THE COMPLEX SYMPLECTIC STRUCTURE OF THE SPACE OF QUASIFUCHSIAN DEFORMATIONS

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ABSTRACT

We study the (complex) symplectic geometry of the space \( Q(S) \) of the quasifuchsian structures of a closed Riemannian surface \( S \) of genus \( g > 1 \). We prove that this space is a flat complex symplectic manifold and we describe the Hamiltonian nature of the quasifuchsian bending vector fields.

A quasifuchsian group \( \Gamma \) is a discrete subgroup of \( \text{PSL}(2, \mathbb{C}) \), which is quasiconformally congruent to a fuchsian group of the first kind. The limit set of \( \Gamma \) is a Jordan curve. If \( \Omega_\Gamma \) is the domain of discontinuity of \( \Gamma \) in the complex plane \( \mathbb{C} \) and \( \mathbb{H}^3 \) is the upper hyperbolic semispace, then the Kleinian 3-manifold \( M_\Gamma = (\mathbb{H}^3 \cup \Omega_\Gamma) / \Gamma \) is diffeomorphic to \( S \times [0,1] \). The space \( Q(S) \) of the quasifuchsian structures on a closed Riemann surface \( S \) of genus \( g > 1 \) can be regarded as the space of marked quasifuchsian manifolds, where a marking of \( M_\Gamma \) is a choice of isomorphism between \( \pi_1(M_\Gamma) \) and \( \pi_1(S) \). This work is divided in three sections:

In section 1 we mostly refer to previous results concerning the geometry and the deformations of \( Q(S) \). For more details about the notions of complex length and bending along a lamination which are widely used here, the reader is referred to the works of Kourouniotis and Epstein-Marden [K1],[K2],[K3],[E-M].

In section 2 we construct a closed holomorphic form \( \Omega \) in \( Q(S) \) using the geometry of \( M_\Gamma \) and the laws that describe the variations of the complex length under bending. In this way the couple \( (Q(S),\Omega) \) turns out to be a complex symplectic manifold, a notion which is an extension in the complex case of the notion of the symplectic manifold. Spaces of the form \( \text{Hom}(\pi_1(S) \to G) \) - where \( G \) is a Lie group with an invariant inner product on its Lie algebra, are already known to be symplectic by a construction of W.Goldman based on strong algebraic topological tools [G1]. In the case where \( G = \text{PSL}(2, \mathbb{C}) \) the symplectic form is described explicitly in [G2] (Theorem p.40). The interested reader can compare Goldman’s formulas with the laws that our construction is based upon, to obtain that the two symplectic forms are the same. Our construction also reveals the Hamiltonian nature of the bending vector fields. A duality formula is given at the end of this section.

In section 3 we use an analytic continuation method to prove that \( Q(S,\Omega) \) is globally a flat complex symplectic manifold. S.Wolpert has proved a similar result for the Weil-Petersson form in the Teichmüller space \( T(S) \); namely that \( T(S) \) is a flat real symplectic manifold [W2].

AMS subject classification (1991) 30F60,30F40
This work is in final form and no version of it will be submitted for publication elsewhere.
1. The space of quasifuchsian structures

We fix a compact Riemann surface $S$ of genus $g > 1$ and identify its fundamental group $\pi_1(S)$ with a Fuchsian group $\Gamma_0$. The space $Q(S)$ is the quotient of the set of discrete faithful representations $\rho \in \text{Hom}(\Gamma_0 \to \text{PSL}(2, \mathbb{C}))$ such that $\Gamma = \rho(\Gamma_0)$ is quasifuchsian, with a left action of $\text{PSL}(2, \mathbb{C})$ via inner automorphisms. This space is a $6g - 6$ complex dimensional complex manifold [B], and complex coordinates for it are provided by the following two theorems [K3]:

**THEOREM A.** Given a partition of the surface $S$ by $3g - 3$ simple closed geodesics $\gamma_i$, there exist complex length functions $\lambda_{\gamma_i} : Q(S) \to \mathbb{C}$ and bending functions $\beta_i : Q(S) \to \mathbb{C}$, $i = 1, \ldots, 3g - 3$ which form a system of global complex coordinates for $Q(S)$.

**THEOREM B.** Given a partition of the surface $S$ by $3g - 3$ simple closed geodesics $\gamma_i$, then for every $\rho \in Q(S)$ there exist a neighborhood $U(\rho)$ and $3g - 3$ simple closed curves $\alpha_i$ with $\gamma_i \cap \alpha_j = \emptyset$ if $i \neq j$ such that the complex length functions $\lambda_{\gamma_i}, \lambda_{\alpha_i}, i = 1, \ldots, 3g - 3$ form a system of local complex coordinates for $Q(S)$.

In [K1] C. Kourouniotis defined a holomorphic transformation of $Q(S)$ called the bending deformation, which is a generalisation on $Q(S)$ of the quasibending deformation defined on the Teichmüller space $T(S)$ [E-M]. We firstly recall the notion of a measured lamination on $S$.

A geodesic lamination $\Lambda_S$ on $S$ is a foliation of a closed subset of $S$ such that the leaves are geodesics. The leaves together with the components of $S - \Lambda_S$ are called the strata of $\Lambda_S$. A geodesic lamination on $S$ lifts to a lamination $\Lambda$ in the hyperbolic plane.

A transverse measure $\mu$ is a finite complex Borel measure on each closed geodesic segment $\alpha$ in the hyperbolic plane, subject to certain conditions:

i) If $\alpha$ is entirely contained in a stratum of $\Lambda$, then the measure of $\alpha$ is zero.

ii) The measures of two geodesic segments with their respective endpoints lying in the same strata of $\Lambda$ are equal.

A measured lamination $(\Lambda, \mu)$ is a geodesic lamination equipped with a transverse measure. From here on we shall denote a measured lamination $(\Lambda, \mu)$ only with the symbol of the measure $\mu$.

Let $\mu$ be a measured lamination on $S$, and $\rho$ a point of $Q(S)$. Then in a neighborhood $W$ of 0 in $\mathbb{C}$ we define a quasiconformal deformation of $\rho$ called bending along $\mu$

$$B_{\mu}(\rho) : W \to Q(S) : t \to B_{\mu}(t, \rho)$$

The mapping $B_{\mu}$ possesses all the properties of a (local) holomorphic flow [K1].

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Let $\alpha$ be a simple geodesic on $S$ and $\lambda_{\alpha}$ its complex length function. The first variation of the complex length of $\alpha$ under bending along $\mu$ is given by:

$$T_\mu \lambda_{\alpha} = \frac{d}{dt} \left[ \lambda_{\alpha}(B_{\mu}(t, r)) \right]$$

If $\nu$ is a measured lamination on $S$ the second variation of $\lambda_{\alpha}$ under bending along $\mu$ and $\nu$ is given by:

$$T_\mu T_\nu \lambda_{\alpha} = \frac{d^2}{dt^2} \left[ \lambda_{\alpha}(B_{\mu}(t, B_\nu(s, r))) \right]$$

Suppose that $\mu, \nu$ are simple closed curves $\alpha, \beta$ respectively equipped with transverse measure the counting one. Then the variations of the complex length have the following geometric interpretation:

$$T_\beta \lambda_{\alpha} = \sum_{\rho \in \alpha \cap \beta} \cosh(\rho(\beta \cdot \rho(\alpha))) \cdot$$

$$T_\beta T_\gamma \lambda_{\alpha} =$$

$$= \frac{1}{2 \sinh \frac{1}{2} \lambda_{\alpha}} \sum_{\rho \in \alpha \cap \beta \cap \gamma} (\sinh(\rho(\alpha \cdot \rho(\beta))) \cdot \sinh(\rho(\alpha \cdot \rho(\gamma))) \cdot \cosh(\frac{1}{2} \lambda_{\alpha} - \sigma_{\rho \gamma}) +$$

$$+ \frac{1}{2 \sinh \frac{1}{2} \lambda_{\beta}} \sum_{\rho \in \alpha \cap \beta \cap \gamma} (\sinh(\rho(\alpha \cdot \rho(\beta))) \cdot \sinh(\rho(\beta \cdot \rho(\gamma))) \cdot \cosh(\frac{1}{2} \lambda_{\beta} - \sigma_{\rho \gamma})$$

where by $\sigma$ we denote the complex distance between geodesics $\rho(\cdot)$ in the upper semispace $\mathcal{H}^2$ (See [K3]).

The following laws are crucial for our construction in section 3:

1. $T_\alpha \lambda_{\beta} + T_\beta \lambda_{\alpha} = 0$
2. $T_\alpha T_\beta \lambda_{\gamma} + T_\beta T_\alpha \lambda_{\gamma} + T_\gamma T_\lambda \lambda_{\alpha} = 0$

Laws I and II are the generalisation on $Q(S)$ of S.Wolpert's laws concerning the first and the second variation of the geodesic length under twisting in $T(S)$ [W1].

**Proposition A.** [K3]. If $\alpha$ is a simple closed curve on $S$, then $T_\alpha$ is a holomorphic vector field on $Q(S)$.

We shall call $T_\alpha$ a $Q$-bending vector field. Let now $\rho$ be a point of $Q(S)$, $\alpha$ be a simple closed curve on $S$, and let also

$$T_\alpha = \frac{1}{2}(F_\alpha - iB_\alpha)$$

the holomorphic tangent $Q$-bending vector at $\rho$. $F_\alpha, B_\alpha$ are real vectors and

$$B_\alpha = J_0 F_\alpha, F_\alpha = -J_0 B_\alpha$$

where $J_0$ is the almost complex structure of $Q(S)$. Let $\beta$ be another simple closed curve on $S$ and $\lambda_{\beta} = l_\beta + i\varphi_{\beta}$ its complex length function ($l_\beta$ is the geodesic length function and $\varphi_{\beta}$ is the angle function [K2], [K3]).
Making some elementary calculations we have:
\[ T_a \lambda_\beta = \frac{1}{2} ( F_a - i B_a \lambda \lambda_\beta + i \varphi_\beta) = \frac{1}{2} ( F_a I_a + B_a \varphi_\beta) + \frac{i}{2} ( F_a \varphi_\beta - B_a I_a) \]
and also
\[ T_a \lambda_\beta = \sum_{\rho \in a \cap \beta} \cos d(\rho(\alpha), \rho(\beta))_\rho = \sum_{\rho \in a \cap \beta} \cos d(\rho(\alpha), \rho(\beta))_\rho \cos \phi(\rho(\alpha), \rho(\beta))_\rho + i \sinh d(\rho(\alpha), \rho(\beta))_\rho \sin \phi(\rho(\alpha), \rho(\beta))_\rho, \]
where by \( d \) we denote the hyperbolic distance between geodesics \( \rho(\cdot) \) in the upper semispace \( \mathcal{H}^J \), and by \( \phi \) their angle \( (\sigma = d + i\phi) \).

Equating real and imaginary parts and applying the Cauchy-Riemann equations we obtain:
\[ F_a I_a = B_a \varphi_\beta = \sum_{\rho \in a \cap \beta} \cos d(\rho(\alpha), \rho(\beta))_\rho \cos \phi(\rho(\alpha), \rho(\beta))_\rho, \]
\[ F_a \varphi_\beta = -B_a I_a = \sum_{\rho \in a \cap \beta} \sinh d(\rho(\alpha), \rho(\beta))_\rho \sin \phi(\rho(\alpha), \rho(\beta))_\rho. \]
When \( \rho \) is a fuchsian point then \( d(\rho(\alpha), \rho(\beta))_\rho = 0 \) since \( \rho(\alpha), \rho(\beta) \) are intersecting geodesics in \( \mathcal{H}^J \) and therefore
\[ T_a \lambda_\beta = F_a I_a = I_a I_a = \sum_{\rho \in a \cap \beta} \cos d(\rho(\alpha), \rho(\beta))_\rho, \]
where \( I_a \) is the Fenchel-Nielsen twist vector field (See [W1] for the definition).

**Lemma 1.1.** Let \( \gamma, \alpha \) be as in theorem B. The Q-bending vector fields \( T_\gamma (= \frac{\partial}{\partial \gamma}) \) and \( T_\alpha \) form a basis of the holomorphic tangent space \( T^{(1,0)}(Q(S)) \).

**Proof:** Let \( V(\rho_0) \) be an open neighborhood of a point \( \rho_0 \in Q(S) \) with local coordinates \((\lambda_1, \ldots, \lambda_{r-1}, \lambda_{a_1}, \ldots, \lambda_{a_{r-1}})\). The Q-bending vectors \( T_\gamma, T_\alpha \) are linear combinations of the vectors \( \frac{\partial}{\partial \lambda_1}, \ldots, \frac{\partial}{\partial \lambda_{r-1}}, \frac{\partial}{\partial \lambda_{a_1}}, \ldots, \frac{\partial}{\partial \lambda_{a_{r-1}}} \). Specifically at each \( \rho \in V(\rho_0) \) we have:
\[
[T] = \begin{bmatrix} 0 & A \\ -A & B \end{bmatrix} \left[ \frac{\partial}{\partial \lambda} \right]
\]
where \([T]\) is the complex matrix \([T_\gamma, \ldots, T_{r-1}, T_\alpha, \ldots, T_{a_{r-1}}]^T\), and \(\left[ \frac{\partial}{\partial \lambda} \right]\) is the complex matrix \(\left[ \frac{\partial}{\partial \lambda_1}, \ldots, \frac{\partial}{\partial \lambda_{r-1}}, \frac{\partial}{\partial \lambda_{a_1}}, \ldots, \frac{\partial}{\partial \lambda_{a_{r-1}}} \right]^T\).

The matrix \( B \) is a non-zero \((3g-3) \times (3g-3)\) complex matrix with entries \( T_\gamma, T_\alpha \), or \( 0 \) and \( A \) is the \((3g-3) \times (3g-3)\) diagonal complex matrix with diagonal entries
\[ [T_\gamma] = T_\gamma, T_\alpha = \sum_{\rho \in V(\rho_0)} \cos d(\rho(\gamma), \rho(\alpha))_\rho. \]

The determinant of the transformation is equal to \((-1)^{r-3}(\det A)^2\) which is different from zero since at every \( \rho \in V \) we have \( T_\gamma, T_\alpha = 0 \) for every \( i = 1, \ldots, 3g-3 \).
2. The complex symplectic structure of $Q(S)$ and the Hamiltonian nature of Q-bending fields.

With the aid of laws I and II we firstly define locally a closed holomorphic 2-form in the holomorphic tangent space $\mathcal{T}^{(1,0)}(Q(S))$. Let $\rho_0$ be a point of $Q(S)$ and $V(\rho_0)$, $T_\gamma$, $T_\alpha$, as in the proceeding section. For indices $\alpha, \beta$ running through $\gamma_i$, $\alpha_i$, $i = 1, \ldots, 3g - 3$ we set for each $\rho \in V$:

$$\Omega_{(\rho)}(T_\alpha, T_\beta) = T_{\alpha(\rho)} - T_{\beta(\rho)} = \sum_{\rho \in a \cap \beta} \cos \beta_0(\rho(\alpha), \rho(\beta)).$$

By law I one obtains that $\Omega$ is skew-symmetric and by the very definition that $\Omega$ is holomorphic since the quantities $T_\alpha \lambda_\beta$ are all holomorphic functions in $V(\rho_0)$.

The form $\Omega$ does not depend on the choice of local coordinates for $V(\rho_0)$ or to be more specific on the choice of the curves $\alpha_i$ of theorem B. One way to see this is to consider $\alpha, \beta$ simple closed curves on $S$ and to establish that the above formula is also valid in this case.

Indeed by lemma 1.1 there exist holomorphic functions $f'_\alpha, g'_\alpha$ and $f'_\beta, g'_\beta$, $i = 1, \ldots, 3g - 3$ defined on $V(\rho_0)$ such that

$$T_\alpha = f'_\alpha T_\gamma + g'_\alpha T_s,$$
$$T_\beta = f'_\beta T_\gamma + g'_\beta T_s,$$

where upper and lower indices denote summation. The calculations now are straightforward:

$$\Omega(T_\alpha, T_\beta) = \Omega(f'_\alpha T_\gamma + g'_\alpha T_s, f'_\beta T_\gamma + g'_\beta T_s) =$$

$$= f'_\alpha f'_\beta \Omega(T_\gamma, T_\gamma) + f'_\alpha g'_\beta \Omega(T_\gamma, T_s) + g'_\alpha f'_\beta \Omega(T_s, T_\gamma) + g'_\alpha g'_\beta \Omega(T_s, T_s) =$$

$$= f'_\alpha f'_\beta T_\gamma + f'_\alpha g'_\beta T_s + g'_\alpha f'_\beta T_\gamma + g'_\alpha g'_\beta T_s =$$

$$= f'_\alpha f'_\beta T_\gamma + g'_\beta g'_\alpha T_s,$$

which by law I is equal to

$$-f'_\alpha f'_\beta T_\gamma - g'_\beta g'_\alpha T_s = -T_\alpha \lambda_\beta = T_\alpha \lambda_\beta.$$

The form $\Omega$ is a closed $(2,0)$ form: Let $d'$ be the holomorphic differential operator on $Q(S)$ and $T_\alpha, T_\beta, T_\gamma$, Q-bending vector fields on curves $\alpha, \beta, \gamma$ of $S$ respectively. Then

$$d'\Omega(T_\alpha, T_\beta, T_\gamma) = T_\alpha \Omega(T_\beta, T_\gamma) - T_\beta \Omega(T_\alpha, T_\gamma) + T_\gamma \Omega(T_\alpha, T_\beta) =$$

$$-\Omega([T_\alpha, T_\beta], T_\gamma) + \Omega([T_\alpha, T_\gamma], T_\beta) - \Omega([T_\beta, T_\gamma], T_\alpha),$$

Using laws I and II we find out that this is equal to:

$$T_\alpha T_\beta \lambda_\gamma - T_\beta T_\alpha \lambda_\gamma + T_\gamma T_\alpha \lambda_\beta =$$

$$-(T_\alpha, T_\beta) \lambda_\gamma + (T_\alpha, T_\gamma) \lambda_\beta + (T_\beta, T_\gamma) \lambda_\alpha =$$

$$T_\alpha T_\beta \lambda_\gamma + T_\beta T_\alpha \lambda_\gamma - T_\gamma T_\alpha \lambda_\beta - T_\alpha T_\beta \lambda_\gamma + T_\beta T_\alpha \lambda_\gamma +$$

$$+ T_\alpha T_\beta \lambda_\gamma - T_\gamma T_\alpha \lambda_\beta - T_\gamma T_\alpha \lambda_\gamma + T_\gamma T_\beta \lambda_\alpha + T_\gamma T_\beta \lambda_\alpha = 0.$$
We can also check that our form $\Omega$ is non-degenerate in the sense that if there exists a holomorphic vector field $Z$ such that $\Omega(Z, W) = 0$ for all holomorphic $W$ then $Z = 0$. Indeed, the equation $\Omega(Z, W) = 0$ for all $W$ is equivalent to the $6g - 6$ equations

$$\Omega(Z, T_{\gamma_i}) = 0$$

where $\alpha$ is an index moving on indices $\gamma_i, \alpha_i, \ i = 1, \ldots, 3g - 3$.

Now since $Z = f^i T_{\gamma_i} + g^i T_{\alpha_i}$ for some holomorphic functions $f^i, g^i$ we have from the equations

$$\Omega(Z, T_{\gamma_i}) = 0$$

that $g^i = 0$ since $\Omega(T_{\gamma_i}, T_{\gamma_j}) = 0$ and $\Omega(T_{\alpha_i}, T_{\gamma_j}) = \sum_{\rho \in A \cap \gamma_i} \cosh \sigma(\rho(\alpha_i), \rho(\gamma_j)) \delta_{ij}$, where $\delta_{ij}$ is the Kronecker delta.

Therefore $Z = f^i T_{\gamma_i}$ and by the same reasoning we get from the equations

$$\Omega(Z, T_{\gamma_i}) = 0$$

that $f^i = 0$. We conclude that $Z = 0$.

**Definition 2.1.** Let $M$ be a $2n$ complex manifold. We say that $M$ carries a complex symplectic structure if there exists a non-degenerate, closed $(2,0)$-form defined on $M$.

A trivial example is $C^{2n}$ with complex coordinates $z_1, \ldots, z_n, w_1, \ldots, w_n$ where we have the standard flat form

$$\Omega_0 = \sum_{i=1}^{n} dz_i \wedge dw_i$$

The couple $(M, \Omega)$ is called a complex symplectic manifold. We recall that a real manifold $M$ is called symplectic if it carries a real symplectic structure, i.e., if there exists a non-degenerate closed 2-form defined on $M$. The following is rather obvious:

**Proposition 2.2.** A complex symplectic manifold is also a real symplectic manifold.

**Proof:** Let $(M, \Omega)$ a complex symplectic manifold. Since $\Omega$ is a complex form we have $\Omega = \omega + i\phi$ where $\omega, \phi$ are non-degenerate real closed forms on $M$ defining symplectic structures on $M$.

From the above discussion we obtain:

**THEOREM 2.3** The space of quasifuchsian structures is a complex symplectic as well as a real symplectic manifold.

We turn now to a more general discussion. Let $(M, \Omega)$ be a complex symplectic manifold. The holomorphic form $\Omega$ defines an isomorphism between holomorphic tangent and cotangent bundles. For every point $p \in M$ and every holomorphic vector $Z \in T^{0,1}_{\gamma_i}(M)$, a holomorphic 1-form $\Omega(Z)$ is describing this isomorphism:

$$(\Omega(Z))_{\gamma_i} = \Omega_{\alpha_i}(Z_{\gamma_i}) = i_{\gamma_i} \Omega_{\gamma_i}$$

where $i_{\gamma_i}$ is the interior product with respect to $Z$.

Let $f$ be a holomorphic function on $M$. We shall call a holomorphic vector field $H_f$ the complex hamiltonian of $f$ if for every holomorphic vector field $X$ defined on $M$ we have:
\[ \Omega(H^C_f \chi, \Xi) = \Omega(H^C_f, \Xi) = -d' f(\Xi) = -\Xi f \]

If \( \varphi \) is the (local) holomorphic flow of \( H^C_f \), then for every \( t \) belonging to a sufficiently small neighborhood of 0 in \( \mathbb{C} \), we have \( \varphi^t \Omega = \Omega \). Also the complex Lie derivative of \( \Omega \) along \( H^C_f \) is zero.

Suppose that \( f = u + iv \) where \( u, v \) are smooth functions of \( M \) satisfying the Cauchy-Riemann equations. Consider the real symplectic manifold \((M, \omega)\) where \( \omega = \text{Re} \Omega \). For the (real) Hamiltonian vector fields \( H_u, H_v \) of \( u, v \) respectively, we have:

\[
\omega(H_u, \xi) = -du(\xi) = -\xi u \\
\omega(H_v, \xi) = -dv(\xi) = -\xi v
\]

for every real vector field \( \xi \) on \( M \).

We can check easily that

\[ J_M H_u = -H_v \text{ and } J_M H_v = H_u \]

where \( J_M \) is the almost complex operator of \( J_M \), the real tangent space of \( M \). Indeed, since \( \omega \) is \( J_M \) anti-invariant being the real part of a holomorphic form, we have:

\[
\omega(J_M H_u, \xi) = \omega(H_u, J_M \xi) = -du(J_M \xi) = -(J_M \xi) u
\]

If \( \Xi = \frac{1}{2} (\xi - i J_M \xi) \) is the holomorphic vector field corresponding to \( \xi \) then

\[
\xi = \Xi + \bar{\Xi} \text{ and } J_M \xi = i (\Xi - \bar{\Xi}) \text{ so } \\
(J_M \xi) u = \frac{i}{2} (\Xi - \bar{\Xi})(f + \bar{f}) = \frac{i}{2} (\Xi f - \bar{\Xi} \bar{f}) = -\text{Im}(\Xi f) = \text{Im}[\Omega(H^C_f, \Xi)].
\]

On the other hand

\[
\omega(H_u, \xi) = -du(\xi) = -\xi u = \frac{i}{2} (\Xi + \bar{\Xi})(f + \bar{f}) = \frac{i}{2} (\Xi f - \bar{\Xi} \bar{f}) = -\text{Im}(\Xi f)
\]

therefore

\[ \omega(J_M H_u, \xi) = \omega(-H_v, \xi) \]

and since \( \omega \) is non-degenerate we have \( J_M H_u = -H_v \), where applying \( J_M \) again we obtain \( J_M H_v = H_u \).

We also note that

\[
\omega(H_u, \xi) = -\xi u = -\frac{1}{2} (\Xi + \bar{\Xi})(f + \bar{f}) = -\text{Re}(\Xi f) = \text{Re}[\Omega(H^C_f, \Xi)].
\]

Thus

\[ \Omega(H^C_f, \Xi) = \text{Re}[\Omega(H^C_f, \Xi)] + i \text{Im}[\Omega(H^C_f, \Xi)] = \omega(H_u, \xi) - i \omega(J_M H_u, \xi) = 
\]

\[ \text{Re}[\Omega(\frac{1}{2} (H_u - i J_M H_u), \Xi)] + i \text{Im}[\Omega(\frac{1}{2} (H_u - i J_M H_u), \Xi)] = \Omega(\frac{1}{2} (H_u - i J_M H_u), \Xi) 
\]

Since \( \Omega \) is non-degenerate we have:

\[
H^C_f = \frac{1}{2} (H_u - i J_M H_u) = \frac{1}{2} (H_u + i H_v)
\]

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Let $a$ be a simple closed curve on $S$ and $\lambda_a = l_a + i\phi_a$ its complex length function defined on $Q(S)$. The geodesic length function $l_a$ and the angle function $\phi_a$ are smooth on $Q(S)$. Applying the results of the previous discussion we obtain the following:

**Proposition 2.4.** The $Q$-bending vector field $T_a = \frac{1}{2}(F_a - iB_a)$ is the complex hamiltonian of the complex length function $\lambda_a : T_a = H_{\lambda_a}^C$. Furthermore

$$H_{l_a} = F_a \quad \text{and} \quad H_{\phi_a} = -B_a$$

**Proposition 2.5.** (Duality formula)

$$\Omega(T_a, \cdot) = -d\lambda_a$$

Observe that when we restrict ourselves to the Teichmüller space, then formula 3.5 is just S.Wolpert's duality formula:

$$\omega_{wp}(T_a, \cdot) = -dl_a$$

where $\omega_{wp}$ is the Weil-Petersson symplectic form (See also lemma 3.3).

3. $Q(S)$ is a flat complex symplectic manifold.

In [W2] S. Wolpert proved the following formula concerning the expression of the Weil-Petersson symplectic form $\omega_{wp}$ of the Teichmüller space in terms of the global real analytic Fenchel-Nielsen coordinates $(l_{\gamma_1}, \ldots, l_{\gamma_{g-1}}, \tau_1, \ldots, \tau_{2g-3})$:

$$\omega_{wp} = \sum_{i=1}^{2g-2} dl_{\gamma_i} \wedge d\tau_i$$

where $l_{\gamma_i}$, $i = 1, \ldots, 3g-3$ are geodesic length functions of closed geodesics $\gamma_i$ forming a partition of $S$ and $\tau_i$, $i = 1, \ldots, 3g-3$ are the twist functions corresponding to these geodesics. According to this formula the form $\omega_{wp}$ is flat symplectic and the Teichmüller space is a (real) flat symplectic manifold. Note here that the term flat refers only to the symplectic nature of $T(S)$ and not to the Kählerian one (where it is known that $T(S)$ has negative holomorphic curvature). In this section we establish the natural extension of Wolpert's result in our case. Namely we prove our main:

**THEOREM 3.1.** $(Q(S), \Omega)$ is a flat complex symplectic manifold. The expression of the form in the global coordinates $\lambda_{\gamma_i}, \beta_i$, $i = 1, \ldots, 3g-3$ of theorem A is

$$\Omega = \sum_{i=1}^{3g-3} d\lambda_{\gamma_i} \wedge d\beta_i$$

Also

$$\Omega(\frac{\partial}{\partial \beta_i}, \cdot) = -d\lambda_{\gamma_i}, \quad \text{and} \quad \Omega(\frac{\partial}{\partial \lambda_{\gamma_i}}, \cdot) = d\beta_i,$$

and therefore the holomorphic vector fields $\frac{\partial}{\partial \beta_i}, \frac{\partial}{\partial \lambda_{\gamma_i}}$ are complex hamiltonian for the form $\Omega$. 

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Our proof is based on an analytic continuation argument. We start with

**Lemma 3.2. (Analytic continuation in several variables)**

Let \( F: \mathbb{C}^n \rightarrow \mathbb{C}, \quad F = F(z_1, \ldots, z_n), z_i = x_i + y_i \) be a holomorphic function and suppose that \( F(x_1, \ldots, x_n, 0, \ldots, 0) = 0 \). Then \( F \) is identically zero in \( \mathbb{C}^n \).

**Proof:** Consider the function \( w_1 = w_1(z_1) = F(x_1, \ldots, x_n, y_1, 0, \ldots, 0) \) which is holomorphic in the variable \( z_1 \). Now \( w_1(x_1, 0) = F(x_1, \ldots, x_n, 0, \ldots, 0) = 0 \) therefore by analytic continuation we have \( w_1(z_1) = 0 \). Consider then \( w_2 = w_2(z_2) = F(x_1, x_2, \ldots, x_n, y_1, y_2, 0, \ldots, 0) \) which is holomorphic in the variable \( z_2 \) (we stabilise all the other variables). We have \( w_2(x_2, 0) = F(x_1, \ldots, x_n, y_1, 0, \ldots, 0) = 0 \) therefore \( w_2(z_2) = 0 \) and we continue with this procedure until we exhaust all variables.

By the duality formula 2.5 we easily see that the following hold:

\[
\Omega \left( \frac{\partial}{\partial \beta_1}, \frac{\partial}{\partial \lambda_1}, \ldots, \frac{\partial}{\partial \beta_N}, \frac{\partial}{\partial \lambda_N} \right) = -\delta_{ij} \quad \text{and} \quad \Omega \left( \frac{\partial}{\partial \beta_i}, \frac{\partial}{\partial \lambda_i} \right) = 0
\]

for \( i, j = 1, \ldots, 3g - 3 \) and thus

\[
\Omega = \sum_{i=1}^{3g-3} d\lambda_i \wedge d\beta_i + \sum_{1 \leq i < j \leq 3g-3} \Omega \left( \frac{\partial}{\partial \lambda_i}, \frac{\partial}{\partial \lambda_j} \right) d\lambda_i \wedge d\lambda_j
\]

where

\[\lambda_i = \alpha_i + i \theta_i, \quad \beta_i = \tau_i + i \psi_i \]

and

\[\frac{\partial}{\partial \lambda_i} = \frac{1}{2} \left( \frac{\partial}{\partial \alpha_i} - i \frac{\partial}{\partial \psi_i} \right), \quad \frac{\partial}{\partial \beta_i} = \frac{1}{2} \left( \frac{\partial}{\partial \tau_i} - i \frac{\partial}{\partial \psi_i} \right) \]

[K3].

After straightforward calculations we obtain

\[
\omega = \text{Re} \Omega = \sum_{i=1}^{3g-3} \left( d\alpha_i \wedge d\tau_i - d\theta_i \wedge d\psi_i \right) + \sum_{1 \leq i < j \leq 3g-3} \text{Re} \left( \Omega \left( \frac{\partial}{\partial \lambda_i}, \frac{\partial}{\partial \lambda_j} \right) \right) (d\alpha_i \wedge d\alpha_j - d\beta_i \wedge d\beta_j)
\]

\[\phi = \text{Im} \Omega = \sum_{i=1}^{3g-3} \left( d\theta_i \wedge d\tau_i + d\beta_i \wedge d\psi_i \right) + \sum_{1 \leq i < j \leq 3g-3} \text{Im} \left( \Omega \left( \frac{\partial}{\partial \lambda_i}, \frac{\partial}{\partial \lambda_j} \right) \right) (d\theta_i \wedge d\beta_j + d\alpha_i \wedge d\theta_j)
\]

and

\[\omega = \text{Re} \Omega = \sum_{i=1}^{3g-3} \left( d\alpha_i \wedge d\tau_i - d\theta_i \wedge d\psi_i \right) + \sum_{1 \leq i < j \leq 3g-3} \text{Re} \left( \Omega \left( \frac{\partial}{\partial \lambda_i}, \frac{\partial}{\partial \lambda_j} \right) \right) (d\alpha_i \wedge d\alpha_j - d\beta_i \wedge d\beta_j)
\]

\[\phi = \text{Im} \Omega = \sum_{i=1}^{3g-3} \left( d\theta_i \wedge d\tau_i + d\beta_i \wedge d\psi_i \right) + \sum_{1 \leq i < j \leq 3g-3} \text{Im} \left( \Omega \left( \frac{\partial}{\partial \lambda_i}, \frac{\partial}{\partial \lambda_j} \right) \right) (d\theta_i \wedge d\beta_j + d\alpha_i \wedge d\theta_j)
\]

where of course

\[\text{Re} \left( \Omega \left( \frac{\partial}{\partial \lambda_i}, \frac{\partial}{\partial \lambda_j} \right) \right) = \omega \left( \frac{\partial}{\partial \alpha_i}, \frac{\partial}{\partial \alpha_j} \right) \quad \text{and} \quad \text{Im} \left( \Omega \left( \frac{\partial}{\partial \lambda_i}, \frac{\partial}{\partial \lambda_j} \right) \right) = \phi \left( \frac{\partial}{\partial \tau_i}, \frac{\partial}{\partial \tau_j} \right)
\]

When restricting ourselves on the tangent bundle of the fuchsian points of \( \mathcal{Q}(S) \) then the functions \( \theta_i, \psi_i, i = 1, \ldots, 3g - 3 \) are all zero, therefore in this case:
\[ \Omega = \omega + i \varphi = \sum_{i=1}^{3g-3} \text{Re}(\Omega \frac{\partial}{\partial \lambda_i \gamma_i}) dl_i \wedge dt_i + \sum_{1 \leq i < j \leq 3g-3} \text{Re}(\Omega(\frac{\partial}{\partial \lambda_i \gamma_i}, \frac{\partial}{\partial \lambda_j \gamma_j})) dl_i \wedge dl_j + \]

\[ + i \sum_{1 \leq i < j \leq 3g-3} \text{Im}(\Omega(\frac{\partial}{\partial \lambda_i \gamma_i}, \frac{\partial}{\partial \lambda_j \gamma_j})) dl_i \wedge dl_j = \]

\[ = \sum_{i=1}^{3g-3} dl_i \wedge dt_i + \sum_{1 \leq i < j \leq 3g-3} \Omega(\frac{\partial}{\partial \lambda_i \gamma_i}, \frac{\partial}{\partial \lambda_j \gamma_j}) dl_i \wedge dl_j, \]

We shall show that the holomorphic functions \( \Omega_i = \Omega(\frac{\partial}{\partial \lambda_i \gamma_i}, \frac{\partial}{\partial \lambda_i \gamma_i}) \) are real valued at fuchsian points: Indeed if \( \rho \) is such a point then in a neighborhood of \( \rho \) we have from lemma 1.1 that

\[ \frac{\partial}{\partial \lambda_i \gamma_i} = f_i^r T_{\gamma_i} + g_i^r T_{\gamma_i} \]

where \( f_i^r, g_i^r \) are holomorphic functions defined on this neighborhood. According to the matrix equation in lemma 1.1 we have by multiplication with the inverse matrix that

\[ \left[ \frac{\partial}{\partial \lambda_i \gamma_i} \right] = [A^{-1}B A^{-1} - A^{-1}] T \]

Therefore

\[ \left[ \frac{\partial}{\partial \lambda_i \gamma_i} \right] = [A^{-1}B A^{-1} - A^{-1}] T \]

where

\[ \left[ \frac{\partial}{\partial \lambda_i \gamma_i} \right] = \left[ \frac{\partial}{\partial \lambda_1 \gamma_1}, \ldots, \frac{\partial}{\partial \lambda_{3g-3} \gamma_{3g-3}} \right]^T \]

Recall that the entries of the (diagonal) matrix \( A \) are of the form \( T_{\gamma_i \lambda_i} \) or 0 where the entries of the matrix \( B \) are of the form \( T_{\gamma_i \lambda_i} \) or 0. All these quantities (and eventually the quantities \( f_i^r, g_i^r \) which are entries of the matrices \( A^{-1}B A^{-1} \) and \( -A^{-1} \) respectively) evaluated on a fuchsian point are real, for if \( \rho \) is such a point then we have

\[ T_{\alpha \beta} = \sum_{r=0}^{g-1} T_{\gamma_i \lambda_i \alpha \beta} \]

where \( \alpha, \beta \) indices running through indices \( \gamma_i, \alpha_i, i = 1, \ldots, 3g-3 \).

Now it is easy to see that

\[ \Omega_i(\rho) = \Omega_{i, \rho}(\frac{\partial}{\partial \lambda_i \gamma_i}, \frac{\partial}{\partial \lambda_i \gamma_i}) = \Omega_{i, \rho}(f_i^r T_{\gamma_i}, T_{\gamma_i} + g_i^r T_{\gamma_i}) = \]

\[ = f_i^r T_{\gamma_i} + f_i^r T_{\gamma_i} + g_i^r T_{\gamma_i} = \lambda_{i, \gamma_i} + g_i^r T_{\gamma_i}, \]

which is clearly real when evaluated at \( \rho \).

Lemma 3.3. At fuchsian points \( \omega = \omega_{\mathcal{P}} \)

**Proof:** Let \( \alpha, \beta \) be simple closed curves on \( \Sigma \) and \( t_{\alpha, \beta} \) the corresponding twist vector fields. Then at a fuchsian \( \rho \) we have by the holomorphicity of bending:

\[ 150 \]
\[ \omega_{W_{\text{pr}}} (t_\alpha, t_\beta) = \frac{d}{dx} (x = 0 \lambda(t_\beta (B_\alpha (x, \rho))) = \]

\[ \Re \left( \frac{d}{dz} (z = 0 \lambda(t_\beta (B_\alpha (z, \rho)))) = \Re [T_{\alpha, \beta}] \lambda_{\beta} = \Re [T_{\alpha, \beta}] (T_{\alpha, \beta}) = \omega(t_\alpha, t_\beta) \]

From Lemma 3.3 and S. Wolpert’s formula for \( \omega_{W_{\text{pr}}} \) we have that

\[ \Re \left( \frac{\partial}{\partial \lambda_{\alpha}}, \frac{\partial}{\partial \lambda_{\beta}} \right) = \omega \left( \frac{\partial}{\partial \lambda_{\alpha}}, \frac{\partial}{\partial \lambda_{\beta}} \right) = \omega_{W_{\text{pr}}}, \quad \frac{\partial}{\partial \lambda_{\alpha}}, \frac{\partial}{\partial \lambda_{\beta}} = 0 \]

and also by the previous discussion

\[ \Im \left[ \Omega \left( \frac{\partial}{\partial \lambda_{\alpha}}, \frac{\partial}{\partial \lambda_{\beta}} \right) = 0 \right. \]

at fuchsians.

Now since each \( \Omega_{\alpha} = \Omega \left( \frac{\partial}{\partial \lambda_{\alpha}}, \frac{\partial}{\partial \lambda_{\beta}} \right) \) is a holomorphic function on \( Q(5) \), that is

\[ \Omega_{\alpha} = \Omega \left( \frac{\partial}{\partial \lambda_{\alpha}}, \frac{\partial}{\partial \lambda_{\beta}} \right) \]

and also at fuchsian points

\[ \Omega_{\alpha} = \Omega \left( \frac{\partial}{\partial \lambda_{\alpha}}, \frac{\partial}{\partial \lambda_{\beta}} \right) = 0 \]

we obtain by lemma 3.2 that \( \Omega_{\alpha} = 0 \) in \( Q(5) \) and thus the proof is complete.

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References


