔΕ, καὶ ἐπεζεύχθωσαν αἱ ΘΒ, ΚΒ, ΖΕ, ΗΕ. καὶ ἐπεὶ ἡ ΖΗ ὀρθή ἐστι πρὸς τὸ ὑποκείμενον ἐπίπεδον, καὶ πρὸς πάσας ἄρα τὰς ἁπτομένας αὐτῆς εὐθείας καὶ οὔσας ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ ὀρθὰς ποιήσει γωνίας· ὀρθὴ ἄρα ἐστὶν ἑκατέρα τῶν ὑπὸ ΖΗΔ, ΖΗΕ γωνιῶν. διὰ τὰ αὐτὰ δὴ καὶ ἑκατέρα τῶν ὑπὸ ΘΚΑ, ΘΚΒ γωνιῶν ὀρθή ἐστιν. καὶ ἐπεὶ δύο αἱ ΚΑ, ΑΒ δύο ταῖς ΗΔ, ΔΕ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ, καὶ γωνίας ἴσας περιέχουσιν, βάσις ἄρα ἡ ΚΒ βάσει τῇ ΗΕ ἴση ἐστίν. ἔστι δὲ καὶ ἡ ΚΘ τῇ ΗΖ ἴση· καὶ γωνίας ὀρθὰς περιέχουσιν· ἴση ἄρα καὶ ἡ ΘΒ τῇ ΖΕ. πάλιν ἐπεὶ δύο αἱ ΑΚ, ΚΘ δυσὶ ταῖς ΔΗ, ΗΖ ἴσαι εἰσίν, καὶ γωνίας ὀρθὰς περιέχουσιν, βάσις ἄρα ἡ ΑΘ βάσει τῇ ΖΔ ἴση ἐστίν. ἔστι δὲ καὶ ἡ ΑΒ τῇ ΔΕ ἴση· δύο δὴ αἱ ΘΑ, ΑΒ δύο ταῖς ΔΖ, ΔΕ ἴσαι εἰσίν. καὶ βάσις ἡ ΘΒ βάσει τῇ ΖΕ ἴση· γωνία ἄρα ἡ ὑπὸ ΒΑΘ γωνίᾳ τῇ ὑπὸ ΕΔΖ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΘΑΛ τῇ ὑπὸ ΖΔΓ ἐστιν ἴση. ἔστι δὲ καὶ ἡ ὑπὸ ΒΑΛ τῇ ὑπὸ ΕΔΓ ἴση.

Πρὸς ἄρα τῇ δοθείσῃ εὐθείᾳ τῇ ΑΒ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α τῇ δοθείσῃ στερεᾷ γωνίᾳ τῇ πρὸς τῷ Δ ἴση συνέσταται· ὅπερ ἔδει ποιῆσαι.



For let some random point F have been taken on DF, and let FG have been drawn from F perpendicular to the plane through ED and DC [Prop. 11.11], and let it meet the plane at G, and let DG have been joined. And let BAL, equal to the angle EDC, and BAK, equal to EDG, have been constructed on the straight-line AB at the point A on it [Prop. 1.23]. And let AK be made equal to DG. And let KH have been set up at the point K at right-angles to the plane through BAL [Prop. 11.12]. And let KH be made equal to GF. And let HA have been joined. I say that the solid angle at A, contained by the (plane) angles BAL, BAH, and HAL, is equal to the solid angle at D, contained by the (plane) angles EDC, EDF, and FDC.

For let AB and DE have been cut off (so as to be) equal, and let HB, KB, FE, and GE have been joined. And since FG is at right-angles to the reference plane (EDC), it will also make right-angles with all of the straight-lines joined to it which are also in the reference plane [Def. 11.3]. Thus, the angles FGD and FGE are right-angles. So, for the same (reasons), the angles HKA and HKB are also right-angles. And since the two (straight-lines) KA and AB are equal to the two (straight-lines) GD and DE, respectively, and they contain equal angles, the base KB is thus equal to the base GE [Prop. 1.4]. And KH is also equal to GF. And they contain right-angles (with the respective bases). Thus, HB (is) also equal to FE [Prop. 1.4]. Again, since the two (straight-lines) AK and KH are equal to the two (straight-lines) DG and GF (respectively), and they contain right-angles, the base AH is thus equal to the base FD [Prop. 1.4]. And AB (is) also equal to DE. So, the two (straight-lines) HA and AB are equal to the two (straight-lines) DF and DE (respectively). And the base HB (is) equal to the base FE. Thus, the angle BAH is equal to the angle EDF [Prop. 1.8]. So, for the same (reasons), HAL is also equal to FDC. And BAL is also equal to EDC.

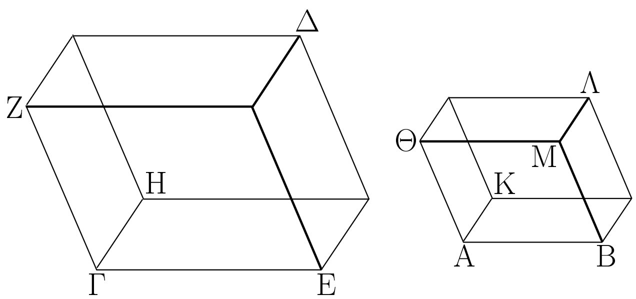
Thus, (a solid angle) has been constructed, equal to the given solid angle at D, on the given straight-line AB, at the given point A on it. (Which is) the very thing it was required to do.

κζʹ.

Ἀπὸ τῆς δοθείσης εὐθείας τῷ δοθέντι στερεῷ παραλληλεπιπέδῳ ὅμοιόν τε καὶ ὁμοίως κείμενον στερεὸν παραλληλεπίπεδον ἀναγράψαι.

Ἔστω ἡ μὲν δοθεῖσα εὐθεῖα ἡ ΑΒ, τὸ δὲ δοθὲν στερεὸν παραλληλεπίπεδον τὸ ΓΔ· δεῖ δὴ ἀπὸ τῆς δοθείσης εὐθείας τῆς ΑΒ τῷ δοθέντι στερεῷ παραλληλεπιπέδῳ τῷ ΓΔ ὅμοιόν τε καὶ ὁμοίως κείμενον στερεὸν παραλληλεπίπεδον ἀναγράψαι.

Συνεστάτω γὰρ πρὸς τῇ ΑΒ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α τῇ πρὸς τῷ Γ στερεᾷ γωνίᾳ ἴση ἡ περιεχομένη ὑπὸ τῶν ΒΑΘ, ΘΑΚ, ΚΑΒ, ὥστε ἴσην εἶναι τὴν μὲν ὑπὸ ΒΑΘ γωνίαν τῇ ὑπὸ ΕΓΖ, τὴν δὲ ὑπὸ ΒΑΚ τῇ ὑπὸ ΕΓΗ, τὴν δὲ ὑπὸ ΚΑΘ τῇ ὑπὸ ΗΓΖ· καὶ γεγονέτω ὡς μὲν ἡ ΕΓ πρὸς τὴν ΓΗ, οὕτως ἡ ΒΑ πρὸς τὴν ΑΚ, ὡς δὲ ἡ ΗΓ πρὸς τὴν ΓΖ, οὕτως ἡ ΚΑ πρὸς τὴν ΑΘ. καὶ δι᾿ ἴσου ἄρα ἐστὶν ὡς ἡ ΕΓ πρὸς τὴν ΓΖ, οὕτως ἡ ΒΑ πρὸς τὴν ΑΘ. καὶ συμπεπληρώσθω τὸ ΘΒ παραλληλόγραμμον καὶ τὸ ΑΛ στερεόν.



Καὶ ἐπεί ἐστιν ὡς ἡ ΕΓ πρὸς τὴν ΓΗ, οὕτως ἡ ΒΑ πρὸς τὴν ΑΚ, καὶ περὶ ἴσας γωνίας τὰς ὑπὸ ΕΓΗ, ΒΑΚ αἱ πλευραὶ ἀνάλογόν εἰσιν, ὅμοιον ἄρα ἐστὶ τὸ ΗΕ παραλληλόγραμμον τῷ ΚΒ παραλληλογράμμῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ μὲν ΚΘ παραλληλόγραμμον τῷ ΗΖ παραλληλογράμμῳ ὅμοιόν ἐστι καὶ ἔτι τὸ ΖΕ τῷ ΘΒ· τρία ἄρα παραλληλόγραμμα τοῦ ΓΔ στερεοῦ τρισὶ παραλληλογράμμοις τοῦ ΑΛ στερεοῦ ὅμοιά ἐστιν. ἀλλὰ τὰ μὲν τρία τρισὶ τοῖς ἀπεναντίον ἴσα τέ ἐστι καὶ ὅμοια, τὰ δὲ τρία τρισὶ τοῖς ἀπεναντίον ἴσα τέ ἐστι καὶ ὅμοια· ὅλον ἄρα τὸ ΓΔ στερεὸν ὅλῳ τῷ ΑΛ στερεῷ ὅμοιόν ἐστιν.

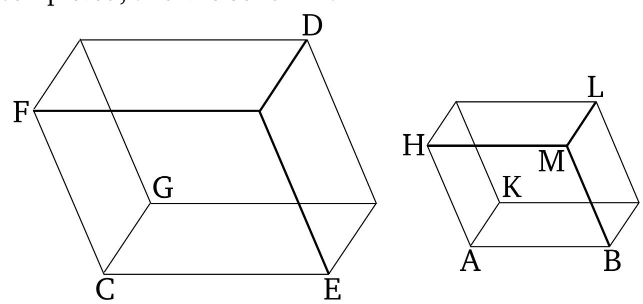
Ἀπὸ τῆς δοθείσης ἄρα εὐθείας τῆς ΑΒ τῷ δοθέντι στερεῷ παραλληλεπιπέδῳ τῷ ΓΔ ὅμοιόν τε καὶ ὁμοίως κείμενον ἀναγέγραπται τὸ ΑΛ· ὅπερ ἔδει ποιῆσαι.

Proposition 27

To describe a parallelepiped solid similar, and similarly laid out, to a given parallelepiped solid on a given straight-line.

Let the given straight-line be AB, and the given parallelepiped solid CD. So, it is necessary to describe a parallelepiped solid similar, and similarly laid out, to the given parallelepiped solid CD on the given straight-line AB.

For, let a (solid angle) contained by the (plane angles) BAH, HAK, and KAB have been constructed, equal to solid angle at C, on the straight-line AB at the point A on it [Prop. 11.26], such that angle BAH is equal to ECF, and BAK to ECG, and KAH to GCF. And let it have been contrived that as EC (is) to CG, so BA (is) to AK, and as GC (is) to CF, so KA (is) to AH [Prop. 6.12]. And thus, via equality, as EC is to CF, so BA (is) to AH [Prop. 5.22]. And let the parallelogram HB have been completed, and the solid AL.



And since as EC is to CG, so BA (is) to AK, and the sides about the equal angles ECG and BAK are (thus) proportional, the parallelogram GE is thus similar to the parallelogram KB. So, for the same (reasons), the parallelogram KH is also similar to the parallelogram GF, and, further, FE (is similar) to HB. Thus, three of the parallelograms of solid CD are similar to three of the parallelograms of solid AL. But, the (former) three are equal and similar to the three opposite, and the (latter) three are equal and similar to the three opposite. Thus, the whole solid CD is similar to the whole solid AL [Def. 11.9].

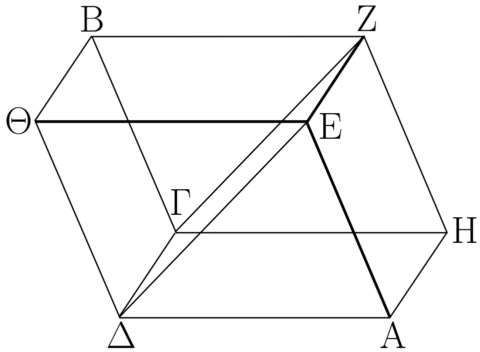
Thus, AL, similar, and similarly laid out, to the given parallelepiped solid CD, has been described on the given straight-lines AB. (Which is) the very thing it was required to do.

κηʹ.

Ἐὰν στερεὸν παραλληλεπίπεδον ἐπιπέδῳ τμηθῇ κατὰ

Proposition 28

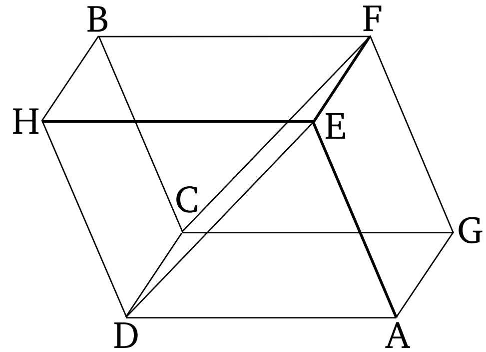
If a parallelepiped solid is cut by a plane (passing) τὰς διαγωνίους τῶν ἀπεναντίον ἐπιπέδων, δίχα τμηθήσεται τὸ στερεὸν ὑπὸ τοῦ ἐπιπέδου.



Στερεὸν γὰρ παραλληλεπίπεδον τὸ ΑΒ ἐπιπέδῳ τῷ ΓΔΕΖ τετμήσθω κατὰ τὰς διαγωνίους τῶν ἀπεναντίον ἐπιπέδων τὰς ΓΖ, ΔΕ· λέγω, ὅτι δίχα τμηθήσεται τὸ ΑΒ στερεὸν ὑπὸ τοῦ ΓΔΕΖ ἐπιπέδου.

Ἐπεὶ γὰρ ἴσον ἐστὶ τὸ μὲν ΓΗΖ τρίγωνον τῷ ΓΖΒ τριγώνῳ, τὸ δὲ ΑΔΕ τῷ ΔΕΘ, ἔστι δὲ καὶ τὸ μὲν ΓΑ παραλληλόγραμμον τῷ ΕΒ ἴσον· ἀπεναντίον γάρ· τὸ δὲ ΗΕ τῷ ΓΘ, καὶ τὸ πρίσμα ἄρα τὸ περιεχόμενον ὑπὸ δύο μὲν τριγώνων τῶν ΓΗΖ, ΑΔΕ, τριῶν δὲ παραλληλογράμμων τῶν ΗΕ, ΑΓ, ΓΕ ἴσον ἐστὶ τῷ πρίσματι τῷ περιεχομένῳ ὑπὸ δύο μὲν τριγώνων τῶν ΓΖΒ, ΔΕΘ, τριῶν δὲ παραλληλογράμμων τῶν ΓΘ, ΒΕ, ΓΕ· ὑπὸ γὰρ ἴσων ἐπιπέδων περιέχονται τῷ τε πλήθει καὶ τῷ μεγέθει. ὥστε ὅλον τὸ ΑΒ στερεὸν δίχα τέτμηται ὑπὸ τοῦ ΓΔΕΖ ἐπιπέδου· ὅπερ ἔδει δεῖξαι.

through the diagonals of (a pair of) opposite planes then the solid will be cut in half by the plane.



For let the parallelepiped solid AB have been cut by the plane $CDEF$ (passing) through the diagonals of the opposite planes CF and DE.† I say that the solid AB will be cut in half by the plane $CDEF$.

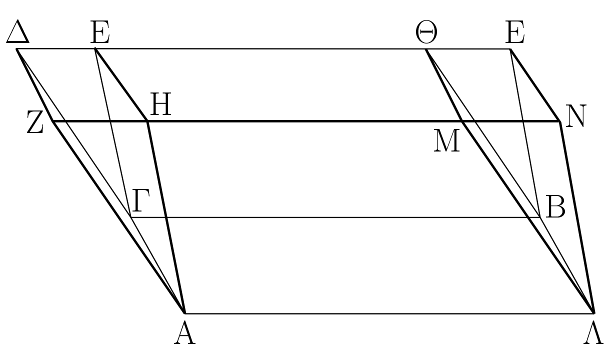
For since triangle CGF is equal to triangle CFB, and ADE (is equal) to DEH [Prop. 1.34], and parallelogram CA is also equal to EB—for (they are) opposite [Prop. 11.24]—and GE (equal) to CH, thus the prism contained by the two triangles CGF and ADE, and the three parallelograms GE, AC, and CE, is also equal to the prism contained by the two triangles CFB and DEH, and the three parallelograms CH, BE, and CE. For they are contained by planes (which are) equal in number and in magnitude [Def. 11.10].‡ Thus, the whole of solid AB is cut in half by the plane $CDEF$. (Which is) the very thing it was required to show.

† Here, it is assumed that the two diagonals lie in the same plane. The proof is easily supplied.

‡ However, strictly speaking, the prisms are not similarly arranged, being mirror images of one another.

κθ´.

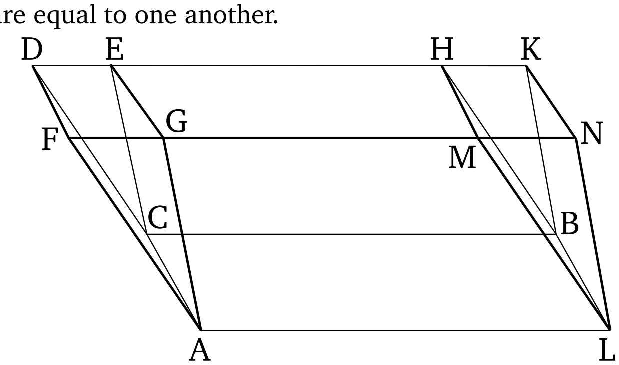
Τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι ἐπὶ τῶν αὐτῶν εἰσιν εὐθειῶν, ἴσα ἀλλήλοις ἐστίν.



Ἔστω ἐπὶ τῆς αὐτῆς βάσεως τῆς ΑΒ στερεὰ παραλλη-

Proposition 29

Parallelepiped solids which are on the same base, and (have) the same height, and in which the (ends of the straight-lines) standing up are on the same straight-lines, are equal to one another.



For let the parallelepiped solids CM and CN be on λεπίπεδα τὰ ΓΜ, ΓΝ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι αἱ ΑΗ, ΑΖ, ΛΜ, ΛΝ, ΓΔ, ΓΕ, ΒΘ, ΒΚ ἐπὶ τῶν αὐτῶν εὐθειῶν ἔστωσαν τῶν ΖΝ, ΔΚ· λέγω, ὅτι ἴσον ἐστὶ τὸ ΓΜ στερεὸν τῷ ΓΝ στερεῷ.

Ἐπεὶ γὰρ παραλληλόγραμμόν ἐστιν ἑκάτερον τῶν ΓΘ, ΓΚ, ἴση ἐστὶν ἡ ΓΒ ἑκατέρᾳ τῶν ΔΘ, ΕΚ· ὥστε καὶ ἡ ΔΘ τῇ ΕΚ ἐστιν ἴση. κοινὴ ἀφῃρήσθω ἡ ΕΘ· λοιπὴ ἄρα ἡ ΔΕ λοιπῇ τῇ ΘΚ ἐστιν ἴση. ὥστε καὶ τὸ μὲν ΔΓΕ τρίγωνον τῷ ΘΒΚ τριγώνῳ ἴσον ἐστίν, τὸ δὲ ΔΗ παραλληλόγραμμον τῷ ΘΝ παραλληλογράμμῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ ΑΖΗ τρίγωνον τῷ ΜΛΝ τριγώνῳ ἴσον ἐστίν. ἔστι δὲ καὶ τὸ μὲν ΓΖ παραλληλόγραμμον τῷ ΒΜ παραλληλογράμμῳ ἴσον, τὸ δὲ ΓΗ τῷ ΒΝ· ἀπεναντίον γάρ· καὶ τὸ πρίσμα ἄρα τὸ περιεχόμενον ὑπὸ δύο μὲν τριγώνων τῶν ΑΖΗ, ΔΓΕ, τριῶν δὲ παραλληλογράμμων τῶν ΑΔ, ΔΗ, ΓΗ ἴσον ἐστὶ τῷ πρίσματι τῷ περιεχομένῳ ὑπὸ δύο μὲν τριγώνων τῶν ΜΛΝ, ΘΒΚ, τριῶν δὲ παραλληλογράμμων τῶν ΒΜ, ΘΝ, ΒΝ. κοινὸν προσκείσθω τὸ στερεὸν, οὗ βάσις μὲν τὸ ΑΒ παραλληλόγραμμον, ἀπεναντίον δὲ τὸ ΗΕΘΜ· ὅλον ἄρα τὸ ΓΜ στερεὸν παραλληλεπίπεδον ὅλῳ τῷ ΓΝ στερεῷ παραλληλεπιπέδῳ ἴσον ἐστίν.

Τὰ ἄρα ἐπὶ τῆς αὐτῆς βάσεως ὄντα στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι ἐπὶ τῶν αὐτῶν εἰσιν εὐθειῶν, ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

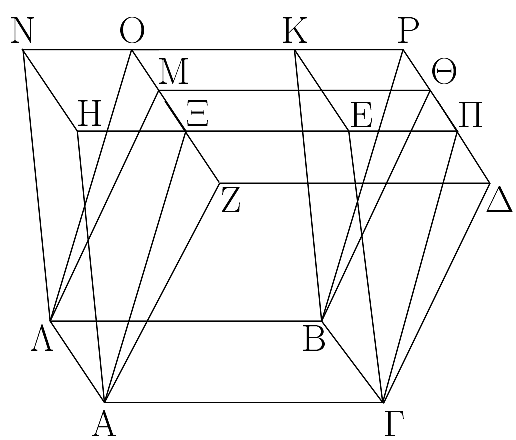
the same base AB, and (have) the same height, and let the (ends of the straight-lines) standing up in them, AG, AF, LM, LN, CD, CE, BH, and BK, be on the same straight-lines, FN and DK. I say that solid CM is equal to solid CN.

For since CH and CK are each parallelograms, CB is equal to each of DH and EK [Prop. 1.34]. Hence, DH is also equal to EK. Let EH have been subtracted from both. Thus, the remainder DE is equal to the remainder HK. Hence, triangle DCE is also equal to triangle HBK [Props. 1.4, 1.8], and parallelogram DG to parallelogram HN [Prop. 1.36]. So, for the same (reasons), traingle AFG is also equal to triangle MLN. And parallelogram CF is also equal to parallelogram BM, and CG to BN [Prop. 11.24]. For they are opposite. Thus, the prism contained by the two triangles AFG and DCE, and the three parallelograms AD, DG, and CG, is equal to the prism contained by the two triangles MLN and HBK, and the three parallelograms BM, HN, and BN. Let the solid whose base (is) parallelogram AB, and (whose) opposite (face is) $GEHM$, have been added to both (prisms). Thus, the whole parallelepiped solid CM is equal to the whole parallelepiped solid CN.

Thus, parallelepiped solids which are on the same base, and (have) the same height, and in which the (ends of the straight-lines) standing up (are) on the same straight-lines, are equal to one another. (Which is) the very thing it was required to show.

λ΄.

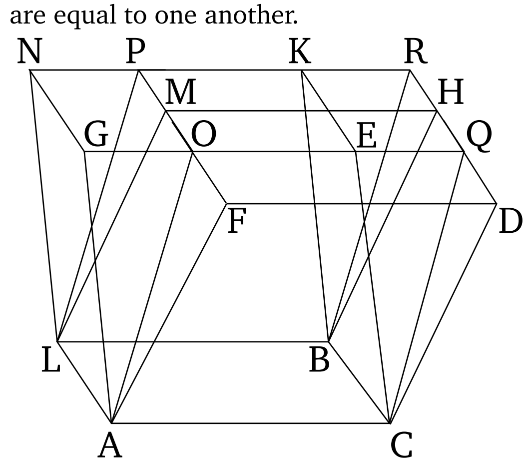
Τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι οὐκ εἰσὶν ἐπὶ τῶν αὐτῶν εὐθειῶν, ἴσα ἀλλήλοις ἐστίν.



Ἔστω ἐπὶ τῆς αὐτῆς βάσεως τῆς ΑΒ στερεὰ παραλληλεπίπεδα τὰ ΓΜ, ΓΝ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι αἱ ΑΖ, ΑΗ, ΛΜ, ΛΝ, ΓΔ, ΓΕ, ΒΘ, ΒΚ μὴ ἔστωσαν ἐπὶ τῶν

Proposition 30

Parallelepiped solids which are on the same base, and (have) the same height, and in which the (ends of the straight-lines) standing up are not on the same straight-lines, are equal to one another.



Let the parallelepiped solids CM and CN be on the same base, AB, and (have) the same height, and let the (ends of the straight-lines) standing up in them, AF, AG, αὐτῶν εὐθειῶν· λέγω, ὅτι ἴσον ἐστὶ τὸ ΓΜ στερεὸν τῷ ΓΝ στερεῷ.

Ἐκβεβλήσθωσαν γὰρ αἱ ΝΚ, ΔΘ καὶ συμπιπτέτωσαν ἀλλήλαις κατὰ τὸ Ρ, καὶ ἔτι ἐκβεβλήσθωσαν αἱ ΖΜ, ΗΕ ἐπὶ τὰ Ο, Π, καὶ ἐπεζεύχθωσαν αἱ ΑΞ, ΛΟ, ΓΠ, ΒΡ. ἴσον δή ἐστι τὸ ΓΜ στερεόν, οὗ βάσις μὲν τὸ ΑΓΒΛ παραλληλόγραμμον, ἀπεναντίον δὲ τὸ ΖΔΘΜ, τῷ ΓΟ στερεῷ, οὗ βάσις μὲν τὸ ΑΓΒΛ παραλληλόγραμμον, ἀπεναντίον δὲ τὸ ΞΠΡΟ· ἐπί τε γὰρ τῆς αὐτῆς βάσεώς εἰσι τῆς ΑΓΒΛ καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι αἱ ΑΖ, ΑΞ, ΛΜ, ΛΟ, ΓΔ, ΓΠ, ΒΘ, ΒΡ ἐπὶ τῶν αὐτῶν εἰσιν εὐθειῶν τῶν ΖΟ, ΔΡ. ἀλλὰ τὸ ΓΟ στερεόν, οὗ βάσις μέν ἐστι τὸ ΑΓΒΛ παραλληλόγραμμον, ἀπεναντίον δὲ τὸ ΞΠΡΟ, ἴσον ἐστὶ τῷ ΓΝ στερεῷ, οὗ βάσις μὲν τὸ ΑΓΒΛ παραλληλόγραμμον, ἀπεναντίον δὲ τὸ ΗΕΚΝ· ἐπί τε γὰρ πάλιν τῆς αὐτῆς βάσεώς εἰσι τῆς ΑΓΒΛ καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι αἱ ΑΗ, ΑΞ, ΓΕ, ΓΠ, ΛΝ, ΛΟ, ΒΚ, ΒΡ ἐπὶ τῶν αὐτῶν εἰσιν εὐθειῶν τῶν ΗΠ, ΝΡ. ὥστε καὶ τὸ ΓΜ στερεὸν ἴσον ἐστὶ τῷ ΓΝ στερεῷ.

Τὰ ἄρα ἐπὶ τῆς αὐτῆς βάσεως στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι οὐκ εἰσὶν ἐπὶ τῶν αὐτῶν εὐθειῶν, ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

LM, LN, CD, CE, BH, and BK, not be on the same straight-lines. I say that the solid CM is equal to the solid CN.

For let NK and DH have been produced, and let them have joined one another at R. And, further, let FM and GE have been produced to P and Q (respectively). And let AO, LP, CQ, and BR have been joined. So, solid CM, whose base (is) parallelogram $ACBL$, and opposite (face) $FDHM$, is equal to solid CP, whose base (is) parallelogram $ACBL$, and opposite (face) $OQRP$. For they are on the same base, $ACBL$, and (have) the same height, and the (ends of the straight-lines) standing up in them, AF, AO, LM, LP, CD, CQ, BH, and BR, are on the same straight-lines, FP and DR [Prop. 11.29]. But, solid CP, whose base is parallelogram $ACBL$, and opposite (face) $OQRP$, is equal to solid CN, whose base (is) parallelogram $ACBL$, and opposite (face) $GEKN$. For, again, they are on the same base, $ACBL$, and (have) the same height, and the (ends of the straight-lines) standing up in them, AG, AO, CE, CQ, LN, LP, BK, and BR, are on the same straight-lines, GQ and NR [Prop. 11.29]. Hence, solid CM is also equal to solid CN.

Thus, parallelepiped solids (which are) on the same base, and (have) the same height, and in which the (ends of the straight-lines) standing up are not on the same straight-lines, are equal to one another. (Which is) the very thing it was required to show.

λαʹ.

Τὰ ἐπὶ ἴσων βάσεων ὄντα στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος ἴσα ἀλλήλοις ἐστίν.

Ἔστω ἐπὶ ἴσων βάσεων τῶν ΑΒ, ΓΔ στερεὰ παραλληλεπίπεδα τὰ ΑΕ, ΓΖ ὑπὸ τὸ αὐτὸ ὕψος. λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΕ στερεὸν τῷ ΓΖ στερεῷ.

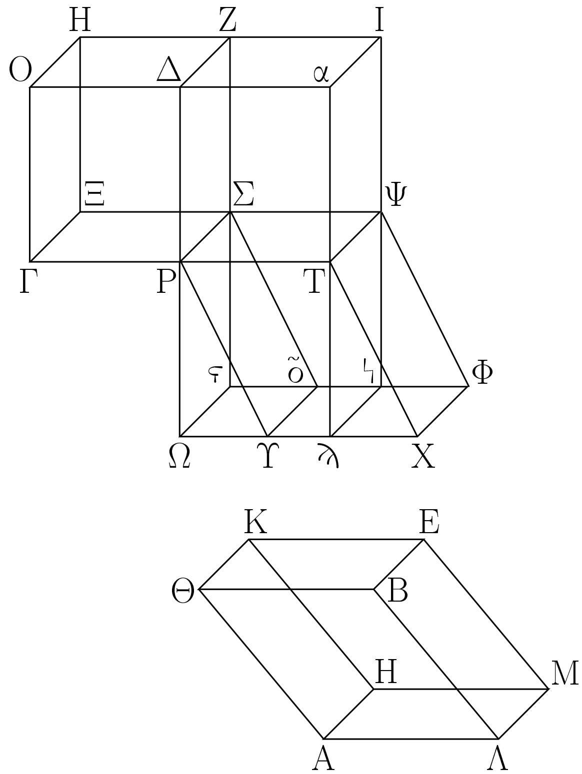
Ἔστωσαν δὴ πρότερον αἱ ἐφεστηκυῖαι αἱ ΘΚ, ΒΕ, ΑΗ, ΛΜ, ΟΠ, ΔΖ, ΓΞ, ΡΣ πρὸς ὀρθὰς ταῖς ΑΒ, ΓΔ βάσεσιν, καὶ ἐκβεβλήσθω ἐπ᾿ εὐθείας τῇ ΓΡ εὐθεῖα ἡ ΡΤ, καὶ συνεστάτω πρὸς τῇ ΡΤ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Ρ τῇ ὑπὸ ΑΛΒ γωνίᾳ ἴση ἡ ὑπὸ ΤΡΥ, καὶ κείσθω τῇ μὲν ΑΛ ἴση ἡ ΡΤ, τῇ δὲ ΛΒ ἴση ἡ ΡΥ, καὶ συμπεπληρώσθω ἥ τε ΡΧ βάσις καὶ τὸ ΨΥ στερεόν.

Proposition 31

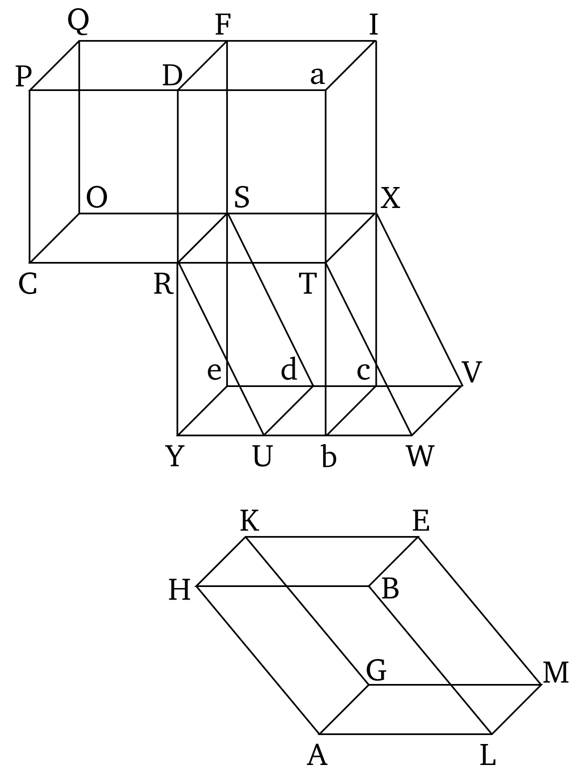
Parallelepiped solids which are on equal bases, and (have) the same height, are equal to one another.

Let the parallelepiped solids AE and CF be on the equal bases AB and CD (respectively), and (have) the same height. I say that solid AE is equal to solid CF.

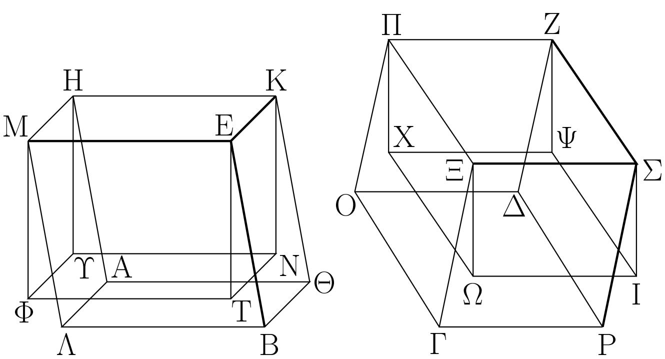
So, let the (straight-lines) standing up, HK, BE, AG, LM, PQ, DF, CO, and RS, first of all, be at right-angles to the bases AB and CD. And let RT have been produced in a straight-line with CR. And let (angle) TRU, equal to angle ALB, have been constructed on the straight-line RT, at the point R on it [Prop. 1.23]. And let RT be made equal to AL, and RU to LB. And let the base RW, and the solid XU, have been completed.



Καὶ ἐπεὶ δύο αἱ ΤΡ, ΡΥ δυσὶ ταῖς ΑΛ, ΛΒ ἴσαι εἰσίν, καὶ γωνίας ἴσας περιέχουσιν, ἴσον ἄρα καὶ ὅμοιον τὸ ΡΧ παραλληλόγραμμον τῷ ΘΛ παραλληλογράμμῳ. καὶ ἐπεὶ πάλιν ἴση μὲν ἡ ΑΛ τῇ ΡΤ, ἡ δὲ ΛΜ τῇ ΡΣ, καὶ γωνίας ὀρθὰς περιέχουσιν, ἴσον ἄρα καὶ ὅμοιόν ἐστι τὸ ΡΨ παραλληλόγραμμον τῷ ΑΜ παραλληλογράμμῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ ΛΕ τῷ ΣΥ ἴσον τέ ἐστι καὶ ὅμοιον· τρία ἄρα παραλληλόγραμμα τοῦ ΑΕ στερεοῦ τρισὶ παραλληλογράμμοις τοῦ ΨΥ στερεοῦ ἴσα τέ ἐστι καὶ ὅμοια. ἀλλὰ τὰ μὲν τρία τρισὶ τοῖς ἀπεναντίον ἴσα τέ ἐστι καὶ ὅμοια, τὰ δὲ τρία τρισὶ τοῖς ἀπεναντίον· ὅλον ἄρα τὸ ΑΕ στερεὸν παραλληλεπίπεδον ὅλῳ τῷ ΨΥ στερεῷ παραλληλεπιπέδῳ ἴσον ἐστίν. διήχθωσαν αἱ ΔΡ, ΧΥ καὶ συμπιπτέτωσαν ἀλλήλαις κατὰ τὸ Ω, καὶ διὰ τοῦ Τ τῇ ΔΩ παράλληλος ἤχθω ἡ αΤϡ, καὶ ἐκβεβλήσθω ἡ ΟΔ κατὰ τὸ α, καὶ συμπεπληρώσθω τὰ ΩΨ, ΡΙ στερεά. ἴσον δή ἐστι τὸ ΨΩ στερεόν, οὗ βάσις μέν ἐστι τὸ ΡΨ παραλληλόγραμμον, ἀπεναντίον δὲ τὸ Ωϟ, τῷ ΨΥ στερεῷ, οὗ βάσις μὲν τὸ ΡΨ παραλληλόγραμμον, ἀπεναντίον δὲ τὸ ΥΦ· ἐπί τε γὰρ τῆς αὐτῆς βάσεώς εἰσι τῆς ΡΨ καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι αἱ ΡΩ, ΡΥ, Τϡ, ΤΧ, Σϛ, Σο͂, Ψϟ, ΨΦ ἐπὶ τῶν αὐτῶν εἰσιν εὐθειῶν τῶν ΩΧ, ϛΦ. ἀλλὰ τὸ ΨΥ στερεὸν τῷ ΑΕ ἐστιν ἴσον· καὶ τὸ ΨΩ ἄρα στερεὸν τῷ ΑΕ στερεῷ ἐστιν ἴσον. καὶ ἐπεὶ ἴσον ἐστὶ τὸ ΡΥΧΤ παραλληλόγραμμον τῷ ΩΤ παραλληλογράμμῳ· ἐπί τε γὰρ τῆς αὐτῆς βάσεώς εἰσι τῆς ΡΤ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΡΤ, ΩΧ· ἀλλὰ τὸ ΡΥΧΤ τῷ ΓΔ ἐστιν ἴσον, ἐπεὶ καὶ τῷ ΑΒ, καὶ τὸ ΩΤ ἄρα παραλληλόγραμμον



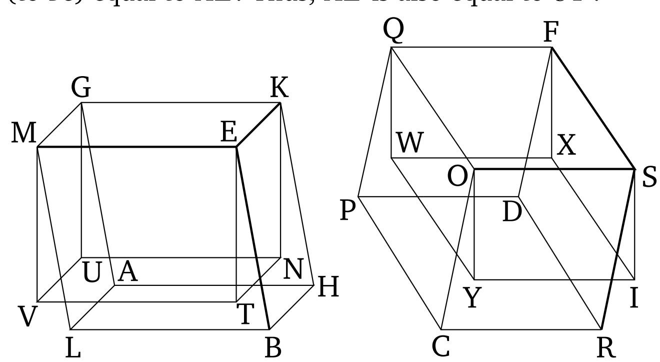
And since the two (straight-lines) TR and RU are equal to the two (straight-lines) AL and LB (respectively), and they contain equal angles, parallelogram RW is thus equal and similar to parallelogram HL [Prop. 6.14]. And, again, since AL is equal to RT, and LM to RS, and they contain right-angles, parallelogram RX is thus equal and similar to parallelogram AM [Prop. 6.14]. So, for the same (reasons), LE is also equal and similar to SU. Thus, three parallelograms of solid AE are equal and similar to three parallelograms of solid XU. But, the three (faces of the former solid) are equal and similar to the three opposite (faces), and the three (faces of the latter solid) to the three opposite (faces) [Prop. 11.24]. Thus, the whole parallelepiped solid AE is equal to the whole parallelepiped solid XU [Def. 11.10]. Let DR and WU have been drawn across, and let them have met one another at Y. And let aTb have been drawn through T parallel to DY. And let PD have been produced to a. And let the solids YX and RI have been completed. So, solid XY, whose base is parallelogram RX, and opposite (face) Yc, is equal to solid XU, whose base (is) parallelogram RX, and opposite (face) UV. For they are on the same base RX, and (have) the same height, and the (ends of the straight-lines) standing up in them, RY, RU, Tb, TW, Se, Sd, Xc and XV, are on the same straight-lines, YW and eV [Prop. 11.29]. But, solid XU is equal to AE. Thus, τῷ ΓΔ ἐστιν ἴσον. ἄλλο δὲ τὸ ΔΤ· ἔστιν ἄρα ὡς ἡ ΓΔ βάσις πρὸς τὴν ΔΤ, οὕτως ἡ ΩΤ πρὸς τὴν ΔΤ. καὶ ἐπεὶ στερεὸν παραλληλεπίπεδον τὸ ΓΙ ἐπιπέδῳ τῷ ΡΖ τέτμηται παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις, ἔστιν ὡς ἡ ΓΔ βάσις πρὸς τὴν ΔΤ βάσιν, οὕτως τὸ ΓΖ στερεὸν πρὸς τὸ ΡΙ στερεόν. διὰ τὰ αὐτὰ δή, ἐπεὶ στερεὸν παραλληλεπίπεδον τὸ ΩΙ ἐπιπέδῳ τῷ ΡΨ τέτμηται παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις, ἔστιν ὡς ἡ ΩΤ βάσις πρὸς τὴν ΤΛ βάσιν, οὕτως τὸ ΩΨ στερεὸν πρὸς τὸ ΡΙ. ἀλλ᾽ ὡς ἡ ΓΔ βάσις πρὸς τὴν ΔΤ, οὕτως ἡ ΩΤ πρὸς τὴν ΔΤ· καὶ ὡς ἄρα τὸ ΓΖ στερεὸν πρὸς τὸ ΡΙ στερεόν, οὕτως τὸ ΩΨ στερεὸν πρὸς τὸ ΡΙ. ἑκάτερον ἄρα τῶν ΓΖ, ΩΨ στερεῶν πρὸς τὸ ΡΙ τὸν αὐτὸν ἔχει λόγον· ἴσον ἄρα ἐστὶ τὸ ΓΖ στερεὸν τῷ ΩΨ στερεῷ. ἀλλὰ τὸ ΩΨ τῷ ΑΕ ἐδείχθη ἴσον· καὶ τὸ ΑΕ ἄρα τῷ ΓΖ ἐστιν ἴσον.



Μὴ ἔστωσαν δὴ αἱ ἐφεστηκυῖαι αἱ ΑΗ, ΘΚ, ΒΕ, ΛΜ, ΓΞ, ΟΠ, ΔΖ, ΡΣ πρὸς ὀρθὰς ταῖς ΑΒ, ΓΔ βάσεσιν· λέγω πάλιν, ὅτι ἴσον τὸ ΑΕ στερεὸν τῷ ΓΖ στερεῷ. ἤχθωσαν γὰρ ἀπὸ τῶν Κ, Ε, Η, Μ, Π, Ζ, Ξ, Σ σημείων ἐπὶ τὸ ὑποκείμενον ἐπίπεδον κάθετοι αἱ ΚΝ, ΕΤ, ΗΥ, ΜΦ, ΠΧ, ΖΨ, ΞΩ, ΣΙ, καὶ συμβαλλέτωσαν τῷ ἐπιπέδῳ κατὰ τὰ Ν, Τ, Υ, Φ, Χ, Ψ, Ω, Ι σημεῖα, καὶ ἐπεζεύχθωσαν αἱ ΝΤ, ΝΥ, ΥΦ, ΤΦ, ΧΨ, ΧΩ, ΩΙ, ΙΨ. ἴσον δή ἐστι τὸ ΚΦ στερεὸν τῷ ΠΙ στερεῷ· ἐπί τε γὰρ ἴσων βάσεών εἰσι τῶν ΚΜ, ΠΣ καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι πρὸς ὀρθάς εἰσι ταῖς βάσεσιν. ἀλλὰ τὸ μὲν ΚΦ στερεὸν τῷ ΑΕ στερεῷ ἐστιν ἴσον, τὸ δὲ ΠΙ τῷ ΓΖ· ἐπί τε γὰρ τῆς αὐτῆς βάσεώς εἰσι καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι οὔκ εἰσιν ἐπὶ τῶν αὐτῶν εὐθειῶν. καὶ τὸ ΑΕ ἄρα στερεὸν τῷ ΓΖ στερεῷ ἐστιν ἴσον.

Τὰ ἄρα ἐπὶ ἴσων βάσεων ὄντα στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

solid XY is also equal to solid AE. And since parallelogram $RUWT$ is equal to parallelogram YT. For they are on the same base RT, and between the same parallels RT and YW [Prop. 1.35]. But, $RUWT$ is equal to CD, since (it is) also (equal) to AB. Parallelogram YT is thus also equal to CD. And DT is another (parallelogram). Thus, as base CD is to DT, so YT (is) to DT [Prop. 5.7]. And since the parallelepiped solid CI has been cut by the plane RF, which is parallel to the opposite planes (of CI), as base CD is to base DT, so solid CF (is) to solid RI [Prop. 11.25]. So, for the same (reasons), since the parallelepiped solid YI has been cut by the plane RX, which is parallel to the opposite planes (of YI), as base YT is to base TD, so solid YX (is) to solid RI [Prop. 11.25]. But, as base CD (is) to DT, so YT (is) to DT. And, thus, as solid CF (is) to solid RI, so solid YX (is) to solid RI. Thus, solids CF and YX each have the same ratio to RI [Prop. 5.11]. Thus, solid CF is equal to solid YX [Prop. 5.9]. But, YX was show (to be) equal to AE. Thus, AE is also equal to CF.

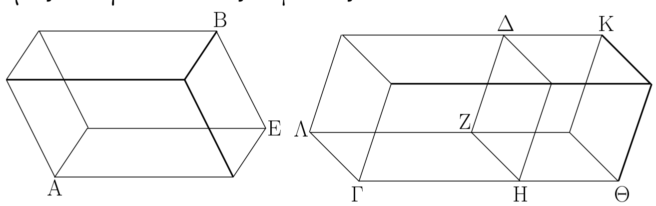


And so let the (straight-lines) standing up, AG, HK, BE, LM, CO, PQ, DF, and RS, not be at right-angles to the bases AB and CD. Again, I say that solid AE (is) equal to solid CF. For let KN, ET, GU, MV, QW, FX, OY, and SI have been drawn from points K, E, G, M, Q, F, O, and S (respectively) perpendicular to the reference plane (*i.e.*, the plane of the bases AB and CD), and let them have met the plane at points N, T, U, V, W, X, Y, and I (respectively). And let NT, NU, UV, TV, WX, WY, YI, and IX have been joined. So solid KV is equal to solid QI. For they are on the equal bases KM and QS, and (have) the same height, and the (straight-lines) standing up in them are at right-angles to their bases (see first part of proposition). But, solid KV is equal to solid AE, and QI to CF. For they are on the same base, and (have) the same height, and the (straight-lines) standing up in them are not on the same straight-lines [Prop. 11.30]. Thus, solid AE is also equal to solid CF.

Thus, parallelepiped solids which are on equal bases, and (have) the same height, are equal to one another. (Which is) the very thing it was required to show.

λβ΄.

Τὰ ὑπὸ τὸ αὐτὸ ὕψος ὄντα στερεὰ παραλληλεπίπεδα πρὸς ἄλληλά ἐστιν ὡς αἱ βάσεις.



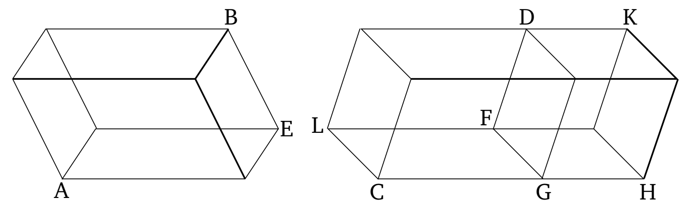
Ἔστω ὑπὸ τὸ αὐτὸ ὕψος στερεὰ παραλληλεπίπεδα τὰ ΑΒ, ΓΔ· λέγω, ὅτι τὰ ΑΒ, ΓΔ στερεὰ παραλληλεπίπεδα πρὸς ἄλληλά ἐστιν ὡς αἱ βάσεις, τουτέστιν ὅτι ἐστὶν ὡς ἡ ΑΕ βάσις πρὸς τὴν ΓΖ βάσιν, οὕτως τὸ ΑΒ στερεὸν πρὸς τὸ ΓΔ στερεόν.

Παραβεβλήσθω γὰρ παρὰ τὴν ΖΗ τῷ ΑΕ ἴσον τὸ ΖΘ, καὶ ἀπὸ βάσεως μὲν τῆς ΖΘ, ὕψους δὲ τοῦ αὐτοῦ τῷ ΓΔ στερεὸν παραλληλεπίπεδον συμπεπληρώσθω τὸ ΗΚ. ἴσον δή ἐστι τὸ ΑΒ στερεὸν τῷ ΗΚ στερεῷ· ἐπί τε γὰρ ἴσων βάσεών εἰσι τῶν ΑΕ, ΖΘ καὶ ὑπὸ τὸ αὐτο ὕψος. καὶ ἐπεὶ στερεὸν παραλληλεπίπεδον τὸ ΓΚ ἐπιπέδῳ τῷ ΔΗ τέτμηται παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις, ἔστιν ἄρα ὡς ἡ ΓΖ βάσις πρὸς τὴν ΖΘ βάσιν, οὕτως τὸ ΓΔ στερεὸν πρὸς τὸ ΔΘ στερεόν. ἴση δὲ ἡ μὲν ΖΘ βάσις τῇ ΑΕ βάσει, τὸ δὲ ΗΚ στερεὸν τῷ ΑΒ στερεῷ· ἔστιν ἄρα καὶ ὡς ἡ ΑΕ βάσις πρὸς τὴν ΓΖ βάσιν, οὕτως τὸ ΑΒ στερεὸν πρὸς τὸ ΓΔ στερεόν.

Τὰ ἄρα ὑπὸ τὸ αὐτὸ ὕψος ὄντα στερεὰ παραλληλεπίπεδα πρὸς ἄλληλά ἐστιν ὡς αἱ βάσεις· ὅπερ ἔδει δεῖξαι.

Proposition 32

Parallelepiped solids which (have) the same height are to one another as their bases.



Let AB and CD be parallelepiped solids (having) the same height. I say that the parallelepiped solids AB and CD are to one another as their bases. That is to say, as base AE is to base CF, so solid AB (is) to solid CD.

For let FH, equal to AE, have been applied to FG (in the angle FGH equal to angle LCG) [Prop. 1.45]. And let the parallelepiped solid GK, (having) the same height as CD, have been completed on the base FH. So solid AB is equal to solid GK. For they are on the equal bases AE and FH, and (have) the same height [Prop. 11.31]. And since the parallelepiped solid CK has been cut by the plane DG, which is parallel to the opposite planes (of CK), thus as the base CF is to the base FH, so the solid CD (is) to the solid DH [Prop. 11.25]. And base FH (is) equal to base AE, and solid GK to solid AB. And thus as base AE is to base CF, so solid AB (is) to solid CD.

Thus, parallelepiped solids which (have) the same height are to one another as their bases. (Which is) the very thing it was required to show.

λγ΄.

Τὰ ὅμοια στερεὰ παραλληλεπίπεδα πρὸς ἄλληλα ἐν τριπλασίονι λόγῳ εἰσὶ τῶν ὁμολόγων πλευρῶν.

Ἔστω ὅμοια στερεὰ παραλληλεπίπεδα τὰ ΑΒ, ΓΔ, ὁμόλογος δὲ ἔστω ἡ ΑΕ τῇ ΓΖ· λέγω, ὅτι τὸ ΑΒ στερεὸν πρὸς τὸ ΓΔ στερεὸν τριπλασίονα λόγον ἔχει, ἤπερ ἡ ΑΕ πρὸς τὴν ΓΖ.

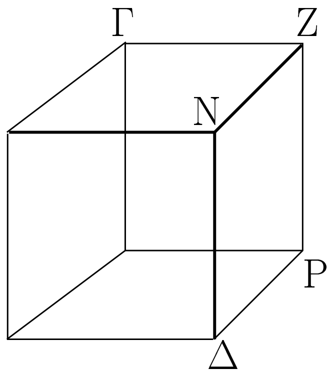
Ἐκβεβλήσθωσαν γὰρ ἐπ᾿ εὐθείας ταῖς ΑΕ, ΗΕ, ΘΕ αἱ ΕΚ, ΕΛ, ΕΜ, καὶ κείσθω τῇ μὲν ΓΖ ἴση ἡ ΕΚ, τῇ δὲ ΖΝ ἴση ἡ ΕΛ, καὶ ἔτι τῇ ΖΡ ἴση ἡ ΕΜ, καὶ συμπεπληρώσθω τὸ ΚΛ παραλληλόγραμμον καὶ τὸ ΚΟ στερεόν.

Proposition 33

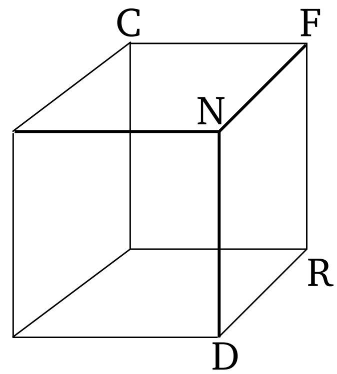
Similar parallelepiped solids are to one another as the cubed ratio of their corresponding sides.

Let AB and CD be similar parallelepiped solids, and let AE correspond to CF. I say that solid AB has to solid CD the cubed ratio that AE (has) to CF.

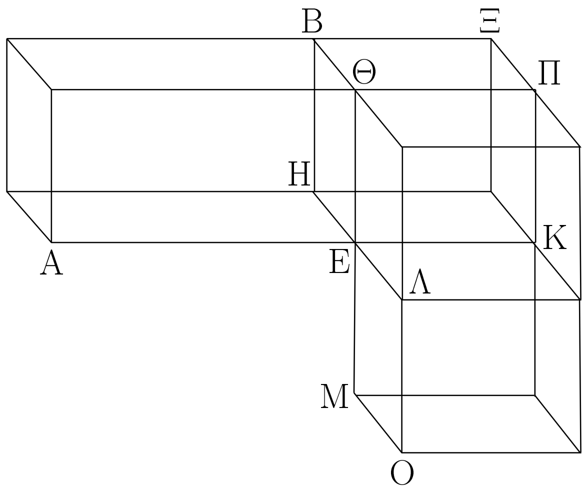
For let EK, EL, and EM have been produced in a straight-line with AE, GE, and HE (respectively). And let EK be made equal to CF, and EL equal to FN, and, further, EM equal to FR. And let the parallelogram KL have been completed, and the solid KP.



Καὶ ἐπεὶ δύο αἱ ΚΕ, ΕΛ δυσὶ ταῖς ΓΖ, ΖΝ ἴσαι εἰσίν, ἀλλὰ καὶ γωνία ἡ ὑπὸ ΚΕΛ γωνίᾳ τῇ ὑπὸ ΓΖΝ ἐστιν ἴση, ἐπειδήπερ καὶ ἡ ὑπὸ ΑΕΗ τῇ ὑπὸ ΓΖΝ ἐστιν ἴση διὰ τὴν ὁμοιότητα τῶν ΑΒ, ΓΔ στερεῶν, ἴσον ἄρα ἐστὶ [καὶ ὅμοιον] τὸ ΚΛ παραλληλόγραμμον τῷ ΓΝ παραλληλογράμμῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ μὲν ΚΜ παραλληλόγραμμον ἴσον ἐστὶ καὶ ὅμοιον τῷ ΓΡ [παραλληλογράμμῳ] καὶ ἔτι τὸ ΕΟ τῷ ΔΖ· τρία ἄρα παραλληλόγραμμα τοῦ ΚΟ στερεοῦ τρισὶ παραλληλογράμμοις τοῦ ΓΔ στερεοῦ ἴσα ἐστὶ καὶ ὅμοια. ἀλλὰ τὰ μὲν τρία τρισὶ τοῖς ἀπεναντίον ἴσα ἐστὶ καὶ ὅμοια, τὰ δὲ τρία τρισὶ τοῖς ἀπεναντίον ἴσα ἐστὶ καὶ ὅμοια· ὅλον ἄρα τὸ ΚΟ στερεὸν ὅλῳ τῷ ΓΔ στερεῷ ἴσον ἐστὶ καὶ ὅμοιον. συμπεπληρώσθω τὸ ΗΚ παραλληλόγραμμον, καὶ ἀπὸ βάσεων μὲν τῶν ΗΚ, ΚΛ παραλληλογράμμων, ὕψους δὲ τοῦ αὐτοῦ τῷ ΑΒ στερεὰ συμπεπληρώσθω τὰ ΕΞ, ΛΠ. καὶ ἐπεὶ διὰ τὴν ὁμοιότητα τῶν ΑΒ, ΓΔ στερεῶν ἐστιν ὡς ἡ ΑΕ πρὸς τὴν ΓΖ, οὕτως ἡ ΕΗ πρὸς τὴν ΖΝ, καὶ ἡ ΕΘ πρὸς τὴν ΖΡ, ἴση δὲ ἡ μὲν ΓΖ τῇ ΕΚ, ἡ δὲ ΖΝ τῇ ΕΛ, ἡ δὲ ΖΡ τῇ ΕΜ, ἔστιν ἄρα ὡς ἡ ΑΕ πρὸς τὴν ΕΚ, οὕτως ἡ ΗΕ πρὸς τὴν ΕΛ καὶ ἡ ΘΕ πρὸς τὴν ΕΜ. ἀλλ᾽ ὡς μὲν ἡ ΑΕ πρὸς τὴν ΕΚ, οὕτως τὸ ΑΗ [παραλληλόγραμμον] πρὸς τὸ ΗΚ παραλληλόγραμμον, ὡς δὲ ἡ ΗΕ πρὸς τὴν ΕΛ, οὕτως τὸ ΗΚ πρὸς τὸ ΚΛ, ὡς δὲ ἡ ΘΕ πρὸς ΕΜ, οὕτως τὸ ΠΕ πρὸς τὸ ΚΜ· καὶ ὡς ἄρα τὸ ΑΗ παραλληλόγραμμον πρὸς τὸ ΗΚ, οὕτως τὸ ΗΚ πρὸς τὸ ΚΛ καὶ τὸ ΠΕ πρὸς τὸ ΚΜ. ἀλλ᾽ ὡς μὲν τὸ ΑΗ πρὸς τὸ ΗΚ, οὕτως τὸ ΑΒ στερεὸν πρὸς τὸ ΕΞ στερεόν, ὡς δὲ τὸ ΗΚ πρὸς τὸ ΚΛ, οὕτως τὸ ΞΕ στερεὸν πρὸς τὸ ΠΛ στερεόν, ὡς δὲ τὸ ΠΕ πρὸς τὸ ΚΜ, οὕτως τὸ ΠΛ στερεὸν πρὸς τὸ ΚΟ στερεόν· καὶ ὡς ἄρα τὸ ΑΒ στερεὸν πρὸς τὸ ΕΞ, οὕτως τὸ ΕΞ πρὸς τὸ ΠΛ καὶ τὸ ΠΛ πρὸς τὸ ΚΟ. ἐὰν δὲ τέσσαρα μεγέθη κατὰ τὸ συνεχὲς ἀνάλογον ᾖ, τὸ πρῶτον πρὸς τὸ τέταρτον τριπλασίονα λόγον ἔχει ἤπερ πρὸς τὸ δεύτερον· τὸ ΑΒ ἄρα στερεὸν πρὸς τὸ ΚΟ τριπλασίονα λόγον ἔχει ἤπερ τὸ ΑΒ πρὸς τὸ ΕΞ. ἀλλ᾽ ὡς τὸ ΑΒ πρὸς τὸ ΕΞ, οὕτως τὸ ΑΗ παραλληλόγραμμον πρὸς τὸ ΗΚ καὶ ἡ ΑΕ εὐθεῖα πρὸς τὴν ΕΚ· ὥστε καὶ τὸ ΑΒ στερεὸν πρὸς τὸ ΚΟ τριπλασίονα λόγον ἔχει ἤπερ ἡ ΑΕ πρὸς τὴν ΕΚ. ἴσον δὲ τὸ [μὲν] ΚΟ στερεὸν τῷ ΓΔ στερεῷ, ἡ δὲ ΕΚ εὐθεῖα τῇ ΓΖ· καὶ τὸ ΑΒ ἄρα στερεὸν πρὸς τὸ ΓΔ στερεὸν τρι-



And since the two (straight-lines) KE and EL are equal to the two (straight-lines) CF and FN, but angle KEL is also equal to angle CFN, inasmuch as AEG is also equal to CFN, on account of the similarity of the solids AB and CD, parallelogram KL is thus equal [and similar] to parallelogram CN. So, for the same (reasons), parallelogram KM is also equal and similar to [parallelogram] CR, and, further, EP to DF. Thus, three parallelograms of solid KP are equal and similar to three parallelograms of solid CD. But the three (former parallelograms) are equal and similar to the three opposite (parallelograms), and the three (latter parallelograms) are equal and similar to the three opposite (parallelograms) [Prop. 11.24]. Thus, the whole of solid KP is equal and similar to the whole of solid CD [Def. 11.10]. Let parallelogram GK have been completed. And let the the solids EO and LQ, with bases the parallelograms GK and KL (respectively), and with the same height as AB, have been completed. And since, on account of the similarity of solids AB and CD, as AE is to CF, so EG (is) to FN, and EH to FR [Defs. 6.1, 11.9], and CF (is) equal to EK, and FN to EL, and FR to EM, thus as AE is to EK, so GE (is) to EL, and HE to EM. But, as AE (is) to EK, so [parallelogram] AG (is) to parallelogram GK, and as GE (is) to EL, so GK (is) to KL, and as HE (is) to EM, so QE (is) to KM [Prop. 6.1]. And thus as parallelogram AG (is) to GK, so GK (is) to KL, and QE (is) to KM. But, as AG (is) to GK, so solid AB (is) to solid EO, and as GK (is) to KL, so solid OE (is) to solid QL, and as QE (is) to KM, so solid QL (is) to solid KP [Prop. 11.32]. And, thus, as solid AB is to EO, so EO (is) to QL, and QL to KP. And if four magnitudes are continuously proportional then the first has to the fourth the cubed ratio that (it has) to the second [Def. 5.10]. Thus, solid AB has to KP the cubed ratio which AB (has) to EO. But, as AB (is) to EO, so parallelogram AG (is) to GK, and the straight-line AE to EK [Prop. 6.1]. Hence, solid AB also has to KP the cubed ratio that AE (has) to EK. And solid KP (is) πλασίονα λόγον ἔχει ἤπερ ἡ ὁμόλογος αὐτοῦ πλευρὰ ἡ ΑΕ πρὸς τὴν ὁμόλογον πλευρὰν τὴν ΓΖ.



Τὰ ἄρα ὅμοια στερεὰ παραλληλεπίπεδα ἐν τριπλασίονι λόγῳ ἐστὶ τῶν ὁμολόγων πλευρῶν· ὅπερ ἔδει δεῖξαι.

Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι ἐὰν τέσσαρες εὐθεῖαι ἀνάλογον ὦσιν, ἔσται ὡς ἡ πρώτη πρὸς τὴν τετάρτην, οὕτω τὸ ἀπὸ τῆς πρώτης στερεὸν παραλληλεπίπεδον πρὸς τὸ ἀπὸ τῆς δευτέρας τὸ ὅμοιον καὶ ὁμοίως ἀναγραφόμενον, ἐπείπερ καὶ ἡ πρώτη πρὸς τὴν τετάρτην τριπλασίονα λόγον ἔχει ἤπερ πρὸς τὴν δευτέραν.

λδ΄.

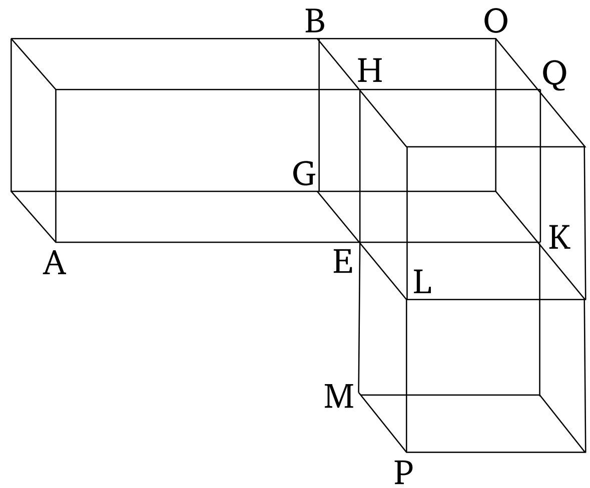
Τῶν ἴσων στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν· καὶ ὧν στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, ἴσα ἐστὶν ἐκεῖνα.

Ἔστω ἴσα στερεὰ παραλληλεπίπεδα τὰ ΑΒ, ΓΔ· λέγω, ὅτι τῶν ΑΒ, ΓΔ στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, καί ἐστιν ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΓΔ στερεοῦ ὕψος πρὸς τὸ τοῦ ΑΒ στερεοῦ ὕψος.

Ἔστωσαν γὰρ πρότερον αἱ ἐφεστηκυῖαι αἱ ΑΗ, ΕΖ, ΛΒ, ΘΚ, ΓΜ, ΝΞ, ΟΔ, ΠΡ πρὸς ὀρθὰς ταῖς βάσεσιν αὐτῶν· λέγω, ὅτι ἐστὶν ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως ἡ ΓΜ πρὸς τὴν ΑΗ.

Εἰ μὲν οὖν ἴση ἐστὶν ἡ ΕΘ βάσιν τῇ ΝΠ βάσει, ἔστι δὲ καὶ τὸ ΑΒ στερεὸν τῷ ΓΔ στερεῷ ἴσον, ἔσται καὶ ἡ ΓΜ τῇ ΑΗ ἴση. τὰ γὰρ ὑπὸ τὸ αὐτὸ ὕψος στερεὰ παραλληλεπίπεδα πρὸς ἄλληλά ἐστιν ὡς αἱ βάσεις. καὶ ἔσται ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ, οὕτως ἡ ΓΜ πρὸς τὴν ΑΗ, καὶ φανερόν, ὅτι

equal to solid CD, and straight-line EK to CF. Thus, solid AB also has to solid CD the cubed ratio which its corresponding side AE (has) to the corresponding side CF.



Thus, similar parallelepiped solids are to one another as the cubed ratio of their corresponding sides. (Which is) the very thing it was required to show.

Corollary

So, (it is) clear, from this, that if four straight-lines are (continuously) proportional then as the first is to the fourth, so the parallelepiped solid on the first will be to the similar, and similarly described, parallelepiped solid on the second, since the first also has to the fourth the cubed ratio that (it has) to the second.

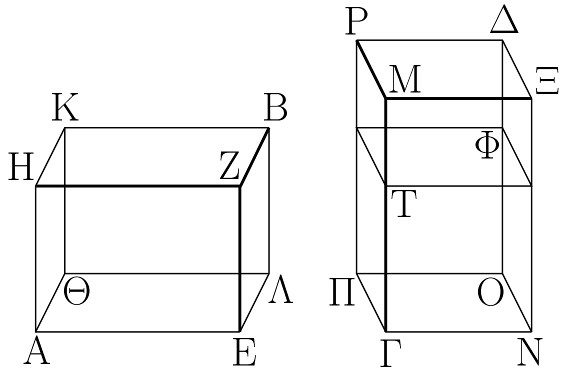
Proposition 34[†]

The bases of equal parallelepiped solids are reciprocally proportional to their heights. And those parallelepiped solids whose bases are reciprocally proportional to their heights are equal.

Let AB and CD be equal parallelepiped solids. I say that the bases of the parallelepiped solids AB and CD are reciprocally proportional to their heights, and (so) as base EH is to base NQ, so the height of solid CD (is) to the height of solid AB.

For, first of all, let the (straight-lines) standing up, AG, EF, LB, HK, CM, NO, PD, and QR, be at right-angles to their bases. I say that as base EH is to base NQ, so CM (is) to AG.

Therefore, if base EH is equal to base NQ, and solid AB is also equal to solid CD, CM will also be equal to AG. For parallelepiped solids of the same height are to one another as their bases [Prop. 11.32]. And as base τῶν ΑΒ, ΓΔ στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν.



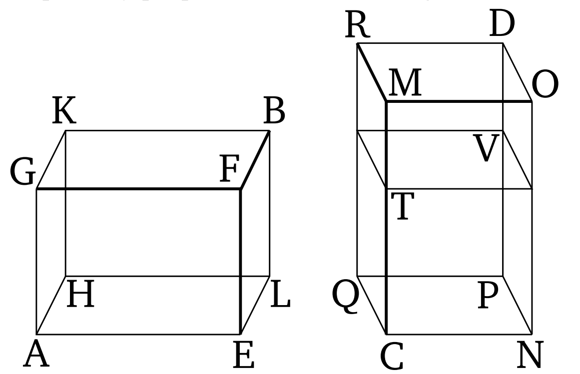
Μὴ ἔστω δὴ ἴση ἡ ΕΘ βάσις τῇ ΝΠ βάσει, ἀλλ᾽ ἔστω μείζων ἡ ΕΘ. ἔστι δὲ καὶ τὸ ΑΒ στερεὸν τῷ ΓΔ στερεῷ ἴσον· μείζων ἄρα ἐστὶ καὶ ἡ ΓΜ τῆς ΑΗ. κείσθω οὖν τῇ ΑΗ ἴση ἡ ΓΤ, καὶ συμπεπληρώσθω ἀπὸ βάσεως μὲν τῆς ΝΠ, ὕψους δὲ τοῦ ΓΤ, στερεὸν παραλληλεπίπεδον τὸ ΦΓ. καὶ ἐπεὶ ἴσον ἐστὶ τὸ ΑΒ στερεὸν τῷ ΓΔ στερεῷ, ἔξωθεν δὲ τὸ ΓΦ, τὰ δὲ ἴσα πρὸς τὸ αὐτὸ τὸν αὐτὸν ἔχει λόγον, ἔστιν ἄρα ὡς τὸ ΑΒ στερεὸν πρὸς τὸ ΓΦ στερεόν, οὕτως τὸ ΓΔ στερεὸν πρὸς τὸ ΓΦ στερεόν. ἀλλ᾽ ὡς μὲν τὸ ΑΒ στερεὸν πρὸς τὸ ΓΦ στερεόν, οὕτως ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν· ἰσοϋψῆ γὰρ τὰ ΑΒ, ΓΦ στερεά· ὡς δὲ τὸ ΓΔ στερεὸν πρὸς τὸ ΓΦ στερεόν, οὕτως ἡ ΜΠ βάσις πρὸς τὴν ΤΠ βάσιν καὶ ἡ ΓΜ πρὸς τὴν ΓΤ· καὶ ὡς ἄρα ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως ἡ ΜΓ πρὸς τὴν ΓΤ. ἴση δὲ ἡ ΓΤ τῇ ΑΗ· καὶ ὡς ἄρα ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως ἡ ΜΓ πρὸς τὴν ΑΗ. τῶν ΑΒ, ΓΔ ἄρα στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν.

Πάλιν δὴ τῶν ΑΒ, ΓΔ στερεῶν παραλληλεπιπέδων ἀντιπεπονθέτωσαν αἱ βάσεις τοῖς ὕψεσιν, καὶ ἔστω ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΓΔ στερεοῦ ὕψος πρὸς τὸ τοῦ ΑΒ στερεοῦ ὕψος· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΒ στερεὸν τῷ ΓΔ στερεῷ.

Ἔστωσαν [γὰρ] πάλιν αἱ ἐφεστηκυῖαι πρὸς ὀρθὰς ταῖς βάσεσιν. καὶ εἰ μὲν ἴση ἐστὶν ἡ ΕΘ βάσις τῇ ΝΠ βάσει, καί ἐστιν ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΓΔ στερεοῦ ὕψος πρὸς τὸ τοῦ ΑΒ στερεοῦ ὕψος, ἴσον ἄρα ἐστὶ καὶ τὸ τοῦ ΓΔ στερεοῦ ὕψος τῷ τοῦ ΑΒ στερεοῦ ὕψει. τὰ δὲ ἐπὶ ἴσων βάσεων στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος ἴσα ἀλλήλοις ἐστίν· ἴσον ἄρα ἐστὶ τὸ ΑΒ στερεὸν τῷ ΓΔ στερεῷ.

Μὴ ἔστω δὴ ἡ ΕΘ βάσις τῇ ΝΠ [βάσει] ἴση, ἀλλ᾽ ἔστω μείζων ἡ ΕΘ· μεῖζον ἄρα ἐστὶ καὶ τὸ τοῦ ΓΔ στερεοῦ ὕψος τοῦ τοῦ ΑΒ στερεοῦ ὕψους, τουτέστιν ἡ ΓΜ τῆς ΑΗ. κείσθω τῇ ΑΗ ἴση πάλιν ἡ ΓΤ, καὶ συμπεπληρώσθω ὁμοίως τὸ ΓΦ στερεόν. ἐπεί ἐστιν ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως ἡ ΜΓ πρὸς τὴν ΑΗ, ἴση δὲ ἡ ΑΗ τῇ ΓΤ,

EH (is) to NQ, so CM will be to AG. And (so it is) clear that the bases of the parallelepiped solids AB and CD are reciprocally proportional to their heights.

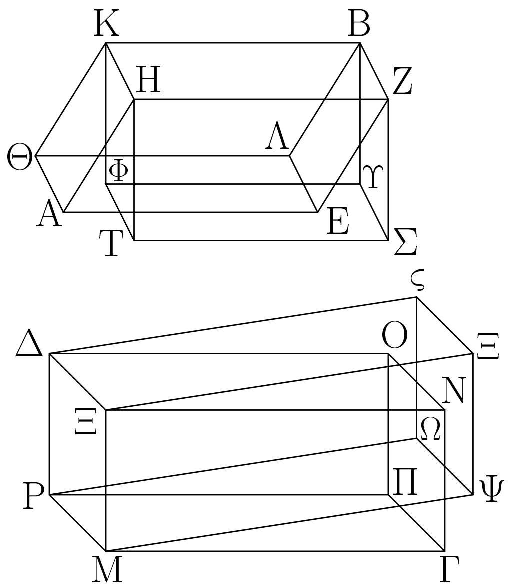


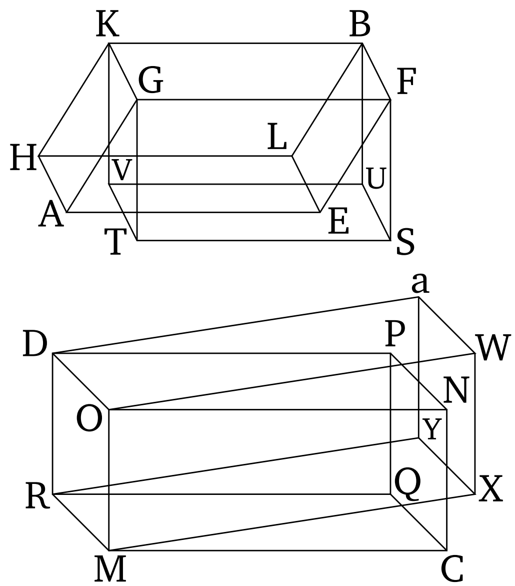
So let base EH not be equal to base NQ, but let EH be greater. And solid AB is also equal to solid CD. Thus, CM is also greater than AG. Therefore, let CT be made equal to AG. And let the parallelepiped solid VC have been completed on the base NQ, with height CT. And since solid AB is equal to solid CD, and CV (is) extrinsic (to them), and equal (magnitudes) have the same ratio to the same (magnitude) [Prop. 5.7], thus as solid AB is to solid CV, so solid CD (is) to solid CV. But, as solid AB (is) to solid CV, so base EH (is) to base NQ. For the solids AB and CV (are) of equal height [Prop. 11.32]. And as solid CD (is) to solid CV, so base MQ (is) to base TQ [Prop. 11.25], and CM to CT [Prop. 6.1]. And, thus, as base EH is to base NQ, so MC (is) to AG. And CT (is) equal to AG. And thus as base EH (is) to base NQ, so MC (is) to AG. Thus, the bases of the parallelepiped solids AB and CD are reciprocally proportional to their heights.

So, again, let the bases of the parallelepipid solids AB and CD be reciprocally proportional to their heights, and let base EH be to base NQ, as the height of solid CD (is) to the height of solid AB. I say that solid AB is equal to solid CD. [For] let the (straight-lines) standing up again be at right-angles to the bases. And if base EH is equal to base NQ, and as base EH is to base NQ, so the height of solid CD (is) to the height of solid AB, the height of solid CD is thus also equal to the height of solid AB. And parallelepiped solids on equal bases, and also with the same height, are equal to one another [Prop. 11.31]. Thus, solid AB is equal to solid CD.

So, let base EH not be equal to [base] NQ, but let EH be greater. Thus, the height of solid CD is also greater than the height of solid AB, that is to say CM (greater) than AG. Let CT again be made equal to AG, and let the solid CV have been similarly completed. Since as base EH is to base NQ, so MC (is) to AG, ἔστιν ἄρα ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως ἡ ΓΜ πρὸς τὴν ΓΤ. ἀλλ᾽ ὡς μὲν ἡ ΕΘ [βάσις] πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ ΑΒ στερεὸν πρὸς τὸ ΓΦ στερεόν· ἰσοϋψῆ γάρ ἐστι τὰ ΑΒ, ΓΦ στερεά· ὡς δὲ ἡ ΓΜ πρὸς τὴν ΓΤ, οὕτως ἥ τε ΜΠ βάσις πρὸς τὴν ΠΤ βάσιν καὶ τὸ ΓΔ στερεὸν πρὸς τὸ ΓΦ στερεόν. καὶ ὡς ἄρα τὸ ΑΒ στερεὸν πρὸς τὸ ΓΦ στερεόν, οὕτως τὸ ΓΔ στερεὸν πρὸς τὸ ΓΦ στερεόν· ἑκάτερον ἄρα τῶν ΑΒ, ΓΔ πρὸς τὸ ΓΦ τὸν αὐτὸν ἔχει λόγον. ἴσον ἄρα ἐστὶ τὸ ΑΒ στερεὸν τῷ ΓΔ στερεῷ.

and AG (is) equal to CT, thus as base EH (is) to base NQ, so CM (is) to CT. But, as [base] EH (is) to base NQ, so solid AB (is) to solid CV. For solids AB and CV are of equal heights [Prop. 11.32]. And as CM (is) to CT, so (is) base MQ to base QT [Prop. 6.1], and solid CD to solid CV [Prop. 11.25]. And thus as solid AB (is) to solid CV, so solid CD (is) to solid CV. Thus, AB and CD each have the same ratio to CV. Thus, solid AB is equal to solid CD [Prop. 5.9].





Μὴ ἔστωσαν δὴ αἱ ἐφεστηκυῖαι αἱ ΖΕ, ΒΛ, ΗΑ, ΚΘ, ΞΝ, ΔΟ, ΜΓ, ΡΠ πρὸς ὀρθὰς ταῖς βάσεσιν αὐτῶν, καὶ ἤχθωσαν ἀπὸ τῶν Ζ, Η, Β, Κ, Ξ, Μ, Ρ, Δ σημείων ἐπὶ τὰ διὰ τῶν ΕΘ, ΝΠ ἐπίπεδα κάθετοι καὶ συμβαλλέτωσαν τοῖς ἐπιπέδοις κατὰ τὰ Σ, Τ, Υ, Φ, Χ, Ψ, Ω, ς, καὶ συμπεπληρώσθω τὰ ΖΦ, ΞΩ στερεά· λέγω, ὅτι καὶ οὕτως ἴσων ὄντων τῶν ΑΒ, ΓΔ στερεῶν ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, καί ἐστιν ὡς ἡ ΕΘ βάσιν πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΓΔ στερεοῦ ὕψος πρὸς τὸ τοῦ ΑΒ στερεοῦ ὕψος.

Ἐπεὶ ἴσον ἐστὶ τὸ ΑΒ στερεὸν τῷ ΓΔ στερεῷ, ἀλλὰ τὸ μὲν ΑΒ τῷ ΒΤ ἐστιν ἴσον· ἐπί τε γὰρ τῆς αὐτῆς βάσεώς εἰσι τῆς ΖΚ καὶ ὑπὸ τὸ αὐτὸ ὕψος· τὸ δὲ ΓΔ στερεὸν τῷ ΔΨ ἐστιν ἴσον· ἐπί τε γὰρ πάλιν τῆς αὐτῆς βάσεώς εἰσι τῆς ΡΞ καὶ ὑπὸ τὸ αὐτὸ ὕψος· καὶ τὸ ΒΤ ἄρα στερεὸν τῷ ΔΨ στερεῷ ἴσον ἐστίν. ἔστιν ἄρα ὡς ἡ ΖΚ βάσις πρὸς τὴν ΞΡ βάσιν, οὕτως τὸ τοῦ ΔΨ στερεοῦ ὕψος πρὸς τὸ τοῦ ΒΤ στερεοῦ ὕψος. ἴση δὲ ἡ μὲν ΖΚ βάσις τῇ ΕΘ βάσει, ἡ δὲ ΞΡ βάσις τῇ ΝΠ βάσει· ἔστιν ἄρα ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΔΨ στερεοῦ ὕψος πρὸς τὸ τοῦ ΒΤ στερεοῦ ὕψος. τὰ δ᾽ αὐτὰ ὕψη ἐστὶ τῶν ΔΨ, ΒΤ στερεῶν καὶ τῶν ΔΓ, ΒΑ· ἔστιν ἄρα ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ

So, let the (straight-lines) standing up, FE, BL, GA, KH, ON, DP, MC, and RQ, not be at right-angles to their bases. And let perpendiculars have been drawn to the planes through EH and NQ from points F, G, B, K, O, M, R, and D, and let them have joined the planes at (points) S, T, U, V, W, X, Y, and a (respectively). And let the solids FV and OY have been completed. In this case, also, I say that the solids AB and CD being equal, their bases are reciprocally proportional to their heights, and (so) as base EH is to base NQ, so the height of solid CD (is) to the height of solid AB.

Since solid AB is equal to solid CD, but AB is equal to BT. For they are on the same base FK, and (have) the same height [Props. 11.29, 11.30]. And solid CD is equal is equal to DX. For, again, they are on the same base RO, and (have) the same height [Props. 11.29, 11.30]. Solid BT is thus also equal to solid DX. Thus, as base FK (is) to base OR, so the height of solid DX (is) to the height of solid BT (see first part of proposition). And base FK (is) equal to base EH, and base OR to NQ. Thus, as base EH is to base NQ, so the height of solid DX (is) to βάσιν, οὕτως τὸ τοῦ ΔΓ στερεοῦ ὕψος πρὸς τὸ τοῦ ΑΒ στερεοῦ ὕψος. τῶν ΑΒ, ΓΔ ἄρα στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν.

Πάλιν δὴ τῶν ΑΒ, ΓΔ στερεῶν παραλληλεπιπέδων ἀντιπεπονθέτωσαν αἱ βάσεις τοῖς ὕψεσιν, καὶ ἔστω ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΓΔ στερεοῦ ὕψος πρὸς τὸ τοῦ ΑΒ στερεοῦ ὕψος· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΒ στερεὸν τῷ ΓΔ στερεῷ.

Τῶν γὰρ αὐτῶν κατασκευασθέντων, ἐπεί ἐστιν ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΓΔ στερεοῦ ὕψος πρὸς τὸ τοῦ ΑΒ στερεοῦ ὕψος, ἴση δὲ ἡ μὲν ΕΘ βάσις τῇ ΖΚ βάσει, ἡ δὲ ΝΠ τῇ ΞΡ, ἔστιν ἄρα ὡς ἡ ΖΚ βάσις πρὸς τὴν ΞΡ βάσιν, οὕτως τὸ τοῦ ΓΔ στερεοῦ ὕψος πρὸς τὸ τοῦ ΑΒ στερεοῦ ὕψος. τὰ δ᾽ αὐτὰ ὕψη ἐστὶ τῶν ΑΒ, ΓΔ στερεῶν καὶ τῶν ΒΤ, ΔΨ· ἔστιν ἄρα ὡς ἡ ΖΚ βάσις πρὸς τὴν ΞΡ βάσιν, οὕτως τὸ τοῦ ΔΨ στερεοῦ ὕψος πρὸς τὸ τοῦ ΒΤ στερεοῦ ὕψος. τῶν ΒΤ, ΔΨ ἄρα στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν· ἴσον ἄρα ἐστὶ τὸ ΒΤ στερεὸν τῷ ΔΨ στερεῷ. ἀλλὰ τὸ μὲν ΒΤ τῷ ΒΑ ἴσον ἐστίν· ἐπί τε γὰρ τῆς αὐτῆς βάσεως [εἰσι] τῆς ΖΚ καὶ ὑπὸ τὸ αὐτὸ ὕψος. τὸ δὲ ΔΨ στερεὸν τῷ ΔΓ στερεῷ ἴσον ἐστίν. καὶ τὸ ΑΒ ἄρα στερεὸν τῷ ΓΔ στερεῷ ἐστιν ἴσον· ὅπερ ἔδει δεῖξαι.

the height of solid BT. And solids DX, BT are the same height as (solids) DC, BA (respectively). Thus, as base EH is to base NQ, so the height of solid DC (is) to the height of solid AB. Thus, the bases of the parallelepiped solids AB and CD are reciprocally proportional to their heights.

So, again, let the bases of the parallelepiped solids AB and CD be reciprocally proportional to their heights, and (so) let base EH be to base NQ, as the height of solid CD (is) to the height of solid AB. I say that solid AB is equal to solid CD.

For, with the same construction (as before), since as base EH is to base NQ, so the height of solid CD (is) to the height of solid AB, and base EH (is) equal to base FK, and NQ to OR, thus as base FK is to base OR, so the height of solid CD (is) to the height of solid AB. And solids AB, CD are the same height as (solids) BT, DX (respectively). Thus, as base FK is to base OR, so the height of solid DX (is) to the height of solid BT. Thus, the bases of the parallelepiped solids BT and DX are reciprocally proportional to their heights. Thus, solid BT is equal to solid DX (see first part of proposition). But, BT is equal to BA. For [they are] on the same base FK, and (have) the same height [Props. 11.29, 11.30]. And solid DX is equal to solid DC [Props. 11.29, 11.30]. Thus, solid AB is also equal to solid CD. (Which is) the very thing it was required to show.

† This proposition assumes that (a) if two parallelepipeds are equal, and have equal bases, then their heights are equal, and (b) if the bases of two equal parallelepipeds are unequal, then that solid which has the lesser base has the greater height.

λεʹ.

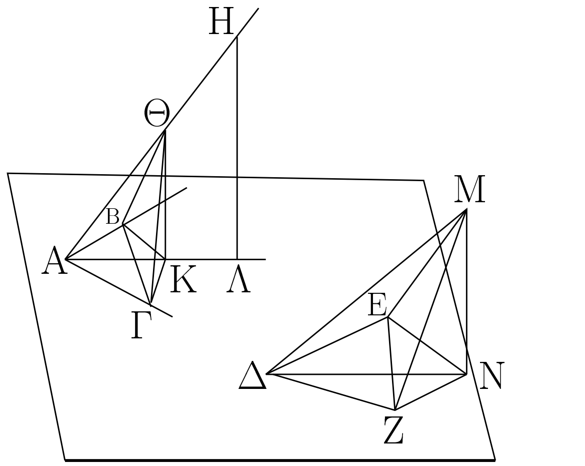
Ἐὰν ὦσι δύο γωνίαι ἐπίπεδοι ἴσαι, ἐπὶ δὲ τῶν κορυφῶν αὐτῶν μετέωροι εὐθεῖαι ἐπισταθῶσιν ἴσας γωνίας περιέχουσαι μετὰ τῶν ἐξ ἀρχῆς εὐθειῶν ἑκατέραν ἑκατέρᾳ, ἐπὶ δὲ τῶν μετεώρων ληφθῇ τυχόντα σημεῖα, καὶ ἀπ᾽ αὐτῶν ἐπὶ τὰ ἐπίπεδα, ἐν οἷς εἰσιν αἱ ἐξ ἀρχῆς γωνίαι, κάθετοι ἀχθῶσιν, ἀπὸ δὲ τῶν γενομένων σημείων ἐν τοῖς ἐπιπέδοις ἐπὶ τὰς ἐξ ἀρχῆς γωνίας ἐπιζευχθῶσιν εὐθεῖαι, ἴσας γωνίας περιέξουσι μετὰ τῶν μετεώρων.

Ἔστωσαν δύο γωνίαι εὐθύγραμμοι ἴσαι αἱ ὑπὸ ΒΑΓ, ΕΔΖ, ἀπὸ δὲ τῶν Α, Δ σημείων μετέωροι εὐθεῖαι ἐφεστάτωσαν αἱ ΑΗ, ΔΜ ἴσας γωνίας περιέχουσιν μετὰ τῶν ἐξ ἀρχῆς εὐθειῶν ἑκατέραν ἑκατέρᾳ, τὴν μὲν ὑπὸ ΜΔΕ τῇ ὑπὸ ΗΑΒ, τὴν δὲ ὑπὸ ΜΔΖ τῇ ὑπὸ ΗΑΓ, καὶ εἰλήφθω ἐπὶ τῶν ΑΗ, ΔΜ τυχόντα σημεῖα τὰ Η, Μ, καὶ ἤχθωσαν ἀπὸ τῶν Η, Μ σημείων ἐπὶ τὰ διὰ τῶν ΒΑΓ, ΕΔΖ ἐπίπεδα κάθετοι αἱ ΗΛ, ΜΝ, καὶ συμβαλλέτωσαν τοῖς ἐπιπέδοις κατὰ τὰ Λ, Ν, καὶ ἐπεζεύχθωσαν αἱ ΛΑ, ΝΔ· λέγω, ὅτι ἴση ἐστὶν ἡ ὑπὸ ΗΑΛ γωνία τῇ ὑπὸ ΜΔΝ γωνίᾳ.

Proposition 35

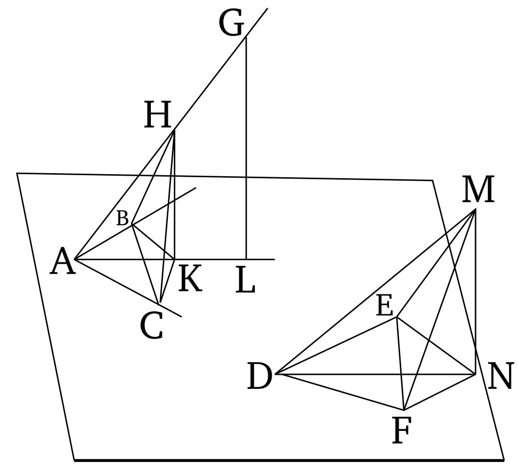
If there are two equal plane angles, and raised straight-lines are stood on the apexes of them, containing equal angles respectively with the original straight-lines (forming the angles), and random points are taken on the raised (straight-lines), and perpendiculars are drawn from them to the planes in which the original angles are, and straight-lines are joined from the points created in the planes to the (vertices of the) original angles, then they will enclose equal angles with the raised (straight-lines).

Let BAC and EDF be two equal rectilinear angles. And let the raised straight-lines AG and DM have been stood on points A and D, containing equal angles respectively with the original straight-lines. (That is) MDE (equal) to GAB, and MDF (to) GAC. And let the random points G and M have been taken on AG and DM (respectively). And let the GL and MN have been drawn from points G and M perpendicular to the planes through



Κείσθω τῇ ΔΜ ἴση ἡ ΑΘ, καὶ ἤχθω διὰ τοῦ Θ σημείου τῇ ΗΛ παράλληλος ἡ ΘΚ. ἡ δὲ ΗΛ κάθετός ἐστιν ἐπὶ τὸ διὰ τῶν ΒΑΓ ἐπίπεδον· καὶ ἡ ΘΚ ἄρα κάθετός ἐστιν ἐπὶ τὸ διὰ τῶν ΒΑΓ ἐπίπεδον. ἤχθωσαν ἀπὸ τῶν Κ, Ν σημείων ἐπὶ τὰς ΑΓ, ΔΖ, ΑΒ, ΔΕ εὐθείας κάθετοι αἱ ΚΓ, ΝΖ, ΚΒ, ΝΕ, καὶ ἐπεζεύχθωσαν αἱ ΘΓ, ΓΒ, ΜΖ, ΖΕ. ἐπεὶ τὸ ἀπὸ τῆς ΘΑ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΘΚ, ΚΑ, τῷ δὲ ἀπὸ τῆς ΚΑ ἴσα ἐστὶ τὰ ἀπὸ τῶν ΚΓ, ΓΑ, καὶ τὸ ἀπὸ τῆς ΘΑ ἄρα ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΘΚ, ΚΓ, ΓΑ. τοῖς δὲ ἀπὸ τῶν ΘΚ, ΚΓ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΘΓ· τὸ ἄρα ἀπὸ τῆς ΘΑ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΘΓ, ΓΑ. ὀρθὴ ἄρα ἐστὶν ἡ ὑπὸ ΘΓΑ γωνία. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΔΖΜ γωνία ὀρθή ἐστιν. ἴση ἄρα ἐστὶν ἡ ὑπὸ ΑΓΘ γωνία τῇ ὑπὸ ΔΖΜ. ἔστι δὲ καὶ ἡ ὑπὸ ΘΑΓ τῇ ὑπὸ ΜΔΖ ἴση. δύο δὴ τρίγωνά ἐστι τὰ ΜΔΖ, ΘΑΓ δύο γωνίας δυσὶ γωνίαις ἴσας ἔχοντα ἑκατέραν ἑκατέρᾳ καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην τὴν ὑποτείνουσαν ὑπὸ μίαν τῶν ἴσων γωνιῶν τὴν ΘΑ τῇ ΜΔ· καὶ τὰς λοιπὰς ἄρα πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξει ἑκατέραν ἑκαρέρᾳ. ἴση ἄρα ἐστὶν ἡ ΑΓ τῇ ΔΖ. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἡ ΑΒ τῇ ΔΕ ἐστιν ἴση. ἐπεὶ οὖν ἴση ἐστὶν ἡ μὲν ΑΓ τῇ ΔΖ, ἡ δὲ ΑΒ τῇ ΔΕ, δύο δὴ αἱ ΓΑ, ΑΒ δυσὶ ταῖς ΖΔ, ΔΕ ἴσαι εἰσίν. ἀλλὰ καὶ γωνία ἡ ὑπὸ ΓΑΒ γωνίᾳ τῇ ὑπὸ ΖΔΕ ἐστιν ἴση· βάσις ἄρα ἡ ΒΓ βάσει τῇ ΕΖ ἴση ἐστὶ καὶ τὸ τρίγωνον τῷ τριγώνῳ καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις· ἴση ἄρα ἡ ὑπὸ ΑΓΒ γωνία τῇ ὑπὸ ΔΖΕ. ἔστι δὲ καὶ ὀρθὴ ἡ ὑπὸ ΑΓΚ ὀρθῇ τῇ ὑπὸ ΔΖΝ ἴση· καὶ λοιπὴ ἄρα ἡ ὑπὸ ΒΓΚ λοιπῇ τῇ ὑπὸ ΕΖΝ ἐστιν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΓΒΚ τῇ ὑπὸ ΖΕΝ ἐστιν ἴση. δύο δὴ τρίγωνά ἐστι τὰ ΒΓΚ, ΕΖΝ [τὰς] δύο γωνίας δυσὶ γωνίαις ἴσας ἔχοντα ἑκατέραν ἑκατέρᾳ καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην τὴν πρὸς ταῖς ἴσαις γωνίαις τὴν ΒΓ τῇ ΕΖ· καὶ τὰς λοιπὰς ἄρα πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξουσιν. ἴση ἄρα ἐστὶν ἡ ΓΚ τῇ ΖΝ. ἔστι δὲ

BAC and EDF (respectively). And let them have joined the planes at points L and N (respectively). And let LA and ND have been joined. I say that angle GAL is equal to angle MDN.



Let AH be made equal to DM. And let HK have been drawn through point H parallel to GL. And GL is perpendicular to the plane through BAC. Thus, HK is also perpendicular to the plane through BAC [Prop. 11.8]. And let KC, NF, KB, and NE have been drawn from points K and N perpendicular to the straight-lines AC, DF, AB, and DE. And let HC, CB, MF, and FE have been joined. Since the (square) on HA is equal to the (sum of the squares) on HK and KA [Prop. 1.47], and the (sum of the squares) on KC and CA is equal to the (square) on KA [Prop. 1.47], thus the (square) on HA is equal to the (sum of the squares) on HK, KC, and CA. And the (square) on HC is equal to the (sum of the squares) on HK and KC [Prop. 1.47]. Thus, the (square) on HA is equal to the (sum of the squares) on HC and CA. Thus, angle HCA is a right-angle [Prop. 1.48]. So, for the same (reasons), angle DFM is also a right-angle. Thus, angle ACH is equal to (angle) DFM. And HAC is also equal to MDF. So, MDF and HAC are two triangles having two angles equal to two angles, respectively, and one side equal to one side—(namely), that subtending one of the equal angles —(that is), HA (equal) to MD. Thus, they will also have the remaining sides equal to the remaining sides, respectively [Prop. 1.26]. Thus, AC is equal to DF. So, similarly, we can show that AB is also equal to DE. Therefore, since AC is equal to DF, and AB to DE, so the two (straight-lines) CA and AB are equal to the two (straight-lines) FD and DE (respectively). But, angle CAB is also equal to angle FDE. Thus, base BC is equal to base EF, and triangle (ACB) to triangle (DFE), and the remaining angles to the remaining angles (respectively) [Prop. 1.4].

καὶ ἡ ΑΓ τῇ ΔΖ ἴση· δύο δὴ αἱ ΑΓ, ΓΚ δυσὶ ταῖς ΔΖ, ΖΝ ἴσαι εἰσίν· καὶ ὀρθὰς γωνίας περιέχουσιν. βάσις ἄρα ἡ ΑΚ βάσει τῇ ΔΝ ἴση ἐστίν. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΘ τῇ ΔΜ, ἴσον ἐστὶ καὶ τὸ ἀπὸ τῆς ΑΘ τῷ ἀπὸ τῆς ΔΜ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΘ ἴσα ἐστὶ τὰ ἀπὸ τῶν ΑΚ, ΚΘ· ὀρθὴ γὰρ ἡ ὑπὸ ΑΚΘ· τῷ δὲ ἀπὸ τῆς ΔΜ ἴσα τὰ ἀπὸ τῶν ΔΝ, ΝΜ· ὀρθὴ γὰρ ἡ ὑπὸ ΔΝΜ· τὰ ἄρα ἀπὸ τῶν ΑΚ, ΚΘ ἴσα ἐστὶ τοῖς ἀπὸ τῶν ΔΝ, ΝΜ, ὧν τὸ ἀπὸ τῆς ΑΚ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΔΝ· λοιπὸν ἄρα τὸ ἀπὸ τῆς ΚΘ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΝΜ· ἴση ἄρα ἡ ΘΚ τῇ ΜΝ. καὶ ἐπεὶ δύο αἱ ΘΑ, ΑΚ δυσὶ ταῖς ΜΔ, ΔΝ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ, καὶ βάσις ἡ ΘΚ βάσει τῇ ΜΝ ἐδείχθη ἴση, γωνία ἄρα ἡ ὑπὸ ΘΑΚ γωνίᾳ τῇ ὑπὸ ΜΔΝ ἐστιν ἴση.

Ἐὰν ἄρα ὦσι δύο γωνίαι ἐπίπεδοι ἴσαι καὶ τὰ ἑξῆς τῆς προτάσεως [ὅπερ ἔδει δεῖξαι].

Thus, angle ACB (is) equal to DFE. And the right-angle ACK is also equal to the right-angle DFN. Thus, the remainder BCK is equal to the remainder EFN. So, for the same (reasons), CBK is also equal to FEN. So, BCK and EFN are two triangles having two angles equal to two angles, respectively, and one side equal to one side—(namely), that by the equal angles—(that is), BC (equal) to EF. Thus, they will also have the remaining sides equal to the remaining sides (respectively) [Prop. 1.26]. Thus, CK is equal to FN. And AC (is) also equal to DF. So, the two (straight-lines) AC and CK are equal to the two (straight-lines) DF and FN (respectively). And they enclose right-angles. Thus, base AK is equal to base DN [Prop. 1.4]. And since AH is equal to DM, the (square) on AH is also equal to the (square) on DM. But, the the (sum of the squares) on AK and KH is equal to the (square) on AH. For angle AKH (is) a right-angle [Prop. 1.47]. And the (sum of the squares) on DN and NM (is) equal to the square on DM. For angle DNM (is) a right-angle [Prop. 1.47]. Thus, the (sum of the squares) on AK and KH is equal to the (sum of the squares) on DN and NM, of which the (square) on AK is equal to the (square) on DN. Thus, the remaining (square) on KH is equal to the (square) on NM. Thus, HK (is) equal to MN. And since the two (straight-lines) HA and AK are equal to the two (straight-lines) MD and DN, respectively, and base HK was shown (to be) equal to base MN, angle HAK is thus equal to angle MDN [Prop. 1.8].

Thus, if there are two equal plane angles, and so on of the proposition. [(Which is) the very thing it was required to show].

Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι, ἐὰν ὦσι δύο γωνίαι ἐπίπεδοι ἴσαι, ἐπισταθῶσι δὲ ἐπ᾿ αὐτῶν μετέωροι εὐθεῖαι ἴσαι ἴσας γωνίας περιέχουσαι μετὰ τῶν ἐξ ἀρχῆς εὐθειῶν ἑκατέραν ἑκατέρᾳ, αἱ ἀπ᾿ αὐτῶν κάθετοι ἀγόμεναι ἐπὶ τὰ ἐπίπεδα, ἐν οἷς εἰσιν αἱ ἐξ ἀρχῆς γωνίαι, ἴσαι ἀλλήλαις εἰσίν. ὅπερ ἔδει δεῖξαι.

Corollary

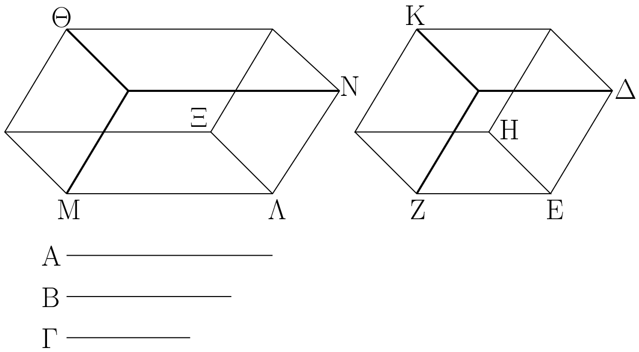
So, it is clear, from this, that if there are two equal plane angles, and equal raised straight-lines are stood on them (at their apexes), containing equal angles respectively with the original straight-lines (forming the angles), then the perpendiculars drawn from (the raised ends of) them to the planes in which the original angles lie are equal to one another. (Which is) the very thing it was required to show.

λϛʹ.

Ἐὰν τρεῖς εὐθεῖαι ἀνάλογον ὦσιν, τὸ ἐκ τῶν τριῶν στερεὸν παραλληλεπίπεδον ἴσον ἐστὶ τῷ ἀπὸ τῆς μέσης στερεῷ παραλληλεπιπέδῳ ἰσοπλεύρῳ μέν, ἰσογωνίῳ δὲ τῷ προειρημένῳ.

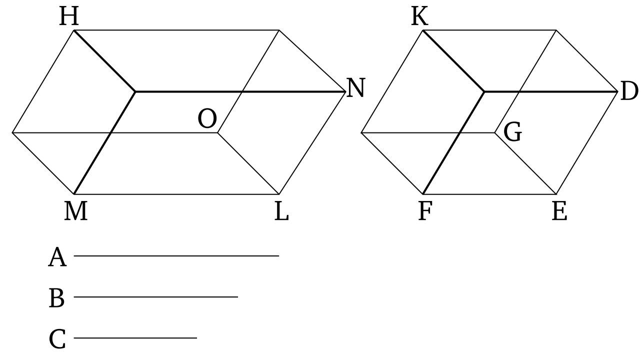
Proposition 36

If three straight-lines are (continuously) proportional then the parallelepiped solid (formed) from the three (straight-lines) is equal to the equilateral parallelepiped solid on the middle (straight-line which is) equiangular to the aforementioned (parallelepiped solid).



Ἔστωσαν τρεῖς εὐθεῖαι ἀνάλογον αἱ Α, Β, Γ, ὡς ἡ Α πρὸς τὴν Β, οὕτως ἡ Β πρὸς τὴν Γ· λέγω, ὅτι τὸ ἐκ τῶν Α, Β, Γ στερεὸν ἴσον ἐστὶ τῷ ἀπὸ τῆς Β στερεῷ ἰσοπλεύρῳ μέν, ἰσογωνίῳ δὲ τῷ προειρημένῳ.

Ἐκκείσθω στερεὰ γωνία ἡ πρὸς τῷ Ε περιεχομένη ὑπὸ τῶν ὑπὸ ΔΕΗ, ΗΕΖ, ΖΕΔ, καὶ κείσθω τῇ μὲν Β ἴση ἑκάστη τῶν ΔΕ, ΗΕ, ΕΖ, καὶ συμπεπληρώσθω τὸ ΕΚ στερεὸν παραλληλεπίπεδον, τῇ δὲ Α ἴση ἡ ΛΜ, καὶ συνεστάτω πρὸς τῇ ΛΜ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Λ τῇ πρὸς τῷ Ε στερεᾷ γωνίᾳ ἴση στερεὰ γωνία ἡ περειχομένη ὑπὸ τῶν ΝΛΞ, ΞΛΜ, ΜΛΝ, καὶ κείσθω τῇ μὲν Β ἴση ἡ ΛΞ, τῇ δὲ Γ ἴση ἡ ΛΝ. καὶ ἐπεί ἐστιν ὡς ἡ Α πρὸς τὴν Β, οὕτως ἡ Β πρὸς τὴν Γ, ἴση δὲ ἡ μὲν Α τῇ ΛΜ, ἡ δὲ Β ἑκατέρᾳ τῶν ΛΞ, ΕΔ, ἡ δὲ Γ τῇ ΛΝ, ἔστιν ἄρα ὡς ἡ ΛΜ πρὸς τὴν ΕΖ, οὕτως ἡ ΔΕ πρὸς τὴν ΛΝ. καὶ περὶ ἴσας γωνίας τὰς ὑπὸ ΝΛΜ, ΔΕΖ αἱ πλευραὶ ἀντιπεπόνθασιν· ἴσον ἄρα ἐστὶ τὸ ΜΝ παραλληλόγραμμον τῷ ΔΖ παραλληλογραμάμμῳ. καὶ ἐπεὶ δύο γωνίαι ἐπίπεδοι εὐθύγραμμοι ἴσαι εἰσὶν αἱ ὑπὸ ΔΕΖ, ΝΛΜ, καὶ ἐπ᾽ αὐτῶν μετέωροι εὐθεῖαι ἐφεστᾶσιν αἱ ΛΞ, ΕΗ ἴσαι τε ἀλλήλαις καὶ ἴσας γωνίας περιέχουσαι μετὰ τῶν ἐξ ἀρχῆς εὐθειῶν ἑκατέραν ἑκατέρᾳ, αἱ ἄρα ἀπὸ τῶν Η, Ξ σημείων κάθετοι ἀγόμεναι ἐπὶ τὰ διὰ τῶν ΝΛΜ, ΔΕΖ ἐπίπεδα ἴσαι ἀλλήλαις εἰσίν· ὥστε τὰ ΛΘ, ΕΚ στερεὰ ὑπὸ τὸ αὐτὸ ὕψος ἐστίν. τὰ δὲ ἐπὶ ἴσων βάσεων στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος ἴσα ἀλλήλοις ἐστίν· ἴσον ἄρα ἐστὶ τὸ ΘΛ στερεὸν τῷ ΕΚ στερεῷ. καί ἐστι τὸ μὲν ΛΘ τὸ ἐκ τῶν Α, Β, Γ στερεόν, τὸ δὲ ΕΚ τὸ ἀπὸ τῆς Β στερεόν· τὸ ἄρα ἐκ τῶν Α, Β, Γ στερεὸν παραλληλεπίπεδον ἴσον ἐστὶ τῷ ἀπὸ τῆς Β στερεῷ ἰσοπλεύρῳ μέν, ἰσογωνίῳ δὲ τῷ προειρημένῳ· ὅπερ ἔδει δεῖξαι.



Let A, B, and C be three (continuously) proportional straight-lines, (such that) as A (is) to B, so B (is) to C. I say that the (parallelepiped) solid (formed) from A, B, and C is equal to the equilateral solid on B (which is) equiangular with the aforementioned (solid).

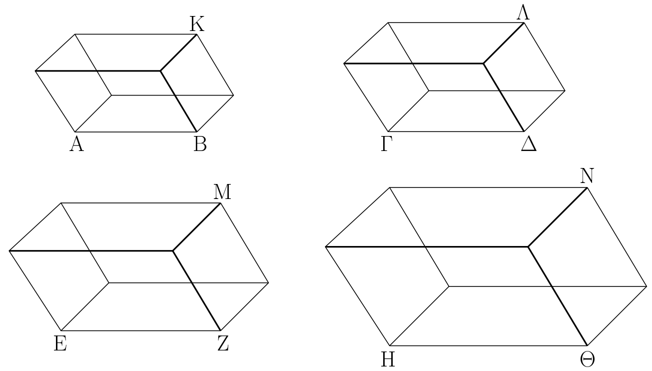
Let the solid angle at E, contained by DEG, GEF, and FED, be set out. And let DE, GE, and EF each be made equal to B. And let the parallelepiped solid EK have been completed. And (let) LM (be made) equal to A. And let the solid angle contained by NLO, OLM, and MLN have been constructed on the straight-line LM, and at the point L on it, (so as to be) equal to the solid angle E [Prop. 11.23]. And let LO be made equal to B, and LN equal to C. And since as A (is) to B, so B (is) to C, and A (is) equal to LM, and B to each of LO and ED, and C to LN, thus as LM (is) to EF, so DE (is) to LN. And (so) the sides around the equal angles NLM and DEF are reciprocally proportional. Thus, parallelogram MN is equal to parallelogram DF [Prop. 6.14]. And since the two plane rectilinear angles DEF and NLM are equal, and the raised straight-lines stood on them (at their apexes), LO and EG, are equal to one another, and contain equal angles respectively with the original straight-lines (forming the angles), the perpendiculars drawn from points G and O to the planes through NLM and DEF (respectively) are thus equal to one another [Prop. 11.35 corr.]. Thus, the solids LH and EK (have) the same height. And parallelepiped solids on equal bases, and with the same height, are equal to one another [Prop. 11.31]. Thus, solid HL is equal to solid EK. And LH is the solid (formed) from A, B, and C, and EK the solid on B. Thus, the parallelepiped solid (formed) from A, B, and C is equal to the equilateral solid on B (which is) equiangular with the aforementioned (solid). (Which is) the very thing it was required to show.

λζʹ.

Ἐὰν τέσσαρες εὐθεῖαι ἀνάλογον ὦσιν, καὶ τὰ ἀπ᾽ αὐτῶν

Proposition 37†

If four straight-lines are proportional then the similar, στερεὰ παραλληλεπίπεδα ὅμοιά τε καὶ ὁμοίως ἀναγραφόμενα ἀνάλογον ἔσται· καὶ ἐὰν τὰ ἀπ᾽ αὐτῶν στερεὰ παραλληλεπίπεδα ὅμοιά τε καὶ ὁμοίως ἀναγραφόμενα ἀνάλογον ᾖ, καὶ αὐταὶ αἱ εὐθεῖαι ἀνάλογον ἔσονται.



Ἔστωσαν τέσσαρες εὐθεῖαι ἀνάλογον αἱ ΑΒ, ΓΔ, ΕΖ, ΗΘ, ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν ΗΘ, καὶ ἀναγεγράφθωσαν ἀπὸ τῶν ΑΒ, ΓΔ, ΕΖ, ΗΘ ὅμοιά τε καὶ ὁμοίως κείμενα στερεὰ παραλληλεπίπεδα τὰ ΚΑ, ΛΓ, ΜΕ, ΝΗ· λέγω, ὅτι ἐστὶν ὡς τὸ ΚΑ πρὸς τὸ ΛΓ, οὕτως τὸ ΜΕ πρὸς τὸ ΝΗ.

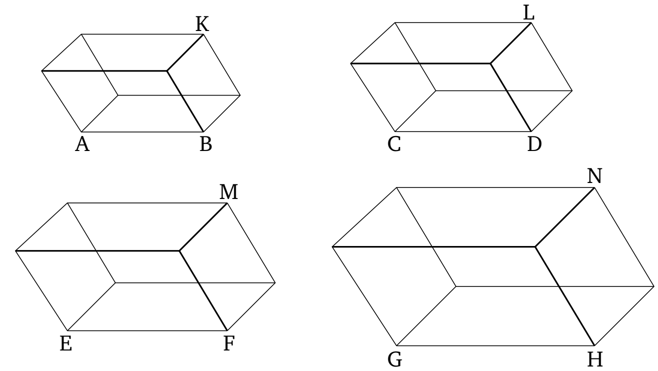
Ἐπεὶ γὰρ ὅμοιόν ἐστι τὸ ΚΑ στερεὸν παραλληλεπίπεδον τῷ ΛΓ, τὸ ΚΑ ἄρα πρὸς τὸ ΛΓ τριπλασίονα λόγον ἔχει ἤπερ ἡ ΑΒ πρὸς τὴν ΓΔ. διὰ τὰ αὐτὰ δὴ καὶ τὸ ΜΕ πρὸς τὸ ΝΗ τριπλασίονα λόγον ἔχει ἤπερ ἡ ΕΖ πρὸς τὴν ΗΘ. καί ἐστιν ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν ΗΘ. καὶ ὡς ἄρα τὸ ΑΚ πρὸς τὸ ΛΓ, οὕτως τὸ ΜΕ πρὸς τὸ ΝΗ.

Ἀλλὰ δὴ ἔστω ὡς τὸ ΑΚ στερεὸν πρὸς τὸ ΛΓ στερεόν, οὕτως τὸ ΜΕ στερεὸν πρὸς τὸ ΝΗ· λέγω, ὅτι ἐστὶν ὡς ἡ ΑΒ εὐθεῖα πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν ΗΘ.

Ἐπεὶ γὰρ πάλιν τὸ ΚΑ πρὸς τὸ ΛΓ τριπλασίονα λόγον ἔχει ἤπερ ἡ ΑΒ πρὸς τὴν ΓΔ, ἔχει δὲ καὶ τὸ ΜΕ πρὸς τὸ ΝΗ τριπλασίονα λόγον ἤπερ ἡ ΕΖ πρὸς τὴν ΗΘ, καί ἐστιν ὡς τὸ ΚΑ πρὸς τὸ ΛΓ, οὕτως τὸ ΜΕ πρὸς τὸ ΝΗ, καὶ ὡς ἄρα ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν ΗΘ.

Ἐὰν ἄρα τέσσαρες εὐθεῖαι ἀνάλογον ὦσι καὶ τὰ ἑξῆς τῆς προτάσεως· ὅπερ ἔδει δεῖξαι.

and similarly described, parallelepiped solids on them will also be proportional. And if the similar, and similarly described, parallelepiped solids on them are proportional then the straight-lines themselves will be proportional.



Let AB, CD, EF, and GH, be four proportional straight-lines, (such that) as AB (is) to CD, so EF (is) to GH. And let the similar, and similarly laid out, parallelepiped solids KA, LC, ME and NG have been described on AB, CD, EF, and GH (respectively). I say that as KA is to LC, so ME (is) to NG.

For since the parallelepiped solid KA is similar to LC, KA thus has to LC the cubed ratio that AB (has) to CD [Prop. 11.33]. So, for the same (reasons), ME also has to NG the cubed ratio that EF (has) to GH [Prop. 11.33]. And since as AB is to CD, so EF (is) to GH, thus, also, as AK (is) to LC, so ME (is) to NG.

And so let solid AK be to solid LC, as solid ME (is) to NG. I say that as straight-line AB is to CD, so EF (is) to GH.

For, again, since KA has to LC the cubed ratio that AB (has) to CD [Prop. 11.33], and ME also has to NG the cubed ratio that EF (has) to GH [Prop. 11.33], and as KA is to LC, so ME (is) to NG, thus, also, as AB (is) to CD, so EF (is) to GH.

Thus, if four straight-lines are proportional, and so on of the proposition. (Which is) the very thing it was required to show.

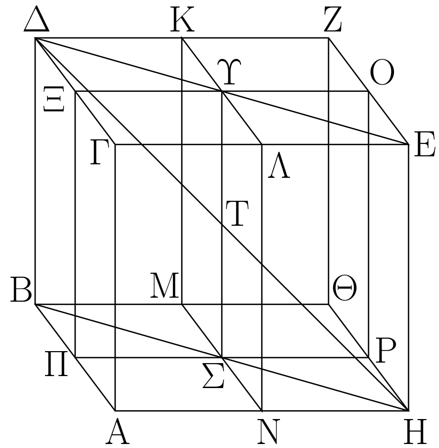
† This proposition assumes that if two ratios are equal then the cube of the former is also equal to the cube of the latter, and *vice versa*.

λη′.

Ἐὰν κύβου τῶν ἀπεναντίον ἐπιπέδων αἱ πλευραὶ δίχα τμηθῶσιν, διὰ δὲ τῶν τομῶν ἐπίπεδα ἐκβληθῇ, ἡ κοινὴ τομὴ τῶν ἐπιπέδων καὶ ἡ τοῦ κύβου διάμετρος δίχα τέμνουσιν ἀλλήλας.

Proposition 38

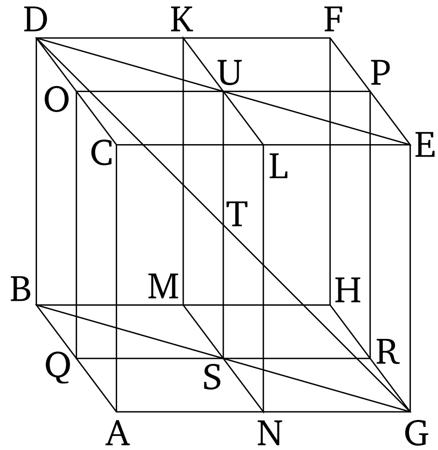
If the sides of the opposite planes of a cube are cut in half, and planes are produced through the pieces, then the common section of the (latter) planes and the diameter of the cube cut one another in half.



Κύβου γὰρ τοῦ ΑΖ τῶν ἀπεναντίον ἐπιπέδων τῶν ΓΖ, ΑΘ αἱ πλευραὶ δίχα τετμήσθωσαν κατὰ τὰ Κ, Λ, Μ, Ν, Ξ, Π, Ο, Ρ σημεῖα, διὰ δὲ τῶν τομῶν ἐπίπεδα ἐκβεβλήσθω τὰ ΚΝ, ΞΡ, κοινὴ δὲ τομὴ τῶν ἐπιπέδων ἔστω ἡ ΥΣ, τοῦ δὲ ΑΖ κύβου διαγώνιος ἡ ΔΗ. λέγω, ὅτι ἴση ἐστὶν ἡ μὲν ΥΤ τῇ ΤΣ, ἡ δὲ ΔΤ τῇ ΤΗ.

Ἐπεζεύχθωσαν γὰρ αἱ ΔΥ, ΥΕ, ΒΣ, ΣΗ. καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΔΞ τῇ ΟΕ, αἱ ἐναλλὰξ γωνίαι αἱ ὑπὸ ΔΞΥ, ΥΟΕ ἴσαι ἀλλήλαις εἰσίν. καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν ΔΞ τῇ ΟΕ, ἡ δὲ ΞΥ τῇ ΥΟ, καὶ γωνίας ἴσας περιέχουσιν, βάσις ἄρα ἡ ΔΥ τῇ ΥΕ ἐστιν ἴση, καὶ τὸ ΔΞΥ τρίγωνον τῷ ΟΥΕ τριγώνῳ ἐστὶν ἴσον καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι· ἴση ἄρα ἡ ὑπὸ ΞΥΔ γωνία τῇ ὑπὸ ΟΥΕ γωνίᾳ. διὰ δὴ τοῦτο εὐθεῖά ἐστιν ἡ ΔΥΕ. διὰ τὰ αὐτὰ δὴ καὶ ΒΣΗ εὐθεῖά ἐστιν, καὶ ἴση ἡ ΒΣ τῇ ΣΗ. καὶ ἐπεὶ ἡ ΓΑ τῇ ΔΒ ἴση ἐστὶ καὶ παράλληλος, ἀλλὰ ἡ ΓΑ καὶ τῇ ΕΗ ἴση τέ ἐστι καὶ παράλληλος, καὶ ἡ ΔΒ ἄρα τῇ ΕΗ ἴση τέ ἐστι καὶ παράλληλος. καὶ ἐπιζευγνύουσιν αὐτὰς εὐθεῖαι αἱ ΔΕ, ΒΗ· παράλληλος ἄρα ἐστὶν ἡ ΔΕ τῇ ΒΗ. ἴση ἄρα ἡ μὲν ὑπὸ ΕΔΤ γωνία τῇ ὑπὸ ΒΗΤ· ἐναλλὰξ γάρ· ἡ δὲ ὑπὸ ΔΤΥ τῇ ὑπὸ ΗΤΣ. δύο δὴ τρίγωνά ἐστι τὰ ΔΤΥ, ΗΤΣ τὰς δύο γωνίας ταῖς δυσὶ γωνίαις ἴσας ἔχοντα καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην τὴν ὑποτείνουσαν ὑπὸ μίαν τῶν ἴσων γωνιῶν τὴν ΔΥ τῇ ΗΣ· ἡμίσειαι γάρ εἰσι τῶν ΔΕ, ΒΗ· καὶ τὰς λοιπὰς πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἕξει. ἴση ἄρα ἡ μὲν ΔΤ τῇ ΤΗ, ἡ δὲ ΥΤ τῇ ΤΣ.

Ἐὰν ἄρα κύβου τῶν ἀπεναντίον ἐπιπέδων αἱ πλευραὶ δίχα τμηθῶσιν, διὰ δὲ τῶν τομῶν ἐπίπεδα ἐκβληθῇ, ἡ κοινὴ τομὴ τῶν ἐπιπέδων καὶ ἡ τοῦ κύβου διάμετρος δίχα τέμνουσιν ἀλλήλας· ὅπερ ἔδει δεῖξαι.



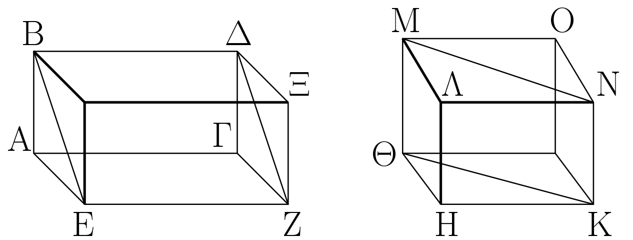
For let the opposite planes CF and AH of the cube AF have been cut in half at the points K, L, M, N, O, Q, P, and R. And let the planes KN and OR have been produced through the pieces. And let US be the common section of the planes, and DG the diameter of cube AF. I say that UT is equal to TS, and DT to TG.

For let DU, UE, BS, and SG have been joined. And since DO is parallel to PE, the alternate angles DOU and UPE are equal to one another [Prop. 1.29]. And since DO is equal to PE, and OU to UP, and they contain equal angles, base DU is thus equal to base UE, and triangle DOU is equal to triangle PUE, and the remaining angles (are) equal to the remaining angles [Prop. 1.4]. Thus, angle OUD (is) equal to angle PUE. So, for this (reason), DUE is a straight-line [Prop. 1.14]. So, for the same (reason), BSG is also a straight-line, and BS equal to SG. And since CA is equal and parallel to DB, but CA is also equal and parallel to EG, DB is thus also equal and parallel to EG [Prop. 11.9]. And the straight-lines DE and BG join them. DE is thus parallel to BG [Prop. 1.33]. Thus, angle EDT (is) equal to BGT. For (they are) alternate [Prop. 1.29]. And (angle) DTU (is equal) to GTS [Prop. 1.15]. So, DTU and GTS are two triangles having two angles equal to two angles, and one side equal to one side—(namely), that subtended by one of the equal angles—(that is), DU (equal) to GS. For they are halves of DE and BG (respectively). (Thus), they will also have the remaining sides equal to the remaining sides [Prop. 1.26]. Thus, DT (is) equal to TG, and UT to TS.

Thus, if the sides of the opposite planes of a cube are cut in half, and planes are produced through the pieces, then the common section of the (latter) planes and the diameter of the cube cut one another in half. (Which is) the very thing it was required to show.

λθ΄.

Ἐὰν ᾖ δύο πρίσματα ἰσοϋψῆ, καὶ τὸ μὲν ἔχῃ βάσιν παραλληλόγραμμον, τὸ δὲ τρίγωνον, διπλάσιον δὲ ᾖ τὸ παραλληλόγραμμον τοῦ τριγώνου, ἴσα ἔσται τὰ πρίσματα.



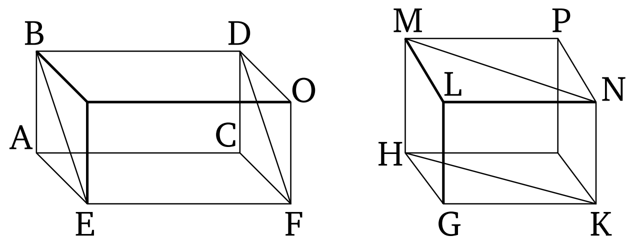
Ἔστω δύο πρίσματα ἰσοϋψῆ τὰ ΑΒΓΔΕΖ, ΗΘΚΛΜΝ, καὶ τὸ μὲν ἐχέτω βάσιν τὸ ΑΖ παραλληλόγραμμον, τὸ δὲ τὸ ΗΘΚ τρίγωνον, διπλάσιον δὲ ἔστω τὸ ΑΖ παραλληλόγραμμον τοῦ ΗΘΚ τριγώνου· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΒΓΔΕΖ πρίσμα τῷ ΗΘΚΛΜΝ πρίσματι.

Συμπεπληρώσθω γὰρ τὰ ΑΞ, ΗΟ στερεά. ἐπεὶ διπλάσιόν ἐστι τὸ ΑΖ παραλληλόγραμμον τοῦ ΗΘΚ τριγώνου, ἔστι δὲ καὶ τὸ ΘΚ παραλληλόγραμμον διπλάσιον τοῦ ΗΘΚ τριγώνου, ἴσον ἄρα ἐστὶ τὸ ΑΖ παραλληλόγραμμον τῷ ΘΚ παραλληλογράμμῳ. τὰ δὲ ἐπὶ ἴσων βάσεων ὄντα στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος ἴσα ἀλλήλοις ἐστίν· ἴσον ἄρα ἐστὶ τὸ ΑΞ στερεὸν τῷ ΗΟ στερεῷ. καί ἐστι τοῦ μὲν ΑΞ στερεοῦ ἥμισυ τὸ ΑΒΓΔΕΖ πρίσμα, τοῦ δὲ ΗΟ στερεοῦ ἥμισυ τὸ ΗΘΚΛΜΝ πρίσμα· ἴσον ἄρα ἐστὶ τὸ ΑΒΓΔΕΖ πρίσμα τῷ ΗΘΚΛΜΝ πρίσματι.

Ἐὰν ἄρα ᾖ δύο πρίσματα ἰσοϋψῆ, καὶ τὸ μὲν ἔχῃ βάσιν παραλληλόγραμμον, τὸ δὲ τρίγωνον, διπλάσιον δὲ ᾖ τὸ παραλληλόγραμμον τοῦ τριγώνου, ἴσα ἐστὶ τὰ πρίσματα· ὅπερ ἔδει δεῖξαι.

Proposition 39

If there are two equal height prisms, and one has a parallelogram, and the other a triangle, (as a) base, and the parallelogram is double the triangle, then the prisms will be equal.



Let $ABCDEF$ and $GHKLMN$ be two equal height prisms, and let the former have the parallelogram AF, and the latter the triangle GHK, as a base. And let parallelogram AF be twice triangle GHK. I say that prism $ABCDEF$ is equal to prism $GHKLMN$.

For let the solids AO and GP have been completed. Since parallelogram AF is double triangle GHK, and parallelogram HK is also double triangle GHK [Prop. 1.34], parallelogram AF is thus equal to parallelogram HK. And parallelepiped solids which are on equal bases, and (have) the same height, are equal to one another [Prop. 11.31]. Thus, solid AO is equal to solid GP. And prism $ABCDEF$ is half of solid AO, and prism $GHKLMN$ half of solid GP [Prop. 11.28]. Prism $ABCDEF$ is thus equal to prism $GHKLMN$.

Thus, if there are two equal height prisms, and one has a parallelogram, and the other a triangle, (as a) base, and the parallelogram is double the triangle, then the prisms are equal. (Which is) the very thing it was required to show.