6
Projective Geometry

The invention of projective geometry led to great improvements in railway safety
Biff

Geometries are defined by their objects of study. Euclidean geometry is naturally preoccupied with distance and angle. Otherwise how could we distinguish between an acute and an obtuse triangle? In the discipline of topology a straight line is as good as a curved one. Projective geometry falls between these two extremes. We have already met projections; projective geometry studies properties unchanged by these.

Under a projection from one plane to another, a straight line will stay a straight line. However the distance between two points or the angle between two lines can vary. Even more disconcertingly, parallel lines may project to lines which meet. To view this, look along a straight railway track or road. The sides will appear to converge at the horizon even though we know they cannot meet. Alternatively look at the sky on a day with broken cloud. Rays of light from the sun are essentially parallel but the clouds, if in the right place, will break the sun's rays into a Λ shape.

In this chapter we consider only the linear properties of the projective plane, that is points, lines and their properties. Conics and higher order curves in the projective plane would take us beyond the scope of this book.

6.1 The Projective Plane

In order to translate the above considerations into a mathematical theory we will describe a model of the real projective plane, written as $\mathbb{R}P^2$.

The *projective plane* $\mathbb{R}P^2$ is the set of lines in space $\mathbb{R}^3$ which pass through the origin.

Since a line in $\mathbb{R}^3$ which passes through the origin is determined by any of its points other than O we see that an element of $\mathbb{R}P^2$ can be considered as the set of points $\{rA\}$ in $\mathbb{R}^3$ where A is a point other than the origin and r varies over all non-zero real numbers. So any non-zero point $A = (x, y, z)$ in $\mathbb{R}^3$ determines a point in the projective plane. We can specify this point by three *homogeneous coordinates* or ratios, written $[x, y, z]$ where $[x', y', z']$ represents the same point provided there is a non-zero coefficient r such that

$$x' = rx, \ y' = ry, \ z' = rz.$$

For example the following,

$$[3, -5, 2], \ [-9, 15, -6], \ [27, -45, 18], \ [-3/5, 1, -2/5]$$

all represent the same point in $\mathbb{R}P^2$.

The projective plane is the set of pairs of antipodal points on the two dimensional sphere. This is because a line in $\mathbb{R}^3$ through the origin meets the two dimensional sphere, S^2, in an antipodal pair A, $-A$ where $|A| = 1$. We can discard the southern point of all antipodal pairs and we can think of the projective plane as the points in the northern hemisphere in S^2 together with pairs of antipodal points on the equator. We may now flatten the northern hemisphere by an orthogonal projection. This means that the projective plane is also the set of points in a plane disc provided pairs of antipodal points on the boundary circle are identified.

The use of homogeneous coordinates also suggests the definition RP^2 for the R projective plane with points specified by three ratios $[x, y, z]$, where x, y, z lie in a division ring R. However some care must be taken if R is non-commutative. In this case the ratios must all be taken on one side.

Exercise 6.1

How many points are represented by $[\pm 1, \pm 1, \pm 1]$?

We shall use the general notation $\mathsf{A} = [a_1, a_2, a_3] = [A]$ for the point in $\mathbb{R}P^2$ represented by the point $A = (a_1, a_2, a_3)$ in $\mathbb{R}^3$. The points

$$\mathsf{X} = [1, 0, 0], \ \mathsf{Y} = [0, 1, 0], \ \mathsf{Z} = [0, 0, 1]$$

are called the vertices of the *triangle of reference*. The point $\mathsf{U} = [1, 1, 1]$ is called the *unit point*.

There is no need to restrict attention to three homogeneous coordinates. Define the projective space of dimension n to be the set of points $[a_1, \ldots, a_{n+1}]$ where as before $[a_1, \ldots, a_{n+1}] = [ra_1, \ldots, ra_{n+1}]$. The coefficients a_i may be real numbers or complex numbers or members of other algebraic systems such as the quaternions.

6.2 Lines in the Projective Plane

What! Will the line stretch
out to the crack of doom?
Macbeth

A *line* in the projective plane is the set of points $[x, y, z]$ satisfying a homogeneous linear equation

$$ax + by + cz = 0 \qquad (a, b, c) \neq (0, 0, 0).$$

Notice that this equation is well defined in the projective plane because non-zero multiples of coordinates can be cancelled in the above equation.

In space, $\mathbb{R}^3$, the above equation represents a plane, ϖ say, through the origin in space. Any point in the projective plane on the line, $ax + by + cz = 0$, will correspond to a line in space which passes through the origin and lies in the plane ϖ.

We can identify the projective plane as an extension of the euclidean plane as follows. Let $\mathbb{E}$ be the set of points in $\mathbb{R}P^2$ with last coordinate non-zero. Dividing the coefficients by this last element means that any point in $\mathbb{E}$ can be written uniquely as $[x, y, 1]$ and hence can be identified with the point (x, y) in the euclidean plane.

The remaining points in $\mathbb{R}P^2$ have last coordinate zero and so lie on the line $z = 0$. This line is called the *line at infinity* with respect to $\mathbb{E}$. A point on the line at infinity is called a *vanishing point.*

To get a feel for this extension of the euclidean plane, imagine yourself as an observer in a boat on a calm sea far from the sight of land. Think of your eye as the origin O, the sea as the plane $z = -1$ and the sky as the euclidean plane $z = 1$. The sea also corresponds to the euclidean plane because the points $[x, y, 1]$ and $[-x, -y, -1]$ are one and the same. A point in the sky and the corresponding point in the sea lie on a line through your eye.

The horizon corresponds twice over to the line at infinity $z = 0$. Antipodal points on the horizon are identified. In particular north and south are identified and east and west are identified. The line at infinity is a circle on which the horizon circle is wrapped round twice.

Analytically points on the line at infinity may be written

$$[x, y, 0] = [r\cos\theta, r\sin\theta, 0].$$

Since $r \neq 0$ this point corresponds to the point on the unit circle $(\cos\theta, \sin\theta)$ and since $[\cos\theta, \sin\theta, 0] = [-\cos\theta, -\sin\theta, 0]$ the angle θ is an absolute angle modulo π.

Any line may be taken as the line at infinity by a suitable rotation of space. For example the model for the euclidean plane could be $2x - 7y + 345z = 1$. In that case the line at infinity would be $2x - 7y + 345z = 0$.

6.3 Incidence and Duality

Two points determine a unique line containing them. For example the points $[1,2,3]$ and $[3,2,1]$ lie on the line $x-2y+z = 0$. Interpreting this in space $\mathbb{R}^3$, the lines through $(1,2,3)$ and $(3,2,1)$ and the origin lie in the plane $x - 2y + z = 0$.

In addition two lines meet in a unique point. For example the lines $x + 2y + 3z = 0$ and $3x + 2y + z = 0$ meet in the point $[1, -2, 1]$.

The above examples illustrate the notion of *duality*. The line $x - 2y + z = 0$ can be represented by the homogeneous coordinates $[1, -2, 1]$ which represents a point of the projective line called the *dual point*. Conversely the *dual line* of the point $[1,2,3]$ is the line $x + 2y + 3z = 0$. The dual of the line containing two points is the point where the dual lines meet. More generally any statement of incidence of points and lines is converted to a statement of incidence of lines and points.

If $A = (a, b, c)$ and $X = (x, y, z)$ then the line $ax + by + cz = 0$ can be written as $A \cdot X = 0$ and the dual point is $[A]$. This is represented in space as the line through the origin at right angles to the plane $ax + by + cz = 0$.

To simplify calculations let us introduce the notation

$$\begin{vmatrix} \mathbf{a_1} & \mathbf{a_2} & \mathbf{a_3} \\ \mathbf{b_1} & \mathbf{b_2} & \mathbf{b_3} \end{vmatrix} = [\mathbf{a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1}].$$

Then the dual coordinates of the line through $[a_1, a_2, a_3]$ and $[b_1, b_2, b_3]$ are $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ which are also the coordinates of the intersection point of the lines $a_1x + a_2y + a_3z = 0$ and $b_1x + b_2y + b_3z = 0$.

Using the notation of the previous chapter the line through the points $[A]$ and $[B]$ has dual coordinates $[A \times B]$. Dually the lines $A \cdot X = 0$ and $B \cdot X = 0$ meet at $[A \times B]$.

Example 6.1

To find the dual coordinates of the line through $[1, 1, -1]$ and $[2, 0, 1]$ consider

$$\begin{vmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \end{vmatrix} = [1-0, -2-1, 0-2] = [1, -3, -2].$$

The equation of the line is $x - 3y - 2z = 0$.

Exercise 6.2

Show that the line through the vertices X and Y of the triangle of reference has equation $z = 0$.

Exercise 6.3

Find the equation of the line through the points $[0, 1, 1]$, $[2, -1, 0]$ and find the line's dual coordinates.

Exercise 6.4

Find the line coordinates of the line through $[1, \theta, \theta^2]$ and $[1, \phi, \phi^2]$, $(\theta \neq \phi)$. What happens when ϕ tends to θ?

If the point $[A]$ lies on the line determined by the points $[B]$ and $[C]$ then $A \cdot (B \times C) = 0$. This can be reinterpreted as

■ The points A, B and C in $\mathbb{R}P^2$ are collinear if and only if

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$$ □

Exercise 6.5

What is the dual of the above result?

Exercise 6.6

Interpret the above result in terms of volume.

Exercise 6.7

Let P be a variable point on the line $z = 0$ and let A, B be fixed points on the line $y = 0$. Let L, M be the intersections of the lines PA, PB with the line $x = 0$ respectively and let P′ be the intersection of the lines AM and BL. Show that P′ lies on a fixed line through Y= $[0, 1, 0]$.

6.4 Desargues' Theorem

We now come to the first major theorem of projective geometry. The famous theorem of Girard Desargues (1591–1661) was published in a book by Abraham Bosse (1648). The importance of the theorem was not noted until the 19th century.

Desargues' Theorem If two triangles are in perspective then the points of intersection of corresponding sides are collinear.

Proof Being in perspective means that the lines defined by corresponding vertices are concurrent. The reader should consult Fig. 6.1 where the triangles are A, B, C and A′, B′, C′ and the perspective point is P. The points of intersection of the corresponding sides are L, M, N and it will be our task to prove that they are collinear using the techniques described above.

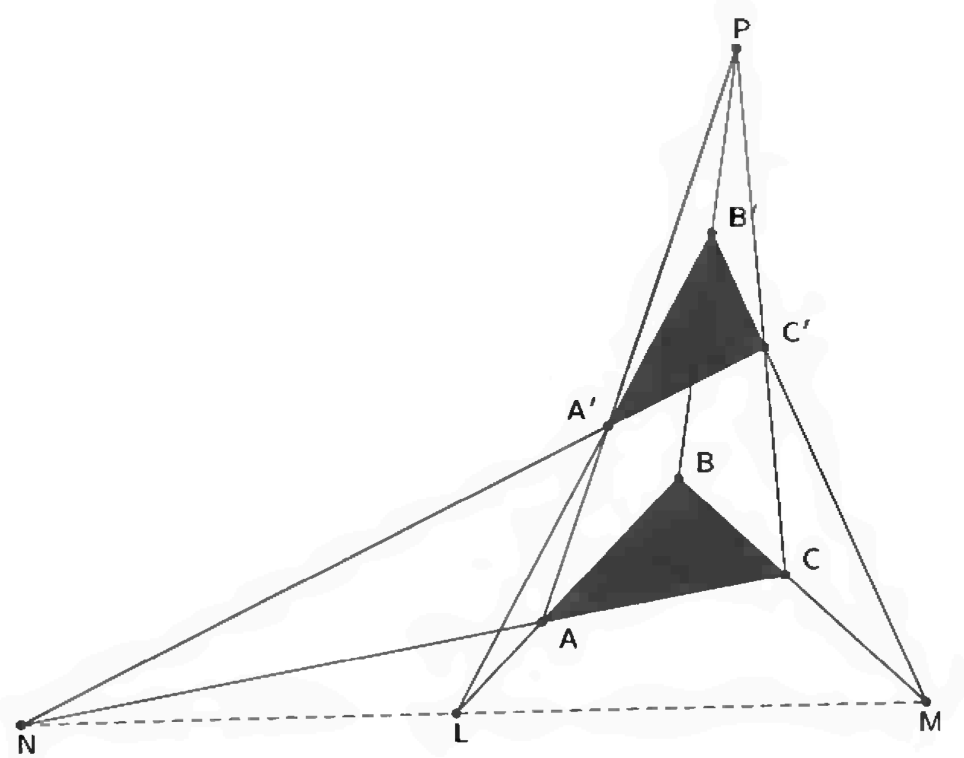

Fig. 6.1

Since $[A], [A'], [P]$ are collinear there are real numbers a, a' (not equal to 0) such that

$$P = aA + a'A'.$$

By choosing the representatives of A, A′ as $aA, a'A'$ we can rewrite this equation as

$$P = A + A'.$$

Similarly

$$P = B + B'$$
$$P = C + C'.$$

So $A - B = B' - A'$. It follows that $[A - B]$ is the intersection of the line $[A][B]$ with the line $[A'][B']$, that is L. Similarly M= $[B - C]$ and N= $[C - A]$. Since $(A - B) + (B - C) + (C - A) = 0$ it follows that L, M, N are collinear. □

The purely algebraic methods of the proof show that Desargues' theorem holds whenever the coefficients lie in a division ring. Plane geometries where Desargues' theorem does not hold are called non-desarguian planes.

By 'lifting' Desargues' theorem into three-dimensional space a much easier proof can be given. Consider Fig. 6.2.

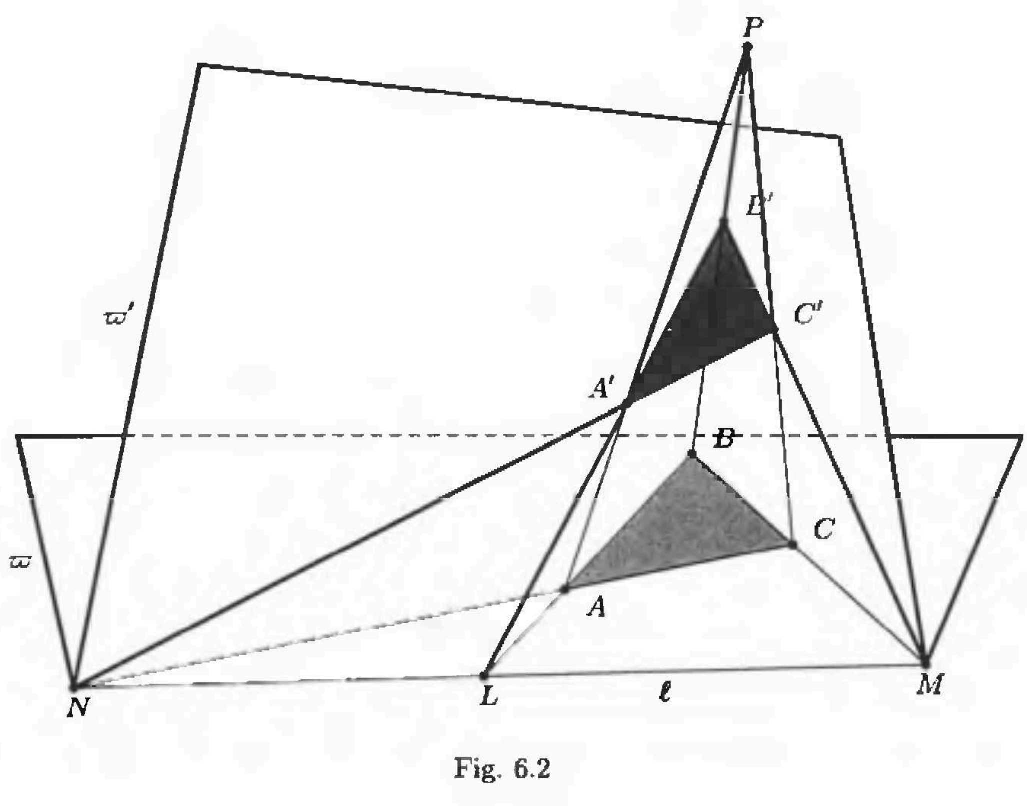

Fig. 6.2

The triangles which are in perspective from P are now assumed to lie in planes ϖ and ϖ' respectively. Let the planes be distinct and meet in the line ℓ. The planes PAB and $PA'B'$ coincide by hypothesis. This common plane meets ϖ in the line AB and meets ϖ' in the line $A'B'$. So these lines, which meet at L must also meet at a point on ℓ. Similarly, so do the points M and N.

If $\varpi = \varpi'$ then the above argument fails completely since the line ℓ common to the two planes is not defined. In order to prove the theorem in this case take a line ℓ through P, the point of perspective, suppose that ℓ is not in ϖ and let P_1, P_2 be points on ℓ other than P.

Then the lines P_1A and P_2A' lie in the plane containing the lines, P_1PP_2 and $PA'A$ and so meet in a point A'' say. Similarly let P_1B and P_2B' meet at B'' and let P_1C and P_2C' meet at C''. Then $A''B''C''$ is a triangle which is in perspective with ABC from P_1 and which is in perspective with $A'B'C'$ from P_2.

Let ϖ' be the plane of $A''B''C''$. Then $\varpi \neq \varpi'$ and we can apply Desargues' theorem in this case. But the line of perspective of the sides is then the line common to ϖ and ϖ' which contains L, M and N. □

The three-dimensional nature of this proof means that non-desarguian planes cannot be embedded in a three-dimensional geometry.

6.5 Cross Ratios Again

It is clear that angles and distance, the familiar invariants of the euclidean plane, are not projective invariants. Projective transformations stretch and change them both. In fact there are precious few numerical projective invariants. However one such is the cross ratio. We will define this initially in terms of the projective plane and then relate this definition to that given in Chapter 4.

Let $[A], [B], [C], [D]$ be four points, no three of which are equal, lying on a projective line. This means that the four points A, B, C, D of $\mathbb{R}^3$ lie in a plane through the origin. If $[A] \neq [B]$ the vectors C, D can be written uniquely as linear combinations of A and B. Suppose

$$C = pA + qB, \ D = rA + sB.$$

Then their ***cross ratio*** is defined to be

$$(\mathrm{AB,CD}) = qr/ps.$$

Before we can be sure that this makes sense we should see if it is unchanged for different representatives of the same points. This is not difficult. If aA, bB, cC

and dD are different representatives of A, B, C, D then

$$cC = (cp/a)(aA) + (cq/b)(bB), \ dD = (dr/a)(aA) + (ds/b)(bB)$$

and the new multiples cancel out in the quotient.

Example 6.2

Let $[1,2,3]$, $[1,1,2]$, $[3,5,8]$, $[1,-1,0]$ be four points in the projective plane. It happens that they all lie on a projective line. Let us find their cross ratio.

The equations

$$(3,5,8) = p(1,2,3) + q(1,1,2), \ (1,-1,0) = r(1,2,3) + s(1,1,2)$$

have the unique solution $p = 2, \ q = 1 \, r = -2, \ s = 3$ so the cross ratio is

$$\frac{1 \times (-2)}{2 \times 3} = -\frac{1}{3}.$$

Exercise 6.8

Find the cross ratio of the points $[2,1,3], [1,2,3], [8,1,9]$ and $[4,-1,3]$.

Exercise 6.9

Show that

$$(\mathsf{AB,CC}) = 1, \ (\mathsf{AB,AD}) = 0, \ (\mathsf{AB,BD}) = \infty.$$

The cross ratio is like a coordinate determining the fourth member from a triple. It is clear that any real number can and ∞ occur. Moreover the answer is unique in the following sense.

▮ (Unique fourth point theorem) Let A, B, C, X, Y be collinear points such that

$$(\mathsf{AB,CX}) = (\mathsf{AB,CY}).$$

Then X=Y.

Proof Suppose that $C = sA + tB$, $X = \alpha A + \beta B$ and $Y = \gamma A + \delta B$. Then the left-hand cross ratio is $t\alpha/s\beta$ and the right-hand cross ratio is $t\gamma/s\delta$. So $\alpha/\beta = \gamma/\delta$. It follows that $Y = (\gamma/\alpha)(\alpha A + \beta B) = (\gamma/\alpha)X$ and this represents the same point in the projective plane as X. □

In Chapter 4 the cross ratio was defined in terms of complex numbers. Recall that the cross ratio was real if the points all belonged to a (euclidean) line or circle.

Our next task is to show that the two definitions of cross ratio coincide in any mutually meaningful situation. Take two points A and B in some euclidean space. Then for all real numbers t, the point $C = tA + (1-t)B$ lies on the line through A and B, and has the property that the ratio of distances AC/CB is the fraction $(1-t)/t$. It follows that if $D = sA + (1-s)B$ is another such point then the cross ratio, defined in terms of distances, is

$$(AB, CD) = \frac{AC.BD}{AD.BC} = \frac{(1-t)s}{t(1-s)}.$$

Suppose now that the points A, B, C, D are in $\mathbb{R}^3$. Then the corresponding points A, B, C, D in the projective plane are collinear and have the same cross ratio $(1-t)s/t(1-s)$ according to this chapter's definition.

Exercise 6.10

* Let $[A], [B], [C], [D]$ be four distinct points lying on a projective line. Show that their cross ratio is given by the formula

$$\pm\sqrt{\frac{(A\cdot A)(C\cdot C)-(A\cdot C)^2}{(B\cdot B)(C\cdot C)-(B\cdot C)^2}\cdot\frac{(B\cdot B)(D\cdot D)-(B\cdot D)^2}{(A\cdot A)(D\cdot D)-(A\cdot D)^2}}.$$

As in the complex number definition the cross ratio satisfies a number of identities as the points are permuted. These can be summarised as follows.

■ The cross ratio satisfies,

$$(\mathsf{AB,CD}) = (\mathsf{BA,DC}) = (\mathsf{CD,AB}) = (\mathsf{DC,BA})$$

and

$$\begin{array}{lllll} (\mathsf{AB,CD}) & = & c & (\mathsf{AB,DC}) & = & 1/c \\ (\mathsf{AC,BD}) & = & 1-c & (\mathsf{AC,DB}) & = & 1/(1-c)\,. \\ (\mathsf{AD,BC}) & = & 1-1/c & (\mathsf{AD,CB}) & = & c/(c-1) \end{array}$$

Proof For example, to see that (AB,CD)= 1−(AC,BD) we must interchange the rôle of B and C. Let $C = pA + qB$, $D = rA + sB$. Then $B = (-p/q)A + (1/q)C$ and $D = rA + s((-p/q)A + (1/q)C) = (r - sp/q)A + (s/q)C$ and so

$$(\mathsf{AC,BD}) = \frac{(1/q)(r-sp/q)}{(-p/q)(s/q)} = 1 - qr/ps = 1 - (\mathsf{AB,CD}).$$

The other identities can be safely left as an exercise. □

6.6 Cross Ratios and Duality

A collection of points on a line is called a *range* and the line is called the *axis* of the range. The dual of a range is called a *pencil* of lines. If the range has axis ℓ then the lines of the dual pencil pass through the point dual to the line ℓ. This common point of the lines of the pencil is called the *apex* of the pencil.

Since the dual coordinates of four lines of a pencil are dependent, a cross ratio can be defined for a pencil of four lines in exactly the same way as one was defined for four collinear points.

Exercise 6.11

Show that the four lines

$$x + y + 2z = 0,\ 3x - y + 4z = 0,\ 5x + y + 8z = 0,\ 2x + 3z = 0$$

are concurrent and find their cross ratio.

Any line not through the apex of a pencil of four points (called a *transversal*) will meet the pencil in four collinear points (Fig. 6.3). It turns out that their cross ratio is the same as that of the pencil.

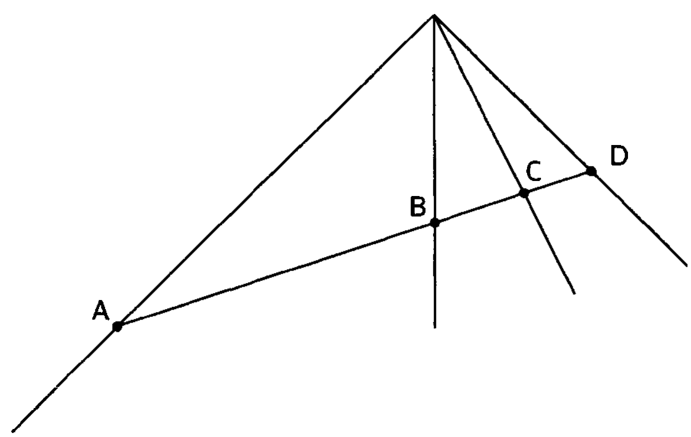

Fig. 6.3 Four collinear points determined by a transversal to a pencil

■ The four collinear points determined by a transversal to a pencil have the same cross ratio as the pencil itself.

Proof Suppose that the lines of the pencil have dual coordinates

$$[A], [B], [C], [D] \text{ where } C = pA + qB,\ D = rA + sB.$$

Let the transversal have dual coordinates $[L]$. Then the points of intersection of the transversal with the pencil have coordinates $[L\times A]$, $[L\times B]$, $[L\times C]$, $[L\times D]$. Since $L \times C = pL \times A + qL \times B$ and $L \times D = rL \times A + sL \times B$ we see that the cross ratio is the same. □

Since the calculation of the cross ratio only depends on the pencil we see that any another transversal will meet the pencil in four points with the same cross ratio. This may be summed up by the following result.

■ Let a transversal meet a pencil in four points A, B, C, D and let another transversal meet the pencil in four points A′, B′, C′, D′ (Fig. 6.4).

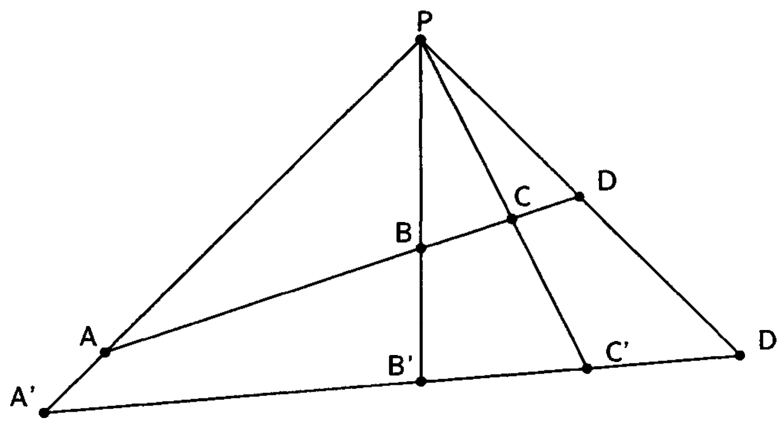

Fig. 6.4

Then (AB,CD)= (A′B′,C′D′). □

Here is a partial converse to the above result.

■ Let ℓ and ℓ' be two lines meeting at a point P. Let A, B, C be three points on ℓ and let A′, B′, C′ be three points on ℓ' such that (PA,BC)=(PA′,B′C′). Then the lines AA′, BB′ and CC′ are concurrent.

Proof Let the lines AA′, BB′ meet at Q and let QC meet ℓ' at C″ (see Fig. 6.5). Our aim will be to show that C″=C′. As P, A, B, C and P, A′, B′, C″ are in perspective from Q it follows that the cross ratios are equal. So (PA,BC)=(PA′,B′C″). But (PA,BC)=(PA′,B′C′) and so (PA′,B′C″)=(PA′,B′C′). By the unique fourth point theorem C″=C′. □

Exercise 6.12

Let A, B, C, X be points on a line ℓ and let A′, B′, C′, X′ be points on a line ℓ' such that the points of intersection of the lines {AB′, BA′}, {AC′, CA′} and {AX′, XA′} are collinear. Show that (AB,CX)=(A′B′,C′X′).

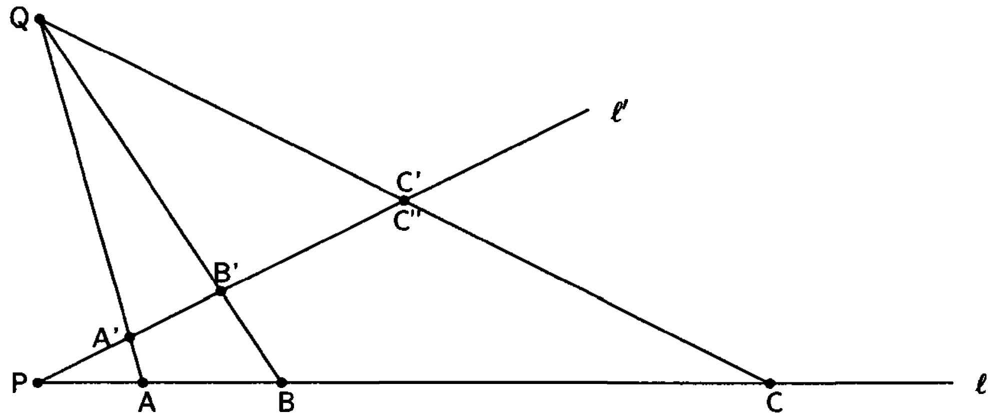

Fig. 6.5

6.7 Projectivities and Perspectivities

Let A_1, A_2, A_3, A be four points on a line a and let B_1, B_2, B_3, B be four points on a line b such that $(A_1A_2, A_3A)=(B_1B_2, B_3B)$. Then A is uniquely determined by B and conversely. The relation between A and B is called a *projectivity* between the points of a and the points of b.

If (A) is a range of points with axis a projectively related to a range of points (B) with axis b then we write

$$(A) \wedge (B).$$

The relation

$$(A_1, A_2, A_3, A_4) \wedge (B_1, B_2, B_3, B_4)$$

is equivalent to the equality

$$(A_1A_2, A_3A_4) = (B_1B_2, B_3B_4).$$

A special example of a projectivity is a *perspectivity* from a point P, since perspective points have the same cross ratio. In that case we write

$$(A) \overset{P}{\wedge} (B).$$

We will now give a geometric construction for any projectivity. Let (A_1, A_2, A_3) be three distinct points on a line a and let (B_1, B_2, B_3) be three distinct points on another line b. Suppose under a projectivity between a and b that

$$(A_1, A_2, A_3, A) \wedge (B_1, B_2, B_3, B).$$

We will assume that A is given and then construct the corresponding point B geometrically.

Let A_1B_1 meet A_2B_2 in the point O and A_3B_3 in the point O'. There are two cases to consider.

(i) If O and O' coincide let B be the point where the line OA meets b.

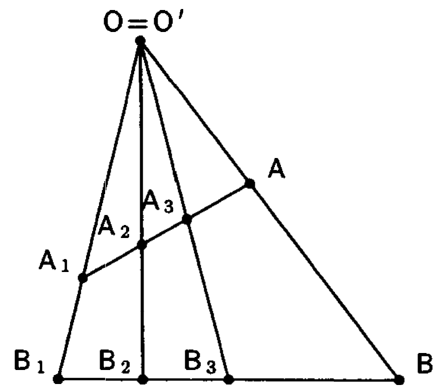

Fig. 6.6

Then

$$(A_1, A_2, A_3, A) \overset{O}{\wedge} (B_1, B_2, B_3, B).$$

That is, the projectivity is a perspectivity from O.

(ii) If O and O' are distinct let OO' meet A_3B_2 in X, let OA meet A_3B_2 in Y and let $O'Y$ meet b in B (see Fig. 6.7).

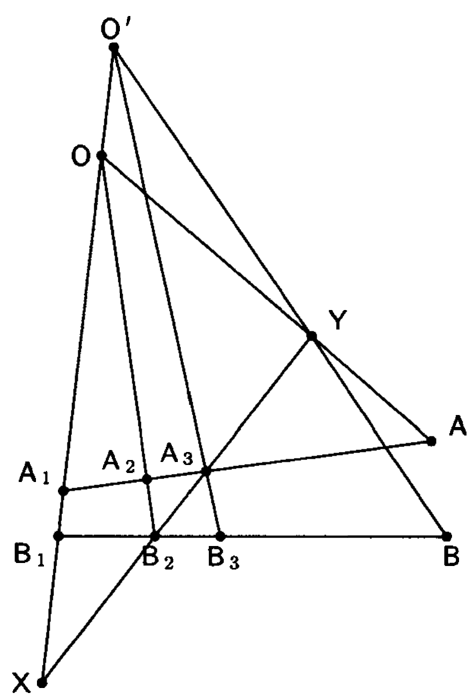

Fig. 6.7

Then

$$(A_1, A_2, A_3, A) \overset{O}{\wedge} (X, B_2, A_3, Y) \overset{O'}{\wedge} (B_1, B_2, B_3, B).$$

This writes the projectivity as the composition of the perspectivities from O and O′. So we have proved the following.

▮ Any projectivity between two lines is either a perspectivity or the composition of two perspectivities. □

We can give a simple rule for when a projectivity is a perspectivity as follows.

▮ Let a, b be two lines meeting at the point P. Then a projectivity between the points of a and b is a perspectivity if and only if P corresponds to itself under the projectivity.

Proof Clearly under a perspectivity P is self-corresponding. For the converse use the notation of the construction of the two perspectivities given above. Let OP meet B_2A_3 in the point Z. Since P is self-corresponding O′ must lie in the line OZ. But O, O′ also lie on the distinct line A_1B_1. So O, O′ coincide and the projectivity is a perspectivity from O. □

It is useful at this stage to pause and consider the dual of the above results. The dual of a range of points on a line is a pencil of lines through a point. A projectivity between two pencils is a correspondence which preserves the cross ratio. Two pencils, (a) and (b) are in perspective from a line c if any line a from the first pencil meets the corresponding line b from the second pencil on the line c. Any projectivity between two pencils is either a perspectivity or the product of two perspectivities. A projectivity is a perspectivity if and only if the line joining the apexes (common points) of the two pencils is self-corresponding.

The Pappus Line

Let the range of points (A_i) be projectively related to the range of points (B_i) under the correspondence $A_i \leftrightarrow B_i$. Fix i: then the pencil of lines through A_i is projectively related to the pencil of lines through B_i by the sequence

$$(A_iB_j) \overset{A_i}{\wedge} B_j \wedge A_j \overset{B_i}{\wedge} (B_iA_j).$$

The line A_iB_i is self-corresponding and so this projectivity is a perspectivity from a line c_i. If we interchange the rôle of i and j we see that c_i is independent of i. This line is called the *Pappus line* of the projectivity. It has the following property: the intersection of A_iB_j with A_jB_i lies on the Pappus line. In fact once the Pappus line is known then it can be used as an alternative geometric construction of the point B corresponding to A.

Exercise 6.13

If the range of points (A_i) on the line a is projectively related to the range of points (B_i) on the line b under a perspectivity, show that the Pappus line passes through the point common to a and b. What happens otherwise?

Pappus' theorem Let A_1, A_2, A_3 and B_1, B_2, B_3 be two sets of collinear points (lying on distinct lines). Let A_2B_3, A_3B_2 meet in C_1, let A_3B_1, A_1B_3 meet in C_2 and let A_1B_2, A_2B_1 meet in C_3. Then C_1, C_2, C_3 are collinear.

Proof

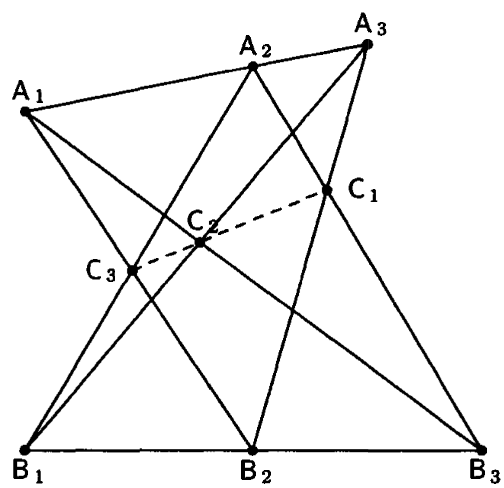

Fig. 6.8

A projectivity is uniquely determined by three pairs of corresponding points. So $A_i \to B_i$, $i = 1, 2, 3$ determines a unique projectivity. The points C_1, C_2, C_3 all lie on the Pappus line of this projectivity (Fig. 6.8). □

Exercise 6.14

State the dual of Pappus' theorem.

6.8 Quadrilaterals

A quadrilateral is defined by four general points A, B, C, D in the projective plane. The four points determine six lines which fall into three pairs of opposite sides {AC, BD}, {AD, BC} and {AB, CD} which meet in three diagonal points X, Y, Z (see Fig. 6.9).

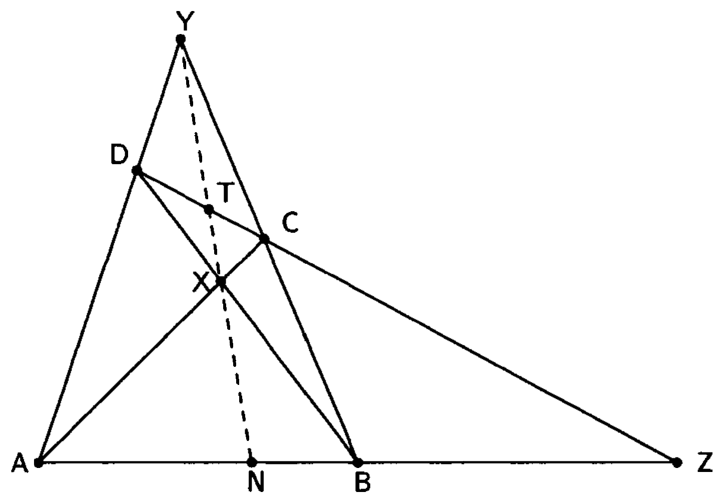

Fig. 6.9

Exercise 6.15

If the coordinates of A, B, C, D are $[\lambda, \mu, \nu]$, $[\lambda, \mu, -\nu]$, $[-\lambda, \mu, \nu]$, $[\lambda, -\mu, \nu]$, show that the three diagonal points are the vertices of the triangle of reference.

There is no particular order to the points of a quadrilateral in the projective plane.

Four collinear points A, B, A', B' are said to be ***harmonic*** if (AB,A'B')= -1.

Exercise 6.16

Show that distinct points A, B, A', B' are harmonic if and only if (AB,A'B')=(BA,A'B'.)

▮ In Fig. 6.9 let AB meet XY in N. Then A, B, N, Z are harmonic.

Proof Let DC meet XY in T. Then (AB,NZ)=(DC,TZ) by perspectivity from Y. But (DC,TZ)=(BA,NZ) by perspectivity from X. By the results of Exercise 6.16 A, B, N, Z are harmonic. □

Exercise 6.17

If the coordinates of A, B, C, D are $[\lambda,\mu,\nu]$, $[\lambda,\mu,-\nu]$, $[-\lambda,\mu,\nu]$, $[\lambda,-\mu,\nu]$, find the coordinates of N and show that A, B, N, Z are harmonic by direct calculation.

6.9 Projective Transformations

We have already met projectivities, that is maps from a line to a line which preserve the only one-dimensional projective feature, the cross ratio. Let us now consider transformations of the projective plane which preserve the essential two-dimensional features. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be an invertible linear transformation. Then T defines a transformation, denoted by $[T]$, of the projective plane, $\mathbb{R}P^2$, by the rule

$$[T]([X]) = [T(X)].$$

Such a transformation is called a *projective transformation.*

If T is defined by a matrix $\mathbf{M}$ then $[T]$ is defined by any non-zero multiple of $\mathbf{M}$. Because the linear transformation is invertible the determinant of $\mathbf{M}$ will be non-zero.

An invertible linear transformation of $\mathbb{R}^3$ takes planes through the origin (subspaces of dimension 2) to planes through the origin. It follows that the corresponding projective transformation will take a line in the projective space to another line. A projective transformation will also preserve incidence. That is, it takes points on a line through two points into points on the line through the corresponding points. As a consequence we say that properties of incidence are projective invariants.

Exercise 6.18

Show that the line $2x+y-3z = 0$ is transformed into the line $5x-5y+z = 0$ under the projective transformation with matrix

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ -2 & 0 & 1 \end{pmatrix}.$$

Exercise 6.19

Show that the projective transformation with matrix $\mathbf{M}$ takes the line with dual coordinates A into the line with dual coordinates $A(\mathbf{M}^T)^{-1}$.

We now show that the cross ratio is an invariant of projective transformations.

▮ The cross ratio of four collinear points is unchanged by a projective transformation.

Proof Suppose that the four points are $[A], [B], [C], [D]$ where $C = pA + qB$, $D = rA + sB$. Let $[P] \to [P\mathbf{M}]$ be a projective transformation. Then the four collinear points are transformed to

$$[A\mathbf{M}], \quad [B\mathbf{M}], \quad [C\mathbf{M}], \quad [D\mathbf{M}].$$

Since $C\mathbf{M} = pA\mathbf{M} + qB\mathbf{M}$ and $D\mathbf{M} = rA\mathbf{M} + sB\mathbf{M}$ the cross ratio is unchanged. □

Exercise 6.20

* Show that any continuous transformation of $\mathbb{R}P^2$ which takes lines into lines and preserves incidence is determined by a linear transformation of $\mathbb{R}^3$ as above.

In fact the projective transformation is determined by what happens to four general points, that is four points no three of which are collinear.

▮ There is a unique projective transformation taking four general points to four other general points.

Proof Since a projective transformation is invertible these points might as well be the vertices of the triangle of reference, $\mathsf{X} = [1,0,0]$, $\mathsf{Y} = [0,1,0]$, $\mathsf{Z} = [0,0,1]$ and the unit point $\mathsf{U} = [1,1,1]$. Let the other four points be $\mathsf{A} = [A]$, $\mathsf{B} = [B]$, $\mathsf{C} = [C]$, $\mathsf{D} = [D]$.

For arbitrary non-zero real numbers λ, μ, ν, the matrix

$$\mathbf{M} = \begin{pmatrix} \lambda A \\ \mu B \\ \nu C \end{pmatrix}$$

defines a projective transformation $[T]$ taking X to A, Y to B, Z to C and U to $[\lambda A + \mu B + \nu C]$. Since A, B, C form a basis of $\mathbb{R}^3$ we can find unique λ, μ, ν so that $D = \lambda A + \mu B + \nu C$ and then $[T]$ takes U to D. □

Exercise 6.21

Find the matrix (up to a non-zero multiple) of the projective transformation which takes $[1,0,0]$ to $[0,0,1]$, $[0,1,0]$ to $[0,1,1]$, $[0,0,1]$ to $[1,1,1]$ and $[1,1,1]$ to $[3,2,4]$.

6.10 Fixed Points and Eigenvectors

If A is a (real) eigenvector of the matrix $\mathbf{M}$ corresponding to an eigenvalue λ then $A\mathbf{M} = \lambda A$ and so $[A]$ is a fixed point of the corresponding projective transformation and conversely. Since a 3×3 real matrix always has at least one real eigenvalue every projective transformation has at least one fixed point.

Example 6.3

Let us find the fixed points of the projective transformation corresponding to the matrix

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix}.$$

The eigenvalues are given by the equation

$$\begin{vmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 1 & 1 \\ 0 & -2 & \lambda - 4 \end{vmatrix} = (\lambda - 2)^2(\lambda - 3) = 0.$$

So there are two eigenvalues 2 (twice) and 3. The corresponding eigenvectors are (up to a non-zero multiple) $(0, 2, 1)$ and $(0, 1, 1)$. Hence the fixed points of the projective transformation are $[0, 2, 1]$ and $[0, 1, 1]$. Notice that the line $x = 0$ is invariant in the sense that any point with x-coordinate zero is transformed into another point with x-coordinate zero.

Exercise 6.22

Find the fixed points of the projective transformation corresponding to the matrix

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & -1 & 4 \end{pmatrix}.$$

6.11 Pappus' Theorem

We will now use the above result to prove the famous theorem of Pappus using homogeneous coordinates. Earlier, the proof used the properties of projectivities. Because this proof uses coordinates the dependence of the proof on the commutativity of the real numbers will be made clear.

▮ Let A_1, A_2, A_3 and B_1, B_2, B_3 be two sets of collinear points (lying on distinct lines). Let A_2B_3, A_3B_2 meet in C_1; let A_3B_1, A_1B_3 meet in C_2; and let A_1B_2, A_2B_1 meet in C_3. Then C_1, C_2, C_3 are collinear.

Proof By the above discussion we can apply a projective transformation to simplify the algebra. This will not effect the essential features of the situation. In essence we are choosing the triangle of reference and we take $A_1A_2A_3$ as the line $y = 0$ and $B_1B_2B_3$ as the line $z = 0$. Let

$$\begin{aligned} A_1 &= [p,0,1], \quad & A_2 &= [q,0,1], \quad & A_3 &= [r,0,1], \\ B_1 &= [l,1,0], \quad & B_2 &= [m,1,0], \quad & B_3 &= [n,1,0]. \end{aligned}$$

Then the line A_2B_3 has dual coordinates

$$\begin{vmatrix} q & 0 & 1 \\ n & 1 & 0 \end{vmatrix} = [-1,n,q]$$

and the line A_3B_2 has dual coordinates

$$\begin{vmatrix} r & 0 & 1 \\ m & 1 & 0 \end{vmatrix} = [-1,m,r].$$

These lines meet at C_1 which therefore has coordinates

$$\begin{vmatrix} -1 & n & q \\ -1 & m & r \end{vmatrix} = [nr - qm, r - q, n - m].$$

Similarly

$$C_2 = [lp - rn, p - r, l - n], \quad C_3 = [mq - pl, q - p, m - l].$$

Since

$$\begin{vmatrix} nr - qm & r - q & n - m \\ lp - rn & p - r & l - n \\ mq - pl & q - p & m - l \end{vmatrix} = 0$$

(the rows sum to zero), it follows that C_1, C_2, C_3 are collinear. □

The alert reader will have noted that the vanishing of the determinant which proves the collinearity of C_1, C_2, C_3 depends on the fact that $nr = rn$ etc. With more general non-commuting algebraic systems such as the quaternions, Pappus' theorem may fail to hold.

6.12 Perspective Drawing: Tricks of the Trade

Before the Renaissance, most paintings were for the use of the church and reflected its priorities. So a saint would be depicted larger than a mere mortal irrespective of their actual positions in space. With the rise of rich and powerful patrons whose interests were more secular a greater realistic representation was necessary. Many artists and mathematicians such as Albrecht Dürer (1471–1528) in Germany and Leonardo da Vinci (1452–1519) in Italy applied themselves to the problem of greater spatial reality in a painting or drawing. The interest that these inquiries generated, fed back to mathematics and to geometry in particular.

The task was to represent three-dimensional space onto a two dimensional space by stereographic projection with the eye as projection point. Dürer had a glass screen with a grid etched on it. The screen was placed between himself and the subject matter so that its appearance on the grid could be transfered to a similar grid on his work surface.

Of course in such a projection parallel lines may appear to meet at a vanishing point. The standard example is railway lines which we know are always a constant distance apart. Nevertheless they appear to meet at a vanishing point on the horizon.

The perspective view of a box shown in Fig. 6.10 is a typical example. The horizontal parallel lines meet at two vanishing points on the horizon.

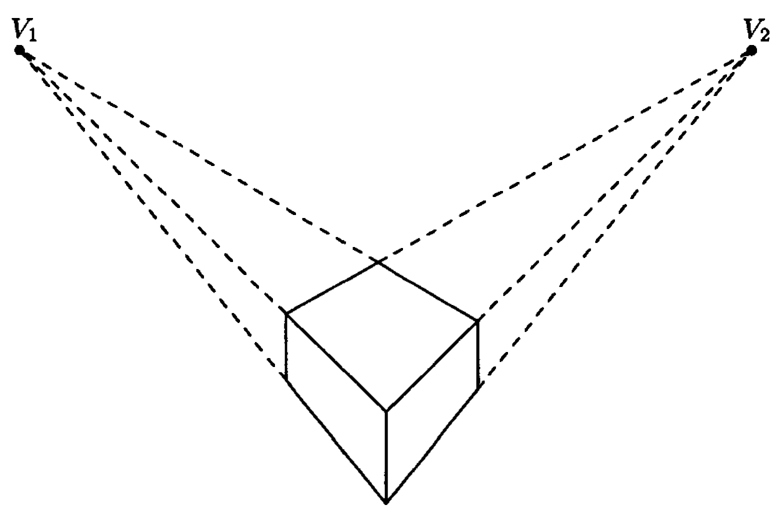

Fig. 6.10 A box in perspective

It must not be thought that paintings with "correct" perspective are necessarily superior to the older type. Perspective was just another technique for the painter to use. With the appearance of photography it was realised that the exact representation of the physical world was not necessary for great art and our views are now in some respects more in tune with medieval ideas.

A Brick in the Wall:

Very often a regular repeating pattern needs to be represented. Obvious examples are telegraph poles, railway sleepers and bricks in a wall. In Figs. 6.11

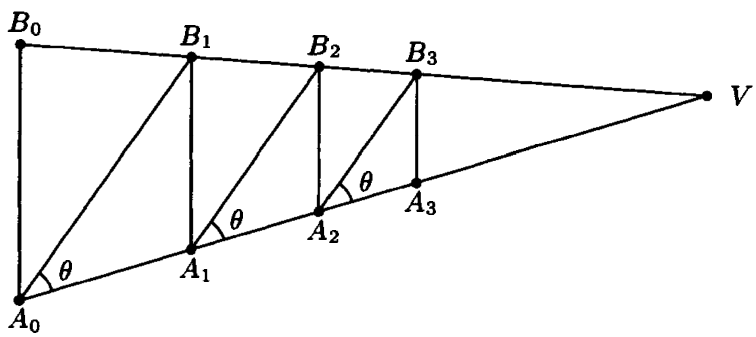

Fig. 6.11 Bricklaying by the constant angle method

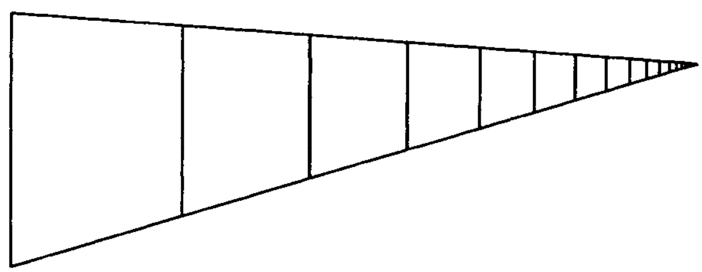

Fig. 6.12 The completed course

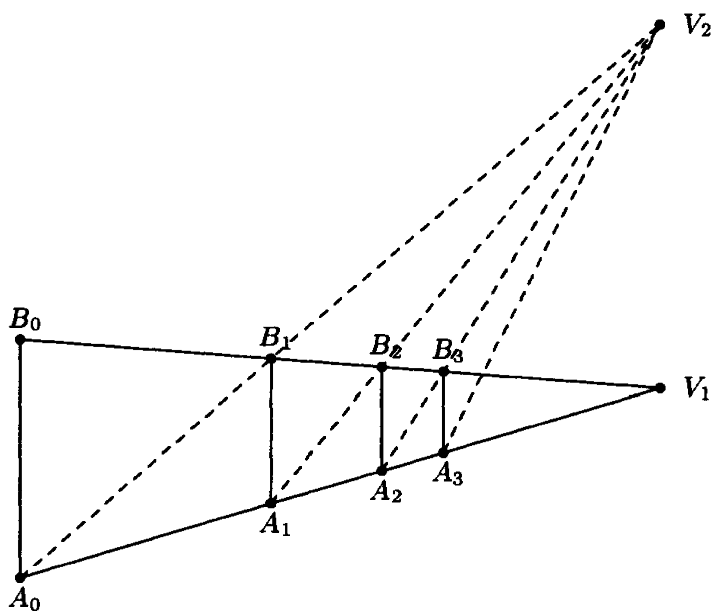

Fig. 6.13 Putting the diagonals in perspective

and 6.12 all the diagonals of the bricks are represented as parallel. This means that they make a constant angle θ with the base line. This means that once the first brick is drawn the positions of the next bricks are determined. Another idea is to make the diagonals meet at a vanishing point V_2 (Fig. 6.13). Now two bricks must be drawn to determine V_2. The rest are determined.

Leon Battista Alberti (1404–1472) described a method for representing a set of squares in a horizontal ground plane, for example a chess board or tiled floor, in the vertical plane of a painting. The initial line AB is divided into equal parts (Fig. 6.14). The sides of the squares are either horizontal or meet at a vanishing point V_2 on the horizon. The diagonals meet at another vanishing point V_1 on the horizon. The method of determining the position of the squares is clear from Fig. 6.14. To find the position of points in general we can use the

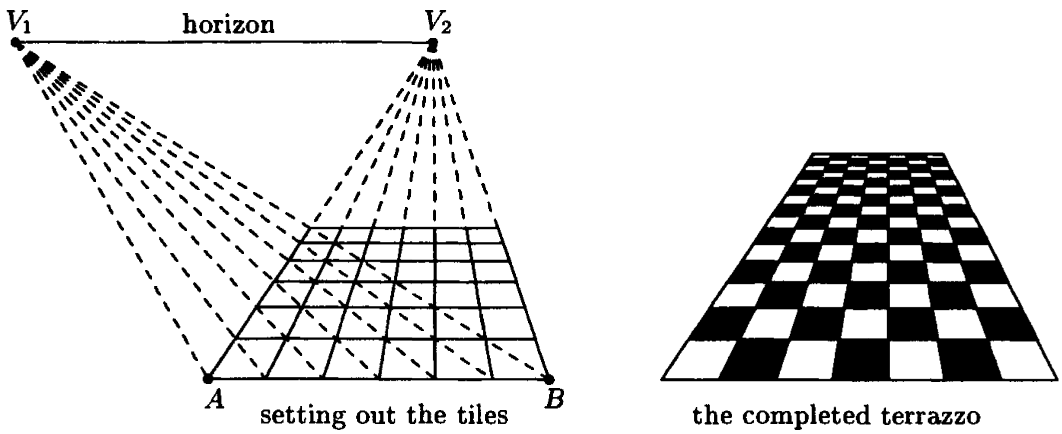

Fig. 6.14

fact that the cross ratio is invariant under a perspectivity. Here is a typical application.

Example 6.4

On a straight road approaching traffic lights there are "slow down" signs, 400 m and 200 m from the traffic lights, and a warning sign 100 m from the traffic lights. A town planner makes a perspective drawing in which the "slow down" signs are 3 cm and 1 cm from the traffic lights. Where should the warning sign be placed on the drawing? Let the slow down signs be S and S' and the warning sign be at W. If T is the traffic lights then the cross ratio is

$$\frac{(S-W)(S'-T)}{(S-T)(S'-W)} = \frac{300 \times 200}{400 \times 100} = \frac{3}{2}.$$

If in the drawing the warning sign is placed x cm from traffic lights then this is equal to

$$\frac{(3-x)(1)}{3(1-x)}.$$

So $9(1-x) = 2(3-x)$ which when solved gives $x = 3/7$ cm.

Exercise 6.23

The Doge's palace in Venice has two pillars and a statue in a line. They are 4 m, 3 m, and 1 metre respectively, from a wall. Leonardo has been asked to make a perspective drawing. He places the pillars 3 cm and 1 cm from the wall in the drawing. Where should the statue be placed in the drawing?

6.13 The Fano Plane

We have previously discussed the possibility of using different algebraic objects for coordinates. Here we illustrate using the integers modulo 2, $\mathbb{Z}_2 = \{0, 1\}$. We can think of 0 as the collection of all even integers and 1 as the collection of all odd integers and add and multiply accordingly. For example an even integer plus an odd integer is odd so $0+1 = 1$ but an even integer times an odd integer is even so $0 \times 1 = 0$.

The *Fano plane*, $\mathbb{Z}_2P^2$, is the projective plane defined by 3 homogeneous $\mathbb{Z}_2$ coordinates. So an element of the Fano plane is specified by a triple (x, y, z) where x, y, z are 0 or 1. Since $(0, 0, 0)$ is excluded $\mathbb{Z}_2P^2$ has seven points. By duality there are seven lines and these are illustrated in Fig. 6.15.

The only possible conceptual difficulty might be the line $x + y + z = 0$ passing through the points $(0, 1, 1), (1, 0, 1), (1, 1, 0)$ and represented by a circle in Fig. 6.15.

Exercise 6.24

What is the maximum number of points in $\mathbb{Z}_2P^2$, no three being collinear? How many (non-degenerate) triangles are there in the Fano plane? How many (non-degenerate) quadrilaterals?

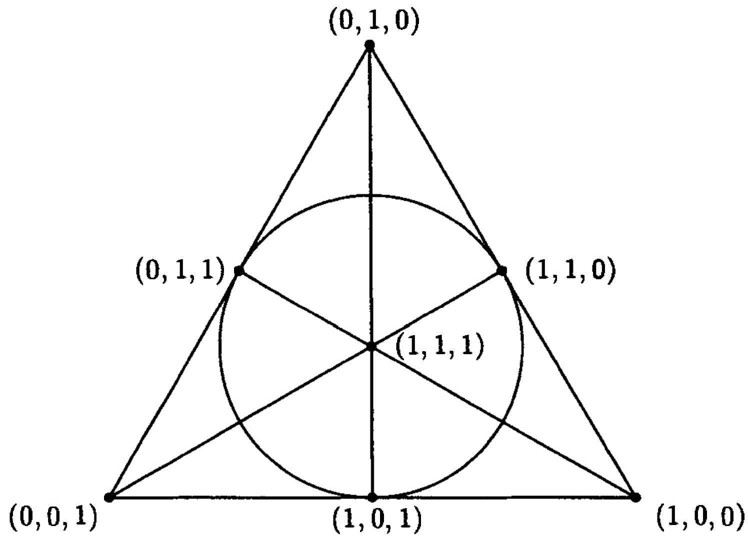

Fig. 6.15 The Fano plane

Answers to Selected Questions in Chapter 6

6.1 4.

6.3

$$\begin{vmatrix} 0 & 1 & 1 \\ 2 & -1 & 0 \end{vmatrix} = [1, 2, -2].$$

So the equation of the line is $x + 2y - 2z = 0$ and the dual coordinates of the line are $[1, 2, -2]$.

6.4

$$\begin{vmatrix} 1 & \theta & \theta^2 \\ 1 & \phi & \phi^2 \end{vmatrix} = [\theta\phi^2 - \theta^2\phi, \theta^2 - \phi^2, \phi - \theta] = [\theta\phi, -\theta - \phi, 1], \qquad \theta \neq \phi.$$

If $\phi = \theta$ then the line has coordinates $[\theta^2, -2\theta, 1]$. (This is the tangent line of a parabola.)

6.7 Let P= $[a, b, 0]$, A= $[a_1, 0, a_2]$, B= $[b_1, 0, b_2]$, then the line through PA has equation

$$\begin{vmatrix} x & y & z \\ a_1 & 0 & a_2 \\ a & b & 0 \end{vmatrix} = -xa_2b + ya_2a + za_1b = 0.$$

It meets the line $x = 0$ at the point L= $[0, a_1b, -a_2a]$. Similarly M is the point given by M= $[0, b_1b, -b_2a]$. So the line AM has equation

$$\begin{vmatrix} x & y & z \\ a_1 & 0 & a_2 \\ 0 & b_1b & -b_2a \end{vmatrix} = -xa_2b_1b + ya_1b_2a + za_1b_1b = 0.$$

Similarly the line BL has equation

$$-xa_1b_2b + ya_2b_1a + za_1b_1b = 0.$$

They meet at the point

$$\begin{vmatrix} -a_2b_1b & a_1b_2a & a_1b_1b \\ -a_1b_2b & a_2b_1a & a_1b_1b \end{vmatrix} = [a_1b_1a, -a_1b_1b, a(a_1b_2 + a_2b_1)]$$

which lies on the line

$$a_1b_1z = (a_1b_2 + a_2b_1)x.$$

This line clearly passes through $[0,1,0]$.

6.8 $(8,1,9) = 5(2,1,3) - 2(1,2,3)$ and $(4,-1,3) = 3(2,1,3) - 2(1,2,3)$. So the cross ratio is $3/5$.

6.10 Use the angle formula given in Chapter 4 and the fact that, for example, $B \cdot C = |B||C| \cos\phi$ where ϕ is the angle $\angle BOC$.

6.11 The dual coordinates are

$$[1,1,2],\ [3,-1,4],\ [5,1,8],\ [2,0,3].$$

Since $(5,1,8) = 2(1,1,2) + (3,-1,4)$ and $(2,0,3) = \frac{1}{2}(1,1,2) + \frac{1}{2}(3,-1,4)$ the lines are concurrent with cross ratio $\frac{1\times1/2}{2\times1/2} = 1/2$.

6.12 Let the points of intersection of the lines {AB', BA'}, {AC', CA'} and {AX', XA'} be L, M, N and let A" be the point of intersection of the line containing L, M, N with the line AA'. Perspective from A' gives (AB,CX)=(A"L,MN) and perspective from A gives (A"L,MN)=(A'B'C'X').

6.13 Let the point common to a and b be P. Then by construction the Pappus line passes through P since it is self-corresponding. Otherwise suppose as a point of a the point P corresponds to L and as a point of b it corresponds to M. Then the Pappus line is LM.

6.14 Let a_1, a_2, a_3 and b_1, b_2, b_3 be two sets of concurrent lines. Let the points of intersection a_2b_3, a_3b_2 define the line c_1. Define c_2 and c_3 similarly. Then c_1, c_2, c_3 all pass through a common point.

6.15 The dual coordinates of the line AC are $[0, -\nu, \mu]$ and the dual coordinates of the line BD are $[0, \nu, \mu]$. These lines meet in the point $X = [1,0,0]$. A similar calculation proves that for the other diagonal points $Y = [0,1,0]$ and $Z = [0,0,1]$.

6.16 If (AB,A'B')= c then (BA,A'B')= $1/c$. So $c^2 = 1$. Since $c \neq 1$ if the points are distinct we must have (AB,A'B')= -1.

6.17 The point N is where the line $z = 0$ meets the line AB which has dual coordinates $[-\mu, \lambda, 0]$. It follows that N$= [\lambda, \mu, 0]$. So N$= [A + B]$ and Z$= [A - B]$. Harmony is now clear.

6.18 The point with coordinates $[x, y, z]$ is transformed into $[a, b, c] = [x + y - 2z, y, x + 3y + z]$ by the matrix. If we write $[x, y, z]$ in terms of $[a, b, c]$ we get $[x, y, z] = [a - 7b + 2c, 3b, c - a - 2b]/3 = [a - 7b + 2c, 3b, c - a - 2b]$. So $2x + y - 3z = 0$ becomes $5a - 5b + c = 0$ after simplification.

6.19 Suppose $Y = X\mathbf{M}$. Then $X = Y\mathbf{M}^{-1}$. The line $AX^T = 0$ with dual coordinates A becomes $A(\mathbf{M}^T)^{-1}Y = 0$ with dual coordinates $A(\mathbf{M}^T)^{-1}$. To see how this compares with the previous question note that

$$\mathbf{M}^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ -7 & 3 & -2 \\ 2 & 0 & 1 \end{pmatrix} /3.$$

So $(2, 1, -3) \to (2, 1, -3)(\mathbf{M}^T)^{-1} = (5, -5, 1)/3$.

6.21 Since

$$(3, 2, 4) = 2(0, 0, 1) - (0, 1, 1) + 3(1, 1, 1)$$

the matrix

$$\begin{pmatrix} 0 & 0 & 2 \\ 0 & -1 & -1 \\ 3 & 3 & 3 \end{pmatrix}$$

defines the transformation which does the job.

6.22 Note that the matrix is the transpose of the previous matrix and so has the same eigenvalues 2 and 3. The corresponding eigenvectors are (up to a non-zero multiple) $(1, 0, 0)$ and $(1, 1, -2)$.

6.23 Suppose in the drawing he places the statue x cm from the wall. The important thing to remember is that the cross ratio is preserved so

$$\frac{4-1}{4} \cdot \frac{3}{3-1} = \frac{3}{4} \cdot \frac{3}{2} = \frac{9}{8} = \frac{3-x}{3} \cdot \frac{1}{1-x},$$

giving $x = 3/19$.

6.24 Four.

6.25 The number of triangles is $\binom{7}{3} = 35$. Of these, 7 lie in a line. So the number of non-degenerate triangles is 28. The are $\binom{7}{4} = 35$ quadrilaterals. The degenerate ones consist of three points in a line and another point. There are $7 \times (7 - 3) = 28$ of these. So there are 7 non-degenerate quadrilaterals. Alternatively: a non-degenerate quadrilateral is the complement of a line and there are seven of these.