# QUASICONFORMAL MAPPINGS IN THE HEISENBERG GROUP

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## Lectures in Hunan University, Changsha, Hunan, People’s Republic of China

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*To the people of the*
*Department of*
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# FOREWORD

*Applicants for wisdom do what I have done: inquire within.*

**– Heraclitus 535-475 B.C.**

*Knowledge is a treasure, but practice is the key to it.*

**– Lao Zi**

# PREFACE

The text on hand is the bulk of the lecture notes of a one month course which I taught for the graduate students of the Department of Mathematics, Hunan University, Changsha, PRC, in April 2018. Tt constitutes a brief but rather concise introduction to the theory of quasiconformal mappings of the Heisenberg group as this was established in the seminal works of Korányi-Reimann and Pansu. They also contain some recent developments based mostly in personal taste. The purpose of these notes is to excite the minds of the interested readers so that they will further investigate the beautiful Korányi-Reimann theory and its many developments by themselves in the future.

I.D. PLATIS

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I.D.P.

# CHAPTER 1

# INTRODUCTION

## 1.1 Conformal mappings

Let $U$ and $U'$ be domains (that is, open, connected sets) of $\mathbb{R}^n$. A *conformal* mapping $f : U \to U'$ is a sense-preserving diffeomorphism such that $f_*$, the derivative of $f$, is a scalar multiple of an orthogonal transformation at each point of $U$. In other words, $f$ is angle preserving and thus in the tangent space it maps infinitesimal balls to infinitesimal balls.

Of special interest is the case $n = 2$: Let $f(x, y) = (u(x, y), v(x, y))$. Then

$$f_* = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

and suppose that $\det(f_*) = u_x v_y - u_y v_x > 0$ so that $f$ is sense-preserving. If $f$ is conformal and $p \in U$, there exists an element $R_\phi$ of $\mathrm{SO}^+(2)$, say

$$R_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}, \quad \phi \in \mathbb{R}$$

so that at each point $p \in U$, $f_* = c(p)R_\phi$ for some scalar $c(p)$ depending on $p$. That is equivalent to say that at $p$ we must have

$$u_x = v_y \quad \text{and} \quad u_y = -v_x,$$

that is, $f$ is conformal if at each point the Cauchy-Riemann equations hold.

Conformal mappings in $\mathbb{R}^2 = \mathbb{C}$ (also known by their German name as *schlicht* mappings) are extremely important in the context of complex analysis. Several important results appear from their study; we mention here the following

**Theorem 1.1 Riemann Mapping Theorem:** *If $U$ is a non-empty simply connected*[1] *open subset of the complex plane $\mathbb{C}$ which is not all of $\mathbb{C}$, then there exists a biholomorphic mapping f (i.e., a bijective holomorphic mapping whose inverse is also holomorphic) from $U$ onto the open unit disk*

$$D = \{z \in \mathbb{C} \ : \ |z| < 1\}.$$

This is a really striking theorem; it actually tells us that all simply connected domains of $\mathbb{C}$ (besides $\mathbb{C}$ itself) are *conformally equivalent* and thus the study of geometric and analytic properties of any such random domain may be reduced to the study of the open unit disk.

The complex case though is somewhat idyllic; for $n > 2$ a conformal mapping is necessarily a Möbius transformation. We have the following:

**Theorem 1.2 Liouville's Theorem:** *Conformal mappings of the one-point compactification $\hat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ of $\mathbb{R}^n$, $n > 2$, form a Lie group which is isomorphic to a connected component of the identity of* $\mathrm{SO}^+(n, 1)$.

We conclude that the class of conformal mappings is quite rigid; a new class had to be introduced and in what follows we will briefly describe how this was happened.

## 1.2 Gauss and smooth quaasiconformal maps

The first hint of what is known as a *quasiconformal mapping of the plane* $\mathbb{C}$ appeared in the work of C.F. Gauss. Let $U \subset \mathbb{R}^2$ be an open set and let $\sigma : U \to \mathbb{R}^3$,

$$\sigma(u, v) = (x(u, v), y(u, v), z(u, v)),$$

be a smooth surface patch: that is, $\sigma$ is adequately smooth and $\mathrm{rank}(\sigma_*) = 2$ everywhere in $U$. The first fundamental form of the surface patch $\sigma$ is then

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2,$$

where

$$E = \sigma_u \cdot \sigma_u, \quad F = \sigma_u \cdot \sigma_v, \quad G = \sigma_v \cdot \sigma_v.$$

[1]Recall here that a simply connected subset of $\mathbb{C}$ is a subset which has no holes. Strictly speaking, the fundamental group at each point of the subset is trivial.

(Here, $\cdot$ denotes scalar multiplication). The coordinates $(u, v)$ of the patch $\sigma$ are called *isothermal* if

$$E = G \quad \text{and} \quad F = 0;$$

in this case, $ds^2 = \lambda^2(u, v)(du^2 + dv^2)$, a positive multiple of the first fundamental form of a piece of the plane. The question arising now is if we can always introduce isothermal coordinates on a surface, or in other words, if a surface is locally diffeomorphic to the plane. For this, let $z = u + iv$ be a complex coordinate; after pretty tedious but straightforward calculations we may write

$$ds = \lambda(z)|dz + \mu(z)d\overline{z}|,$$

where

$$\mu = \frac{E - G + 2iF}{E + G + 2\sqrt{EG - F^2}}.$$

Note that $\|\mu\| < 1$. Thus we have isothermal coordinates iff $\mu = 0$ and furthermore we may find an admissible change of coordinates $f$ to isothermal coordinates if we can solve the partial differential equation[2]

$$f_{\overline{z}} = \mu f_z. \tag{1.2.1}$$

This equation is the famous *Beltrami equation* and once it is solved, the isothermal coordinates are provided by $\sigma \circ f^{-1}$. In this way we have our first pretty strong definition for quasiconformal mappings:

**Definition 1.3 Quasiconformal diffeomorphisms:** *A $C^1$ diffeomorphic solution $f$ of the Beltrami equation 1.2.1 with $\|\mu\|_\infty \leq k < 1$ is called a $K$-quasiconformal mapping with dilation $\mu$. Here, $K = (1+k)/(1-k)$.*

Some remarks follow: First, the *maximal distortion* $K = K_f$ of $f$ is defined as

$$K = \frac{1 + \|\mu|_\infty}{1 - \|\mu\|_\infty}$$

and $f$ is also called $K$-quasiconformal. We will elaborate on the definition further in the next section; for the moment we stress that although we have spoken about sufficiently smooth maps (in fact, $C^1$ maps) we will eventually see below why smoothness is not necessary for a map to be quasiconformal.

[2]Recall here that if $z = u + iv$ then

$$f_z = \frac{1}{2}(f_u - if_v), \quad f_{\overline{z}} = \frac{1}{2}(f_u + if_v).$$

### 1.2.1 Grötzsh's definition for smooth quasiconformal maps

Before we go onto actually solving Grötzsch's problem in the next section, we now discuss Definition 1.3 and give an equivalent definition due to Grötzsch below. Let $f : U \to U'$ be a smooth ($C^1$ at least) diffeomorphism and let $w = f(z)$. We have

$$dw = df = f_z dz + f_{\overline{z}} d\overline{z}.$$

Being a diffeomorphism, $f$ is well approximated at each point $z_0$ of $U$ by the differential $df$ at $z_0$, which is of course a linear map. Now the linear map $df(z_0)$ maps the unit circle of the $z$-plane onto an ellipse of the $w$-plane such that its major and its minor axes have lengths $\alpha$ and $\beta$ respectively. Since the Jacobian $J_f$ of $f$ is

$$J_f = |f_z|^2 - |f_{\overline{z}}|^2,$$

assuming that $f$ is sense-preserving we obtain $|f_z| > |f_{\overline{z}}|$. Now the *distortion* $K_f(z_0)$ is defined by

$$K_f(z_0) = \frac{|f_z(z_0)| + |f_{\overline{z}}(z_0)|}{|f_z(z_0)| - |f_{\overline{z}}(z_0)|} = \frac{\alpha}{\beta}.$$

Let the complex dilation $\mu_f(z_0)$ be

$$\mu_f(z_0) = \frac{f_{\overline{z}}(z_0)}{f_z(z_0)}.$$

Clearly,

$$K_f(z_0) = \frac{1 + |\mu_f(z_0)|}{1 - |\mu_f(z_0)|}.$$

We may now give the following equivalent version of the definition for $C^1$ qc maps; this is Grötzsch's definition:

**Definition 1.4** *A $C^1$ diffeomorphism is quasiconformal if its distortion function $K_f(z)$ is bounded in $U$. It is $K$-quasiconformal if $K_f(z) \leq K$.*

The *maximal distortion* $K_f$ of $f$ is thus the infimum of all $K$ for which $f$ is $K$-quasiconformal.

Definitions 1.3 and 1.4 are equivalent: If a $C^1$ $f : U \to U'$ is quasiconformal according to Definition 1.3 then $K_f(z)$ is bounded in $U$. The bound is the real number $K = (1 + k)/(1 - k) = K_f \geq 1$, the maximal distortion of $f$. On the other hand, if the distortion of $f$ is bounded by $K > 1$, then the complex dilation

$$\mu_f(z) = f_{\overline{z}}/f_z$$

satisfies

$$|\mu_f(z)| = |f_{\overline{z}}|/|f_z| \leq (K - 1)/(K + 1) = k \in [0, 1).$$

We conclude these comments by stating the following results whose proof msy br found for instance in [11]:

- A $C^1$ diffeomorphism is conformal if and only if is 1-quasiconformal as this follows by the Cauchy-Riemann equations: $f$ is conformal implies $f_{\overline{z}} = 0$ which is the complex version of the Cauchy-Riemann equations, therefore $K_f \equiv 1$ and vice-versa.
- If $f$ is $C^1$ $K$-quasiconformal, then $f^{-1}$ is $K$-quasiconformal as well.
- If $f, g$ are $C^1$ $K_1$ and $K_2$ quasiconformal mappings, respectively, then $f \circ g$ is $K_1 K_2$-quasiconformal.

## 1.3 Grötzsch's problem

A corollary of the Riemann Mapping Theorem is the following:

**Proposition 1.5** *Let $U$ and $U'$ be simply connected subsets of $\mathbb{C}$ other than $\mathbb{C}$ itself and let $z_1, z_2, z_3$ and $w_1, w_2, w_3$ two triples of pairwise distinct points of $\partial U$ and $\partial U'$, respectively. Then there exists a unique conformal mapping $f$ from $U$ onto $U'$ such that $f(z_i) = w_i$, $i = 1, 2, 3$.*

Uniqueness follows here by pre-composing (or post-composing) our Riemann map with the unique Möbius transformation of the plane which maps $z_i$ to $w_i$, $i = 1, 2, 3$. But how about if we have two specific *quadruples* $\mathbf{z} = (z_1, \dots, z_4)$ and $\mathbf{w} = (w_1, \dots, w_4)$ on the respective boundaries of our sets? Those can be mapped onto each other by a Möbius transformation if and only if their respective cross-ratios

$$\mathbb{X}(\mathbf{z}) = \frac{(z_4 - z_2)(z_3 - z_1)}{(z_4 - z_1)(z_3 - z_2)}, \quad \mathbb{X}(\mathbf{w}) = \frac{(w_4 - w_2)(w_3 - w_1)}{(w_4 - w_1)(w_3 - w_2)}$$

are equal. Taking this under account, Grötzsch addressed in 1928 the following problem:

**Grötzsch's problem:** Let $R$ and $R'$ be two rectangles with sides parallel to the coordinate axes and with side lengths $a, b$ and $a', b'$ respectively. Find the closets to a conformal map $f$ that maps $R$ onto $R'$ in a manner so that if $\mathbf{z} = (z_1, \dots, z_4)$ and $\mathbf{z}' = (z'_1, \dots, z'_4)$ are the quadruples of their respective vertices taken in the counter-clockwise manner, then $f(z_i) = z'_i$, $i = 1, \dots, 4$.

### 1.3.1 Solution to Grötzsch's problem

In the formulation given in the previous section, consider the affine map

$$f(z) = \frac{1}{2}\left(\frac{a'}{a} + \frac{b'}{b}\right) z + \frac{1}{2}\left(\frac{a'}{a} - \frac{b'}{b}\right) \overline{z}.$$

This maps the rectangle $R$ with vertices $(0, a, a+bi, bi)$ to the rectangle $R'$ with vertices $(0, a, a'+b'i, b'i)$ so that

$$f(0)=0, \quad f(a)=a', \quad f(a+bi)=a'+b'i, \quad f(bi)=b'i.$$

We also have

$$f_{\overline{z}} = \frac{1}{2}\left(\frac{a'}{a} - \frac{b'}{b}\right), \quad f_z = \frac{1}{2}\left(\frac{a'}{a} + \frac{b'}{b}\right)$$

an thus the linear map has constant maximal distortion

$$K_f = \frac{a'/b'}{a/b}.$$

We will show in what follows that this is the solution to Grötzsch's problem. The method is standard and we will come back to it later in the course. Suppose that a mapping $g$ has the required properties and choose a horizontal line segment $\gamma \in R$, $\gamma = \gamma(s)$, $s \in [0, a]$, joining its vertical sides. Then if $\ell$ denotes curve length we have by chain rule

$$\|\dot{g}(\gamma(s))\| = |g_z(\gamma(s))\dot{\gamma}(s) + g_{\overline{z}}(\gamma(s))\overline{\dot{\gamma}(s)}|$$

$$\begin{aligned} a' \leq \ell(g(\gamma)) &= \int_0^a \|\dot{g}(\gamma(s))\| ds \\ &= \int_0^a |g_z(\gamma(s))\dot{\gamma}(s) + g_{\overline{z}}(\gamma(s))\overline{\dot{\gamma}(s)}| ds \\ &= \int_0^a |g_z||1+\mu_g| dx \end{aligned}$$

because $\gamma$ is a straight line ($s = x$). By integrating with respect to $y$,

$$\begin{aligned} a'b &\leq \iint_R |g_z||1+\mu_g| dxdy \\ &= \iint_R |g_z||1+\mu_g| \cdot \frac{J_g^{1/2}}{J_g^{1/2}} dxdy \\ &= \iint_R \frac{|1+\mu_g|}{(1-|\mu_g|^2)^{1/2}} \cdot J_g^{1/2} dxdy. \end{aligned}$$

At this point, we apply Cauchy-Schwarz Inequality to obtain

$$a'b \leq \left(\iint_R \frac{|1+\mu_g|^2}{1-|\mu_g|^2} dxdy\right)^{1/2} \left(\iint_R J_g dxdy\right)^{1/2}.$$

This gives

$$\frac{a'b^2}{b'} \leq \iint_R \frac{|1+\mu_g|^2}{1-|\mu_g|^2} dxdy \leq \iint_R K_g dxdy$$

and thus

$$\frac{a'/b'}{a/b} \leq K_g.$$

## 1.4 From Lavrentiev and Teichmüller to Ahlfors-bers Theory

Grötzsch's work of quasiconformal mappings remained pretty much in obscurity until 1937 when Teichmüller used it in an essential manner for his study of the famous

**Riemann Moduli Problem:** Let $\Sigma$ be a closed (compact and without boundary) surface and consider the set of its conformal equivalence classes.[3] Describe this set by a finite set of parameters.

It had also appeared in the work of Lavrentiev (1936) towards the direction of elliptic partial differential equations. The work of Ahlfors and Bers spanned almost four decades starting from the 1930's and it was culminated in the

**Theorem 1.6 Measurable Riemann Mapping Theorem:** *Let the Beltrami equation 1.2.1 with $\mu$ measurable and essentially bounded by 1. Then it has a quasiconformal[4] solution which is unique u to composition with conformal mappings.*

This theorem is the cornerstone for the staudy of Teichmüller spaces. The theory based on and also around it is now known as Ahlfors-Bers theory of quasiconformal mappings of the plane and it is a basic tool for the study of the properties of Teichmüller spaces. For the interested reader, the books of Ahlfors, [1], and also of Lehto-Virtanen, [27], which appeared in that period remain seminal in this context.

## 1.5 Quasiconformal mappings in Euclidean spaces

Concentrated in the context of Analysis, the theory of quasiconformal mappings was initiated by Gehring and Väisälä around the beginning of the 1960's. We have already given a definition of quasiconformal mappings which, however, depends strongly on the analytic nature of the complex plane. It is transparent that moving to higher dimensions need a more flexible definition. This is known as the metric definition:

**Definition 1.7 Metric definition of quasiconformal mappings in $\mathbb{R}^n$:** *Let $U, U' \subset \mathbb{R}^n$ be open sets and let also $f : U \to U'$ be a homeomorphism. If for each $x \in U$*

$$H(x) = \lim_{r\to 0} \sup \frac{\sup_{|x-y|=r} |f(x) - f(y)|}{\inf_{|x-y|=r} |f(x) - f(y)|}$$

*is uniformly bounded in $U$, then $f$ is quasiconformal.*

Compare that definition with the remarks we have given after the analytic definition in the complex plane case. This definition is very important and it has been

[3] This set would thereafter be named the *Teichmüller space of* $\Sigma$.

[4] Here quasiconformal is meant according to the *metric* definition, see next sextion and also next chapter. It will turn out that such a solution lies in the Sobolev space $W^{1,2}_{\text{loc}}$, it is a.e. differentiable and also its distortion is bounded by some $K$.

shown that is equivalent to an analytic definition (setting a distrortion condition on the derivative) and also to a geometric definition (using capacities or moduli) as we shall see further on.

The suspected reader must have already observed that the metric definition above can be easily extended to *any* metric space. We shall come back to it later in our course.

## 1.6 From Mostow Rigidity to Pansu and Korányi-Reimann theory

We state the celebrated 1968 rigidity result of Mostow.

**Theorem 1.8 Mostow Rigidity Theorem:** *In dimensions $n > 2$ diffeomorphic compact Riemennian manifolds with constant negative curvature are isometric, in particular they are conformally equivalent.*

The proof of this result relies heavily in the use of quasiconformal mappings of $\mathbb{R}^n$. The beginning of the study of quasiconformal mappings in a non-Riemannian setting may be considered as Mostow's attempt (see [29]) to extended this result to the setting of *symmetric spaces of rank 1 of non compact type*: those are hyperbolic spaces $\mathbf{H}^n_{\mathbb{K}}$ where $\mathbb{K}$ can be the set of real numbers $\mathbb{R}$ (except when $n = 2$), the set of complex numbers $\mathbb{C}$, the set of quaternions $\mathbb{H}$, or the set of octonions $\mathbb{O}$ (the latter only when $n = 2$). To obtain this, Mostow had to develop quasiconformal mappings on the boundary of these spaces as an indispensable tool. In rough lines, Mostow's proof in the case $\mathbb{K} = \mathbb{C}$ and $n = 2$ goes as follows. Let $G$ and $G'$ be two cocompact lattices, i.e. $M = \mathbf{H}^2_{\mathbb{C}}/G$ and $M' = \mathbf{H}^2_{\mathbb{C}}/G'$ are compact and suppose that $\rho : G \to G'$ is an isomorphism. From $\rho$ it is possible to define a quasi-isometric self map $F$ of $\mathbf{H}^2_{\mathbb{C}}$ which is equivariant; this map need not to be even continuous but has the property that it takes geodesics to quasi-geodesics. Due to a fundamental result in Gromov hyperbolic spaces, from this property $F$ is extended to a boundary map $F_\infty : \partial\mathbf{H}^2_{\mathbb{C}} \to \mathbf{H}^2_{\mathbb{C}}$ which is in fact a quasiconformal homeomorphism of $\partial\mathbf{H}^2_{\mathbb{C}}$ with respect to a metric comparable with the *Korányi–Cygan metric*. After showing that $F_\infty$ has enough smoothness, Mostow proves that a $(G, G')-$equivariant quasiconformal self map of the boundary is associated with the action of an element of the isometry group of $\mathbf{H}^2_{\mathbb{C}}$ and the proof is concluded from the equivariance of the resulting isometry. We refer the interested reader to [7], pp. 135–140, for a short but more detailed description of Mostow's proof.

It was Pansu who obtained an even stronger rigidity statement for the cases $\mathbb{K} = \mathbb{H}$ and $\mathbb{K} = \mathbb{O}$ in [31]. By using Mostow's methods, Pansu proved the following property which does not hold for real and complex hyperbolic spaces:

**Theorem 1.9 Pansu Rigidity Theorem:** *Every quasi-isometry of quaternionic or octonionic spaces has bounded distance from an isometry.*

From this result, by using the conformal geometry of the boundary which is modelled on a nilpotent group $\mathfrak{H}_{\mathbb{K}}$ with a Carnot–Carathéodory metric (i.e. a Carnot group), and general properties of Loewner spaces, (we postpone the definitions of all these for later), he proved that *any* quasiconformal (in fact quasisymmetric) homeomorphism of $\mathfrak{H}_{\mathbb{K}}$ is actually conformal.

This result does not apply for the case $\mathbb{K} = \mathbb{C}$ and it is an open problem to understand the intrinsic reason for this phenomenon.

Mostow's rigidity had serious consequences; perhaps two of them are the most significant:

- The moduli space of hyperbolic metrics on a surface, i.e. the *Teichmüller space* of the surface, which is the case $\mathbb{K} = \mathbb{R}$ and $n = 2$ in the above setting, is just a counterexample of rigidity. The proof fails there, since it involves absolute continuity in measure of the boundary quasiconformal (actually quasisymmetric) mappings: in $S^1 = \partial\mathbf{H}^2_{\mathbb{R}}$ this does not hold.

- The theory of quasiconformal mappings of the Heisenberg group emerged, after the pioneering articles [22], [23] of Korányi–Reimann and Pansu, [31]. These works ignited the research of quasiconformal mappings in various other spaces; only some of them are $\mathbb{CR}-$manifolds, Carnot groups, metric spaces with controlled geometry, etc. We will give further details about them throughout the course.

### 1.6.1 Measurable Riemann Mapping Theorem in the non-complex case

We have already highlighted the Measurable Mapping Theorem as the cornerstone of Ahlfors-Bers theory. Unfortunately, such a result is not available to Euclidean dimensions greater than 3, or to non-Reimannian cases like the Heisenberg group case. However, in the Korányi-Reimann theory of quasiconformal mappings of the Heisenberg group (which for the moment we should think as the boundary of complex hyperbolic plane minus one point) there exists a less powerful but extremely useful result; this is an infinitesimal version of Measurable Riemann Mapping Theorem and states that particular vector fields generate flows of quasiconformal mappings. Those vector fields are called quasiconformal deformations and the huge importance of this result lies on the fact that they actually exist non-trivial quasiconformal mappings in the Heisenberg group.

## 1.7 Heinonen-Koskela and beyond

Recently, the theory of quasiconformal mappings revolves around to what is called metric spaces with controlled geometry. We have already underlined that we can define quasiconformal mappings in any metric space; the question really is if any of such mapping do actually exist! For this, if you pick your favourite metric space, then your first step is to find the proper analytic definition of quasiconformal mappings and show its equivalence with the metric one.

The work of Heinonen and Koskela is about these questions; they consider spaces where the metric definition is equivalent to an analytic definition which in turn is equivalent to a geometric definition which uses capacities (or moduli of curve families).

# CHAPTER 2

# QUASICONFORMAL MAPPINGS IN THE COMPLEX PLANE

We recall the metric definition for quasiconformal mappings in $\mathbb{C}$:

**Definition 2.1 Metric definition of quasiconformal mappings in $\mathbb{C}$:** *Let $U, U' \subset \mathbb{R}^n$ be open subsets of the complex plane $\mathbb{C}$ and let also $f : U \to U'$ be a homeomorphism. If for each $z \in U$*

$$H(z) = \lim_{r \to 0} \sup \frac{\sup_{|z-w|=r} |f(z) - f(w)|}{\inf_{|z-w|=r} |f(z) - f(w)|}$$

*is uniformly bounded in $U$, then $f$ is quasiconformal. If also $H(z) \leq K$ then $f$ is called $K$-quasiconformal.*

In this chapter we are going to describe the manner of how from that from that definition all analytic properties of a quasiconformal mapping are revealed. We start from a geometric definition first.

## 2.1 Moduli of curve familes and the geometric definition

### 2.1.1 Absolute continuity

Let $I = [a, b]$ be an interval in the real line $\mathbb{R}$. A function $f\colon I \to \mathbb{R}$ is *absolutely continuous on* $I$ if for every positive number $\epsilon$, there is a positive number $\delta$ such that whenever a finite sequence of pairwise disjoint sub-intervals $(x_k, y_k)$ of $I$ with $x_k, y_k \in I$ satisfies

$$\sum_k (y_k - x_k) < \delta$$

then

$$\sum_k |f(y_k) - f(x_k)| < \epsilon.$$

The collection of all absolutely continuous functions on $I$ is usually denoted by $AC(I)$.

The following conditions on a real-valued function $f$ on a closed interval $[a, b]$ are equivalent:

1. $f$ is absolutely continuous;
2. $f$ has a derivative $f'$ a.e., the derivative is Lebesgue integrable, and
$$f(x) = f(a) + \int_a^x f'(t)\,dt \quad \text{for all } x \in [a, b];$$
3. there exists a Lebesgue integrable function $g$ defined on $[a, b]$ such that
$$f(x) = f(a) + \int_a^x g(t)\,dt$$
for all $x \in [a, b]$.

If these equivalent conditions are satisfied then necessarily $g = f$ a.e..

Equivalence between (1) and (3) is known as the *fundamental theorem of Lebesgue integral calculus*, which is due to Lebesgue. The notion of absolute continuity is generalised in an obvious way to *any* metric space. For instance if we replace $\mathbb{R}$ by $\mathbb{C}$, then $f$ is absolutely continuous if and only if the coordinate functions are absolutely continuous. We further note that every absolutely continuous function is uniformly continuous and, therefore, continuous. Every Lipschitz-continuous function is absolutely continuous.

### 2.1.2 Modulus of curve families

Let $\gamma : [a, b] \to \mathbb{C}$ be a curve. If

$$a = t_0 \le t_1 \le \cdots \le t_n = b$$

is a subdivision of $[a, b]$ and if

$$\ell(\gamma) = \sup \sum_{k=1}^{n} |\gamma(t_k) - \gamma(t_{k-1})| \leq \infty,$$

where the supremum is taken over all subdivisions of $[a, b]$, then $\gamma$ is called rectifiable. It is called locally rectifiable if every closed sub-curve is rectifiable.

If $\gamma$ is a rectifiable curve, then it can be parametrised by arc-length

$$s(t) = \ell(\gamma[a, t]),$$

and $s$ is an increasing continuous function. It can be shown that if $\gamma$ is a closed rectifiable curve then $s$ is absolutely continuous if and only if $\gamma$ is absolutely continuous.[1] In this manner, suppose that $\rho : \mathbb{C} \to \mathbb{R}^+$ is Borel,we can define its curve integral by

$$\int_\gamma \rho ds = \int_0^{\ell(\gamma)} \rho(\gamma(s))ds = \int_a^b \rho(\gamma(t)|\dot{\gamma}(t)|dt,$$

where $\gamma$ is a closed rectifiable curve.

Suppose now that $\Gamma$ is a family of (at least absolutely continuous) curves whose images lie in a region $U \subset \mathbb{C}$. A Borel function $\rho : U \to \mathbb{R}^+$ shall be called admissible for $\Gamma$ if

$$\int_\gamma \rho ds \geq 1$$

and the set of all admissible functions for $\Gamma$ shall be denoted by $\mathrm{Adm}(\Gamma)$.

**Definition 2.2** *The modulus* $\mathrm{Mod}(\Gamma)$ *of* $\Gamma$ *is*

$$\mathrm{Mod}(\Gamma) = \inf_{\rho \in \mathrm{Adm}(\Gamma)} \iint_U \rho^2 d\mu_L$$

*where* $d\mu_L$ *is the standard Lebesgue measure in the complex plane.*[2]

We further notice that if there is a $\rho$ which realises the infimum in the definition, then it is called an *extremal density* for $\Gamma$.

We give below some examples of moduli of curve families inside specific regions. To calculate moduli we follow a certain strategic which will become transparent from the examples.

[1]Recall that an absolutely continuous map is differentiable a.e.

[2]For any $p \geq 1$ we can actually define the $p$-modulus by

$$\mathrm{Mod}_p(\Gamma) = \inf_{\rho \in \mathrm{Adm}(\Gamma)} \iint_U \rho^p d\mu_L.$$

Thus our definition concerns the 2-modulus.

### 2.1.3 Modulus of a rectangle

We consider the rectangle $R(0, a, a+bi, bi)$ and the family $\Gamma$ of curves which connect the sides $(0, bi)$ and $(a, a+bi)$. We consider the sub-family $\Gamma_1$ comprising of line segments connecting the two sides, that is,

$$\Gamma_1 = \{\gamma_y \ : \ \gamma_y(t) = t + iy,\ t \in [0, a],\ y \in [0, b]\}.$$

If $\rho \in \mathrm{Adm}(\Gamma_1)$ then for each $\gamma_y$ we have $\dot{\gamma}_y(t) = 1$ and by putting $t = x$ we have

$$1 \leq \int_{\gamma_y} \rho ds = \int_0^a \rho(x + iy)dx.$$

We know apply Cauchy-Schwarz inequality in the right hand-side to the functions $\rho$ and $1$ to obtain:[3]

$$1 \leq \left( \int_0^a \rho^2(x + iy)dx \right)^{1/2} \cdot a^{1/2},$$

which we may write as

$$1/a \leq \int_0^a \rho^2(x + iy)dx.$$

By integrating the above inequality with respect to $y$ we have

$$b/a \leq \iint_R \rho^2 dxdy.$$

By then taking the infimum over all $\rho \in \mathrm{Adm}(\Gamma_1)$ we obtain

$$\mathrm{Mod}(\Gamma_1) \geq b/a$$

and since $\Gamma_1 \subset \Gamma$ we also have

$$\mathrm{Mod}(\Gamma) \geq b/a.$$

This concludes the first step of our calculation; in order to show equality in the above inequality we trace an extremal desnsity for $\Gamma$. To do so, we must think of a function $\rho_0$ that realises equality in the above Cauchy-Schwarz inequality. Indeed, this function is

$$\rho_0(x + iy) = 1/a.$$

In the first place,

$$\iint_R (1/a^2)dxdy = b/a$$

[3] Cauchy-Schwarz Inequality: if $f, g \in L^2([a, b])$ then

$$\int_a^b |f \cdot g| \leq \left( \int_a^b |f|^2 \right)^{1/2} \left( \int_a^b |g|^2 \right)^{1/2}.$$

Equality holds if and only if $f$ is a constant multiple of $g$.

and moreover, $\rho_0 \in \mathrm{Adm}(\Gamma)$: If $\gamma \in \Gamma$ then

$$\int_\gamma \rho_0 ds = (1/a)\ell(\gamma) \geq 1,$$

since the length $\ell(\gamma)$ of $\gamma$ is always greater or equal than the length of a line segment parallel to the $x$-axis that joins the two sides. This concludes our proof because we have

$$\mathrm{Mod}(\Gamma) \leq \iint_R \rho^2 d\mu_L = b/a.$$

In the same lines of the above proof the interested reader may show that the modulus of the family of curves inside the rectangle that connect the sides $(0, a)$ and $(bi, a + bi)$ is $a/b$ with extremal density $1/b$.

We comment at this point the geometrical meaning of this example: The modulus equals to the similarity ratio of the rectangle $R$. It is worth to notice that most books on the quasiconformal mappings of the plane take a considerable amount of effort to give the geometric definition of quasiconformal mappings by dealing first with moduli of *quadrilaterals*. By a guadrilateral $Q(z_1, z_2, z_3, z_4)$ here we mean a simply connected domain in $\mathbb{C}$ with four mutually disjoint prescribed points $z_1, z_2, z_3, z_4$ (the vertices of the quadrilateral) on its boundary. Due to standard complex analytic arguments any such quadrilateral can be mapped into a rectangle in a manner that the vertices of the quadrilateral are mapped onto the vertices of the rectangle. Since this is something that can happen only in the complex plane, we shall use a more general route.

### 2.1.4 Moduli of curve families inside the annulus

Let $0 < a < b$ and consider the annulus

$$A_{a,b} = \{z \in \mathbb{C} \ : \ a < |z| < b\}.$$

By a modification of the Riemann mapping Theorem, each doubly connected region in $\mathbb{C}$ can be conformally mapped to an annulus $A_{a,b}$.[4] We consider two families of curves inside the annulus: First, the family $\Gamma$ of curves that connect the two boundary circles. To calculate its modulus, also called the *capacity* $\mathrm{Cap}(A_{a,b})$ we consider the sub-family of radial curves

$$\Gamma_1 = \{\gamma_t \ : \ \gamma_t(r) = re^{it}, \ r \in [a, b], \ t \in [0, 2\pi)\}.$$

If $\rho \in \mathrm{Adm}(\Gamma_1)$ then $\dot{\gamma}_t(r) = e^{it}$ and

$$1 \leq \int_{\gamma_t} \rho ds = \int_a^b \rho(re^{it}) dr = \int_a^b \rho(re^{it}) r^{1/2} \cdot \frac{1}{r^{1/2}} dr.$$

[4] We stress here that the annulus in question is unique in the following sense: If $\Gamma$ is the family of boundary connected curves, then its modulus has to be equal to $\pi/\log(b/a)$.

Applying Cauchy-Schwarz inequality and taking it to the square we have

$$1/\log(b/a) \leq \int_a^b \rho(re^{it}) r dr.$$

By integrating with respect to $t$ and using the Change of Variables Theorem we have after taking the infimum over all $\rho \in \mathrm{Adm}(\Gamma_1)$ that

$$2\pi/\log(b/a) \leq \mathrm{Mod}(\Gamma_1) \leq \mathrm{Mod}(\Gamma).$$

Our task now is to find an extremal density for $\Gamma$. By inspecting the above Cauchy-Schwarz inequality we take

$$\rho_0 = c/r, \quad c = 1/\log(b/a).$$

Indeed,

$$\iint_{A_{a,b}} \rho_0^2(re^{it}) r dr dt = c^2 \cdot 2\pi \log(b/a) = 2\pi/\log(b/a).$$

On the other hand, if $\gamma \in \Gamma$, we may suppose $\gamma(s) = r(s)e^{it(s)}$, $s \in [0, \ell(\gamma)]$ and such that $r(0) = a$ and $r(\ell(\gamma)) = b$. We may also suppose that $\dot{r}(s) \geq 0$. Then

$$|\dot{\gamma}(s)| = |\dot{r}(s) + i\dot{t}(s)| \geq \dot{r}(s).$$

Then

$$\int_\gamma \rho_0 ds = \frac{1}{\log(b/a)} \int_0^{\ell(\gamma)} \frac{|\dot{\gamma}(s)|}{r(s)} ds \geq \frac{1}{\log(b/a)} \int_0^{\ell(\gamma)} \frac{\dot{r}(s)}{r(s)} ds = 1.$$

This shows that

$$\mathrm{Mod}(\Gamma) = 2\pi \log(b/a).$$

We next consider the family $\Gamma'$ of curves inside the annulus that wind around the origin. To calculate $\mathrm{Mod}(\Gamma')$ we take the sub-family of circles

$$\Gamma_2 = \{\gamma_r \ : \ \gamma_r(t) = re^{it}, \ r \in [a,b], \ t \in [0, 2\pi)\}.$$

If $\rho \in \mathrm{Adm}(\Gamma_2)$ then $\dot{\gamma}_r(t) = ire^{it}$ and

$$1 \leq \int_\gamma \rho ds = \int_0^{2\pi} \rho(re^{it}) r dr = \int_0^{2\pi} \rho(re^{it}) r^{1/2} \cdot r^{1/2} dt.$$

Again, by applying Cauchy-Schwarz inequality and taking the inequality to the square we have

$$1/(2\pi r) \leq \int_0^{2\pi} \rho(re^{it}) r dt.$$

Now by integrating with respect to $r$ and by using again the Change of Variables Theorem we have after taking the infimum over all $\rho \in \mathrm{Adm}(\Gamma_2)$ that

$$\frac{\log(b/a)}{2\pi} \leq \mathrm{Mod}(\Gamma_2) \leq \mathrm{Mod}(\Gamma').$$

By inspecting equality in the above Cauchy-Schwarz inequality we take

$$\rho_0 = 1/(2\pi r).$$

Then

$$\iint_{A_{a,b}} \rho_0^2(re^{it})rdrdt = \log(b/a)/(2\pi).$$

Finally, if $\gamma \in \Gamma'$, we may suppose $\gamma(s) = r(s)e^{it(s)}$, $s \in [0, 2\pi]$ and such that $\gamma(0) = \gamma(2\pi)$. Now,

$$\int_\gamma \rho_0 ds = \frac{1}{2\pi}\int_0^{2\pi} \frac{|\dot\gamma(s)|}{r(s)}ds \geq \left|\frac{1}{2\pi}\int_0^{2\pi} \frac{\dot\gamma(s)}{\gamma(s)}ds\right|,$$

which is the winding number of $\gamma$, that is greater or equal to 1. Our proof is complete.

### 2.1.5 Modulus Inequality

It is now time to show that the modulus of curve families is a conformal invariant. For this we shal need the following Lemma.

**Lemma 2.3** *Let $f : U \to U'$ be a $C^1$ sense-preserving diffeomorphic map of domains of $\mathbb{C}$. Then for each absolutely continuous curve $\gamma$ lying inside $U$ we have*

$$(|f_z(\gamma(t))| - |f_{\overline{z}}(\gamma(t))|)|\dot\gamma(t)| \leq |(f \dot\circ \gamma)(t)| \leq (|f_z(\gamma(t))| + |f_{\overline{z}}(\gamma(t))|)|\dot\gamma(t)|.$$

*Proof:* By Chain Rule,

$$(f \dot\circ \gamma)(t) = f_z(\gamma(t))\dot\gamma(t) + f_{\overline{z}}(\gamma(t))\overline{\dot\gamma(t)}.$$

By taking absolute velues and using triangle Inequality we obtain the desired result. ■

**Theorem 2.4 Modulus Inequality:** *Let $f : U \to U'$ be a $C^1$ sense-preserving $K$-quasiconformal map of domains of $\mathbb{C}$. Let $\Gamma$ be a family of curves inside $U$ and let also $K_f(z)$ be the distortion function of $f$. Then if $\rho \in \mathrm{Adm}(\Gamma)$ we have*

$$\mathrm{Mod}(f(\Gamma)) \leq \iint_U K_f(z)\rho^2(z)d\mu_L(z).$$

*Moreover,*

$$(1/K)\mathrm{Mod}(\Gamma) \leq \mathrm{Mod}(f(\Gamma)) \leq K\mathrm{Mod}(\Gamma).$$

*Proof:* If $\rho \in \mathrm{Adm}(\Gamma)$ we consider the push-forward density

$$\rho' = \frac{\rho}{|f_z| - |f_{\overline{z}}|} \circ f^{-1}.$$

In the first place $\rho' \in \mathrm{Adm}(f(\Gamma))$: if $f(\gamma) \in f(\Gamma)$, then by supposing that $\gamma$ is parametrised in some $[a, b]$ we get by using the left hand-side of Lemma that

$$\int_{f(\gamma)} \rho' ds = \int_a^b \frac{\rho(\gamma(t))|(f \dot\circ \gamma)(t)|}{|f_z(\gamma(t))| - |f_{\overline{z}}(\gamma(t))|}dt \geq \int_a^b \rho(\gamma(t))|\dot\gamma(t)|dt = \int_\gamma \rho ds \geq 1.$$

Now

$$\mathrm{Mod}(f(\Gamma)) \leq \iint_{U'} (\rho')^2(w)d\mu_L(w) = \iint_{U'} \left(\frac{\rho}{|f_z| - |f_{\bar{z}}|}\right)^2 (f^{-1}(w))d\mu_L(w).$$

We set $w = f(z)$ and use the Change of Variables Theorem:

$$\mathrm{Mod}(f(\Gamma)) \leq \iint_U \frac{\rho^2(z)}{(|f_z| - |f_{\bar{z}}|)^2} J_f(z)d\mu_L(z) = \iint_U K_f(z)\rho^2(z)d\mu_L(z)$$

since $J_f(z) = |f_z|^2 - |f_{\bar{z}}|^2$. Since also $K_f(z) \leq K$ we have

$$\mathrm{Mod}(f(\Gamma)) \leq K \iint_U \rho^2 d\mu_L$$

and by taking infima in both sides we obtain

$$\mathrm{Mod}(f(\Gamma)) \leq K\mathrm{Mod}(\Gamma).$$

We leave as an exercise to the reader to show that

$$(1/K)\mathrm{Mod}(\Gamma) \leq \mathrm{Mod}(f(\Gamma)).$$

(Hint: If $f$ is $K$-quasiconformal, then $f^{-1}$ is $K$-quadiconformal.) ■

**Definition 2.5 Geometric definition:** *A $C^1$ diffeomorphism $f : U \to U'$ is $K$-quasiconformal if for every curve family $\Gamma$ lying inside $U$ we have*

$$(1/K)\mathrm{Mod}(\Gamma) \leq \mathrm{Mod}(f(\Gamma)) \leq K\mathrm{Mod}(\Gamma).$$

It may be proved that Definitions 1.4 and 2.5 are equivalent but we shall not be concerned with this. The interested reader may concult [11], Theorem 3.2.1.

Here, we remark that Theorem 2.4 actually shows that Definition 1.4 implies Definition 2.5. Now we will show that if we assume that the geometric definition holds for *homeomorphisms*, then it implies the metric definition. We start with the following.

**Definition 2.6** *A generalised annulus $A_{C_0,C_1}$ (or, condenser) is a domain whose complement consists of two connected components $C_0$ and $C_1$. The modulus* $\mathrm{Mod}$ $(A_{C_0,C_1})$ *is the modulus of the family of curves which connect $\partial C_0$ and $\partial C_1$.*

It can be shown (see [35]) that $\mathrm{Mod}(A_{C_0,C_1})$ equals to the capacity $\mathrm{Cap}(C_0, C_1)$ which is defined as follows. Consider $C^1$ functions $u : \mathbb{C} \to \mathbb{R}$ such that $u \equiv 0$ on $\partial C_0$ and $u \equiv 1$ on $\partial C_1$. If $\nabla$ denotes the gradient, then

$$\mathrm{Cap}(C_0, C_1) = \inf \iint_{A_{C_0,C_1}} \|\nabla u\|^2 d\mu_L,$$

where the infimum is taken over all admissible functions.

We shall make use of the following Lemma which we will use without proof.

**Lemma 2.7** *There is a constant $k$ with the following property: if for the generalised annulus $A_{C_0,C_1}$ the component $C_0$ contains 0 and a point in the unit circle and if $C_1$ contains $\infty$ and a point in the unit circle, then*

$$\mathrm{Mod}(A_{C_0,C_1}) \geq k/2.$$

**Theorem 2.8** *Assume that there exists a constant $K \geq 1$ such that for the homeomorphism $f : U \to U'$ between domains of $\mathbb{C}$ the Modulus Inequality holds. Then $f$ is quasiconformal according to the Metric Definition.*

*Proof:* Let $z \in U$ and consider the annulus $A_{\beta,\alpha}$ with centre $f(z)$ and radii

$$\alpha = \sup_{|z-w|=r} |f(z) - f(w)| \quad \text{and} \quad \beta = \inf_{|z-w|=r} |f(z) - f(w)|.$$

If $r$ is sufficiently small, this annulus lies entirely in $U'$. Its $f$-pre-image is a generalised annulus $A_{C_0,C_1}$ such that $C_0$ comprises $z$ and a path which connects it with the circle $S = \{w : |z - w| = r\}$ and $C_1$ comprises $\infty$ and a path which connects it with $S$. Since

$$\mathrm{Mod}(A_{\beta,\alpha}) = \frac{2\pi}{\log(\alpha/\beta)},$$

we use the previous lemma and Modulus Inequality to obtain

$$k/2 \leq \mathrm{Mod}(A_{C_0,C_1}) \leq K\mathrm{Mod}(A_{\beta,\alpha}) = \frac{2\pi K}{\log(\alpha/\beta)},$$

therefore

$$\log(\alpha/\beta) \leq 4\pi K/k$$

and thus

$$H(z) \leq e^{4\pi K/k}$$

and $f$ is quasiconformal according to the Metric Definition. ∎

**Remark 2.9** *It can be shown (see [15]) that the mappings of the above theorem are actually $K$-quasiconformal.*

## 2.2 Analytic definition

### 2.2.1 Weak (generalised) derivatives and Sobolev spaces

Let $u$ be a function in $L^1([a, b])$. We say that $v \in L^1([a, b])$ is a *weak derivative of* $u$ if,

$$\int_a^b u(t)\varphi'(t)dt = -\int_a^b v(t)\varphi(t)dt$$

for all infinitely differentiable functions $\varphi$ with $\varphi(a) = \varphi(b) = 0$. This definition is of course motivated by the integration by parts formula.

We may generalise to $n$ dimensions: if $u$ and $v$ are in the space $L^1_{\text{loc}}(U)$ of locally integrable functions for some open set $U \subset \mathbb{R}^n$, and if $\alpha$ is a multi-index, then $v$ is the $\alpha^{\text{th}}$-weak derivative of $u$ if

$$\int_U u D^\alpha \varphi = (-1)^{|\alpha|} \int_U v\varphi,$$

for all $\varphi \in C_c^\infty(U)$. That is, for all infinitely differentiable functions $\varphi$ with *compact support in* $U$. The multi-index partial derivative $D^\alpha \varphi$ is defined as

$$D^\alpha \varphi = \frac{\partial^{|\alpha|}\varphi}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}.$$

If $u$ u has a weak derivative, we denote it by $D^\alpha u$; this is reasonable since weak derivatives are uniquely defined a.e.

If a function $f$ is defined in a subset of $\mathbb{R}$ we say that it belongs to the *Sobolev space* $W^{k,p}$ if $f$ and its weak derivatives up to some order $k$ are in $L^p$ norm, $1 \leq p \leq \infty$. It turns out that it is enough to suppose only that the weak $(k-1)$-th derivative of $f$ is differentiable a.e. and is equal a.e. to the Lebesgue integral of its derivative.

Thus instead of the norm

$$\|f\|_{k,p} = \left(\sum_{i=0}^{k} \left\|f^{(i)}\right\|_p^p\right)^{\frac{1}{p}} = \left(\sum_{i=0}^{k} \int \left|f^{(i)}(t)\right|^p \, dt\right)^{\frac{1}{p}},$$

which turns $W^{k,p}$ into s a Banach space, we can use the equivalent norm which is defined by

$$\left\|f^{(k)}\right\|_p + \|f\|_p.$$

This definition generalises to higher dimensions: If $U \subset \mathbb{R}^n$, the Sobolev space $W^{k,p}(U)$ is defined to be the set of all functions $f : U \to \mathbb{R}$ such that for every multi-index $\alpha$ with $|\alpha| \leq k$, the mixed partial derivative

$$\frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$$

exists in the weak sense and is in $L^p(U)$, therefore

$$W^{k,p}(U) = \{f \in L^p(\Omega) : D^\alpha f \in L^p(\Omega) \ \forall |\alpha| \leqslant k\}.$$

### 2.2.2 Absolute continuity in lines

**Definition 2.10** *A continuous real-valued function* u *is said to be absolutely continuous on lines (ACL) in the domain* $U \subset \mathbb{C}$ *if for each closed rectangle contained in* $U$*, almost every restriction of the function to each horizontal and vertical closed line* *segment of the rectangle is absolutely continuous. A complex valued function is ACL in $U$ if its real and imaginary parts are ACL in $U$.*

Nikodym established in 1933 the following ACL characterisation of Sobolev spaces: Let $1 \leq p \leq \infty$. If $f \in W^{1,p}(U)$, then perhaps except on a set of measure zero, the restriction to almost every line parallel to the coordinate directions in $\mathbb{R}^n$ is absolutely continuous, i.e. $f$ is ACL. Moreover, the usual derivative along the lines parallel to the coordinate directions are in $L^p(U)$. Conversely, if the restriction of $f$ to a.e. line parallel to the coordinate directions is absolutely continuous, then $f \in W^{1,p}(U)$ provided $f$ and all the weak first derivatives are all in $L^p(U)$. In particular, in this case the weak partial derivatives of $f$ and pointwise partial derivatives of $f$ are the same a.e.

We now have the following theorem which is mainly due to Pflüger and Morrey:

**Theorem 2.11** *If $f : U \to U$ is quasiconformal according to the metric definition, then it is ACL in $U$.*

The proof of this theorem is rather long and pretty technical. For a nice exposition, see [1] for the complex case and also for greater dimensions see the notes of Butler,

`http://math.uchicago.edu/~cbutler/Quasiconformality.pdf`

There, it is exposed the variant of Gehring, which constitutes the proof for $\mathbb{R}^n$. Mostow proved in 1973 the ACL property for quasiconformal mappings between boundaries of symmetric spaces of rank one-that is, between mappings on the Heisenberg group. But, it wasn't only till 1990 when it was realised that Mostow's proof contained an error; this was fixed by Mostow, Korányi and Reimann.

Recall now

**Rademacher's Theorem:** *If $U$ is an open subset of $R^n$ and $f : U \to \mathbb{R}^m$ is Lipschitz continuous, then $f$ is differentiable a.e. in $U$.*

The following generalisation is due to Stepanov:

**Stepanov's Theorem:** *Let $U \subseteq \mathbb{R}^n$ be measurable and let also $f : U \to \mathbb{R}^m$ be a measurable function. Then $f$ is a.e. differentiable on the set*

$$\left\{ x \in U : \mathrm{limsup}_{y \to x} \frac{|f(x) - f(y)|}{|x - y|} < \infty \right\}.$$

Proofs of Rademacher-Stepanov Theorem may be found in standard books of analysis; see for instance [36].

Now, what we have from the above in other words, is that the function $I_f : U \to \mathbb{R}$, where

$$I_f(x) = \mathrm{limsup}_{|x-y|=r} \frac{|f(x) - f(y)|}{r}$$

is bounded a.e. in $U$. Suppose now that $f : U \to U'$ is a quasiconformal mapping and for each $z \in U$ and $r > 0$ set $B(z, r)$ to be the open disk centred at $z$ and with radius $r$. Let

$$J_f(z) = \text{limsup}_{r\to 0} \frac{\mu_L(f(B(z,r)))}{\mu_L(B(z,r))}.$$

It is known by Lebesque's density theorem that $J_f$ is bounded. Now

$$I_f(z)^2 \leq H(z)^2 J_f(z)$$

and since $H(z) \leq H$ we obtain from Stepanov's Theorem the following:

**Theorem 2.12** *Quasicconformal mappings between domains of $\mathbb{C}$ are a.e. differentiable.*

Note that then

$$J_f = |f_{\overline{z}}|^2 - |f_z|^2$$

is the Jacobian determinant of $f$.

Suppose now that the quasiconformal $f$ is differentiable at $z \in U$. Denote by $f_*(z)$ the derivative of $f$ at $z$. Then we can show that, see [27]

$$K_f(z) = \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|} \leq K,$$

where $K = H^2$. In a more general way,

$$\|f_*\|^2 \leq K J_f,$$

where

$$\|f_*\|^2 = \text{tr}^2(f_*) = (|f_z| + |f_{\overline{z}}|)^2.$$

### 2.2.3 Beltrami equation

We finally state again:

**Theorem 2.13 Measurable Riemann Mapping Theorem:** *Suppose that $U$ is a simply connected domain in $\mathbb{C}$ that is not equal to $\mathbb{C}$, and suppose that $\mu : U \to \mathbb{C}$ is Lebesgue measurable and satisfies $\|\mu\|_\infty < 1$. Then there is a quasiconformal homeomorphism $f : U \to D$ where $D$ is the unit disk which is in the Sobolev space $W^{1,2}(U)$ and satisfies the corresponding Beltrami equation in the distributional sense. The map $f$ is unique up to Möbius transformations.*

The proof (by Ahlfors and Bers) can be found in many standard books, see for instance [1] and [11]. This is of central importance in the theory of quasiconformal mappings in the plane: It generalizes the Riemann Mapping Theorem from conformal to quasiconformal homeomorphisms. There is no analogue of this theorem to greater dimensions. In the next paragraph we shall see how we can compensate with this fact. For the moment we state

**Definition 2.14 Analytic definition:** *A homeomorphism $f : U \to U'$, between domains in $\mathbb{C}$ is an orientation-preserving $K$-quasiconformal mapping if $f$*

*(i) is ACL;*

*(ii) is a.e. differentiable, and*

*(iii) satisfies a.e. a Beltrami equation of the form 1.2.1, where $\mu$ is a complex function in $U$ such that $\|\mu\|_\infty < 1$.*

An analogous definition holds for orientation-reversing quasiconformal mappings. An equivalent definition is:

**Definition 2.15 (Analytic definition)** *A homeomorphism $f : U \to U'$, between domains in $\mathbb{C}$ is an orientation-preserving $K-$quasiconformal mapping if $f \in HW^{1,2}(U, \mathbb{C})$ and if*

$$\|f_{*,z}\|^2 \leq K J_f(z)$$

*for almost all $z \in U$.*

## 2.3 Quasiconformal deformations

By $f^\mu$ we shall denote the mapping with complex dilatation $\mu$, normalized so that it fixes 0, 1 and $\infty$. From such an $f^\mu$ we may define a family of complex dilatations by

$$\mu_t(z) = \frac{\mu(z)}{|\mu(z)|} \tanh\left(\frac{t}{\log K} \tanh^{-1} |\mu(z)|\right).$$

According to the Measurable Riemann Mapping Theorem, for each $t \in \mathbb{R}$ there exists a unique normalised solution given by $f(t,z) = f^{\mu_t}(z)$. We comment on the definition of the homotopy $\mu_t(z)$: $\mu_t$ is chosen so that $\mu_t(z)$ is the point on the radius through $\mu(z)$ such that in the unit hyperbolic disk the hyperbolic distance $d_h(0, \mu_t(z))$ is the hyperbolic distance $d_h(0, \mu(z))$ multiplied by the factor $t/\log K$. It takes little effort to show that

$$\mu_t(z) = \frac{\mu(z)}{|\mu(z)|} \cdot \frac{K(z,t) - 1}{K(z,t) + 1}, \quad K(z,t) = \left(\frac{1 + |\mu(z)|}{1 - |\mu(z)|}\right)^{t/\log K}.$$

Note here that by taking derivatives[5] with respect to $t$ we have

$$\frac{\partial \mu_t(z)}{\partial t} = \frac{2}{(K(z,t) + 1)^2} \cdot K_t(z,t),$$

[5] We assume here that $\mu$ and thus $\mu_t(z)$ is $C^1$. If this does not happen, there exists a sequence $\mu_n$ of $C^1$ and essentially bounded by $k = (K-1)/(K+1)$ complex dilations which converge a.e. to $\mu$ and are such that the sequence of corresponding normalised solutions $f^{\mu_n}$ converge locally uniformly to $f$, see [27].

and

$$K_t(z,t) = \frac{K(z,t)\cdot \log(K(z,\log K))}{\log K} \leq K(z,t).$$

Therefore

$$\left|\frac{\partial \mu_t(z)}{\partial t}\right| = \frac{\partial |\mu_t(z)|}{\partial t} \leq \frac{2K(z,t)}{(K(z,t)+1)^2} = \frac{1}{2}(1-|\mu_t(z)|^2). \tag{2.3.1}$$

We now define a vector field $v(z,t)$ by the differential equation

$$V(f(z,t),t) = \frac{\partial f(z,t)}{\partial t},$$

with initial condition $f(z,\log K) = f^{\mu}(z)$. We will show now using Eq. 2.3.1 that

$$|V_{\overline{z}}(z,t)| \leq 1/2.$$

In fact, by differentiating $f_{\overline{z}}(z,t) = \mu_t(z) f_z(z,t)$ with respect to $t$ we have

$$\frac{\partial \mu_t}{\partial t} f_z(z,t) = \frac{\partial^2 f(z,t)}{\partial \overline{z}\partial t} - \mu_t(z)\frac{\partial^2 f(z,t)}{\partial z \partial t}.$$

We now use the differential equation to write

$$\frac{\partial^2 f(z,t)}{\partial \overline{z}\partial t} = V_z f_{\overline{z}}(z,t) + V_{\overline{z}}\overline{f(z,t)}_{\overline{z}},$$
$$\frac{\partial^2 f(z,t)}{\partial z\partial t} = V_z f_z(z,t) + V_{\overline{z}}\overline{f(z,t)}_z.$$

Hence

$$\frac{\partial \mu_t}{\partial t} f_z(z,t) = V_{\overline{z}}(\overline{f(z,t)}_{\overline{z}} - \mu_t(z)\overline{f(z,t)}_z) = V_{\overline{z}}\overline{f(z,t)}_{\overline{z}}(1-|\mu_t(z)|^2)$$

and by taking absolute values we have

$$|V_{\overline{z}}| = \frac{\left|\frac{\partial \mu_t(z)}{\partial t}\right|}{1-|\mu_t(z)|^2}.$$

Our desired result now follows from Eq. 2.3.1. We have the following important theorem due to Gehring and Reich, [14] which summarises the above discussion:

**Theorem 2.16 Parametric representation:** *If $f:\mathbb{C}\to\mathbb{C}$ is $K$-quasiconformal then there exists a vectorfield $V(z,t)$, $t\in[0,\log K]$ with bounded $\overline{z}$-derivative by 1/2 a.e. such that its flow $f(z,t)$ comprises of $e^t$-quasiconformal mappings with $f(z,0) = id.$ and $f(z,\log K) = f(z)$.*

We have already noted in the introduction the importance of this theorem: it lies in the fact that it admits generalisations to higher dimensions (and to the Heisenberg group!) whereas there is no known equivalent of the Existence Theorem neither in $\mathbb{R}^n$, $n>2$ nor in other spaces.

## 2.4 Extremal problems and Modulus method

In this section we are concerned with the following problem: Let $U$ be a domain in $U$, $\rho : U \to \mathbb{R}^+$ a Borel function and let $K_f(z) = K(f,z)$ be the distortion functions of quasiconformal mappings $f : U \to U'$.

**Problem:** To find, if it exists, a quasiconformal $f_0 : U \to U'$ such that its maximal distortion is minimal among the maximal distortions of all quasiconformal $f : U \to U'$.

A solution $f_0$ for that problem is called *extremal*. The theory of extremal mappings is rich, starting from the works of Teichmüller and later of Strebel and Reich among others, see [37].

The *mean distortion integral* is defined by

$$\mathfrak{M}(f,\rho) = \frac{\iint_U K_f(z)\rho^2(z)d\mu_L(z)}{\iint_U \rho^2(z)d\mu_L(z)}.$$

**Problem:** To find, if it exists, a quasiconformal $f_0 : U \to U'$ such that

$$\mathfrak{M}(f_0,\rho) \leq \mathfrak{M}(f,\rho),$$

for each quasiconformal $f : U \to U'$.

A method of solving that problem is described below; it is known as the *Modulus Method* for the detection of minimisers of the mean distortion integral.

*Step 1.* Assume that there exists a curve family $\Gamma_0$ such that $\rho \in \mathrm{Adm}(\Gamma_1)$ and it satisfies

$$\mathrm{Mod}(\Gamma_0) = \iint_U \rho^2 d\mu_L.$$

*Step 2.* Assume that there exists a quasiconformal $f_0 : U \to U'$ such that:

$$\mathrm{Mod}(f_0(\Gamma_0)) = \iint_U K_{f_0}(z)\rho^2(z)d\mu_L(z).$$

*Step 3.* Assume that there exists a curve family $\Gamma$ containing $\Gamma_0$ such that

$$\rho \in \mathrm{Adm}(\Gamma_0)$$

and moreover,

$$\mathrm{Mod}(f_0(\Gamma_0)) \leq \mathrm{Mod}(f(\Gamma)),$$

for each quasiconformal $f : U \to U'$.

Then $f_0$ is the desired minimiser: From Modulus inequality and our assumptions we have

$$\iint_U K_{f_0}(z)\rho^2(z)d\mu_L(z) = \mathrm{Mod}(f_0(\Gamma_0)) \leq \mathrm{Mod}(f(\Gamma)) \leq \iint_U K_f(z)\rho^2(z)d\mu_L(z).$$

Dividing by $\iint_U \rho^2 d\mu_L$ we obtain then

$$\mathfrak{M}(f_0, \rho) \leq \mathfrak{M}(f, \rho)$$

for all $\rho$, and thus $f_0$ is a minimiser.

We remark that whereas the Modulus Method is elementary, it has a lot of assumptions which are not obvious at all in most cases. However, the method can be applied also to mappings that have only bounded distortion (they are not necessarily ACL). For an implementation of this method, see [2] and the references therein.

Note that if a minimiser $f_0$ for the mean distortion functional has constant maximal distortion $K_{f_0}$, then due to the Modulus Inequality we have that $f_0$ is a minimiser for the maximal distortion among quasiconformal maps $f : U \to U'$.

**Exercise!** Formulate and solve the Grötzsch problem using the Modulus Method.

(Hint: If your rectangle is $R(0, a, a + bi, bi)$ take $\rho = 1/a$, $\Gamma_0$ the family of line segments paralle to the $x$-axis and $f_0$ the linear map.

We state two questions which appear here:

- When a minimiser for the mean distortion exists? Is it unique?
- When a minimiser for the maximal distortion is a minimiser for the mean distortion integral as well?

## 2.5 Additional remarks

We summarise below some important properties of quasiconformal mappings of the complex plane. The interested reader should refer to [11] for instance, for further details.

### 2.5.1 Hölder continuity

The following theorem is due to Mori and it proves Hölder continuity of quasiconformal maps:

**Theorem 2.17** *Let $f$ be a $K$-quasiconformal self-mapping of the unit disk $D$ with $f(0) = 0$. Then for each $z_1, z_2 \in D$,*

$$|f(z_1 0 - f(z_2)| \leq 16|z_1 - z_2|^{1/K},$$

*and therefore $f$ is $1/K$-Hölder continuous. The constant 16 is sharp.*

### 2.5.2 Compactness properties

Suppose $\mathcal{F}$ is a family of normalised quasiconformal mappings of the extended plane $\hat{\mathbb{C}}$. It can be proved that $\mathcal{F}$ is a compact subset of the set of homeomorphisms of $\hat{\mathbb{C}}$ with respect to the $C^0$ topology.

### 2.5.3 Quasisymmetric and quasi-isometric mappings

If $f : \mathbb{H} \to \mathbb{H}$ is a quasiconformal self-mapping of the upper half-plane

$$\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\},$$

then it can be extended to $\mathbb{C} \setminus \mathbb{R}$ to a quasiconformal map $\tilde{f}$ defined by

$$\tilde{f}(z) = \begin{cases} f(z) & z \in \mathbb{H} \\ \overline{f(\overline{z})} & z \in \mathbb{C} \setminus \overline{\mathbb{H}} \end{cases}$$

One can check that $f$ and $\tilde{f}$ have the same maximal distortion. The quastion is now if $\tilde{f}$ can be extended to the whole plane $\mathbb{C}$. The answer is positive: Given a quasiconformal mapping of $\hat{\mathbb{C}} \setminus \{\gamma\}$, where $\gamma$ is a Jordan curve, then the mapping can be extended on the whole sphere, in other words, Jordan curves are removable singularities for quasiconformal mappings. The extension also fixes $\infty$. A nice geometrical property hold for the restriction of the extended mapping on the real line $\mathbb{R}$: There exists a constant $M > 0$ such that for all $x \in \mathbb{R}$ and $t > 0$,

$$\frac{1}{M} \leq \frac{\hat{f}(x+t) - \hat{f}(x)}{\hat{f}(x) - \hat{f}(x-t)} \leq M. \tag{2.5.1}$$

This property follows from the consequent lemma:

**Lemma 2.18** *If four pairwise distinct points $z_1, z_2, z_3, z_4$ lie in a compact subset $E$ of the sphere $\hat{\mathbb{C}}$ which does not contain $0, 1, \infty$, then there exists a compact $F \subset \hat{\mathbb{C}}$ which does not contain $0, 1, \infty$ and is such that the cross-ratio $\mathbb{X}(z_1, z_2, z_3, z_4) \in F$.*

**Definition 2.19** *A homeomorphism $\hat{f}$ of the real line which satisfies 2.5.1 is called a quasisymmetric mapping.*

Quasisymmetric mappings are very important: It has been proved by Beurling and Ahlfors that they can be extended to quasiconformal mappings of the upper half-plane (Beurling-Ahlfors extension). Another notable extension is the barycentric extension done by Douady-Earle.

**Definition 2.20** *A homeomorphism $f : \mathbb{H} \to \mathbb{H}$ is called $(K, a)$-quasi-isometric, if there exists positive $K$ and $a$ such that for each $z, w \in \mathbb{H}$ we have*

$$\frac{d_{\mathbb{H}}(z,w)}{K} - a \leq d_{\mathbb{H}}(z,w) \leq K d_{\mathbb{H}}(z,w) + a.$$

It can be proved that the extension of a quasi-isometric map which preserves $\infty$ to the real line is a quasisymmetric map. Moreover, a $K$-quasiconformal self-map of the upper half-plane is a $(K, a(K))$ quasi-isometric map, where $a(K)$ is a constant depending only on the constant of quasiconformality $K$.

# CHAPTER 3

# THE HEISENBERG GROUP

## 3.1 Complex hyperbolic plane and the Heisenberg group

The Heisenberg group appears naturally in certain ways; the one we are interested in is as the boundary of complex hyperbolic plane minus one point. This is in analogy of the real line $\mathbb{R}$ (viewed as the additive group $(\mathbb{R},+)$ appearing as the boundary of real hyperbolic plane minus one point. The material in this section concerning the complex hyperbolic plane is standard; the definitions and results presented here can be found in [12], [29], [32] and [7].

### 3.1.1 Complex hyperbolic plane

We revisit the definition of complex hyperbolic plane first. Following Mostow [29], we define $\mathbb{C}^{2,1}$ to be the vector space $\mathbb{C}^3$ with the Hermitian form of signature $(2,1)$ given by

$$\langle \mathbf{z},\mathbf{w}\rangle = \mathbf{w}^*J\mathbf{z} = z_1\overline{w}_3 + z_2\overline{w}_2 + z_3\overline{w}_1$$

with matrix

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Motivated by relativity, we consider the following subspaces of $\mathbb{C}^{2,1}$:

$$\begin{aligned} V_- &= \Big\{ \mathbf{z} \in \mathbb{C}^{2,1} \ : \ \langle \mathbf{z},\, \mathbf{z} \rangle < 0 \Big\}, \\ V_0 &= \Big\{ \mathbf{z} \in \mathbb{C}^{2,1} \setminus \{\mathbf{0}\} \ : \ \langle \mathbf{z},\, \mathbf{z} \rangle = 0 \Big\}. \end{aligned}$$

If $\mathbb{P} : \mathbb{C}^{2,1} \setminus \{0\} \longrightarrow \mathbb{C}P^2$ is the canonical projection onto complex projective space, then we define *complex hyperbolic plane* $\mathbf{H}^2_{\mathbb{C}}$ to be $\mathbb{P}V_-$. Its boundary $\partial \mathbf{H}^2_{\mathbb{C}}$ is $\mathbb{P}V_0$.

Specifically, $\mathbb{C}^{2,1} \setminus \{\mathbf{0}\}$ may be covered with three charts $H_1$, $H_2$, $H_3$ where $H_j$ comprises those points in $\mathbb{C}^{2,1} \setminus \{\mathbf{0}\}$ for which $z_j \neq 0$. It is clear that $V_-$ is contained in $H_3$. The canonical projection from $H_3$ to $\mathbb{C}^2$ is given by $\mathbb{P}(\mathbf{z}) = (z_1/z_3,\, z_2/z_3) = z$. Therefore we can write $\mathbf{H}^2_{\mathbb{C}} = \mathbb{P}(V_-)$ as

$$\mathbf{H}^2_{\mathbb{C}} = \big\{ (z_1,\, z_2) \in \mathbb{C}^2 \ : \ 2\Re(z_1) + |z_2|^2 < 0 \big\}.$$

In this manner, $\mathbf{H}^2_{\mathbb{C}}$ is the *Siegel domain* in $\mathbb{C}^2$; see [12]. There are distinguished points in $V_0$ which we denote by $\mathbf{o}$ and $\infty$:

$$\mathbf{o} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \infty = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Then $V_0 \setminus \{\infty\}$ is contained in $H_3$ and $V_0 \setminus \{\mathbf{o}\}$ (in particular $\infty$) is contained in $H_1$. Let $\mathbb{P}\mathbf{o} = o$ and $\mathbb{P}\infty = \infty$. Then we can write $\partial \mathbf{H}^2_{\mathbb{C}} = \mathbb{P}(V_0)$ as

$$\partial \mathbf{H}^2_{\mathbb{C}} \setminus \{\infty\} = \big\{ (z_1,\, z_2) \in \mathbb{C}^2 \ : \ 2\Re(z_1) + |z_2|^2 = 0 \big\}.$$

In particular $o = (0, 0) \in \mathbb{C}^2$.

Conversely, we may lift a point $z = (z_1, z_2) \in \mathbb{C}^2 = \mathbb{C}P^2$ to a point $\mathbf{z}$ in $\mathbb{C}^{2,1}$called the *standard lift* of $z$, by writing $\mathbf{z}$ in non-homogeneous coordinates as

$$\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \\ 1 \end{pmatrix}.$$

The *Bergman metric* on $\mathbf{H}^2_{\mathbb{C}}$ is defined by the distance function $\rho$ given by the formula

$$\cosh^2\left(\frac{\rho(z, w)}{2}\right) = \frac{\langle \mathbf{z}, \mathbf{w} \rangle \langle \mathbf{w}, \mathbf{z} \rangle}{\langle \mathbf{z}, \mathbf{z} \rangle \langle \mathbf{w}, \mathbf{w} \rangle} = \frac{\big|\langle \mathbf{z}, \mathbf{w} \rangle\big|^2}{|\mathbf{z}|^2 |\mathbf{w}|^2}$$

where $\mathbf{z}$ and $\mathbf{w}$ in $V_-$ are the standard lifts of $z$ and $w$ in $\mathbf{H}^2_{\mathbb{C}}$ and $|\mathbf{z}| = \sqrt{-\langle \mathbf{z}, \mathbf{z}\rangle}$. Alternatively,

$$ds^2 = -\frac{4}{\langle \mathbf{z}, \mathbf{z}\rangle^2} \det \begin{bmatrix} \langle \mathbf{z}, \mathbf{z}\rangle & \langle d\mathbf{z}, \mathbf{z}\rangle \\ \langle \mathbf{z}, d\mathbf{z}\rangle & \langle d\mathbf{z}, d\mathbf{z}\rangle \end{bmatrix}.$$

The holomorphic sectional curvature of $\mathbf{H}^2_{\mathbb{C}}$ equals to $-1$ and its real sectional curvature is pinched between $-1$ and $-1/4$.

### 3.1.2 Isometries, complex lines, Lagrangian planes

Let $\mathrm{U}(2,1)$ be the group of unitary matrices for the Hermitian form $\langle \cdot, \cdot \rangle$. Each such matrix $A$ satisfies the relation $A^{-1} = JA^*J$ where $A^*$ is the Hermitian transpose of $A$.

The full group of holomorphic isometries of complex hyperbolic space is the *projective unitary group* $\mathrm{PU}(2,1) = \mathrm{U}(2,1)/\mathrm{U}(1)$, where $\mathrm{U}(1) = \{e^{i\theta}I, \theta \in [0, 2\pi)\}$ and $I$ is the $3 \times 3$ identity matrix. It is convenient sometimes to consider instead the group $\mathrm{SU}(2,1)$ of matrices which are unitary with respect to $\langle \cdot, \cdot \rangle$, and have determinant 1. This is in accordance with the standard fact $\mathbf{H}^2_{\mathbb{C}} = \mathrm{SU}(2,1)/\mathrm{U}(2)$. We have $\mathrm{PU}(2,1) = \mathrm{SU}(2,1)/\{I, \omega I, \omega^2 I\}$, where $\omega$ is a non real cube root of unity, and so $\mathrm{SU}(2,1)$ is a 3-fold covering of $\mathrm{PU}(2,1)$. This is the direct analogue of the fact that $\mathrm{SL}(2,\mathbb{C})$ is the double cover of $\mathrm{PSL}(2,\mathbb{C})$.

We note here that $A \in \mathrm{SU}(2,1)$ is of the form

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}$$

with inverse

$$A^{-1} = \begin{pmatrix} \overline{j} & \overline{f} & \overline{c} \\ \overline{h} & \overline{e} & \overline{b} \\ \overline{g} & \overline{d} & \overline{a} \end{pmatrix}$$

There are various relations between the entries of $A$; these follow from the relations

$$AA^{-1} = A^{-1}A = I,$$

where $I$ is the $3 \times 3$ identity matrix.

Isometric embeddings of $\mathbf{H}^2_{\mathbb{R}}$ (that is the Beltrami–Klein model for the usual hyperbolic plane) and $\mathbf{H}^1_{\mathbb{C}}$ (that is the Poincaré model for the usual hyperbolic plane) induce 2–dimensional geodesic submanifolds of the complex hyperbolic plane: a complex line is an isometric image of the embedding of $\mathbf{H}^1_{\mathbb{C}}$ into $\mathbf{H}^2_{\mathbb{C}}$. A Lagrangian plane is an isometric image of $\mathbf{H}^2_{\mathbb{R}}$ into $\mathbf{H}^2_{\mathbb{C}}$. In fact, and in contrast to the case of real hyperbolic spaces, complex lines and Lagrangian planes are the only 2–dimensional geodesic submmanifolds and there are no geodesic submanifolds of dimension 3.

### 3.1.3 Group structure of the boundary

A finite point $z$ is in the boundary of the Siegel domain if its standard lift to $\mathbb{C}^{2,1}$ is $\mathbf{z}$ where

$$\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \\ 1 \end{pmatrix} \qquad \text{where} \qquad z_1 + \bar{z}_1 + |z_2|^2 = 0.$$

We write $z = z_2/\sqrt{2} \in \mathbb{C}$ and this condition becomes $2\Re(z_1) = -2|z|^2$. Hence we may write $z_1 = -|z|^2 + it$ for $t \in \mathbb{R}$. That is for $z \in \mathbb{C}$ and $t \in \mathbb{R}$:

$$\mathbf{z} = \begin{pmatrix} -|z|^2 + it \\ \sqrt{2}z \\ 1 \end{pmatrix}.$$

Therefore we may identify the boundary of the Siegel domain with the one point compactification of $\mathbb{C} \times \mathbb{R}$. It is clear that the identification is via the mapping $\Phi : \partial\mathbf{H}^2_{\mathbb{C}} \to \mathbb{C} \times \mathbb{R}$ where

$$\Phi(z_1, z_2) = (z_2/\sqrt{2}, \Im(z_1)).$$

Its inverse $\Phi^{-1} : \mathbb{C} \times \mathbb{R} \to \partial\mathbf{H}^2_{\mathbb{C}}$ is given exactly by

$$\Phi^{-1}(z,t) = (-|z|^2 + it, \sqrt{2}z).$$

The map $\Phi$ is called the parametrisation of the Heisenberg group by the boundary of the siegel domain. Let

$$\{\partial_z,\ \partial_{\overline{z}}, \partial_t\},$$

be the natural basis of the tangent space of $\mathbb{C} \times \mathbb{R}$. Then its derivative is given by the matrix

$$(\Phi^{-1})_*(z,t) = \begin{pmatrix} -\overline{z} & -z & i \\ -z & -\overline{z} & -i \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}.$$

Therefore

$$\begin{aligned}
(\Phi^{-1})_*\left(\frac{\partial}{\partial z}\right) &= -\frac{\overline{z_2}}{\sqrt{2}}\left(\frac{\partial}{\partial z_1}\right) - \frac{z_2}{\sqrt{2}}\left(\frac{\partial}{\partial \overline{z_1}}\right) + \sqrt{2}\left(\frac{\partial}{\partial z_2}\right),\\
(\Phi^{-1})_*\left(\frac{\partial}{\partial \overline{z}}\right) &= -\frac{z_2}{\sqrt{2}}\left(\frac{\partial}{\partial z_1}\right) - \frac{\overline{z_2}}{\sqrt{2}}\left(\frac{\partial}{\partial \overline{z_1}}\right) + \sqrt{2}\left(\frac{\partial}{\partial \overline{z_2}}\right),\\
(\Phi^{-1})_*\left(\frac{\partial}{\partial t}\right) &= i\left(\frac{\partial}{\partial z_1}\right) - i\left(\frac{\partial}{\partial \overline{z_1}}\right).
\end{aligned}$$

For further use, we make here the following observation: if

$$Z = \partial_z + i\overline{z}\partial_t, \quad \overline{Z} = \partial_{\overline{z}} - iz\partial_t, \quad T = \partial_t,$$

then

$$(\Phi^{-1})_*(Z) = \sqrt{2}\,(\partial_{z_2} - \overline{z_2}\partial_{z_1})\,, \tag{3.1.1}$$
$$(\Phi^{-1})_*(\overline{Z}) = \sqrt{2}\,(\partial_{\overline{z_2}} - z_2\partial_{\overline{z_1}})\,, \tag{3.1.2}$$
$$(\Phi^{-1})_*(T) = i\partial_{z_1} - i\partial_{\overline{z_1}}. \tag{3.1.3}$$

We shall come back to those vector fields in the next section. We now turn our attention to the way that $\mathbb{C}\times\mathbb{R}$ is endowed with a natural group structure. Consider the stabiliser of infinity:

$$\mathrm{Stab}(\infty) = \{A \in \mathrm{SU}(2,1) : A(\infty) = \infty\}.$$

One can show that this set comprises elements of $\mathrm{SU}(2,1)$ that are upper triangular with diagonal entries all equal to 1. The action of the stabiliser of infinity $\mathrm{Stab}(\infty)$ gives to the set of these points the structure of a non Abelian group as follows. We identify $(z,t)$ with the element

$$T(z,t) = \begin{pmatrix} 1 & -\sqrt{2}\overline{z} & -|z|^2 + it \\ 0 & 1 & \sqrt{2}z \\ 0 & 0 & 1 \end{pmatrix}.$$

Via this identification we obtain the group law $*$ in $\mathbb{C}\times\mathbb{R}$:

$$(z,t) * (w,s) = T^{-1}(T(z,t)T(w,s)) = (z+w, t+s+2\Im(z\overline{w})).$$

**Definition 3.1** *The Heisenberg group* $\mathfrak{H}$ *is the group* $(\mathbb{C}\times\mathbb{R}, *)$.

From the definition it follows that $(z,t)^{-1} = (-z,-t)$, the identity element is $(0,0)$ and $\mathfrak{H}$ is non-Abelian. But the Heisenberg group is a *2-step nilopotent* group: Recall that a group $G$ is an $n$-step nilpotent if it has a central series of finite length $n$. That is, a series of normal subgroups

$$\{1\} = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

where $G_{i+1}/G_i \le Z(G/G_i)$, or equivalently $[G, G_{i+1}] \le G_i$.

Elementary calculations show that $Z(\mathfrak{H}) = \{0\}\times\mathbb{R}$ and we have the central series

$$\{1\} \lhd Z(\mathfrak{H}) \lhd \mathfrak{H}.$$

## 3.2 Korányi-Cygan metric structure

The Heisenberg norm (Korányi gauge) is given by

$$|(z,t)| = \left||z|^2 - it\right|^{1/2}.$$

Note that despite its name the Heisenberg norm is not a norm in the usual sense. From this norm we obtain a metric, the Korányi-Cygan (K-C) (or Heisenberg) metric on $\mathfrak{H}$, by the relation

$$d_{\mathfrak{H}}\left((z_1,t_1),\,(z_2,t_2)\right) = \left|(z_1,t_1)^{-1} * (z_2,t_2)\right|.$$

Or, in other words[1]

$$d_{\mathfrak{H}}\left((z_1,t_1),\,(z_2,t_2)\right) = \left||z_1-z_2|^2 - it_1 + it_2 - 2i\Im(z_1\bar{z}_2)\right|^{1/2}.$$

We consider the following transformations of $\mathfrak{H}$; they all extend trivially to $\partial\mathbf{H}^2_{\mathbb{C}}$:

1. *Left translations*: for $(\zeta,s)\in\mathfrak{H}$ we define

$$T_{(\zeta,s)}(z,t) = (\zeta,s)*(z,t).$$

   These include Heisenberg vertical translations $T_{(0,s)}$.

2. *Rotations about the vertical axis*: for $\theta\in\mathbb{R}$ we define

$$R_\theta(z,t) = (ze^{i\theta},t).$$

3. Conjugation: This is defined by

$$j(z,t) = (\overline{z},-t).$$

Left translations are left actions of $\mathfrak{H}$ onto itself and rotations are induced by the action of $\mathrm{U}(1)$ on $\mathfrak{H}$. It is easy to see that the metric $d_{\mathfrak{H}}$ is invariant by both left translations, rotations and conjugation. These transformations form the group $\mathrm{Isom}(\mathfrak{H},d_{\mathfrak{H}})$ of *Heisenberg isometries*.

There are two other kinds of transformations that we consider, namely

(3) *Dilations*: For $\delta>0$ we define

$$D_\delta(z,t) = (\delta z,\delta^2 t).$$

For every $(z,t),(z',t')\in\partial\mathbf{H}^2_{\mathbb{C}}$ we have

$$d_{\mathfrak{H}}\left(D_\delta(z,t),D_\delta(z',t')\right) = \delta\; d_{\mathfrak{H}}\left((z,t),(z',t')\right)$$

and thus the metric $d_{\mathfrak{H}}$ is scaled up to multiplicative constants by the action of dilations.

[1] This is actually given by the usual formula

$$d_{\mathfrak{H}}\left((z_1,t_1),\,(z_2,t_2)\right) = \left|\left\langle \begin{pmatrix} -|z_1|^2+it_1 \\ \sqrt{2}z_1 \\ 1 \end{pmatrix}, \begin{pmatrix} -|z_2|^2+it_2 \\ \sqrt{2}z_2 \\ 1 \end{pmatrix} \right\rangle\right|^{1/2}.$$

(4) *Inversion*: It is the transformation $R$ given by

$$R(z,t) = \left( \frac{z}{-|z|^2 + it} \, , \, \frac{-it}{|-|z|^2 + it|^2} \right), \quad \text{if } (z,t) \neq o, \infty; ;$$

it can be extended to the whole boundary by setting

$$R(o) = \infty, \quad R(\infty) = o.$$

Inversion $R$ is a (holomorphic) involution of $\partial \mathbf{H}^2_{\mathbb{C}}$. Moreover, for all $p = (z,t), p' = (z',t') \in \mathfrak{H} \setminus \{o\}$ we have the inversion formula

$$d_{\mathfrak{H}}(R(p), o) = \frac{1}{d_{\mathfrak{H}}(p,o)}, \quad d_{\mathfrak{H}}(R(p), R(p')) = \frac{d_{\mathfrak{H}}(p,p')}{d_{\mathfrak{H}}(p,o)\; d_{\mathfrak{H}}(o,p')}.$$

In analogy to the classical case, it is proved that the action of $\mathrm{SU}(2,1)$ on the boundary $\partial \mathbf{H}^2_{\mathbb{C}}$ is completely described by (compositions of the above transformations, that is, if $g$ is an isometry of complex hyperbolic plane, then it is the composition of transformations of the form (1)–(6).

We finally remark that the K-C metric is not a path metric, that is, there exist pairs of points such that the distance between them is strictly shorter than the length of any path joining those points.

## 3.3 Lie group structure

The Heisenberg group is a 3-dimensional Lie group: Its underlying manifold is $\mathbb{C} \times \mathbb{R}$ and the mapping $\mathfrak{H} \times \mathfrak{H} \to \mathfrak{H}$ where

$$((z,t),(w,s)) \mapsto (z,t)^{-1} * (w,s)$$

is differemtiable. A left-invariant basis for its Lie algebra $\mathfrak{h}$ (that is, its tangent space) comprises the vector fields

$$X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}.$$

As before, instead of $X$ and $Y$ we will mainly consider the complex fields

$$Z = \frac{1}{2}(X - iY) = \frac{\partial}{\partial z} + i\overline{z}\frac{\partial}{\partial t}, \quad \overline{Z} = \frac{1}{2}(X + iY) = \frac{\partial}{\partial \overline{z}} - iz\frac{\partial}{\partial t}.$$

The only non-trivial Lie bracket equation is

$$[X, Y] = -4T.$$

We deduce that if $\mathfrak{h}$ is the Lie algebra of $\mathfrak{H}$, then

$$[\mathfrak{h}, \mathfrak{h}] = [\mathfrak{h}_1], \quad \mathfrak{h}_1 = \langle \partial_t \rangle,$$

and

$$[\mathfrak{h}, \mathfrak{h}_1] = 0$$

This is to say that the Heisenberg group is a 2-step nilpotent Lie group: Recall that a $k$-step nilpotent Lie group is a Lie group $G$ which is connected and whose Lie algebra is a $k$-step nilpotent Lie algebra $\mathfrak{g}$. That is, its Lie algebra lower central series

$$\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}], \quad \mathfrak{g}_2 = [\mathfrak{g}, \mathfrak{g}_1], ...$$

eventually vanishes in the $k$-th step: $\mathfrak{g}_k = 0$. Actually, it can be proved that the Heisenberg Lie algebra $\mathfrak{h}$ is the unique simply connected 2-step nilpotent Lie algebra; it follows that the Heisenberg group can be recovered in a unique way from its Lie algebra: In Lie group theory, simply connected nilpotent Lie groups $G$ are identified via the exponential map

$$\mathfrak{g} \ni X \mapsto \exp(tX) \in G$$

with their Lie algebras (and hence with some $\mathbb{R}^m$. The group law is then obtained by the Campbell-Hausdorff formula

$$\exp X * \exp Y = \exp(X + Y + (1/2)[X, Y] + R(X, Y)),$$

where $R(X, Y)$ is a polunomial of the commutators of $X$, $Y$ of order at least 3. In the case of the Heisenberg group we have $R(X, Y) = 0$ and therefore

$$\exp X * \exp Y = \exp(X + Y + (1/2)[X, Y]) = \exp(X + Y - 2T).$$

But there is more to it: the Heisenberg group is a *Carnot group*: Recall that a Carnot group $G$ is a simply connected nilpotent Lie group with a derivation $\alpha$ on its Lie algebra $\mathfrak{g}$ such that $V^1 = \ker(\alpha - 1)$ generates $\mathfrak{g}$. By setting $V^{i+1} = [V^1, V^i]$ we obtain a grading:

$$\mathfrak{g} = \oplus_{i=1}^{r} V^i, \quad [V^i, V^j] \subset V^{i+j}$$

and $\alpha \mid_{V^j} = j\mathrm{id}_{V^j}$. In the Heisenberg group case the derivation $\alpha$ is given in terms of the basis by

$$\alpha(X) = X, \quad \alpha(Y) = Y, \quad \alpha(T) = 2T$$

and thus

$$\mathfrak{h} = V^1 \oplus V^2,$$

where

$$V^1 = \mathrm{span}_{\mathbb{R}}\{X, Y\} \quad \text{and} \quad V^2 = \mathrm{span}_{\mathbb{R}}\{T\}.$$

We now show why the dilations, the Heisenberg norm and the Korányi distance are actually Carnot group aspects in the Heisenberg group. Now in a Carnot Group $G$ with coordinates

$$(x_{11}, \ldots, x_{1n_1}, \ldots x_{r1}, \ldots x_{rn_r})$$

which follow from the identification with $\mathbb{R}^m$, $m = n_1 + \ldots n_r$ via the derivation $\alpha$, gives rise to dilations $D_s$ where

$$D_s(x_{11}, \ldots x_{1n_1}, \ldots, x_{r1_r} \ldots x_{rn_r}) = (sx_{11}, \ldots sx_{1n_1}, \ldots, s^r x_{r1_r} \ldots s^r x_{rn_r}).$$

Then the expression

$$\|x\|^{2r!} = \sum_{i=1}^{r} \left( \sum_{j=1}^{n_i} |x_{ij}|^2 \right)^{\frac{r!}{i}}$$

is homogeneous with respect to $D_s$, that is

$$\|D_s(x)\| = s\|x\|.$$

The left-invariant distance function defined by

$$d(x,y) = \|x^{-1} * y\|$$

satisfies

$$d(D_s(x), D_s(y)) = s\ d(x,y).$$

In general, this does not satisfy the Triangle Inequality; in the Heisenberg group case though the Korányi distance does this exactly.

In a Carnot group $G$ the subspace $V^1$ of $\mathfrak{g}$ is called the *horizontal space* of $G$ and is of fundamental importance. If we fix a metric $|\cdot|$ for $V^1$, a curve $\gamma : [0,1] \to G$ is called horizontal if $\dot\gamma(t) \in V^1$ for all $t$. Then distance $d_{CC}$ can be defined in $G$ as follows: For every $x, y \in G$ set

$$d_{CC}(x,y) = \inf \int_0^1 |\dot\gamma(t)| dt,$$

where the infimum is taken over all horizontal $\gamma$ joining $x$ and $y$. Such curves always exist: This fact was first proven by Carathéodory in the beginning of the 20th century and by Chow in its full generality; it constitutes the cornerstone of sub-Riemannian geometry.

The horizontal length $\ell_h(\gamma)$ is

$$\ell(\gamma) = \int_0^1 |\dot\gamma(t)| dt.$$

The distances $d$ and $d_{CC}$ on a Carnot group are equivalent: They are both left-invariant and are homogeneous with respect to dilations. We shall come back to the CC distance in the Heisenberg group below. For the moment we state our final facts for carnot groups:

- The Haar measure on $G$ is bi-invariant and nothing but a (constant multiple of the) Lebesque measure $\mu_L$ of the underlying space $R^m$.

- The Jacobian determinant of the dilation $D_s$ is $s^Q$,

$$Q = \sum_{j=1}^{m} jr^j, \quad r_j = dim(V^j).$$

From this we obtain that

$$\mu_L(B(r)) = r^Q \mu_L(B(1)),$$

where $B(r)$ can be the metric ball of radius $r$ for either of the two metrics $d$ or $d_{CC}$. We will see later that this tells us exactly that $G$ is a $Q$-regyular space.

The number $Q$ is called the homogeneous or Hausdorff dimension of $G$ and in the case of the Heisenberg group $Q = 4$.

## 3.4 More structures of the Heisenberg group

The Heisenberg group $\mathfrak{H}$ is also the prototype for a CR manifold of codimension 1. In the next section we recall some standard facts about CR structures.

### 3.4.1 CR structures

There are two equivalent definitions of an *abstract CR structure*. Suppose first that $M$ is a $(2p+s)$-dimensional real manifold. A *CR structure of codimension $s$ in $M$* is a pair $(\mathcal{D}, J)$ where $\mathcal{D}$ is a $2p$-dimensional smooth subbundle of $T(M)$ and $J$ is a bundle automorphism of $\mathcal{D}$ such that:

(i) $J^2 = -id.$ and

(ii) if $X$ and $Y$ are sections of $\mathcal{D}$ then the same holds for $[X, Y] - [JX, JY]$, $[JX, Y] + [X, JY]$ and moreover

$$J\left([X, Y] - [JX, JY]\right) = [JX, Y] + [X, JY].$$

On the other hand, let $M$ be a $(2p+s)$-dimensional real manifold and let $T_{\mathbb{C}}(M)$ be its complexified tangent bundle. A CR structure of codimension $s$ in $M$ is a complex $p$-complex dimensional smooth subbundle $\mathcal{H}$ of $T^{\mathbb{C}}(M)$ such that:

(i) $\mathcal{H} \cap \overline{\mathcal{H}} = \{0\}$ and

(ii) $\mathcal{H}$ is involutive, that is for any vector fields $Z$ and $W$ in $\mathcal{H}$ we have

$$[Z, W] \in \mathcal{H}.$$

The two definitions are equivalent; see for instance Theorem 1.1, Chpt. VI of [5]. A manifold endowed with a CR structure is called a *CR manifold*. A special class of CR manifolds are the CR submanifolds: Suppose that $N$ be a complex manifold of complex dimension $n$ with complex structure $J$, and let $M$ be a submanifold of $N$ of real dimension $m$. Then

$$\mathcal{H} = T(M) \cap J(T(M)),$$

is the maximal invariant subspace of $T(M)$ under the action of $J$, it is also a smooth subbundle on $M$ and $M$ is called a *CR submanifold of* $(N, J)$. A CR submanifold is in fact a CR manifold (see for instance Theorem 2.1, p.135 of [5]). The CR structure is $(\mathcal{H}, J)$, where here by $J$ we denote the bundle automorphism induced by the restriction of $J$ in $\mathcal{H}$. The corresponding complex subbundle is

$$\mathcal{H}^{(1,0)} = \{Z \in T^{\mathbb{C}}(M) \mid Z = X - iJX,\ X \in \mathcal{H}\},$$

and we have

$$X \in \mathcal{H} \quad \text{if and only if} \quad Z = X - iJX \in \mathcal{H}^{(1,0)}.$$

Suppose now that $M$ is a CR submanifold of the $n$-complex dimensional complex manifold $N$ with $n = p + s$, such that $\dim_{\mathbb{R}}(M) = 2p + s$, where $2p = \dim_{\mathbb{R}} \mathcal{H}$; that is, $M$ is a codimension $s$ CR submanifold of $N$. [2]

CR diffeomorphisms are defined as follows: Let $M$ and $M'$ be CR manifolds of the same dimension $m = 2p + s$ with CR structures $\mathcal{H}$ and $\mathcal{H}'$ respectively, of the same dimension $s$. A diffeomorphism $F : M \to M'$ is a CR *diffeomorphism* if it preserves CR structures; that is $F_*\mathcal{H} = \mathcal{H}'$. In other words, $F$ is a CR diffeomorphism if and only if for each $Z \in \mathcal{H}$ we have $F_*Z \in \mathcal{H}'$. In terms of the corresponding real distributions $(\mathcal{D}, J)$ and $(\mathcal{D}', J')$ we may say that $F$ is CR if for each $X \in \mathcal{D}$ we have $F_*(JX) = J'(F_*X)$.

For our purposes we are going to be concerned in particular with CR structures in subvarieties of $\mathbb{C}^2$. We consider the manifold $\mathbb{C}^2$, with the natural complex coordinates $(z_1, z_2)$, $z_i = x_i + iy_i$, $i = 1, 2$. Denote also by $\mathbb{J}$ the natural complex structure of $\mathbb{C}^2$. An 3-dimensional smooth subvariety of $\mathbb{C}^2$ is defined by an equation

$$\rho(z_1, z_2) = 0.$$

The set $M$ consisting of points of the subvariety at which the matrix

$$D = \begin{pmatrix} \frac{\partial \rho}{\partial x_1} & \frac{\partial \rho}{\partial x_2} & \frac{\partial \rho}{\partial y_1} & \frac{\partial \rho}{\partial y_2} \end{pmatrix},$$

is of constant rank 1 is a real submanifold of $\mathbb{C}^2$ with $\dim(M) = 3$. Its tangent space $T_x(M)$ at a point $x \in M$ is identified to the set

$$T_x(M) = \{X \in T_x(\mathbb{C}^2) \mid (d\rho)_x(X) = 0.\}.$$

The maximal complex subspace $\mathcal{H}_x$ at each $x \in M$ comprises of $X \in T_x(\mathbb{C}^2)$ such that

$$(d\rho)_x(X) = 0 \ \text{ and } \ (d^c\rho)_x(X) = 0,$$

[2] Let $\mathcal{V}$ be a complementary to $\mathcal{H}$ subbundle of $M$:

$$T(M) = \mathcal{H} \oplus \mathcal{V},$$

Note that $\dim_{\mathbb{R}} \mathcal{V}_x = s$. If

$$J(\mathcal{V}) \cap T(M) = \{0\},$$

we call $M$ an *antiholomorphic* CR submanifold of $N$.

where
$$(d^c\rho)_x(X) = -(d\rho)_x(\mathbb{J}X).$$
Let
$$\mathcal{H}_x^{(1,0)} = \{Z = X - i\mathbb{J}X \in T^{(1,0)}(\mathbb{C}^2) \mid X \in \mathcal{H}_x\}.$$
Then $\mathcal{H}_x^{(1,0)}$ comprises of $Z \in T_x^{(1,0)}(\mathbb{C}^2)$ such that
$$(\partial\rho)_x(Z) = 0,$$
and one verifies that
$$X \in \mathcal{H}_x \text{ if and only if } Z = X - i\mathbb{J}X \in \mathcal{H}_x^{(1,0)}.$$
Denote by $\mathcal{H}^{(1,0)}$ the complex subbundle comprising of $\mathcal{H}_x^{(1,0)}$, $x \in M$. At points $x \in M$ consider the matrix
$$D^{(1,0)} = \begin{pmatrix} \frac{\partial\rho}{\partial z_1} & \frac{\partial\rho}{\partial z_2} \end{pmatrix},$$
and let $M' \subset M$ be the set at which $D^{(1,0)}$ is of constant rank $l$. Then $\mathcal{H}^{(1,0)}$ is defined at $M'$, $\dim_{\mathbb{C}} \mathcal{H}^{(1,0)} = 3 - l = 2$. Moreover, the integrability condition holds obviously ($[Z, Z] = 0$). Therefore, $\mathcal{H}^{(1,0)}$ is a CR structure of codimension $1 = 4 - 3$.[3]

The single vector field generating the CR structure is
$$Z = -\frac{\partial\rho}{\partial z_2} \cdot \frac{\partial}{\partial z_1} + \frac{\partial\rho}{\partial z_1} \cdot \frac{\partial}{\partial z_2}.$$
Now, the *Levi form* $(L)_p : \mathcal{H}_p^{(1,0)} \to \mathbb{R}^2$ is defined in $M'$ by
$$Z_p \mapsto L(p) = dd^c\rho(Z, \overline{Z})_p,$$
where
$$L(p) = \begin{pmatrix} -\frac{\partial\rho}{\partial z_2} & \frac{\partial\rho}{\partial z_1} \end{pmatrix}_p \cdot \begin{pmatrix} \frac{\partial^2\rho}{\partial z_1 \partial \overline{z_1}} & \frac{\partial^2\rho}{\partial z_1 \partial \overline{z_2}} \\ \frac{\partial^2\rho}{\partial z_2 \partial \overline{z_1}} & \frac{\partial^2\rho}{\partial z_2 \partial \overline{z_2}} \end{pmatrix}_p \cdot \begin{pmatrix} -\frac{\partial\rho}{\partial \overline{z_2}} \\ \frac{\partial\rho}{\partial \overline{z_1}} \end{pmatrix}_p.$$
An important case occurs when $L(p) > 0$ for each $p$. Then we call the CR structure *strongly pseudoconvex*. The reason for that name is justified from complex analysis in several variables: A domain $U \subset \mathbb{C}^2$ with defining equation $\rho$, that is, the set of points $(z_1, z_2)$ such that
$$\rho(z_1, z_2) < 0,$$
is called strongly pseudoconvex, if the Levi form is positive definite in the boundary set of points defined by $\rho(z_1, z_2) = 0$.

[3] We call the set $\mathcal{S} = M \setminus M'$ the *singular set* of the CR structure.

**Proposition 3.2** *The boundary of the Siegel domain*

$$\mathcal{S} = \{(z_1, z_2) \in \mathbb{C}^2 : 2\Re(z_1) + |z_2|^2 < 0\}$$

*admits a strongly pseudoconvex CR structure.*

*Proof:* Let

$$\rho(z_1, z_2) = 2\Re(z_1) + |z_2|^2 = 0.$$

Using the notation of the above discussion we have

$$D = \begin{pmatrix} 1 & 2x_2 & 0 & 2y_2 \end{pmatrix},$$

which is of constant rank 1 everywhere,

$$D^{(1,0)} = \begin{pmatrix} 1 & \overline{z_2} \end{pmatrix}$$

and therefore

$$Z = -\overline{z_2} \cdot \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}$$

generates the CR structure. Note that at each $p = (z_1, z_2) \in \partial\mathcal{S}$ we have

$$L(p) = \begin{pmatrix} -\overline{z_2} & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -z_2 \\ 1 \end{pmatrix} = 1.$$

■

Suppose

## 3.5 Contact structure

There are two ways to reach the contact structure of the Heisenberg group $\mathbb{H}$: By regarding $\mathfrak{H}$ as a 2-step nilpotent Lie group with underlying manifold $\mathbb{C} \times \mathbb{R}$ first, we consider its left invariant vector fields $X, Y, T$ (or alternatively, $Z, \overline{Z}, T$) and its Lie bracket relations. Then the contact form $\omega$ of $\mathfrak{H}$ is defined as the unique 1-form satisfying $X, Y \in \ker\omega$, $\omega(T) = 1$. Explicitly, in Heisenberg coordinates $z = x + iy, t$,

$$\omega = dt + 2(xdy - ydx) = dt + 2i\Im(\overline{z}dz).$$

Uniqueness here is modulo change of coordinates as it follows by the Darboux's Theorem. The distribution $\mathcal{H}$ in $\mathfrak{H}$ defined by $\mathcal{H} = V^1 = \ker\ \omega$ is called the *horizontal distribution*. The vector field $T$ whiuch generates $V^2$ is called the Reeb vector field and satisfies $[X, Y] = -4T$ (or equivalently, $[Z, \overline{Z}] = -2iT$), $\omega(T) = 1, T \in \ker\ d\omega$.

The contact structure of $\mathfrak{H}$ (and of $\partial\mathbf{H}^2_{\mathbb{C}}$) may be also obtained by the strongly pseudoconvex CR structure of $\mathfrak{H}$. Consider the 1-form $\omega = -\frac{1}{2}d^c\rho$, where $\rho$ is the defining function of the Siegel domain $\mathcal{S}$,

$$\rho(z_1, z_2) = 2\Re(z_1) + |z_2|^2.$$

Then through the map $\Phi$ we may identify $\partial S$ and $\mathfrak{H}$.

The volume form on $\mathfrak{H}$ is $\omega \wedge d\omega = 4dx \wedge dy \wedge dt$, i.e. a multiple of the usual volume form obtained by the Lebesgue measure in $\mathbb{C} \times \mathbb{R}$.

Finally, we make some remarks about the horizontal curves and the CC distance. An absolutely continuous curve $\gamma : [a, b] \to \mathfrak{H}$ (in the Euclidean sense) with

$$\gamma(s) = (z(s), t(s)) \in \mathbb{C} \times \mathbb{R}.$$

is *horizontal* if

$$\dot\gamma(s) \in \mathcal{H}_{\gamma(s)}(\mathfrak{H}) \quad \text{for almost every } t \in [a, b].$$

This is equivalent to

$$\dot t(s) = -2\Im(\overline{z(s)}\dot z(s)),$$

which gives

$$t(s) = -2\int_0^s \Im(\overline{z(u)}\dot z(u))du$$

It can be proved that a curve $\gamma : [a, b] \to \mathfrak{H}$ is absolutely continuous with respect to the Korányi distance $d_{\mathfrak{H}}$ if and only if it is a horizontal curve. Moreover, the horizontal length of a smooth rectifiable curve $\gamma(s) = (z(s), t(s))$ with respect to $d_{\mathfrak{H}}$ is given by the integral over the (Euclidean) norm of the horizontal part of the tangent vector,

$$\ell_h(\gamma) = \int_a^b |\dot z(s)|\, ds,$$

see [23].

Horizontal curves are *geodesics* for the *Carnot-Caratheodorý* metric. It can be shown that the CC-length of a horizontal curve is its horizontal length defined above. Like the metric $d_{\mathfrak{H}}$, the CC-metric $d_{CC}$ is invariant under left translations and rotations, and is also scaled up to the positive constant $\delta$ under dilations $D_\delta$. The relation of $d_{\mathfrak{H}}$ (which is not a geodesic metric) and $d_{CC}$ is given as follows: there exist universal constants $C_1, C_2 > 0$ so that

$$C_1 d_{\mathfrak{H}}(p, 0) \le d_{CC}(p, 0) \le C_2 d_{\mathfrak{H}}(p, 0)$$

for each $p \in \mathfrak{H}$.

# CHAPTER 4

# QUASICONFORMAL MAPPINGS IN THE HEISENBERG GROUP

## 4.1 Metric definition

**Definition 4.1 Metric definition:** *Given two domains $\Omega, \Omega'$ in $\mathfrak{H}$, a homeomorphism $f : \Omega \to \Omega'$ is called* quasiconformal *if*

$$H_f(p) = \limsup_{r \to 0} \frac{\max_{d_{\mathfrak{H}}(p,q)=r} d_{\mathfrak{H}}(f(p), f(q))}{\min_{d_H(p,q)=r} d_{\mathfrak{H}}(f(p), f(q))}, \quad p \in \Omega, \tag{4.1.1}$$

*is uniformly bounded from above. It is called $K$-quasiconformal if there is a constant $K \geq 1$ such that*

$$\text{ess sup}_{p \in \Omega} H(p) \leq K.$$

We note that we obtain an equivalent definition if we substitute $d_{\mathfrak{H}}$ by the $CC$-metric; the constant of quasiconformality does not change. We will see additionally that conformal mappings (i.e., elements of $\mathrm{SU}(2,1)$ acting on $\mathfrak{H}$) are 1-quasiconformal and the he converse is also true: a 1-quasiconformal mapping is necessarily an element of $\mathrm{SU}(2,1)$.

## 4.2 Quasiconformal contact transformations

Contact transformations between domains of $\mathfrak{H}$ play an important role in the theory of quasiconformal mappings of $\mathfrak{H}$.

**Definition 4.2** *A contact transformation $f : \Omega \to \Omega'$ on $\mathfrak{H}$ is a $C^1$ diffeomorphism between domains $\Omega$ and $\Omega'$ in $\mathfrak{H}$ which preserves the contact structure, i.e.*

$$f^*\omega = \lambda\omega \tag{4.2.1}$$

*for some non-vanishing real valued function $\lambda$.*[1]

Therefore contact transformations map hrizontal vector fields to horizontal vector fields: If $V \in \mathcal{H}(\Omega)$ then since

$$\omega(f_*V) = (f^*\omega)(V) = \lambda\omega(V) = 0,$$

it follows that $f_*V \in \mathcal{H}(\Omega')$.

We usually write $f = (f_I, f_3)$, $f_I = f_1 + \mathrm{i}f_2$ for maps in $\mathfrak{h}$. If such a map is differentiable, its derivative $f_*$ is terms of the basis

$$Z, \overline{Z}, T$$

and its dual basis

$$dz, d\overline{z}, \omega$$

is given by

$$f_* = \begin{pmatrix} d(f_I)(Z) & d(f_I)(\overline{Z}) & d(f_I)(T) \\ d(\overline{f_I})(Z) & d(\overline{f_I})(\overline{Z}) & d(\overline{f_I})(T) \\ (f^*\omega)(Z) & (f^*\omega)(\overline{Z}) & f^*(\omega)(T) \end{pmatrix}.$$

We have

$$\begin{aligned} d(f_I)(Z) = Zf_I, \quad d(f_I)(\overline{Z}) = \overline{Z}f_I, \quad d(f_I)(T) = Tf_I, \\ d(\overline{f_I})(Z) = Z\overline{f_I}, \quad d(\overline{f_I})(\overline{Z}) = \overline{Z}\overline{f_I}, \quad d(\overline{f_I})(T) = T\overline{f_I}, \end{aligned}$$

[1]Notation clarification: Let $F : M \to N$ be a differentiable mapping between manifolds $M$ and $N$, $\dim(M) = m$ and $\dim(N) = n$. Let also $\omega$ be a differential 1-form in $N$. The pullback $F^*\omega$ of $\omega$ is the differential 1-form on $M$ defined by the formula:

$$(F^*\omega)_p(X) = \omega_{F(p)}(dF_p(X)),$$

for any $p \in M$ and $X \in T_pM$. Here, $dF_p$ is the derivative of $F$ at $p$. It follows that if $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_n)$ are coordinate neighbourhoods around $p$ and $F(p)$, respectively, so that $\omega_{F(p)} = \sum_{i=1}^n a_i(dy_i)_{F(p)}$, then

$$(F^*\omega)_p = \sum_{i=1}^n \sum_{j=1}^m a_i \frac{\partial y_i}{\partial x_j}(p)(dx_j)_p.$$

and if $f$ is contact then

$$(f^*\omega)(Z) = \lambda\omega(Z) = 0, \quad (f^*\omega)(\overline{Z}) = \lambda\omega(\overline{Z}) = 0, \quad (f^*\omega)(\overline{T}) = \lambda\omega(T) = 1.$$

Therefore for $f$ contact we have that its derivative is given by /6+ 8*e

$$f_* = \begin{pmatrix} Zf_I & \overline{Z}f_I & Tf_I \\ Z\overline{f_I} & \overline{Z}\overline{f_I} & T\overline{f_I} \\ 0 & 0 & \lambda \end{pmatrix}. \tag{4.2.2}$$

Now since

$$\begin{aligned} 0 = (f^*\omega)(Z) = \omega(f_*Z) = (df_3 + 2\Im(\overline{f_I}df_I)))(Z), \\ 0 = (f^*\omega)(\overline{Z}) = \omega(f_*Z) = (df_3 + 2\Im(\overline{f_I}df_I)))(\overline{Z}), \\ \lambda = (f^*\omega)(T) = \omega(f_*Z) = (df_3 + 2\Im(\overline{f_I}df_I)))(T), \end{aligned}$$

we obtain

$$\overline{f}_I Zf_I - f_I Z\overline{f}_I + iZf_3 = 0 \tag{4.2.3}$$
$$f_I \overline{Z}\overline{f}_I - \overline{f}_I \overline{Z}f_I - i\overline{Z}f_3 = 0 \tag{4.2.4}$$
$$-i(\overline{f}_I Tf_I - f_I T\overline{f}_I + iTf_3) = \lambda. \tag{4.2.5}$$

Thus a contact map $f$ is completely determined by $f_I$ in the sense that the contact condition 4.2.1 is equivalent to the above (overdetermined) system of partial differential equations.

Observe now that the jacobian $J_f$ of $f$ is

$$J_f = \lambda J_f^h,$$

where

$$J_f^h = \det \begin{pmatrix} Zf_I & \overline{Z}f_I \\ Z\overline{f_I} & \overline{Z}\overline{f_I} \end{pmatrix}.$$

**Proposition 4.3** *If the contact map $f$ is sense-preserving and $C^2$ then*

$$\det J_f = \lambda^2,$$

*Proof:* It suffices to prove that $J_f^h = \lambda$. We have

$$d\omega = 2idz \wedge d\overline{z}$$

hence

$$f^*(d\omega) = 2idf_I \wedge d\overline{f_I}.$$

Therefore

$$\begin{aligned} (f^*(d\omega))(Z, \overline{Z}) &= 2i(df_I \wedge d\overline{f_I})(Z, \overline{Z}) \\ &= 2i(Zf_I \cdot \overline{Z}\overline{f_I} - Z\overline{f_I} \cdot \overline{Z}f_I)) \\ &= 2iJ_f^h. \end{aligned}$$

On the other hand,[2]

$$f^*(d\omega) = d(f^*\omega) = d(\lambda\omega) = d\lambda \wedge \omega + \lambda d\omega$$

and so

$$(f^*(d\omega))(Z, \overline{Z}) = \lambda d\omega(Z, \overline{Z}) = 2i\lambda.$$

The proof is complete. ∎

**Proposition 4.4** *Left-translations, rotations, dilations, conjugation and inversion are all contact transformations of* $\mathfrak{h}$.

The proof of this proposition is left as an exercise.

We now consider the Pansu or horizontal derivative $f_*^h$ of $f$ given by the matrix

$$f_* = \begin{pmatrix} Zf_I & \overline{Z}f_I \\ Z\overline{f_I} & \overline{Zf_I} \end{pmatrix}. \tag{4.2.6}$$

By requiring $f$ to be $C^2$ and sense-preserving we have that $\det(f_*^h) = \lambda$. At each $p$, the horizontal (sub-Riemannian) norm in $\mathcal{H}_p$ is given for each vector $V_p = aX_p + bY_p = 2\Re(a+bi)Z_p)$ by

$$\|V_p\| = (a^2 + b^2)^{1/2}.$$

We define

$$\lambda_1(p) = \sup_{\|V_p\|=1} |f_* V_p|,$$
$$\lambda_2(p) = \inf_{\|V_p\|=1} |f_* V_p|.$$

Here,

$$\begin{aligned}
|f_*V_p| &= |f_*((a+bi)Z_p + (a-bi)\overline{Z}_p)| \\
&= |(a+bi)(f_*(Z_p)) + (a-bi)(f_*(\overline{Z}_p)| \\
&= |(a+bi)(Zf_I \cdot Z_p + Z\overline{f_I} \cdot \overline{Z}_p) + (a-bi)(\overline{Z}f_I \cdot Z_p + \overline{Zf_I} \cdot \overline{Z}_p)| \\
&= |[(a+ib)Zf_I + (a-ib)\overline{Z}f_)] \cdot Z_p + [(a+ib)Z\overline{f_I} - (a-ib)\overline{Zf_I}] \cdot \overline{Z}_p)| \\
&= |(a+ib)Zf_I + (a-ib)\overline{Z}f_I|.
\end{aligned}$$

Since $a^2+b^2 = 1$ and $|Zf_I(p)| > |\overline{Z}f_I(p)|$ ($J_f^h(p) = \lambda(p) > 0$) we have by triangle inequality that

$$|Zf_I(p)| - |\overline{Z}f_I(p)| \leq |f_*V_p| \leq |Zf_I(p)| + |\overline{Z}f_I(p)|.$$

We examine when equality is attained in both sides of the above inequality. Set $a + bi = e^{i\phi}$. Then

$$|e^{i\phi}Zf_I(p) + e^{-i\phi}\overline{Z}f_I(p)|^2 = |Zf_I(p)|^2 + |\overline{Z}f_I(p)|^2 + 2\Re\left(e^{2i\phi}Zf_I(p) \cdot Z\overline{f}_I(p)\right).$$

[2]The $C^2$ requirement for $f$ lies exactly in this equation.

Equality is attained in the right hand-side when

$$e^{2i\phi} Zf_I(p) \cdot Z\overline{f}_I(p) \geq 0,$$

that is,

$$2\phi = \arg\left(\frac{\overline{Z} f_I(p)}{Z f_I(p)}\right),$$

whereas equality is attained in the left hand-side when

$$e^{2i\phi} Zf_I(p) \cdot Z\overline{f}_I(p) \leq 0,$$

that is,

$$2\phi = \pi + \arg\left(\frac{\overline{Z} f_I(p)}{Z f_I(p)}\right).$$

Therefore if $f$ is contact,

$$\lambda_1(p) = |Zf_I(p)| + |\overline{Z} f_I(p)|,$$
$$\lambda_2(p) = |Zf_I(p)| - |\overline{Z} f_I(p)|,$$

and the *distortion function* $K_f(p)$ of $f$ is just

$$K_f(p) = \frac{\lambda_1(p)}{\lambda_2(p)} = \frac{|Zf_I(p)| + |\overline{Z} f_I(p)|}{|Zf_I(p)| - |\overline{Z} f_I(p)|}.$$

**Definition 4.5 Quasiconformal $C^2$ contact transformations:** *A sense-preserving $C^2$ contact map $f : \Omega \to \Omega'$ is $K$-quasiconformal ($K \geq 1$) if*

$$K_f(p) \leq K, \quad \textit{a.e. in } \Omega.$$

*The infimum $K_f$ of all $K$ such that $f$ is $K$-quasiconformal shall be called the maximal distortion of $f$.*

Quasiconformal mappings according to the metric definition which are also $C^2$ have to be contact transformations. This is verified via the following theorems, see [22].

**Theorem 4.6** *A $C^2$ diffeomorphism $f : \Omega \to \Omega'$ between domains in $\mathfrak{H}$ which is quasiconformal according to the metric definition satisfies*

$$\frac{\lambda_1(p)}{\lambda_2(p)} \leq H_f(p).$$

*Proof:* If $p \in \Omega$, by pre-composing or post-composing by a Heisenberg similarity we may assume that $p = (0,0)$ and $F(0,0) = (0,0)$. Thus it suffices to prove

$$\frac{\lambda_1(0,0)}{\lambda_2(0,0)} \leq H_f(0,0).$$

We consider the Taylor expansion of second order of $f_3 := g$ at $(0,0)$:

$$\begin{aligned} g(z,t) &= g_z(0,0)z + \overline{g_z(0,0)z} + g_t(0,0)t \\ &+ \frac{1}{2}\big(g_{zz}(0,0)z^2 + g_{\overline{z}\overline{z}}(0,0)\overline{z}^2 + 2g_{z\overline{z}}(0,0)|z|^2 \\ &+ g_{zt}(0,0)zt + g_{\overline{z}t}(0,0)\overline{z}t + g_{tt}(0,0)t^2\big) + O(|z|^2+t^2). \end{aligned}$$

From the contact conditions we then have

$$\begin{aligned} &g_z(0,0) = Zf_3(0,0) = 0, \\ &\overline{g_z(0,0)} = \overline{Zf_3(0,0)} = 0, \\ &g_t(0,0) = Tf_3(0,0) = \lambda(0,0). \end{aligned}$$

For the second order terms we shall express the usual derivatives in terms of the derivatives w.r.t. $Z, \overline{Z}$ and $T$. For instance,

$$\begin{aligned} \frac{\partial^2}{\partial z^2} &= (Z - i\overline{z}T)(Z - i\overline{z}T) \\ &= Z(Z - i\overline{z}T) - i\overline{z}T(Z - i\overline{z}T) \\ &= ZZ - i\overline{z}ZT - i\overline{z}TZ - \overline{z}^2TT \end{aligned}$$

One then uses again the contact conditions and shows that the only possibly non-vanishing partial derivatives of second order are

$$g_{zt}(0,0), \quad g_{\overline{z}t}(0,0), \quad g_{tt}(0,0).$$

We conclude that

$$f(z,t) = \lambda(0,0)\cdot t + \frac{1}{2}\left(g_{zt}(0,0)z + g_{\overline{z}t}(0,0)\overline{z} + g_{tt}(0,0)t\right)\cdot t + O(|z|^2+t^2),$$

and thus the plane $t = 0$ is preserved up to terms of second order. ■

**Theorem 4.7** *A contact transformation which is $C^2$ and satisfies*

$$\frac{\lambda_1(p)}{\lambda_2(p)} \le K,$$

*is a K-quasiconformal mapping.*

*Proof:* See the proof of Theorem 4 in [22]. ■

**Corollary 4.8** *The group* $\mathrm{SU}(2,1)$ *acts on* $\mathfrak{H}$ *as a group of* 1*-quasiconformal (conformal) mappings.*

### 4.3 Analytic definition

Before stating the analytic definition of qc mappings we need to define the notions of the $P$-*differentiability* of mappings between domains of $\mathfrak{H}$ and that of the *absolute continuity in lines* (ACL). The notion of $P$-differentiability is due to Pansu [32], and is the natural generalisation of differentiability in Euclidean spaces to the Heisenberg setting.

**Definition 4.9** *A mapping $f : \Omega \to \Omega'$ between domains of $\mathfrak{H}$ is called $P$-differentiable at $p \in \Omega$ if for $c \to 0$ the mappings*

$$D_c^{-1} \circ T_{f(p)}^{-1} \circ f \circ T_{f(p)} \circ D_c$$

*converge locally uniformly to a homomorphism $f_{*,p}^P$ from $T_p(\mathfrak{H})$ to $T_{f(p)}(\mathfrak{H})$ which preserves the horizontal space $\mathcal{H}(\mathfrak{H})$. Here $D$ and $T$ are dilations and left translations, respectively.*

In terms of the standard basis comprising $Z, \overline{Z}, T$, the $P$-derivative of $f = (f_I, f_3)$ at a point $p$ is in matrix form

$$f_{*,p}^P = \begin{pmatrix} Zf_I & \overline{Z}f_I & 0 \\ Z\overline{f_I} & \overline{Z}\overline{f_I} & 0 \\ 0 & 0 & |Zf_I|^2 - |\overline{Z}f_I|^2 \end{pmatrix}_p ,$$

where here all derivatives are in the distributional sense. Pansu proved in [31] that quasiconformal mappings between domains in $\mathfrak{H}$ are a.e. $P$-differentiable. In particular, (see Proposition 6 of [23]), we have the following:

**Proposition 4.10** *If $f$ is $P$-differentiable at $p \in \mathfrak{H}$ with derivative $f_{*,p}^P$, then the restriction of $f$ to the plane*

$$\left\{ p \exp(xX + yY) \mid (x, y) \in \mathbb{R}^2 \right\}$$

*is differentiable at $p$ in the Euclidean sense and its derivative $f_{*,p}$ is the restriction of $f_{*,p}^P$ in horizontal spaces; in matrix form, it is given by*

$$f_{*,p}^P = \begin{pmatrix} Zf_I & \overline{Z}f_I \\ Z\overline{f_I} & \overline{Z}\overline{f_I} \end{pmatrix}_p ,$$

The following also hold for a $P$-differentiable at $p$ quasiconformal mapping $f$:

1. $\|f_{*,p}^P\| := \max\left\{ \|f_{*,p}^h(V)\| \mid |V| = 1 \right\} = |Zf_I(p)| + |\overline{Z}f_I(p)|$ a.e.;

2. $J_f(p) = \det f_{*,p}^P = (\det f_{*,p})^2 = \left( |Zf_I(p)|^2 - |\overline{Z}f_I(p)|^2 \right)^2$;

3.

$$K(p,f)^2 = \frac{\|f_{*,p}\|^4}{J_f(p)} = \left(\frac{|Zf_I(p)| + |\overline{Z}f_I(p)|}{|Zf_I(p)| - |\overline{Z}f_I(p)|}\right)^2 \leq K_f^2 = K.$$

The function $p \mapsto K(p,f) \in [1,\infty)$ is the *distortion function* of $f$ and the constant of quasiconformality $K = K_f^2$ is also called the *maximal distortion* of $f$.

We have seen that the basic property concerning the regularity of quasiconformal mappings on the complex plane (and more generally, on Euclidean spaces of arbitrary dimension) is absolute continuity in lines (ACL): mappings with this property are absolutely continuous on a.e. fiber of any smooth fibration.

In the case of the Heisenberg group $\mathfrak{H}$, absolute continuity holds on almost all fibers of smooth *horizontal* fibrations. For such a fibration, the fibers $\gamma_p$ can be parametrised by the flow $f_s$ of a horizontal *unit* vector field $V$: i.e. $V$ is of the form $aX + bY$ with $|a|^2 + |b|^2 = 1$.

The following theorem is due to Mostow (Theorem A in [23]); its Euclidean counterpart was proved by Gehring in [13]. The proof in the Euclidean case is considerably easier.

**Theorem 4.1** *Quasiconformal mappings are absolutely continuous on a.e. fiber $\gamma$ of any given fibration $\Gamma_V$ determined by a left invariant horizontal vector field $V$.*

About the Beltrami Equations: According to Theorem C in [23], if $f = (f_I, f_3)$ is an orientation preserving $K$-quasiconformal mapping between domains $\Omega$ and $\Omega'$ in $\mathfrak{H}$ then it satisfies a.e. the Beltrami type system of equations

$$\overline{Z}f_I = \mu Z f_I, \tag{4.3.1}$$

$$\overline{Z}f_{II} = \mu Z f_{II}, \tag{4.3.2}$$

where $f_{II} = f_3 + i|f_I|^2$ and $\mu$ is a complex function in $\Omega$ such that

$$\frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty} \leq K \quad \text{a.e.}$$

and

$$\|\mu\|_\infty = \text{esssup}\{|\mu(z,t)| \mid (z,t) \in \Omega\}.$$

For each $p = (z,t) \in \Omega$, the function

$$\mu(p) = \mu_f(p) = \frac{\overline{Z}f_I(p)}{Zf_I(p)}$$

is called the *Beltrami coefficient (complex dilation) of $f$*. Note that if $K = K_f$ is the maximal distortion and $K(p,f)$ is the distortion function of $f$ respectively, then the following hold:

$$|\mu_f(p)| = \frac{K(p,f) - 1}{K(p,f) + 1}, \quad K(p,f) = \frac{1 + |\mu_f(p)|}{1 - |\mu_f(p)|}, \quad \|\mu_f\| = \frac{K_f - 1}{K_f + 1}.$$

We now state the analytic definition of quasiconformality in $\mathfrak{H}$.

**Definition 4.11 Analytic definition:** *A homeomorphism $f:\Omega\to\Omega'$, $f=(f_I,f_3)$, between domains in $\mathfrak{H}$ is an orientation preserving $K$-quasiconformal mapping if $f$*

*(i) is ACL;*

*(ii) is a.e. $P$-differentiable, and*

*(iii) satisfies a.e. a system of Beltrami equations of the form 4.3.1, 4.3.2 where $\mu$ is a complex function in $\Omega$ such that $\|\mu\|_\infty<1$.*

An analogous definition holds for orientation-reversing quasiconformal mappings.

There are various analytic definitions of quasiconformality in $\mathfrak{H}$ which are all equivalent to the metric definition. For instance, we refer the reader to [8], [17] and [42].

For the definition we are about to state, see [3] and the references therein. To state it, we first have to define the *horizontal Sobolev space* $HW^{1,4}(\Omega,\mathfrak{H})$. We say that a function $u:\Omega\to\mathbb{C}$ is in $HW^{1,4}(\Omega,\mathfrak{H})$ if it is in $L^4(\Omega,\mathfrak{H})$ and if there exist functions $v,w\in L^4(\Omega,\mathfrak{H})$ such that

$$\int_\Omega v\phi d\mu_L=-\int_\Omega uZ\phi d\mu_L\quad\text{and}\quad\int_\Omega w\phi d\mu_L=-\int_\Omega u\overline{Z}\phi d\mu_L,$$

for all $\phi\in\mathcal{C}_0^\infty(\Omega,\mathbb{C})$. Now a mapping $f:\Omega\to\mathfrak{H}$, $f=(f_I,f_3)$ is said to be in $HW^{1,4}(\Omega,\mathfrak{H})$ if both $f_I,f_3$ are in $HW^{1,4}(\Omega,\mathfrak{H})$. Such a mapping which also satisfies conditions (4.2.3) and (4.2.4) a.e. is called *weakly contact* and one can define its formal horizontal differential $f^h_{*,p}$ at almost all $p$, which in matrix form is given by

$$f^h_{*,p}=\begin{pmatrix}Zf_I & \overline{Z}f_I\\ Z\overline{f_I} & \overline{Zf_I}\end{pmatrix}_p.$$

This is extended to a Lie algebra homomorphism which is the Pansu derivative $f^P_{*,p}$, as defined above, see [32]:

$$f_{*,p}=\begin{pmatrix}Zf_I & \overline{Z}f_I & 0\\ Z\overline{f_I} & \overline{Zf_I} & 0\\ 0 & 0 & |Zf_I|^2-|\overline{Z}f_I|^2\end{pmatrix}_p.$$

Now let

1. $\|f^h_{*,p}\|:=\max\left\{\|(f^h_{*,p})(V)\|\mid |V|=1\right\}=|Zf_I(p)|+|\overline{Z}f_I(p)|$ a.e.;

2. $J_f(p)=\det(f^h_{*,p})=(\det(f_{*,p})^2=\left(|Zf_I(p)|^2-|\overline{Z}f_I(p)|^2\right)^2$;

3.
$$K(p,f)^2 = \frac{\|f^h_{*,p}\|^4}{J_f(p)} = \left(\frac{|Zf_I(p)| + |\overline{Z}f_I(p)|}{|Zf_I(p)| - |\overline{Z}f_I(p)|}\right)^2.$$

**Definition 4.12 (Analytic definition)** *A homeomorphism $f : \Omega \to \Omega'$, $f = (f_I, f_3)$, between domains in $\mathfrak{H}$ is an orientation preserving $K-$quasiconformal mapping if $f \in HW^{1,4}(\Omega, \mathfrak{H})$ is weakly contact and if*

$$\|f^h_{*,p}\|^4 \leq K J_f(p)$$

*for almost all $p$.*

## 4.4 Geometric definition

Korányi and Reimann proved the equivalence of Metric and Analytic Definitions of quasiconformality in the Heisenberg group by showing that both are equivalent to a third definition which they called the geometric definition. Before going into it we need some notions first.

**Definition 4.13** *If $u : \mathfrak{H} \supset U \to \mathbb{R}$ is a smooth function and $p \in U$, then its horizontal gradient $\nabla_p u$ is defined by*

$$\nabla_p(u) = (X_p u)X_p + (Y_p u)Y_p.$$

The horizontal gradient is the analogue in Heisenberg geometry of the usual gradient of a function in the geometry of Euclidean space.

**Definition 4.14** *By $(E, G)$ we denote the open bounded subset $U = \mathfrak{H} \setminus (E \cup F)$ where $E$ and $F$ are disjoint connected closed subsets of $\mathfrak{H}$ and moreover $E$ is compact. Such a set shall be called a condenser. The capacity $\mathrm{Cap}(E, F)$ of $(E, G)$ in $\mathfrak{H}$ is*

$$\mathrm{Cap}(E, F) = \inf \int_{\mathfrak{H}} |\nabla_h u|^4 d\mu_L,$$

*where the infimum is taken among all smooth functions $u$ in $G$ with $u_{|E} \geq 1$ and $u_{|F} = 0$.*

For details about capacities, see Section 3 in [23].

**Definition 4.15 (Geometric definition I)** *Let $f : \Omega \to \Omega'$ be a homeomorphism between domains in $\mathfrak{H}$. Then $f$ is quasiconformal if there exists a $K \geq 1$ such that for each ring domain $(E, F)$ in $\Omega$ we have the capacity inequality*

$$\mathrm{Cap}(E, F) \leq K^2 \mathrm{Cap} f(E, F).$$

In what follows we will give a geometric definition which involves the notion of the modulus of families of curves. As in the classical case, the geometric definition of quasiconformal mappings that we present here, involves the notion of the *modulus* of a family $\Gamma$ of rectifiable with respect to $d_{\mathfrak{H}}$ curves lying in a domain $\Omega \subset \mathfrak{H}$, i.e. they have finite length with respect to $d_{\mathfrak{H}}$ (or in other words, they have finite horizontal length). If $\rho : \mathfrak{H} \to [0, \infty]$ is a non–negative Borel function and $\gamma$ is a parametrisation of a rectifiable curve $\gamma(t) = (\gamma_I(t), \gamma_3(t))$, $t \in [a, b]$, we define

$$\int_\gamma \rho ds = \int_a^b \rho(\gamma(t))|\dot{\gamma}_I(t)|dt.$$

Let $\mathrm{adm}(\Gamma)$ be the set of these non–negative Borel functions $\rho$ defined in $\mathfrak{H}$ which satisfy

$$\int_\gamma \rho ds \geq 1, \quad \text{for all rectifiable } \gamma \in \Gamma.$$

**Definition 4.16** *The modulus* $\mathrm{Mod}(\Gamma)$ *of* $\Gamma$ *is defined as*

$$\mathrm{Mod}(\Gamma) = \inf_{\rho \in \mathrm{adm}(\Gamma)} \int_{\mathfrak{H}} \rho^4 d\mu_L,$$

*where* $d\mathcal{L}^3$ *is the volume element of the usual Lebesque measure in* $\mathbb{C} \times \mathbb{R}$.

The following inequality may be found in [3].

**Theorem 4.17 (Modulus Inequality)** *Let* $f : \Omega \to \Omega'$ *be a* $K-$*quasiconformal mapping between domains in* $\mathfrak{H}$ *and let* $\Gamma$ *be a family of curves in* $\Omega$. *Then*

$$\frac{1}{K^2}\mathrm{Mod}(\Gamma) \leq \mathrm{Mod}(f(\Gamma)) \leq K^2\mathrm{Mod}(\Gamma). \tag{4.4.1}$$

It follows that the modulus is a conformal invariant; if $f$ is conformal then $\mathrm{Mod}(f(\Gamma)) = \mathrm{Mod}(\Gamma)$.

The following lemma will be useful in establishing the Modulus Inequality in the smooth case.

**Lemma 4.18** *Let* $f : \Omega \to \mathfrak{H}$ *be a* $\mathfrak{H}$ *map on a domain* $\Omega \subset \mathbb{H}^1$ *and let further be* $\gamma : [a, b] \to \Omega$ *a horizontal curve. Then*

$$(f_I \dot{\circ} \gamma)(s) = Zf_I(\gamma(s))\dot{\gamma}_I(s) + \overline{Z}f_I(\gamma(s))\dot{\overline{\gamma}}_I(s) \quad a.e.\ s \in [a, b].$$

*Proof:* Note that a horizontal curve is by our definition absolutely continuous and hence almost everywhere differentiable. Let $s$ be such a point of differentiability. From the complex notation it follows that

$$\begin{aligned} Zf_I(\gamma(s))\dot{\gamma}_I(s) + \overline{Z}f_I(\gamma(s))\dot{\overline{\gamma}}_I(s) =& (\dot{\gamma}_1(s)Xf_1(\gamma(s)) + \dot{\gamma}_2(s)Yf_1(\gamma(s))) \\ &+ \mathrm{i}(\dot{\gamma}_2(s)Yf_2(\gamma(s)) + \dot{\gamma}_1(s)Xf_2(\gamma(s))). \end{aligned}$$

The right-hand side can be further simplified by using the horizontality of $\gamma$. It follows

$$
\begin{aligned}
&Zf_I(\gamma(s))\dot{\gamma}_I(s) + \overline{Z}f_I(\gamma(s))\dot{\overline{\gamma}}_I(s) \\
&\qquad = \frac{\partial f_I}{\partial x}(\gamma(s))\dot{\gamma}_1(s) + \frac{\partial f_I}{\partial y}(\gamma(s))\dot{\gamma}_2(s) + \frac{\partial f_I}{\partial t}(\gamma(s))\dot{\gamma}_3(s).
\end{aligned}
$$

By the chain rule, the last expression is equal to $(f_I \dot{\circ} \gamma)(s)$, which concludes the proof. ∎

We now prove the modulus inequality under additional smoothness assumptions on the curves and on the mapping. We denote by

$$f(\Gamma) := \{f \circ \gamma : \ \gamma \in \Gamma\}$$

the $f$-image of a given family of curves $\Gamma$.

**Proposition 4.19** *Let $f : \Omega \to \Omega'$ be a $C^2$ orientation-preserving quasiconformal map with non-singular derivative between domains in $\mathfrak{H}$. For any family $\Gamma$ of $C^1$ horizontal curves in $\Omega$ we have*

$$\mathrm{Mod}(f(\Gamma)) \leq \int_\Omega K(p,f)^2 \rho^4(p)\, d\mu_L(p) \quad \textit{for all } \rho \in \mathrm{adm}(\Gamma).$$

*Proof:* To each density $\rho \in \mathrm{adm}(\Gamma)$, we assign a push-forward density

$$
\begin{aligned}
\rho'(\zeta,\tau) &:= \frac{\rho(f^{-1}(\zeta,\tau))}{|Zf_I(f^{-1}(\zeta,\tau))| - |\overline{Z}f_I(f^{-1}(\zeta,\tau))|} \\
&= \frac{\rho}{|Zf_I| - |\overline{Z}f_I|} \circ f^{-1}(\zeta,\tau) \quad \text{for } (\zeta,\tau) \in \Omega',
\end{aligned}
$$

and

$$\rho'(\zeta,\tau) = 0 \quad \text{elsewhere.}$$

Since $f$ is an orientation-preserving quasiconformal map, this is a Borel function. We will show that $\rho'$ is admissible for the image family $f(\Gamma)$. To this end, we first observe that for any $\gamma \in \Gamma$, $\gamma(s) = (z(s), t(s))$, the image $f \circ \gamma$ is again a horizontal curve since $f$ is quasiconformal and hence in particular a contact transformation. By Lemma 4.18 it follows

$$|(f_I \dot{\circ} \gamma)| \geq (|Zf_I(\gamma(s))\dot{z}(s)| - |\overline{Z}f_I(\gamma(s))\dot{\overline{z}}(s)|) = (|Zf_I(\gamma(s))| - |\overline{Z}f_I(\gamma(s))|)|\dot{z}(s)|.$$

Therefore,

$$\begin{aligned}\int_{f\circ\gamma}\rho'\ \mathrm{d}\ell &= \int_a^b \rho'(f(\gamma(s)))|(f_I \dot{\circ} \gamma)(s)|\ \mathrm{d}s \\ &\geq \int_a^b \frac{\rho(\gamma(s))}{|Zf_I(\gamma(s))| - |\overline{Z}f_I(\gamma((s))|}(|Zf_I(\gamma(s))| - |\overline{Z}f_I(\gamma(s))|)|\dot{z}(s)|\ \mathrm{d}s \\ &= \int_a^b \rho(\gamma(s))|\dot{z}(s)|\ \mathrm{d}s \\ &= \int_\gamma \rho\ \mathrm{d}\ell \geq 1,\end{aligned}$$

which shows that $\rho'$ is indeed admissible for $f(\Gamma)$.

We now compute $\int_{\Omega'}\rho'^4\ d\mu_L$. This computation will involve the Change of Variables Theorem; for this purpose, recall that the Jacobian determinant of $f$ is given by

$$\det J_f = (|Zf_I|^2 - |\overline{Z}f_I|^2)^2 = \lambda^2.$$

We now have

$$\begin{aligned}\int_{\Omega'}\rho'^4(\zeta,\tau)\ d\mu_L\zeta,\tau) &= \int_{\Omega'}\left(\frac{\rho}{|Zf_I| - |\overline{Z}f_I|}\right)^4 \circ f^{-1}(\zeta,\tau)\ d\mu_L(\zeta,\tau) \\ &= \int_{\Omega}\left(\frac{\rho}{|Zf_I| - |\overline{Z}f_I|}\right)^4 \circ f^{-1}(f(z,t))J_f(z,t)\ d\mu_L(z,t) \\ &\overset{(??)}{=} \int_\Omega \rho^4(z,t)\frac{(|Zf_I(z,t)|^2 - |\overline{Z}f_I(z,t)|)^2}{(|Zf_I(z,t)| - |\overline{Z}f_I(z,t)|)^4}\ d\mu_L(z,t) \\ &= \int_\Omega \rho^4(z,t)K((z,t),f)^2\ d\mu_L(z,t).\end{aligned}$$

Then we can conclude the proof as follows:

$$\begin{aligned}\mathrm{Mod}(f(\Gamma)) &= \inf_{\tilde{\rho}\in\mathrm{adm}(f(\Gamma))}\int_{\Omega'}\tilde{\rho}^4(\zeta,\tau)\ d\mu_L(\zeta,\tau) \\ &\leq \inf_{\rho\in\mathrm{adm}(\Gamma)}\int_{\Omega'}\rho'^4(\zeta,\tau)\ d\mu_L(\zeta,\tau) \\ &= \inf_{\rho\in\mathrm{adm}(\Gamma)}\int_{\Omega}\rho^4(z,t)K((z,t),f)^2\ d\mu_L(z,t) \\ &\leq \int_\Omega \rho^4(z,t)K((z,t),f)^2\ d\mu_L(z,t).\end{aligned}$$

■

**Remark 4.20** *If the map $f$ is conformal, then it is a smooth map with $\overline{Z}f_I = 0$ and it follows from the detail of the proof that* $\mathrm{Mod}(f(\Gamma)) = \mathrm{Mod}(\Gamma)$.

Now the second geometric definition of quasiconformality stands as follows.

**Definition 4.21 (Geometric definition II)** *Let $f : \Omega \to \Omega'$ be a homeomorphism between domains in $\mathfrak{H}$. Then $f$ is $K$-quasiconformal if there exists a $K \geq 1$ such that for each curve family $\Gamma$ in $\Omega$ the modulus inequality 4.4.1 holds.*

Since the capacity of a ring domain $(E, G)$ is equal to the modulus of the family of horizontal curves joining $E$ and $F$ in $U$, see [10] or Proposition 2.4 of [19], geometric definition II implies geometric definition I. The converse may be derived via *quasisymmetric mappings*, see below.

It turns out that alike the classical case, quasiconformal mappings are strongly related to quasisymmetric mppings:.

**Definition 4.22** *A mapping $f : \Omega \to \Omega'$ between domains of $\mathfrak{H}$ is called locally $\eta$-quasisymmetric if there exists an increasing self homeomorphism $\eta$ of $[0, \infty)$ such that for each Whitney ball $B \subset \Omega$,*

$$\frac{d_{\mathfrak{H}}(f(p), f(q))}{d_{\mathfrak{H}}(f(p), f(r))} \leq \eta\left(\frac{d_{\mathfrak{H}}(p, q)}{d_{\mathfrak{H}}(p, r)}\right)$$

*for all $p, q, r \in B$, $p \neq r$. Recall that a Whitney ball $B \subset \Omega$ satisfies $2B \subset \Omega$.*

The next theorem, which in its full generality is in [21], Theorem 9.8, clarifies the equivalence of all the afore stated definitions of quasiconformality. (See also Theorem 6.33 in [7]).

**Theorem 4.23** *Let $f : \Omega \to \Omega'$ be a homeomorphism between domains of $\mathfrak{H}$. The following are equivalent.*

1. *$f$ is quasiconformal according to the metric definition ;*
2. *$f$ is locally $\eta-$quasisymmetric;*
3. *$f$ is quasiconformal according to the geometric definition 4.21.*

It follows that (1)–(3) in Theorem 4.23 are all equivalent to the analytic definition and to geometric definition 4.15 as well.

## 4.5 Quasiconformal deformations

Perhaps one of the most striking results in the original work of Korányi and Reimann, is their generalisation to the Heisenberg setting of the famous measurable Riemann Mapping Theorem in its infinitesimal version, see Theorem 4.4 below. Besides its genuine importance, this theorem enables us to construct as many quasiconformal mappings on $\mathfrak{H}$ as we wish, out of *quasiconformal deformations*. Let $f_s : \mathfrak{H} \to \mathfrak{H}$, $f_s = f_s(z, t)$, $s \in \mathbb{R}$, be a $C^1$ one-parameter group of transformations of $\mathfrak{H}$ with infinitesimal generator $V$, satisfying the initial condition $f_0(z,t) = \mathrm{id}$. Then the following differential equation holds:

$$\frac{d}{ds} f_s(z,t) = V(f_s(z,t)).$$

We are interested primarily in one-parameter groups of *contact* transformations since we have seen that smooth enough quasiconformal mappings are contact. Infinitesimal generators of one-parameter groups of contact transformations have been studied by Liebermann and are of a special form given the following theorem (Theorem 5 in [22]).

**Theorem 4.2** *$C^1$ vector fields of the form*

$$V = -\frac{1}{4}\left[(Yp)X - X(p)Y\right] + pT = \frac{i}{2}\left[(\overline{Z}p)Z - (Zp)\overline{Z}\right] + pT, \qquad (4.5.1)$$

*where $p$ is an arbitrary real valued function, generate local one-parameter groups of contact transformations. Conversely, every $C^1$ vector field $V$ which generates a local one-parameter group of contact transformations is necessarily of this form with $p = \omega(V)$.*

The following theorem (Theorem 6 in [22]), gives a precise estimate for the constant of quasiconformality of a one-parameter group of quasiconformal mappings generated by a $\mathcal{C}^2$ vector field.

**Theorem 4.3** *Let $V$ be a $\mathcal{C}^2$ vector field of the form (4.5.1) generating a one-parameter group $\{f_s\}$ of contact transformations. If*

$$|ZZp| \leq c^2/2$$

*then $f_s$ is $K-$quasiconformal with the constant of quasiconformality $K = K(s)$ of $f_s$ satisfying*

$$K + \frac{1}{K} \leq 2e^{c|s|}.$$

The above result is improved as follows (Theorem H in [**?**]).

**Theorem 4.4** *Let $V$ be a continuous vector field of the form (4.5.1) with compact suport in $\mathfrak{H}$. If the (distributional) derivatives $ZZp$ are bounded in $\mathfrak{H}$ and if*

$$|ZZp| \leq c^2/2$$

*then $f_s$ is $K-$quasiconformal with the constant of quasiconformality $K = K(s)$ of $f_s$ satisfying*

$$K + \frac{1}{K} \leq 2e^{c|s|}.$$

As we already remarked, Theorem 4.4 is the infinitesimal analogue of the measurable Riemann mapping theorem of Ahlfors and Bers in the Heisenberg setting, but there is no result assuring the existence of a solution to the Beltrami system of equations (4.3.1) and (4.3.2). However, this is also the key step to pass from quasiconformal deformations of the Heisenberg group $\mathfrak{H}$ to quasiconformal deformations of the complex hyperbolic plane $\mathbf{H}^2_{\mathbb{C}}$. In the following we describe this passage, restricting ourselves to the smooth case. Assuming enough smoothness, quasiconformal mappings of $\mathbf{H}^2_{\mathbb{C}}$ are necessarily symplectic transformations, i.e. diffeomorphisms $F$ such that $F * \Omega = \Omega$, where $\Omega$ is the symplectic form in $\mathbf{H}^2_{\mathbb{C}}$ derived by its Kähler metric. If $J$ is the natural complex structure in $\mathbf{H}^2_{\mathbb{C}}$, then $F$ defines another complex structure $J_\mu = F_*^{-1} \circ J \circ F$ in $\mathbf{H}^2_{\mathbb{C}}$ and there is an associated complex antilinear self-mapping of the $(1,0)$-tangent bundle $T^{(1,0)}(\mathbf{H}^2_{\mathbb{C}})$ such that the $(1,0)$-tangent bundle of $J_\mu$ is $\{Z - \overline{\mu Z} \mid Z \in T^{(1,0)}\}$. The map $\mu$ is called the *complex dilation* of $F$ and there is a description of $\mu$ via a Beltrami system of equations, see pp. 401–402 in [9]. The following proposition describes the connection between quasiconformal symplectic transformations of the complex hyperbolic plane and quasiconformal contact transformations of the Heisenberg group, see [24] and [25].

**Proposition 4.24** *(i) A (quasiconformal) symplectic transformation $F$ of the complex hyperbolic plane $\mathbf{H}^2_{\mathbb{C}}$ extends to a (quasiconformal) contact transformation of the boundary.*

*(ii) A quasiconformal deformation of the boundary extends to a quasiconformal deformation in the interior.*

*In both cases, the constant of quasiconformality remains the same.*

**CHAPTER 5**

# ELEMENTS OF HORIZONTAL GEOMETRY OF SURFACES IN $\mathfrak{H}$

## 5.1 Regular Surfaces-Horizontal Normal Vector Field

**Definition 5.1** *Aregular surface $\mathcal{S}$ embedded in the Heisenberg group $\mathfrak{H}$ is an oriented regular surface of $\mathbb{R}^3$, i.e., a countable collection of surface patches $\sigma_\alpha : U_\alpha \to V_\alpha$ where $U_\alpha$ and $V_\alpha$ are open sets of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively, such that*

1. *each $\sigma_\alpha$ is a smooth (at least $C^2$[1]) homeomorphism, and*
2. *the differential $(\sigma_\alpha)_* : \mathbb{R}^2 \to \mathbb{R}^3$ is of rank 2 everywhere.*

Let $\mathcal{S} : U \to \mathbb{R}^3$ be a regular surface and suppose that a surface patch $\sigma$ is defined in an open domain $U \subset \mathbb{R}^2$ by

$$\sigma(u, v) = (x(u, v), y(u, v), t(u, v)),$$

so that its differential $\sigma_*$ is of rank 2. The tangent plane $T_\sigma(\mathcal{S})$ of $\mathcal{S}$ at $\sigma$ is

$$T_\sigma(\mathcal{S}) = \text{span}\left\{\sigma_u = \sigma_* \frac{\partial}{\partial u}, \sigma_v = \sigma_* \frac{\partial}{\partial v}\right\},$$

[1] I know of no definition of surfaces inside $\mathfrak{H}$ whose smoothness is not Euclidean.

which is also defined by the normal vector

$$N_\sigma = \sigma_u \wedge \sigma_v = \frac{\partial(y,t)}{\partial(u,v)}\frac{\partial}{\partial x} + \frac{\partial(t,x)}{\partial(u,v)}\frac{\partial}{\partial y} + \frac{\partial(x,y)}{\partial(u,v)}\frac{\partial}{\partial t},$$

where $\wedge$ is the vector product in $\mathbb{R}^3$. That is

$$T_\sigma(\mathcal{S}) = \{V_\sigma \in T_\sigma(\mathbb{R}^3) \;:\; N_\sigma \cdot V_\sigma = 0\},$$

where the dot is the standard scalar product in $\mathbb{R}^3$. The unit normal vector field of $\mathcal{S}$ is uniquely defined at each local chart by the relation

$$\nu_\sigma = \frac{\sigma_u \wedge \sigma_v}{|\sigma_u \wedge \sigma_v|},$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^3$.

**Definition 5.2** *Let $\mathcal{S}$ be a regular surface and $p \in \mathcal{S}$. The* horizontal plane $\mathrm{H}_p(\mathcal{S})$ *of $\mathcal{S}$ at $p$ is the horizontal plane $\mathrm{H}_p(\mathfrak{H})$.*

For arbitrary $p \in \mathcal{S}$, we wish to find the relation between the horizontal plane $\mathrm{H}_p(\mathcal{S})$ and the tangent plane $T_p(\mathcal{S})$. We begin by defining a suitable wedge product.

**Definition 5.3** *For $p \in \mathfrak{H}$, the Heisenberg wedge product $\wedge_p^{\mathfrak{H}}$ is a mapping $T_p(\mathfrak{H}) \times T_p(\mathfrak{H}) \to T_p(\mathfrak{H})$ which assigns to each two vectors*

$$a = a_1 X + a_2 Y + a_3 T \quad \textit{and} \quad b = b_1 X + b_2 Y + b_3 T$$

*of $T_p(\mathfrak{H})$ the vector $a \wedge^{\mathfrak{H}} b \in T_p(\mathfrak{H})$, which is given by the formal determinant*

$$a \wedge^{\mathfrak{H}} b = \begin{vmatrix} X & Y & T \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} X + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} Y + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} T.$$

Obviously $a \wedge^{\mathfrak{H}} b = -b \wedge^{\mathfrak{H}} a$ and the following clock rule holds:

$$X \wedge^{\mathfrak{H}} Y = T, \quad Y \wedge^{\mathfrak{H}} T = X, \quad T \wedge^{\mathfrak{H}} X = Y.$$

Thus defined,and in accordance to the euclidean case, this wedge product leads to the following:

**Definition 5.4** *If $\sigma : U \to \mathbb{R}^3$ is a surface patch of a regular surface $\mathcal{S}$, the* horizontal normal $N_\sigma^h$ *to $\sigma$ is the horizontal part of*

$$\sigma_u \wedge^{\mathfrak{H}} \sigma_v = \sigma_* \frac{\partial}{\partial u} \wedge^{\mathfrak{H}} \sigma_* \frac{\partial}{\partial v},$$

*that is*

$$N^h_\sigma = (\sigma_u \wedge^{\mathfrak{H}} \sigma_v)^h = \sigma_u \wedge^{\mathfrak{H}} \sigma_v - \omega\left(\sigma_u \wedge^{\mathfrak{H}} \sigma_v\right) T. \tag{5.1.1}$$

One may show that if $\tilde{\sigma} : \tilde{U} \to \mathbb{R}^3$ is a surface patch such that $\sigma(U) = \tilde{\sigma}(\tilde{U})$ and $\phi : \tilde{U} \to U$ is a diffeomorphism, then

$$N^h_{\tilde{\sigma}} = \det(J_\phi) N^h_\sigma,$$

where $\det(J_\phi)$ is the determinant of the Jacobian matrix of $\phi$.

**Definition 5.5** *The unit horizontal normal $\nu^h_\sigma$ to $\sigma$ is*

$$\nu^h_\sigma = \frac{N^h_\sigma}{\|N^h_\sigma\|},$$

*where $\|\cdot\|$ denotes the norm of the product $\langle\cdot,\cdot\rangle$ in $\mathfrak{H}$ (recall relations (**??**).*

We have

$$\nu^h_\sigma = \frac{(\sigma_u \wedge^{\mathfrak{H}} \sigma_v)^h}{\|(\sigma_u \wedge^{\mathfrak{H}} \sigma_v)^h\|}.$$

Observe that $N^h_\sigma$ is *not* the horizontal part of $N_\sigma$.

**Exercise:** Prove the formula:

$$N^h_\sigma = \left(\frac{\partial(y,t)}{\partial(u,v)} + 2y\frac{\partial(x,y)}{\partial(u,v)}\right) X + \left(\frac{\partial(t,x)}{\partial(u,v)} - 2x\frac{\partial(x,y)}{\partial(u,v)}\right) Y.$$

From its very definition, it is immediately derived that the horizontal normal $N^h_p$ at a point $p \in \mathcal{S}$ depends on the choice of the surface patch in the following way: Suppose that $(U,\sigma)$ and $(\tilde{U},\tilde{\sigma})$ are two overlapping patches at $p$. Then if $\Phi = \sigma^{-1}\circ\tilde{\sigma}$ is the transition mapping, we may find from (5.1.1) that around $p$ we have

$$N^h_{\tilde{\sigma}} = \det(J_\Phi) N^h_\sigma,$$

where $\det(J_\Phi) > 0$ since we have already presupposed that $\mathcal{S}$ is oriented. At this point, we would have been ready to define the unit horizontal normal vector field in $\mathcal{S}$ in accordance with the unit normal vector field, which is defined everywhere in a regular surface embedded into Euclidean space, but there is no assurance that

1. $N^h_p \neq 0$ at all $p \in \mathcal{S}$ and
2. $\nu^h_p$ is not in $T_p(\mathcal{S})$.

To this end we give the following definition:

**Definition 5.6** *Let $\mathcal{S}$ be a regular surface. A point $p \in \mathcal{S}$ is called* non characteristic *if $N^h_p \neq 0$. The set of characteristic points*

$$\mathfrak{C}(\mathcal{S}) = \{p \in \mathcal{S} \mid N^h_p = 0\}$$

*is called the* characteristic locus *of* $\mathcal{S}$.

**Corollary 5.7** *The points of* $\mathfrak{C}(\mathcal{S})$ *are given in a local chart* $(U,\sigma)$ *by the equations*

$$\frac{\partial(y,t)}{\partial(u,v)} + 2y\frac{\partial(x,y)}{\partial(u,v)} = 0 \quad \textit{and} \quad \frac{\partial(t,x)}{\partial(u,v)} - 2x\frac{\partial(x,y)}{\partial(u,v)} = 0.$$

An equivalent, but not depending on coordinates definition, will be given in the next section. It remains to show that at non characteristic points of $\mathcal{S}$, $\nu_p^h$ is not in $T_p(\mathcal{S})$:

**Proposition 5.8** *A point* $p = (x,y,t) \in \mathcal{S}$ *is non characteristic if and only if* $N_p \cdot N_p^h \neq 0$*. Moreover,* $N_p = N_p^h$ *as vectors in* $\mathbb{R}^3$ *if and only if* $x = y = 0$ *or* $\partial(x,y) = 0$*. In this case,*

$$N_p = (\partial(y,t), \partial(t,x), 0), \quad N_p^h = \partial(y,t)X + \partial(t,x)Y \quad \textit{and} \quad |N_p| = \|N_p^h\|$$

*and the surface at* $p$ *is tangent to a plane passing through* $p$*, which is orthogonal to the complex plane.*

*Proof:* If $p = \sigma(u,v)$, then $N_p^h$ may be written as a vector of $\mathbb{R}^3$ as follows:

$$N_p^h = \left((\partial(y,t) + 2y\partial(x,y)), (\partial(t,x) - 2x\partial(x,y)), 4(x^2+y^2)\partial(x,y)\right),$$

where we have denoted $\partial(y,t)/\partial(u,v)$ by $\partial(y,t)$ etc. By taking the Euclidean dot product with $N_p$ we find

$$N_p \cdot N_p^h = (\partial(y,t) + 2y\partial(x,y))^2 + (\partial(t,x) - 2x\partial(x,y))^2,$$

and this vanishes if and only if $p$ is characteristic. Our second claim is immediate. ■

**Corollary 5.9** *Let* $\mathcal{S}$ *be a regular surface of* $\mathfrak{H}$*. Then, away from the characteristic locus, (5.1) defines a nowhere vanishing vector field* $\nu_{\mathcal{S}}^h \in \mathrm{H}(\mathcal{S})$*, such that* $\|\nu_{\mathcal{S}}^h\| = 1$.

Denote by $\mathbb{J}$ the complex operator acting in $\mathbb{H}(\mathfrak{H})$ by the relations

$$\mathbb{J}X = Y, \quad \mathbb{J}Y = -X.$$

The operator $\mathbb{J}$ acts in the horizontal space of a regular surface $\mathcal{S}$, and if $\nu_{\mathcal{S}}^h = \nu_1 X + \nu_2 Y$ then

$$\mathbb{J}\nu_{\mathcal{S}}^h = -\nu_2 X + \nu_1 Y.$$

## 5.2 The Induced 1-Form. Contactomorphisms. Horizontal Flow

Let now $\mathcal{S}$ be a regular surface in $\mathfrak{H}$ and denote by $\iota_{\mathcal{S}}$ the inclusion map $\iota_{\mathcal{S}} : \mathcal{S} \hookrightarrow \mathfrak{H}$, given locally by a parametrisation $\sigma(u,v) = (x(u,v), y(u,v), t(u,v))$. Let $\omega = dt + 2xdy - 2ydx$ be the contact form of $\mathfrak{H}$; the pullback $\omega_{\mathcal{S}} = \iota_{\mathcal{S}}^* \omega$ defines a 1-form on $\mathcal{S}$ which in the local parametrisation is given by

$$\omega_{\mathcal{S}} = \sigma^* \omega = (t_u + 2xy_u - 2yx_u)du + (t_v + 2xy_v - 2yx_v)dv.$$

**Proposition 5.10** *The characteristic locus $\mathfrak{C}(\mathcal{S})$ is the (closed) set of points of $\mathcal{S}$ at which $\omega_{\mathcal{S}} = 0$.*

*Proof.* We have:

$$\begin{aligned}
&\omega_{\mathcal{S}}(p) = 0 \quad \text{for some } p \in \mathcal{S} \\
\iff \quad & \omega_p(\sigma_u) = \omega_p(\sigma_v) = 0 \quad \text{for each chart } (U,\sigma) \text{ containing } p \\
\iff \quad & \sigma_u \text{ and } \sigma_v \in \mathbb{H}_p(\mathcal{S}) \quad \text{for each chart } (U,\sigma) \text{ containing } p \\
\iff \quad & (\sigma_u \times \sigma_v)^h = 0 \quad \text{for each chart } (U,\sigma) \text{ containing } p \\
\iff \quad & p \in \mathfrak{C}(\mathcal{S}).
\end{aligned}$$

The proof is complete. □

Regular surfaces in $\mathfrak{H}$ with empty characteristic locus is of special interest but we will not go into details here. We refer for instance to [33].

**Definition 5.11** *Let $\mathcal{S}$ and $\tilde{\mathcal{S}}$ be regular surfaces and $f : \mathcal{S} \to \tilde{\mathcal{S}}$ be a smooth diffeomorphism. We may assume a weaker condition, that is we will require $f$ to be a local diffeomorphism outside the characteristic loci of $\mathcal{S}$ and $\tilde{\mathcal{S}}$. The mapping $f$ is called a local contactomorphism of $\mathcal{S}$ and $\tilde{\mathcal{S}}$ if there exists a smooth function $\lambda$ so that*

$$f^* \omega_{\tilde{\mathcal{S}}} = \lambda \omega_{\mathcal{S}}.$$

Since $f$ is a local diffeomorphism, if $\sigma : U \to \mathbb{R}^3$ is a surface patch for $\mathcal{S}$, then $\tilde{\sigma} = f \circ \sigma$ is a surface patch for $\tilde{\mathcal{S}}$ (with the possible exception of characteristic points). It follows that $f : \mathcal{S} \to \tilde{\mathcal{S}}$ is a contactomorphism if and only if

$$\omega_{\tilde{\sigma}}(u,v) = \lambda(u,v)\omega_{\sigma}(u,v) \quad \text{for almost all } (u,v) \in U. \tag{5.2.1}$$

**Definition 5.12** *A* surface curve *on a regular surface $\mathcal{S}$ is a smooth mapping $\gamma : I \to \mathcal{S}$, where $I$ is an open interval of $\mathbb{R}$.*

We wish to find conditions, so that a surface curve is horizontal, i.e., its horizontal tangent $\dot{\gamma}_h(s) \in \mathbb{H}_{\gamma(s)}(\mathcal{S})$.

**Proposition 5.13** *Suppose that $\sigma : U \to \mathfrak{H}$ is a surface patch and*

$$\gamma(s) = \sigma(u(s), v(s)), \quad s \in I,$$

*is a smooth surface curve (that is $\tilde{\gamma}(s) = (u(s), v(s))$ is a smooth curve in $U$). Then away from the characteristic locus $\gamma(s)$ is horizontal if and only if*

$$\dot{\tilde{\gamma}} \in \ker \omega_{\mathcal{S}},$$

*or in other words,*

$$(t_u + 2xy_u - 2yx_u)\dot{u} + (t_v + 2xy_v - 2yx_v)\dot{v} = 0,$$

*where the dot denotes $d/ds$. In this case,*

$$\dot{\gamma} = (x_u\dot{u} + x_v\dot{v})X + (y_u\dot{u} + y_v\dot{v})Y.$$

*Proof:* We only prove the first statement; the other two are derived immediately. We have

$$\begin{aligned} \gamma \text{ horizontal} &\iff \omega(\dot{\gamma}_h) = 0 \\ &\iff \omega(\sigma_*\dot{\tilde{\gamma}}) = 0 \\ &\iff (\sigma^*\omega)(\dot{\tilde{\gamma}}) = 0 \\ &\iff \dot{\tilde{\gamma}} \in \ker \omega_{\mathcal{S}}. \end{aligned}$$

This completes the proof. ■

The following Proposition indicates the importance of the unit horizontal normal vector field $\mathbb{J}\nu_{\mathcal{S}}$.

**Proposition 5.14** *The 1-form $\omega_{\mathcal{S}}$ defines an integrable foliation of $\mathcal{S}$ (with singularities at characteristic points) by horizontal surface curves. These curves are tangent to $\mathbb{J}\nu_{\mathcal{S}}^h$.*

*Proof:* Integrability is obvious: $\omega_{\mathcal{S}}$ is a 1-form defined in a two-dimensional manifold. For the second statement we set

$$\alpha = \frac{1}{\|N^h\|}(t_u - 2yx_u + 2xy_u), \quad \beta = \frac{1}{\|N^h\|}(t_v - 2yx_v + 2xy_v), \tag{5.2.2}$$

where $\|N^h\| = \|(\sigma_u \wedge^{\mathfrak{H}} \sigma_v)^h\|$, and consider

$$J\mathcal{V} = \beta\frac{\partial}{\partial u} - \alpha\frac{\partial}{\partial v} \in \ker\omega_{\mathcal{S}}. \tag{5.2.3}$$

By observing that

$$\begin{aligned} \beta y_u - \alpha y_v &= \frac{\partial(y,t) + 2y\partial(x,y)}{\|N^h\|} = \nu_1, \\ \beta x_u - \alpha x_v &= -\frac{\partial(t,x) - 2x\partial(x,y)}{\|N^h\|} = -\nu_2, \end{aligned}$$

we obtain

$$\begin{aligned}
\sigma_*(J\mathcal{V}) &= \beta\sigma_u - \alpha\sigma_v \\
&= \beta(x_u X + y_u Y + \|N^h\|\alpha T) - \alpha(x_v X + y_v Y + \|N^h\|\beta T) \\
&= (\beta x_u - \alpha x_v)X + (\beta y_u - \alpha y_v)Y \\
&= -\nu_2 X + \nu_1 Y = \mathbb{J}\nu_{\mathcal{S}}.
\end{aligned}$$

Note finally that by (5.2.3) the integral curves of $\mathbb{J}\nu_{\mathcal{S}}$ are the solutions of the system of differential equations $\dot{u} = \beta$ and $\dot{v} = -\alpha$. ∎

We remark for later use that when $D = \partial(x, y) \neq 0$ we also have the following expressions for $\alpha$ and $\beta$:

$$\alpha = -\frac{\nu_1 x_u + \nu_2 y_u}{D}, \quad \beta = -\frac{\nu_1 x_v + \nu_2 y_v}{D}. \tag{5.2.4}$$

**Definition 5.15** *The foliation of $\mathcal{S}$ by the integrable curves of $\mathbb{J}\nu_{\mathcal{S}}$ is called the horizontal flow of $\mathcal{S}$.*

## 5.3 Horizontal Mean Curvature

Horizontal mean curvature is defined as follows:

**Definition 5.16** *Let $\mathcal{S}$ be a non characteristic point of a regular surface $\mathcal{S}$ and let also $\nu_p^h = \nu_1 X + \nu_2 Y$ be the unit horizontal normal of $\mathcal{S}$ at $p$. The* horizontal mean curvature *$H^h(p)$ of $\mathcal{S}$ at $p$ is given by*

$$H^h(p) = X_p \nu_1 + Y_p \nu_2.$$

A more geometric but equivalent definition follows from the next proposition according to which the horizontal mean curvature at non characteristic points of a regular surface may be defined as the signed curvature of the projection to $\mathbb{C}$ of the leaf of the horizontal flow passing from $p$ (see also Proposition 4.24 of [7]).

**Proposition 5.17** *Let $\mathcal{S}$ be a regular surface and $p \in \mathcal{S}$ a non characteristic point. Let $\nu_{\mathcal{S}}^h = \nu_1 X + \nu_2 Y$ be the unit horizontal normal vector field of $\mathcal{S}$, and $\gamma$ the unique unit speed surface curve passing from $p$, which is tangent to $\mathbb{J}\nu_p^h$ at $p$. If $\pi = pr_{\mathbb{C}}\gamma$ is the projection of $\gamma$ on $\mathbb{C}$, $p'$ is the projection of $p$ and $\kappa_s$ is the signed curvature of $\pi$, then*

$$\kappa_s(p') = X_p \nu_1 + Y_p \nu_2.$$

*Proof:* Let $p \in \mathcal{S}$ and let $\gamma(s)$ be the unit speed horizontal surface curve passing from $p$. Let $\pi(s)$ be the projection of $\gamma(s)$ in $\mathbb{C} = \mathbb{R}^2$; then its tangent is

$$\dot{\pi}(s) = (-\nu_2(s), \nu_1(s))$$

and is of unit speed. We have by applying the chain rule that

$$\begin{aligned}
\dot{\nu}_1 &= (\nu_1)_x\dot{x} + (\nu_1)_y\dot{y} + (\nu_1)_t\dot{t} \\
&= (X\nu_1 - 2yT\nu_1)(x_u\dot{u} + x_v\dot{v}) + (Y\nu_1 + 2xT\nu_1)(y_u\dot{u} + y_v\dot{v}) + T\nu_1(t_u\dot{u} + t_v\dot{v}) \\
&= -(X\nu_1 - 2yT\nu_1)\nu_2 + (Y\nu_1 + 2xT\nu_1)\nu_1 + T\nu_1(-2y\nu_2 - 2x\nu_1) \\
&= -\nu_2(X\nu_1 + Y\nu_2),
\end{aligned}$$

where we have used

$$\dot{\gamma}(s) = (x_u\dot{u} + x_v\dot{v})X + (y_u\dot{u} + y_v\dot{v})Y = -\nu_2 X + \nu_1 Y,$$

and the relation $\nu_1 Y\nu_1 = -\nu_2 Y\nu_2$ which follows from $\nu_1^2 + \nu_2^2 = 1$. Working analogously for $\dot{\nu}_2$ (using $\nu_1 X\nu_1 = -\nu_1 X\nu_2$ this time), we have $\dot{\nu}_2 = \nu_1(X\nu_1 + Y\nu_2)$, hence

$$\begin{aligned}
\ddot{\pi} = (-\dot{\nu}_2, \dot{\nu}_1) &= (-\nu_1(X\nu_1 + Y\nu_2), -\nu_2(X\nu_1 + Y\nu_2)) \\
&= \kappa_s(-\nu_1, -\nu_2),
\end{aligned}$$

where $\kappa_s$ is the signed curvature of the curve $\pi$. This yields $\kappa_s = X\nu_1 + Y\nu_2$. ∎

A local expression for $H^h$ is in order:

**Proposition 5.18** *Let $\mathcal{S}$ be a regular surface in $\mathfrak{H}$. In every surface patch $\sigma = (x, y, t)$ with $\partial(x, y) \neq 0$ and sufficiently away from the characteristic locus, the horizontal mean curvature is given by*

$$H^h(\sigma) = \frac{\partial(\nu_1, y) + \partial(x, \nu_2)}{\partial(x, y)},$$

*where $\nu_i$, $i = 1, 2$, are the components of the unit horizontal normal vector $\nu$ of $\mathcal{S}$.*

*Proof:* Suppose first that $\partial(x, y) \neq 0$. Using the chain rule we write

$$\begin{aligned}
(\nu_1)_u &= (\nu_1)_x x_u + (\nu_1)_y y_u + (\nu_1)_t t_u = (X\nu_1)x_u + (Y\nu_1)y_u + (t_u - 2yx_u + 2xy_u)T\nu_1, \\
(\nu_2)_u &= (\nu_2)_x x_u + (\nu_2)_y y_u + (\nu_2)_t t_u = (X\nu_2)x_u + (Y\nu_2)y_u + (t_u - 2yx_u + 2xy_u)T\nu_2, \\
(\nu_1)_v &= (\nu_1)_x x_v + (\nu_1)_y y_v + (\nu_1)_t t_v = (X\nu_1)x_v + (Y\nu_1)y_v + (t_v - 2yx_v + 2xy_v)T\nu_1, \\
(\nu_2)_v &= (\nu_2)_x x_v + (\nu_2)_y y_v + (\nu_2)_t t_v = (X\nu_2)x_v + (Y\nu_2)y_v + (t_v - 2yx_v + 2xy_v)T\nu_2.
\end{aligned}$$

The first and the third equations are written as

$$\begin{aligned}
(X\nu_1)x_u + (Y\nu_1)y_u &= (\nu_1)_u - \alpha\|N^h\|T\nu_1, \\
(X\nu_1)x_v + (Y\nu_1)y_v &= (\nu_1)_v - \beta\|N^h\|T\nu_1,
\end{aligned}$$

where we have used Equations (5.2.2). Solving the system we obtain

$$X\nu_1 = \frac{\partial(\nu_1, y) + \|N^h\|\nu_1 T\nu_1}{\partial(x, y)}, \quad Y\nu_1 = \frac{\partial(x, \nu_1) + \|N^h\|\nu_2 T\nu_1}{\partial(x, y)},$$

where we have used Equations 5.2.4. In an analogous manner, we obtain the following from the second and the fourth equations:

$$X\nu_2 = \frac{\partial(\nu_2, y) + \|N^h\|\nu_1 T\nu_2}{\partial(x,y)}, \quad Y\nu_2 = \frac{\partial(x, \nu_2) + \|N^h\|\nu_2 T\nu_2}{\partial(x,y)}.$$

Therefore

$$\begin{aligned} X\nu_1 + Y\nu_2 &= \frac{\partial(\nu_1, y) + \partial(x, \nu_2) + \|N^h\|(\nu_1 T\nu_1 + \nu_2 T\nu_2)}{\partial(x,y)} \\ &= \frac{\partial(\nu_1, y) + \partial(x, \nu_2)}{\partial(x,y)}, \end{aligned}$$

since $\nu_1^2 + \nu_2^2 = 1$ and hence $\nu_1 T\nu_1 + \nu_2 T\nu_2 = 0$. ■

**Remark 5.19** *In case $\partial(x, y) = 0$ it is deduced from Proposition 5.8, that the horizontal normal vector field $\nu_\sigma^h$ is orthogonal to a plane vertical to the complex plane. Then we may assume that $\sigma$ is of the form*

$$\sigma(u, v) = (x(v), y(v), u),$$

*i.e., a generalised cylinder with profile curve $(x(v), y(v), 0)$. Therefore the horizontal mean curvature $H^h(p)$ at a point $p$ equals to $\kappa_s(p')$, where $\kappa_s$ is the signed curvature of the profile curve of the cylinder and $p'$ is the projection of $p$ on the profile curve .*

**Proposition 5.20** *If a regular surface $\mathcal{S}$ in $\mathfrak{H}$ is locally contactomorphic to the complex plane, then it is $H$-minimal.*

*Proof:* First we prove the statement for graphs $G_f$ of smooth functions $t = f(x, y)$ over $\mathbb{C}$. Here $(x, y)$ lies in an open subset of the plane. Let

$$\sigma(x, y) = (x, y, f(x, y)), \quad (x, y) \in U.$$

The induced 1-form is $\omega_{G_f} = (f_x - 2y)dx + (f_y + 2x)dy$. From the contactomorphism condition we also have

$$f_x - 2y = -2\lambda y \quad \text{and} \quad f_y + 2x = 2\lambda x$$

for some non zero function $\lambda$. Moreover,

$$N^h = (-f_x + 2y)X + (-f_y - 2x)Y = 2\lambda(yX - xY),$$

and therefore

$$\nu_{G_f} = \nu_1 X + \nu_2 Y = \pm\frac{yX - xY}{(x^2 + y^2)^{1/2}}.$$

Using Proposition 5.18 we have for the positive sign case (the other case is treated analogously):

$$\begin{aligned}
H^h &= \partial(\nu_1, y) + \partial(x, \nu_2) \quad (\partial(x,y) = 1),\\
&= \partial\left(\frac{y}{(x^2+y^2)^{1/2}}, y\right) + \partial\left(\frac{x}{(x^2+y^2)^{1/2}}, x\right)\\
&= y\partial\left(\frac{1}{(x^2+y^2)^{1/2}}, y\right) + x\partial\left(\frac{1}{(x^2+y^2)^{1/2}}, x\right)\\
&= y\partial_x\left(\frac{1}{(x^2+y^2)^{1/2}}\right) - x\partial_y\left(\frac{1}{(x^2+y^2)^{1/2}}\right)\\
&= y \cdot \frac{-x}{(x^2+y^2)^{3/2}} - x \cdot \frac{-y}{(x^2+y^2)^{3/2}}\\
&= 0.
\end{aligned}$$

Next we show that all coordinate planes are locally contactomorphic; we will treat the case of the planes $x = 0$ and $t = 0$ and leave the other cases to the reader. We parametrise the plane $x = 0$ by $\sigma(u,v) = (0,u,v)$ and consider the map $f : \{x = 0\} \to \{t = 0\}$ given by

$$(0, u, v) \mapsto (uv, v, 0).$$

Denote by $\tilde{\sigma}$ the surface patch $f \circ \sigma$. Then

$$\omega_\sigma = dv \quad \text{and} \quad \omega_{\tilde{\sigma}} = -2u^2 dv,$$

which, by the contact condition (5.2.1) proves our assertion.

If now $\sigma(u,v) = (x(u,v), y(u,v), t(u,v))$ is an arbitrary surface patch for $\mathcal{S}$, from regularity we have that at least one among $\partial(x,y)$, $\partial(y,t)$ and $\partial(t,x)$ is different from zero. We may now assume that $\partial(x,y) \neq 0$ and reparametrise if necessary by

$$\tilde{u} = x(u,v), \quad \tilde{v} = y(u,v)$$

to obtain the regular surface patch $\sigma(\tilde{u}, \tilde{v}, t(\tilde{u}, \tilde{v}))$, which is a local graph of a function over the complex plane. ■

## 5.4 Horizontal Area and Horizontal Area Integral

In an arbitrary regular surface $\mathcal{S}$, the notion of the area $\mathcal{A}$ is given by integrating at each coordinate neighborhood $(U, \sigma)$ the length of the normal vector $N_\sigma = \sigma_u \times \sigma_v$. Accordingly, we define the *horizontal area* (elsewhere called the *perimeter*) of $\mathcal{S}$.

**Definition 5.21** *Let $\mathcal{S}$ be a regular surface in $\mathfrak{H}$ and suppose that $\sigma : U \to \mathfrak{H}$ is any surface patch. Let $N^h_\sigma = (\sigma_u \wedge^{\mathfrak{H}} \sigma_v)^h$. If $R$ is a domain in $U$ then, its* horizontal area *is given by*

$$\mathcal{A}^h_\sigma(R) = \iint_R \|N^h_\sigma(u,v)\| du dv. \tag{5.4.1}$$

The above integral may of course be infinite; however, assuming that $R$ is contained in a rectangle whose closure lies inside $U$, then the integral is finite. Furthermore, a reparametrisation does not change the value of the integral. Finally, in the case where $\mathcal{S}$ is compact, the horizontal area of $\mathcal{S}$ is well defined and will be denoted by

$$\mathcal{A}^h(\mathcal{S}) = \iint_{\mathcal{S}} d\mathcal{S}^h.$$

Here $d\mathcal{S}^h$ is the *horizontal area element* of $\mathcal{S}$; at each surface patch $(U, \sigma)$,

$$d\mathcal{S}^h = \|N^h_\sigma(u,v)\| du dv.$$

With the assumptions of Definition 5.21 suppose also that $\rho : \mathcal{S} \to \mathbb{R}$ is a function. The *horizontal area integral of* $\rho$ in $R$ is defined by

$$\iint_{\sigma(R)} \rho d\mathcal{S}^h = \iint_U \rho(\sigma(u,v))\|N^h_\sigma(u,v)\| du dv, \tag{5.4.2}$$

if $\rho(\sigma(u,v)\|N^h_\sigma(u,v)\| \in L^1(R)$. Again, a reparametrisation does not change the integral and in the case where $\mathcal{S}$ is compact the horizontal area integral of $\rho$ is defined globally as follows. Suppose that $\sigma_i : U_i \to \mathcal{S}$, $i \in I$ is a finite covering of $\mathcal{S}$ by surface patches and $\rho\sigma_i\|N^h_{\sigma_i}\| \in L^1(U_i)$ for each $i \in I$. Then

$$\iint_{\mathcal{S}} \rho d\mathcal{S}^h = \sum_{i \in I} \iint_{U_i} \rho(\sigma_i(u_i, v_i))\|N^h_{\sigma_i}(u_i, v_i) du_i dv_i. \tag{5.4.3}$$

## 5.5 Regular Surfaces and Contact–Quasiconformal Transformations

Let $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ be two regular oriented surfaces in $\mathfrak{H}$. We shall consider mappings $\mathcal{S} \to \widetilde{\mathcal{S}}$ that are induced by $\mathcal{C}^2$ orientation preserving contact transformations $f = (f_1, f_2, f_3)$ of $\mathfrak{H}$: $f^*\omega = \lambda\omega$ where $\lambda = J_f^{1/2} > 0$. Let $f$ be such a transformation with the property $f(\mathcal{S}) = \widetilde{\mathcal{S}}$. Since $f$ is a $\mathcal{C}^2$ diffeomorphism and both $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ are $\mathcal{C}^2$ embedded submanifolds of $\mathfrak{H}$, it follows that the restriction $f_{\mathcal{S}} : \mathcal{S} \to \widetilde{\mathcal{S}}$ of $f$ to $\mathcal{S}$ is also a $\mathcal{C}^2$ diffeomorphism between $\mathcal{S}$ and $\widetilde{\mathcal{S}}$. In particular, for every local charts $(U, \sigma)$ and $(\widetilde{U}, \tilde{\sigma})$ of $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ respectively, the mapping $\tilde{\sigma}^{-1} \circ f \circ \sigma$ is a $\mathcal{C}^2$ diffeomorphism in its domain of definition. Being also contact, the transformation $f$ adds something more to this, i.e. the surfaces $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ are locally contactomorphic.

**Proposition 5.22** *Let $\mathcal{S}, \widetilde{\mathcal{S}}$ be two regular oriented surfaces in $\mathfrak{H}$ and $f = (f_1, f_2, f_3)$ be a $\mathcal{C}^2$ orientation preserving contact transformation of $\mathfrak{H}$ such $f(\mathcal{S}) = \widetilde{\mathcal{S}}$. Then $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ are locally contactomorphic.*

*Proof:* If $\iota_{\mathcal{S}}$ and $\iota_{\widetilde{\mathcal{S}}}$ are the inclusions of $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ respectively in $\mathfrak{H}$ then $f \circ \iota_{\mathcal{S}} = \iota_{\widetilde{\mathcal{S}}} \circ f$. The result follows. ∎

The next lemma is useful for our subsequent discussion.

**Lemma 5.23** *Let $\mathcal{S}$ be a regular oriented surface in $\mathfrak{H}$ and $f = (f_1, f_2, f_3)$ be a $\mathcal{C}^2$ orientation preserving contact transformation of $\mathfrak{H}$ such that $f^*\omega = \lambda\omega$ where $\lambda = J_f^{1/2}$ and $J_f$ is the Jacobian determinant of $f$. Then the following hold.*

1. *If $(U, \sigma)$ is a surface patch of $\mathcal{S}$, then $(U, f\circ\sigma)$ is a surface patch for $\widetilde{\mathcal{S}} = f(\mathcal{S})$.*
2. *If $N^h_\sigma = n_1X + n_2Y$ is the horizontal normal vector of $\sigma$, then,*

$$N^h_{f\circ\sigma} = \lambda\left((n_1Yf_2 - n_2Xf_2)X + (n_2Xf_1 - n_1Yf_1)Y\right). \quad (5.5.1)$$

*Proof:* The proof of (1) is immediate since the restriction of $f$ in $\mathcal{S}$ is a $\mathcal{C}^2$ diffeomorphism. To prove (2) we first write the matrices of $f_*$ and $\sigma_*$ with respect to the basis $\{X, Y, T\}$. Those are

$$\begin{pmatrix} Xf_1 & Yf_1 & Tf_1 \\ Xf_2 & Yf_2 & Tf_2 \\ 0 & 0 & \lambda \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_u & x_v \\ y_u & y_v \\ \alpha\|N^h_\sigma\| & \beta\|N^h_\sigma\| \end{pmatrix},$$

respectively, where $\alpha$ and $\beta$ are as in 5.2.2. Therefore, from chain rule we have

$$\begin{aligned} f_*\sigma_u \;=\; & \left(x_uXf_1 + y_uYf_1 + \alpha\|N^h_\sigma\|Tf_1\right)X \\ & + \left(x_uXf_2 + y_uYf_2 + \alpha\|N^h_\sigma\|Tf_2\right)Y \\ & + \lambda\alpha\|N^h_\sigma\|T, \end{aligned}$$

and

$$\begin{aligned} f_*\sigma_v \;=\; & \left(x_vXf_1 + y_vYf_1 + \beta\|N^h_\sigma\|Tf_1\right)X \\ & + \left(x_vXf_2 + y_vYf_2 + \beta\|N^h_\sigma\|Tf_2\right)Y \\ & + \lambda\beta\|N^h_\sigma\|T. \end{aligned}$$

The desired Equation 5.5.1 now follows from formula 5.1.1. ∎

**Proposition 5.24** *With the hypotheses of Lemma 5.23, in surface patches $(U, \sigma)$ and $(U, f\circ\sigma)$ of $\mathcal{S}$ and $f(\mathcal{S})$ respectively and at non characteristic points, the following inequality holds.*

$$\lambda(|Zf_I| - |\overline{Z}f_I|)\|N^h_\sigma\| \le \|N^h_{f\circ\sigma}\| \le \lambda(|Zf_I| + |\overline{Z}f_I|)\|N^h_\sigma\|, \quad (5.5.2)$$

*where $\lambda = |Zf_I|^2 - |\overline{Z}f_I|^2 = J_f^{1/2}$ and $J_f$ is Jacobian determinant of $f$.*

*Proof:* We engage complex terminology and we write $m = n_1 + in_2$. In this manner,

$$\begin{aligned} n_1Yf_2 - n_2Xf_2 &= \Re\left(m(Zf_I - Z\overline{f}_I)\right), \\ n_2Xf_1 - n_1Yf_1 &= \Im\left(m(Zf_I + Z\overline{f}_I)\right), \end{aligned}$$

and therefore, Equation 5.5.1 may be written equivalently as

$$N^h_{f\circ\sigma} = 2\lambda\Re\left((mZf_I - \overline{m}\overline{Z}f_I)\cdot Z\right)$$

and subsequently,

$$\|N^h_{f\circ\sigma}\| = \lambda|Zf_I - e^{-2i\arg(m)}\overline{Z}f_I|\|N^h_\sigma\|.$$

Inequality 5.5.2 follows by applying the triangle inequality. ■

**Corollary 5.25** *With the hypotheses of Proposition 5.24, suppose also that $f$ is quasiconformal with Beltrami coefficient $\mu$. Then:*

1. *The right inequality in 5.5.2 is attained as an equality if and only if*

$$\mu e^{-2i\arg(m)} < 0, \text{ equivalently } \arg\mu = \pi + 2\arg m. \tag{5.5.3}$$

2. *The left inequality in 5.5.2 is attained as an equality if and only if*

$$\mu e^{-2i\arg(m)} > 0, \text{ equivalently } \arg\mu = 2\arg m. \tag{5.5.4}$$

*Proof:* If $f$ is quasiconformal with Beltrami coefficient $\mu$, then $\overline{Z}f_I/Zf_I = \mu$, with $\mu$ essentially bounded by a constant less than 1. Therefore,

$$\|N^h_{f\circ\sigma}\| = \lambda|Zf_I||1 - \mu e^{-2i\arg(m)}|\cdot\|N^h_\sigma\|$$

and inequality 5.5.2 may be written as

$$|1 - |\mu|| \le \left|1 - \mu e^{-2i\arg(m)}\right| \le |1 + |\mu||\,.$$

The proof follows.[2] ■

## 5.6 Modulus of Surface Families

By $\Sigma$ we shall denote a family of regular surfaces in $\mathfrak{H}$. The set $\mathrm{Adm}(\Sigma)$ comprises of non negative Borel functions $\rho$ in $\mathfrak{H}$ such that for every $\mathcal{S} \in \Sigma$ we have

$$\iint_{\mathcal{S}} \rho d\mathcal{S}^h \ge 1.$$

[2] In case 5.5.3 we say that $f$ has the *maximal stretching property (MSP)* for $\mathcal{S}$ and in case 5.5.4 we say that $f$ has the *minimal stretching property (mSP)* for $\mathcal{S}$. The latter is very useful when we apply the Modulus Method to solve extremal problems.

**Definition 5.26** *The modulus of a family $\Sigma$ of regular surfaces in $\mathfrak{H}$ is defined by*

$$\mathrm{Mod}(\Sigma) = \inf_{\rho\in \mathrm{Adm}(\Sigma)} \iiint_{\mathfrak{H}} \rho^{4/3} d\mathcal{L}^3,$$

*where by $\mathcal{L}^3$ we denote the Lebesgue measure in $\mathbb{R}^3$.*

If the infimum is attained by a function $\rho_0 \in \mathrm{Adm}(\Sigma)$, that is

$$\mathrm{Mod}(\Sigma) = \iiint_{\mathfrak{H}} \rho_0^{4/3} d\mathcal{L}^3,$$

then we call $\rho_0$ an *extremal density for* $\Sigma$.

## 5.7 The Modulus Inequality

**Theorem 5.27** *Let $\Omega$ and $\Omega'$ be domains in $\mathfrak{H}$ and $f : \Omega \to \Omega'$ be a $\mathcal{C}^2$ orientation preserving contact quasiconformal transformation. For any family of oriented regular surfaces inside $\Omega$ we have*

$$\mathrm{Mod}(f(\Sigma)) \le \iiint_{\Omega} K_f^{2/3}(p)\rho^{4/3}(p) d\mathcal{L}^3(p) \quad \textit{for each} \quad \rho \in \mathrm{Adm}(\Sigma). \tag{5.7.1}$$

*If moreover $K_f$ is the maximal distortion of $f$ ($K_f(p) \le K_f$ for all $p$), then*

$$\frac{1}{K_f^{2/3}}\mathrm{Mod}(\Sigma) \le \mathrm{Mod}(f(\Sigma)) \le K_f^{2/3}\mathrm{Mod}(\Sigma). \tag{5.7.2}$$

*Proof:* For every $\rho \in \mathrm{Adm}(\Sigma)$ we define a non negative Borel function in $\Omega'$ by the relation

$$\rho' = \begin{cases} \frac{\rho}{\lambda(|Zf_I|-|\overline{Z}f_I|)} \circ f^{-1} & \text{in } \Omega, \\ \\ 0 & \text{elsewhere,} \end{cases}$$

where $\lambda = |Zf_I|^2 - |\overline{Z}f_I|^2 = J_f^{1/2}$. Then for each $\mathcal{S} \in \Sigma$ with $f(\mathcal{S}) = \mathcal{S}'$ we have by the left hand side of inequality 5.5.2 that

$$\iint_{\mathcal{S}'} \rho' d\mathcal{S}'^h \ge \iint_{\mathcal{S}} \rho d\mathcal{S}^h.$$

Therefore by changing the variables $q = f(p)$ we obtain

$$\begin{aligned} \iiint_{\Omega'} (\rho')^{4/3}(q) d\mathcal{L}^3(q) &= \iiint_{\Omega} (\rho'(f(p))^{4/3} J_f(p) d\mathcal{L}^3(p) \\ &= \iiint_{\Omega} \rho^{4/3}(p) \left(\frac{|Zf_I(p)| + |\overline{Z}f_I(p)|}{|Zf_I(p)| - |\overline{Z}f_I(p)|}\right)^{2/3} d\mathcal{L}^3(p) \\ &= \iiint_{\Omega} K_f^{2/3}(p)\rho^{4/3}(p) d\mathcal{L}^3(p). \end{aligned}$$

By taking the infimum over all functions in $\mathrm{Adm}(f(\Sigma))$ we obtain 5.7.1. Also the right hand side of 5.7.2 is obtained by 5.7.1 and the relation

$$K_f(p) \leq K_f \quad \text{for all} \quad p \in \Omega.$$

To obtain the left hand side of 5.7.2 we consider the inverse transformation $f^{-1} : \Omega' \to \Omega$ which is also quasiconformal with maximal distortion $K_f$. Thus, by applying 5.7.1 we have

$$\begin{aligned}\mathrm{Mod}(\Sigma) = \mathrm{Mod}(f^{-1}(\Sigma')) &\leq \iiint_{\Omega'} K_f^{2/3}(q)\rho^{4/3}(q)d\mathcal{L}^3(q) \\ &\leq K_f^{2/3} \iiint_{\Omega'} \rho^{4/3}(q)d\mathcal{L}^3(q) \quad \text{for all} \quad \rho \in \mathrm{Adm}(\Sigma')\end{aligned}$$

and the inequality follows after taking the infimum over all $\rho \in \mathrm{Adm}(\Sigma')$. ■

**Corollary 5.28** *The modulus of surface families is a conformal invariant.*

**CHAPTER 6**

# FURTHER DEVELOPMENTS AND SOME OPEN PROBLEMS

The study of quasiconformal mappings in the Heisenberg group $\mathfrak{H}$ motivated the study of quasiconformal mappings to larger and more abstract spaces; some of which are CR manifolds, metric spaces with controlled geometry and Carnot groups. It also gave rise to questions concerning the comparison between the classical Ahlfors-Bers and the Korányi–Reimann theory. It is equally fascinating to detect the points where similarities do exist, but also the points where they break down. All these are shortly addressed below.

## 6.1 Gromov hyperbolic spaces

As we have emphasized already, quasiconformal maps on Heisenberg groups and two-step nilpotent Lie groups are important for the Mostow strong rigidity. It is perhaps helpful to point out that quasiconformal maps on more general spaces such as the boundary of the Gromov hyperbolic spaces are important for other rigidity properties in geometric group theory.

**Definition 6.1** *Let $(X, d)$ be a metric space. The Gromov product of two points $y, z \in X$ with respect to a third one $x \in X$ is defined by the formula:*

$$(y, z)_x = \frac{1}{2}\left(d(x, y) + d(x, z) - d(y, z)\right).$$

Gromov's definition of a hyperbolic metric space is then as follows:

**Definition 6.2** *$X$ is $\delta$-hyperbolic if and only if all $x, y, z, w \in X$ satisfy the four-point condition*

$$(x, z)_w \geq \min\left((x, y)_w, (y, z)_w\right) - \delta.$$

This condition is called the *hyperbolicity condition*. Note that if this condition is satisfied for all $x, y, z \in X$ and one fixed base point $w_0$, then it is satisfied for all with a constant $2\delta$. Thus the hyperbolicity condition only needs to be verified for one fixed base point; for this reason, the subscript for the base point is often dropped from the Gromov product.

Suppose that $X, X'$ are two Gromov $\delta$-hyperbolic spaces. The following is reasonable and was proved by Gromov:

**Theorem 6.3** *If $X, X'$ are two quasi-isometric Gromov hyperbolic paces, then $\partial X$ and $\partial X'$ can be identified canonically and the quasi-isometric structures on the boundaries $\partial X$ and $\partial X'$ are isomorphic.*

A slightly stronger condition than quasi-conformal maps is that of quasi-symmetric homeomorphisms between $X$ and $X'$ whose boundaries $\partial X$ and $\partial X'$ are also quasi-symmetric.

One interesting result of Tukia proves the converse direction in a special case:

**Theorem 6.4** *If $\Gamma$ is a word hyperbolic group, and its boundary is quasi-symmetric to the standard sphere $S^{n-1}$, $n \geq 3$, then $\Gamma$ is virtually a cocompact lattice acting on $\mathbf{H}^n_{\mathbb{R}}$.*

One conjecture of Cannon asserts: *For a word hyperbolic group $\Gamma$, if its boundary $\partial\Gamma$ is homeomorphic to $S^2$, then $\partial\Gamma$ is quasi-symmetric to $S^2$.* Combined with the result of Tukia above, it implies that $\Gamma$ is a cocompact lattice acting on $\mathbf{H}^3_{\mathbb{R}}$. There are other rigidity results in geometric group theory proved using quasi-conformal and quasi-symmetric maps on metric spaces. See [6] for references of the results discussed here and other results.

## 6.2 Spaces with controlled geometry

We refer the reader to the pioneering work of Heinonen and Koskela [19] and [20], as well as to the notes of Reimann [35]. In general, metric spaces with controlled geometry are the metric spaces which display some kind of regularity with respect to comparison of distance and volume; the latter is the essence of quasiconformal mappings.

**Definition 6.5** *A metric space $(X, d)$ of dimension $Q > 1$ is an Ahlfors-David regular metric space if it is endowed with a Borel measure $\mu$ compatible with the metric $d$ in the following way: there exists a constant $C \geq 1$ such that for all metric balls $B_R$ of radius $R < \operatorname{diam}(X)$ the following inequality holds:*

$$\frac{1}{C} R^Q \leq \mu(B_R) \leq C R^Q.$$

Quasiconformal mappings are defined in such spaces via the metric definition, and the same holds for notions like the modulus of curve families and quasisymmetric mappings. There are well defined notions of $Q-$modulus $\operatorname{Mod}_Q(\Gamma)$ of a family of curves $\Gamma$ and of quasisymmetry, entirely analogous to the definitions we have given in the cases of $\mathbb{C}$ and $\mathfrak{H}$.

**Definition 6.6** *A metric space $(X, d)$ is called a $Q$-Loewner space, if there is a strictly increasing self-mapping $\eta$ of $(0, \infty)$ such that $\operatorname{Mod}_Q(\Gamma) \geq \eta(k)$, where $\Gamma$ is the family of curves connecting two continua $C_0$ and $C_1$ with*

$$\min\{\operatorname{diam}(C_0), \operatorname{diam}(C_1)\} \geq k \operatorname{dist}(C_0, C_1).$$

The Heisenberg group $\mathfrak{H}$ endowed with the $CC$-metric $d_{CC}$ is a 3-regular Loewner space, and a bigger class of Loewner spaces are the Carnot groups. Recall that a Carnot group is a simply connected nilpotent Lie group $G$ with a derivation $\alpha$ on its Lie algebra $\mathfrak{g}$ such that $\ker(\alpha)$ generates $\mathfrak{g}$. Via the exponential map, $N$ and subsequently $\mathfrak{g}$ are identified to $\mathbb{R}^m$ for some $m \in \mathbb{N}$ and the group action is given by the Campbell-Hausdorff formula, The Haar measure of $G$ is just the Lebesgue measure of $R^m$ and a $CC$-distance is well defined. See Pansu's thesis [31] and see also [30].

The primary problem than one is facing in the case of the above spaces is to give proper analytic and geometric definitions of quasiconformality which are equivalent to the general (and applying in all cases) metric definition. In this direction, see the works of Williams [43], Tyson [40] and [41], and Heinonen et al. [21]. On the other hand, the conditions of the metric definition itself can be substantially relaxed and this gives rise to quite striking results, see [4] and the bibliography given there.

## 6.3 Extremal problems

We next examine extremal problems. In the classical theory, *extremal* quasiconformal mappings are the ones minimising the maximal distortion (constant of quasiconformality) within a certain class of mappings in the complex plane or between Riemann surfaces. Since the times of Grötzsch and Teichmüller, a method based on the modulus of curve families has been applied to detect such mappings; it turned out that the very same method applies for the mappings which minimise the *mean distortion functional*

$$\mathfrak{M}(f, \rho) = \frac{\iint_\Omega K(f,p)\rho_0(p)^2 d\mu_L}{\iint_\Omega \rho_0(p)^2 d\mu_L}$$

in the class of mappings between annuli in the complex plane, [2]. Here $\rho_0$ is a certain density corresponding to the geometry of $\Omega$. Recently in [3], a variation of the modulus method has been developed in the Heisenberg group setting to prove that there exists a minimiser of the mean distortion functional

$$\mathfrak{M}(f,\rho) = \frac{\iiint_\Omega K(f,p)^2 \rho_0(p)^4 d\mu_L}{\iiint_\Omega \rho_0(p)^4 d\mu_L}$$

among quasiconformal mappings of Heisenberg spherical rings and this minimiser is an extension of the usual stretch map of the plane. However, it does not minimise the maximal distortion and this is in contrast to the classical situation. The problem of finding such a minimiser is open. We note here that the modulus method is up to now the unique tool for the detection of extremal mappings; in the Heisenberg setting results similar to Teichmüller's Existence and Uniqueness theorems are not available.

## 6.4 CR manifolds

Extremal problems also arise naturally in the theory of quasiconformal mappings of compact psudoconvex $\mathbb{CR}-$manifolds. Such mappings have been defined by Korányi and Reimann in[26] and the extremality problem can be stated as follows. Given two $\mathbb{CR}-$structures on a 3-dimensional contact manifold, determine the quasiconformal homeomorphisms that have the least maximal distortion with respect to the two CR structures. Problems of this type have been studied by various authors, see for instance the works of Miner [28], and Tang [38].

## 6.5 Complex hyperbolic quasi-Fuchsaian space and the holy grail

In contrast to the Teichmüller space case where extremal quasiconformal mappings are used to describe the whole space, it seems that a lot of effort has to be made to obtain (or not!) an analogous result for spaces like the complex hyperbolic quasi-Fuchsian space which is defined now.

Complex hyperbolic quasi-Fuchsian space $\mathcal{Q}_{\mathbb{C}}(\Sigma)$ of a closed surface $\Sigma$ of genus $g > 1$ is perhaps the most natural extension of the Teichmüller space of $\Sigma$: it consists of representations of the fundamental group $\pi_1(\Sigma)$ into the isometry group $\mathrm{SU}(2,1)$ of complex hyperbolic plane which are discrete, faithful, totally loxodromic and geometrically finite. We underline here that those conditions prevent that space (as well as the real quasi-Fuchsian space, a double copy of Teichmüller space) to fall in the Mostow rigidity setting; the representations are convex cocompact and not cocompact as in the assumptions of Mostow' s rigidity theorem. In a convex cocompact representation the quotient of the convex hull of the limit set has finite volume (so it may have infinite funel ends but no cusps). Thus the limit set can never be the entire boundary; there is always a region of discontinuity. In particular, for quasi-Fuchsian or complex hyperbolic quasi-Fuchsian groups the limit set is a topological circle and there is a domain of disconinuity in the boundary.

There is a quite large bibliography on the subject. For a summary of results concerning $\mathcal{Q}_{\mathbb{C}}(\Sigma)$ we refer the reader to [34]. Perhaps the most prominent problem in the subject is to examine the analytical structure of $\mathcal{Q}_{\mathbb{C}}(\Sigma)$. In the case of Teichmüller space this is carried out via the Ahlfors–Bers theory and the challenge here is to use the Korányi-Reimann theory of quasiconformal mappings in the Heisenberg group to obtain similar results. In this direction, and regardeless the lack of an existence theorem for the solution of the Beltrami equation, one is invited to start from an irreducible representation $\rho \in \mathcal{Q}_{\mathbb{C}}(\Sigma)$ and to construct quasiconformal deformations of the Heisenberg group with starting point $\rho$, to determine exactly the tangent space at $\rho$ from the vector fields generating these deformations. The problem is still open (it has been named *the holy grail* by the researchers of the area).

# REFERENCES


1. L.V. Ahlfors; *Lectures on Quasiconformal Mappings.* D. Van Nostrand Company, Inc. Princeton, New Jersey, 1966.
2. Z.M Balogh, K. Fäsler, I.D. Platis; *Modulus of curve families and extremality of spiral–stretch maps.* J. Anal. Math. **113**, (2011), 265–291.
3. Z.M Balogh, K. Fäsler, I.D. Platis; *Modulus method and radial stretch map in the Heisenberg group.* To appear in Ann. Acad. Sci. Fenn., 2013.
4. Z.M. Balogh, P. Koskela, S. Rogovin; *Absolute continuity of quasiconformal mappings on curves.* Geom. Funct. Anal. **17** (2007), no. 3, 645–664.
5. A. Bejancu; *Geometry of CR submanifolds.* Kluwer Acad. Publ. Dordrecht, 1986.
6. M. Bourdon, Structure conforme au bord et flot géodésique d'un CAT(-1)-espace. Enseign. Math. (2) 41 (1995), no. 1-2, 63–102.
7. L. Capogna, D. Danielli, S. Pauls, J. Tyson; *An introduction to the Heisenberg group and the sub–Riemannian Isoperimetric Problem.* Progress in Mathematics vol. 259, Birkháuser, Berlin, 2007.
8. N.S. Dairbekov; *On mappings of bounded distortion on the Heisenberg group.* Sib. Math. Zh. **(41)**, Issue 1, (2000), 49–59.
9. S. Dragomir, G. Tomassini; *Differential Geometry and Analysis on CR manifolds.* Progress in Mathematics vol. 246, Birkháuser, Berlin, 2006.
10. R. Eichmann; *Variationsprobleme auf der Heisenberggruppe.* Lizentiatsarbeit, Universität Bern (1990).

11. A. Fletcher & V. Markovic; *Quasiconformal maps and Teichmüller theory*. Oxford Graduate Text in Math. vol. 11, Oxford Univ. Press Inc., New York, 2007.
12. W. Goldman; *Complex hyperbolic geometry*. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1999.
13. F.W. Gehring; *The definitions and exceptional sets for quasiconformal mapping*. Ann. Acad. Sci. Fen. **(281)**, (1960), 1–28.
14. F.W. Gehring & E. Reich; *Area distortion under quasiconformal mappings*. Ann. Acad. Sci. Fenn. **(338)**, (1966), 3–15.
15. F.W. Gehring & J. Väisälä; . Comment. Math. Helv. **(36)**, (1962), 19–32.
16. H. Grötzsch; *Uber einige Extremalprobleme der konformen Abbildungen*. Ber. Verh. Sächs. Akad. Wiss. Leipzig (**80**), (1928), 367–376.
17. J. Heinonen; *Calculus on Carnot groups*. In: Fall school in Analysis (Jyväskylä, 1994), Report **68**, Univ. Jyväskylä,Jyväskylä, (1995), 1–31.
18. J. Heinonen; *What is a quasiconformal mapping?*. Notices AMS, **53**, no. 11, (2006), 1334–1335.
19. J. Heinonen, P. Koskela; *Definitions of quasiconformality*. Invent. Mat. **(120)**, (1995), 61–79.
20. J. Heinonen, P. Koskela; *Quasiconformal maps in metric spaces with controlled geometry*. Acta Math. **181**, no. 1, (1998), 1–61.
21. J. Heinonen, P. Koskela, N. Shanmugalingam, J. Tyson; *Sobolev classes of Banach space–valued functions and quasiconformal mappings*. J. Anal. Mat. **(85)**, (2001), 87–139.
22. A. Korányi, H.M. Reimann; *Quasiconformal mappings on the Heisenberg groups*. Invent. Math. **(80)** (1985), no.2, 309–338.
23. A. Korányi, H.M. Reimann; *Foundations for the theory of quasiconformal mappings on the Heisenberg groups*. Adv. in Math. **(111)** (1995), no.1, 1–87.
24. A. Korányi, H.M. Reimann; *Contact transformations as limits of symplectomorphisms*. C. R. Acad. Sci. Paris **(318)** (1994), 1119–1124.
25. A. Korányi, H.M. Reimann; *Equivariant extension of quasiconformal deformations into the complex unit ball*. Indiana Univ. Math. J. **(47)** (1998), no. 1, 153–176.
26. A. Korányi, H.M. Reimann; *Quasiconformal mappings on CR manifolds*. Conference in honour of E. Vesentini. Springer Verlag Notes, No. 1422, 59–75, Berlin–Heidelberg–New York, Springer 1988.
27. O. Lehto, K. Virtanen; *Quasikonforme Abbildungen*. Springer, 1966.
28. R.R. Miner; *Quasiconformal equivalence of spherical CR manifolds*. Ann. Acad. Sci. Fenn. Ser. A. I. Math **19**, (1994), 83–93.
29. G.D. Mostow; *Strong rigidity of locally symmetric spaces*. Ann. Math. Stud. **78**, Princeton University Press, New Jersey, 1973.
30. G.A. Margulis, G.D. Mostow; *The differential of a quasi-conformal mapping of a Carnot-Carathodory space*. Geom. Funct. Anal. **5** (1995), no. 2, 402–433.
31. P. Pansu; *Métriques de Carnot–Carathéodory et quasiisométries des espaces symétriques de rang un*. Ann. of Math. **(129)**, (1989), 1–60.

32. J.R. Parker; *Complex hyperbolic Kleinian groups.* Notes, to be published by Cambridge University Press.

33. I.D. Platis; *Straight ruled surfaces in the Heisenberg group.* J. of Geom., (105), Issue 1, 119–138, 2014.

34. J.R. Parker; I.D. Platis; *Complex hyperbolic Quasi–Fuchsian groups.* Geometry of Riemann surfaces, 309–355, London Math. Soc. Lecture Note Ser., 368, Cambridge Univ. Press, Cambridge, 2010.

35. H.M. Reimann; *Quasiconformal mappings on the Heisenberg group.* Lecture Notes, Trento, 2001.

36. E. Stein: *singular integrals and differentiability properties of functions.* Princeton Univ. Press, 1970.

37. K. Strebel & K. Reich; *Extremal quasiconformal mappings with given boundary values.* In Contributions to Analysis. A collection of papers dedicated to Lipman Bers. 375–391 Academic Press, 1974.

38. P. Tang; *Quasiconformal homeomorphisms on CR 3–manifolds with symmetries.* Math. Z. **219**, (1995), 49–69.

39. O. Teichmüller; *Extremale quasikonforme Abbildungen und quadratische Differ- entiale.* Abh. Preuss. Akad. Wiss., Math-Naturw. Kl., 22 (1937), 1–197, or *Gesamelte Abhandlungen.* Collected Papers, ed. by L.V. Ahlfors and F.W. Gehring, Springer-Verlag Berlin (1982), 335–531.

40. J. Tyson; *Quasiconformality and quasisymmetry in metric measure spaces.* Ann. Acad. Sci. Fenn. Ser. A. I. Math. **(23)**, (1998), 525–548.

41. J. Tyson; *Metric and geometric quasiconformality in Ahlfors regular Loewner spaces.* Conf. Geom. Dynam. **(5)**, (2001), 21–73.

42. S.K. Vodop'yanov; *Monotone functions and quasiconformal mappings on Carnot groups.* Sib. Math. Zh. **(37)**, Issue 6, (1996), 1269–1295.

43. M. Williams; *Geometric and analytic quasiconformality in metric measure spaces.* Proc. Amer. Math. Soc. **140** (2012), no. 4, 1251–1266.