LAX PAIRS: INTEGRABIITY AND CHAOS

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Abstract. Completely integrable finite dimensional Hamiltonian systems are well understood thanks to the work of Liouville and Arnold. On the other hand, the Lax Pair formulation of the KdV equation marks the beginning of the extension of the completely integrable theory to infinite dimensional Hamiltonian systems. Solutions of initial value problems for systems that admit a Lax Pair formulation normally have a tame qualitative behavior if Lax Pairs give rise to an infinite complete set of conserved laws. The situation is different for initial-boundary value problems, even in one space dimension. There are problems where integrability persists and others where even chaos can appear. In this short article we review an instance of each case. A more complete understanding of when exactly integrability peristS is still missing.

To the memory of Peter Lax, teacher and mentor

1. Introduction

The Lax Pair formulation of the KdV equation [19] marks the beginning of the extension of the theory of completely integrable systems to Hamiltonian systems of infinite dimension. Initial value problems for systems admitting a Lax Pair formulation are well understood if the initial data satisfy some decay or covergence conditions at infinity or are periodic, enabling the possibility of solution via inverse methods (inverse scattering or inverse spectral). A crucial fact is that Lax Pairs give rise to an infinity of conservation laws. Ensuring appropriate initial data so that the conserved quantites are finite and reducing the system appropriately so that a complete set of (infinitely many) action and angle variables exists, the system is completely integrable in the sense that the solution is reducible to the solution of a (local) Riemann-Hilbert factorization problem (in the case of one space dimension) or a nonlocal Riemann-Hilbert factorization problem or a $\bar{\partial}$-problem (in higher space dimensions). Such problems are amenable to asymptotic analysis. For local problems one has the so-called nonlinear stationary phase and steepest descent methods; there is already a huge literature, see e.g. [7], [18]. For $\bar{\partial}$-problems there are fewer results (e.g.Perry [21]); for nonlocal Riemann-Hilbert factorization problems see Donmazov, Liu and Perry [9].

Things are different for initial-boundary value problems, even in one space dimension. There is an extension fo the inverse method (a "unified transform method", developped by Fokas and his collaborators) [10], [11], [12], [13]. But a crucial feature of this method is that it requires the values of more boundary data than given for a well-posed problem. This has two important consequences. First, it somehow lowers the degree of effectiveness of the asymptotic formulae since they involve knowledge of scattering data associated to both Dirichlet and Neumann data; the Neumann data

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are only very implicitly given. Most crucially however, given some Dirichlet data in an appropriate class (for the unified transform theory to apply) it is not at all clear that the Neumann data also lie in a class that one can work with.

More specifically, in the case of cubic NLS knowledge of the Dirichet data suffices to make the problem well-posed but the unified transform method also requires knowledge of the values of Neumann data. The study of the Dirichlet to Neumann map is thus necessary *before* the application of the unified transform. In the papers [2], [3], we presented a rigorous study of this map for a large class of decaying Dirichlet data. We showed that the Neumann data are also sufficiently decaying and hence that the unified transform method *can* be applied. These results are presented in the next section.

In a later section we present some beautiful numerical experiments by Arthur, Dorey and Parini, which clearly show the existence of chaos in the behavior of the Sine-Gordon initial-boundary value problem with initial data and a Robin boundary condition. Furthermore the Dirichlet boundary data $u(x,0)$ are unbounded. This is a clear instance of a problem which admits a Lax pair formulation but that is far from integrable: adding a boundary and boundary conditions, even ensuring a uniquely solvable problem may or may not preserve integrability! [1]

In the last section we compare the initial-boundary problems to initial value problems with a small perturbation and compare existing results.

2. NLS

Consider the NLS equation with cubic non-linearity, posed on the real positive semi-axis $\mathbb{R}^+$

$$\mathrm{i}q_t + q_{xx} - 2\lambda|q|^2 q = 0, \quad x > 0, \quad 0 < t < +\infty, \tag{2.1}$$

and initial-boundary data

$$\begin{aligned} q(x,0) &= q_0(x), \quad 0 \le x < +\infty \\ q(0,t) &= Q(t), \quad 0 \le t < +\infty, \end{aligned} \tag{2.2}$$

where q_0, Q are classical functions satisfying the compatibility condition $q_0(0) = Q(0)$.

The case $\lambda = 1$ is the defocusing case, while $\lambda = -1$ is the focusing case.

Back in 1991, Carrol and Bu in [5] established the existence of a unique global classical solution $q \in C^1(L^2) \cap C^0(H^2)$ of the problem (2.1)-(2.2), with $q_0 \in H^2$, $Q \in C^2$ and $q_0(0) = Q(0)$, by using PDE theory. Later papers like [15] by Holmer etc. have also provided results in Sobolev cases. For our purposes the classical result in [5] suffices.

On the other hand, it is well-known [22] that the non-linear Schrödinger equation (NLS) with cubic non-linearity can be written as a Lax pair and that, at least the Cauchy problem is 'completely integrable'; this means that there is an infinity of conservation laws which are in Poisson ivolution, and furthermore that the problem can be linearized via the scattering transform. This does not mean that there is a bona fide explicit solution. At best the inverse scattering problem, rewritten as a Riemann-Hilbert factorization problem can be effectively treated asymptotically. *Effective* long time, long range and semiclassical asymtotical formulas can be provided: they depend on the initial data either very explicitly or at worst via the solution of simple linear ODEs.

In [13] the authors use the unified transform method to solve the problem on the real positive semi-axis, given values for the initial data and Dirichlet data (which make the problem well-posed)

[1] This phenomenon should not be confused with the so-called deterministic turbulence [20], [4] appearing in initial value problems that *can* be studied via the inverse scatering transform and Riemann-Hilbert problems[18]. Deterministic turbulence is an integrable phenomenon.

and also the *Neumann* data $P(t) := q_x(0,t)$. What is required for that theory to work is that the Neumann data (as well as the Dirichlet data) live in some class with nice decaying properties such that the unified scattering transform can be properly defined. This is exactly the content of our theorems below: we provide several reasonably inclusive large classes of Dirichlet data, such that both Dirichlet and Neumann data decay as $t \to \infty$ fast enough for the scattering method to work. Hence [13] applies, a Riemann-Hilbert factorization problem is possible, and explicit asymptotics (long time [13], long space, or even semiclassical [16] [14]) are available. Still, these formulae are not as effective as the formulae for the Cauchy problem. The reason is that in general the Dirichlet to Neumann map is very implicit. So some functions appearing in the asymptotic formulae involve scattering data related to the Neumann boundary data; these cannot be effectively computed. [2]

Our main result concerning the defocusing case is the following, see [3].

Theorem 2.1. *Let q be the unique global classical solution $q \in C^1(L^2) \cap C^0(H^2)$ of the initial-value problem for defocusing NLS, with Dirichlet data $Q \in C^2$ and $Q(0) = q_0(0)$.*

Assume that $q_0 \in H^1(0,\infty) \cap L^4(0,\infty)$ and $xq_0 \in L^2(0,\infty)$.

If $q(0,t)$, $q_t(0,t)$ have a sufficiently fast decay as $t \to \infty$, that is $\mathcal{O}(t^{-\alpha})$ and $\mathcal{O}(t^{-\beta})$, for $\alpha > 3/2$ and $\beta > 5/2$ respectively, then

$$\int_0^\infty |q_x(0,t)| dt < \infty.$$

Furthermore, if the Dirichlet data belong in the Schwartz class, then the Neumann data also belong in the Schwartz class.

As stressed above, this implies that a Riemann-Hilbert factorization problem is possible, and explicit asymptotics (long time [13], long space, or even semiclassical [16]) are available. [3] In the next section we present the main long time asymptotics formula for the defocusing case.

We also have a result for the focusing case. Here, we have to assume that some data are small.

Let q be the unique global classical solution $q \in C^1(L^2) \cap C^0(H^2)$ of the initial-value problem for focusing NLS, with Dirichlet data $Q \in C^2$ and $Q(0) = 0$. Assume for simplicity that the initial data is zero.

Also, let

$$\int_0^\infty |Q(t)|^2 dt,$$

be sufficiently small. If as $t \to \infty$

$$Q(t) = \mathcal{O}(t^{-5/2-\varepsilon}), \quad Q_t(t) = \mathcal{O}(t^{-5/2-\varepsilon}), \quad Q_{tt}(t) = \mathcal{O}(t^{-1/2-\varepsilon}),$$

for some small $\varepsilon > 0$, then there exists $c > 0$ independent of t such that for t large

$$\int_0^\infty |q_x(0,t)| dt < \infty. \tag{2.3}$$

Furthermore, if the Dirichlet data belong in the Schwartz class, then the Neumann data also belong in the Schwartz class.

[2]In more complicated problems the dependence can be very unstable ([17] for periodic Dirichlet data) or even chaotic (see the section on Sine-Gordon below).

[3]This is actually what we prefer to call "integrability". This presupposes the existence of Lax pairs AND a valid inverse theory!

3. Long time asymptotics

From the Riemann-Hilbert formulation one can derive precise long time asymptotics. For defocusing NLS this was first done in [6]. Their calculation was for the initial value problem. However, since the Riemann-Hilbert problem for the initial-boundary value problem is actually very similar, the same computation gives rise to the following long time asymptotics, as cited in [13].

$$q(x,t) = \frac{a(\frac{-x}{4t})}{t^{1/2}} e^{ix^2/4t+2i(a(\frac{-x}{4t}))^2 logt + i\phi(\frac{-x}{4t})}(1+O(t^{-1/2})), \tag{3.1}$$

where the functions a, ϕ are given by simple explicit formulae depending on the scattering data corresponding to the Dirichlet and Neumann data. Only the Dirichlet data are given (for a well-posed problem) and the Neumann data are only implicitly determined from the initial and Dirichlet data.

These asymptotics are uniformly valid in any closed linear sector (with half-lines as boundaries) that lies entirely in the open first quadrant $x, t > 0$. The same expression gives the long range asymptotics as $x \to \infty$[4].

A very similar formula also holds in the focusing case if no solitons are present (which is true for zero initial data and small Dirichlet data).

Near the boundary $x = 0$ the analysis is more delicate and depends on the details of the behavior of the Dirichlet data as $t \to \infty$. A careful demonstration has been presented by Lenells for the derivative NLS equation. If the Dirichlet data are unbounded the existing uniform method is not applicable.

4. Sine-Gordon with Robin condition

Following [1] we consider the equation

$$u_{tt} - u_{xx} + sinu = 0 \tag{4.1}$$

in the half plane $x < 0$ with a homogeneous Robin condition

$$u_x + 2ku = 0 \tag{4.2}$$

at $x = 0$. Here k is a given real constant. Note that at $k = 0$ we recover the Neumann condition and at infinity we recover the Dirichlet condition at $x = 0$.

Consider initial data of one-antikink form

$$u(x,0) = 4arctan(e^{-\gamma(v_0)(x-x_0)}), u_t(x,0) = \frac{-4v_0\gamma(v_0)}{1+exp(-2\gamma(v_0)(x-x_0))} exp(-\gamma(v_0)(x-x_0)), \tag{4.3}$$

where $\gamma(v_0) = (1-v_0^2)^{-1/2}$, $v_0 > 0$ being the intial velocity and $x_0 < 0$ the initial antikink center. [5] Careful numerical experiments in [1] (Figures 8 and 14 in particular) consider the boundary-initial value problem and focus on the recovery of the "Dirichlet data" boundary value $u(0,t)$. There are very careful plots $u(0,t)$ in terms of v_0 and k with the choice $x_0 = -30$ (that is away enough from the boundary). The Dirichlet data are necessary for the application of the unified transform even though they are not part of the conditions defining the well-posed problem; they are only implicitly

[4] with small error as $x \to \infty$

[5] This is expected (but never actually rigorously proven as far as we know) to be a uniquely solvable initial-boundary value problem. Of course the numerics of [1] support this fact.

defined by the solution itself. So the understanding of their behavior is crucial for the applicability of the unified transform.

The result of the "reflection" at the boundary $x = 0$ has to consist of breathers, kinks and antikinks (and some decaying "background" term). The intuitive non-rigorous reason for this is that away from the boundary one expects the effect of the initial data to be dominant, because of the expected finite propagation speed (up to smaller error terms). [6] But it is known that any initial data will give rise to a set of kinks, antikinks and breathers[7].

The most striking behavior of $u(0,t)$ is observed for values of the real constant k between 0.05 and 0.07 and v_0 between 0.875 and 0.9 and for large times $t_f = x_0/v_0 + 1000$. The authors observe the existence of breathers and possibly an antikink. Clearly chaotic phenomena occur: very slight changes in the parameters can affect (in a chaotic way) the production (or not) of a reflected antikink!

Furthermore, the function $u(0,t)$ is unbounded for large time.

This means that the map that takes the initial data $u(x,0), u_t(x,0)$ to the "Dirichlet data" $u(0,t)$ depends in a chaotic way on k and v_0. This fact is very interesting (see [17] for a similar result for the periodic NLS problem) but on its own does not exclude the possibility of applying the unified transform method. What *does* render the method inapplicable is the unboundedness of $u(0,t)$. So we have two observations here: chaoticity *and* inapplicability of the unified method. The problem is non-integrable for two reasons!

5. Comparison with the perturbed NLS on the real line

In [8] the authors consider the initial value problem for the defocusing NLS with an extra perturbation term $\epsilon|u|^l u,\ l > 2, \epsilon > 0$ and initial data decaying at infinity. What they discover is that for small ϵ the problem is still integrable! In particular they derive long term asymptotics similar to the unperturbed case.

Now, in a sense the initial-boundary value problem is a forced perturbation of the initial value problem. So it makes sense to compare the results of [8] with our results in [2] and [3]. But it is also interesting that our "forcing" term is not small in the defocusing case; it has to be small only in the focusing case.

Also, no chaotic phenomena are known in the fully non-integrable case of the perturbed NLS on the real line with large positive ϵ. So it seems that the initial-boundary value problem describes a richer set of phenomena!

6. Conclusion. What next?

Initial-boundary value problems for equations that admit a Lax Pair formulation and for which purely initial value problems give rise to completely integrable systems can still be completely integrable and tractable via techniques like Inverse Scattering and Riemann-Hilbert deformation. But there are also initial-boundary value problems for even some of the simplest Lax Pair equations where reasonable seeming data can give rise to chaotic behavior.

Can one understand better when and why this happens? Can one give comprehensive sets of boundary conditions that lead to integrability and chaos respectively?

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[6] Of course the finiteness of the propagation speed can only be proved for intial value problems, using the Riemann-Hilbert formulation. It may only be proved for initial-boundary value problems if one knows already that the unified transform is applicable.

[7] this is the "soliton resolution" for Sine-Gordon

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Figure 8 from ref.1

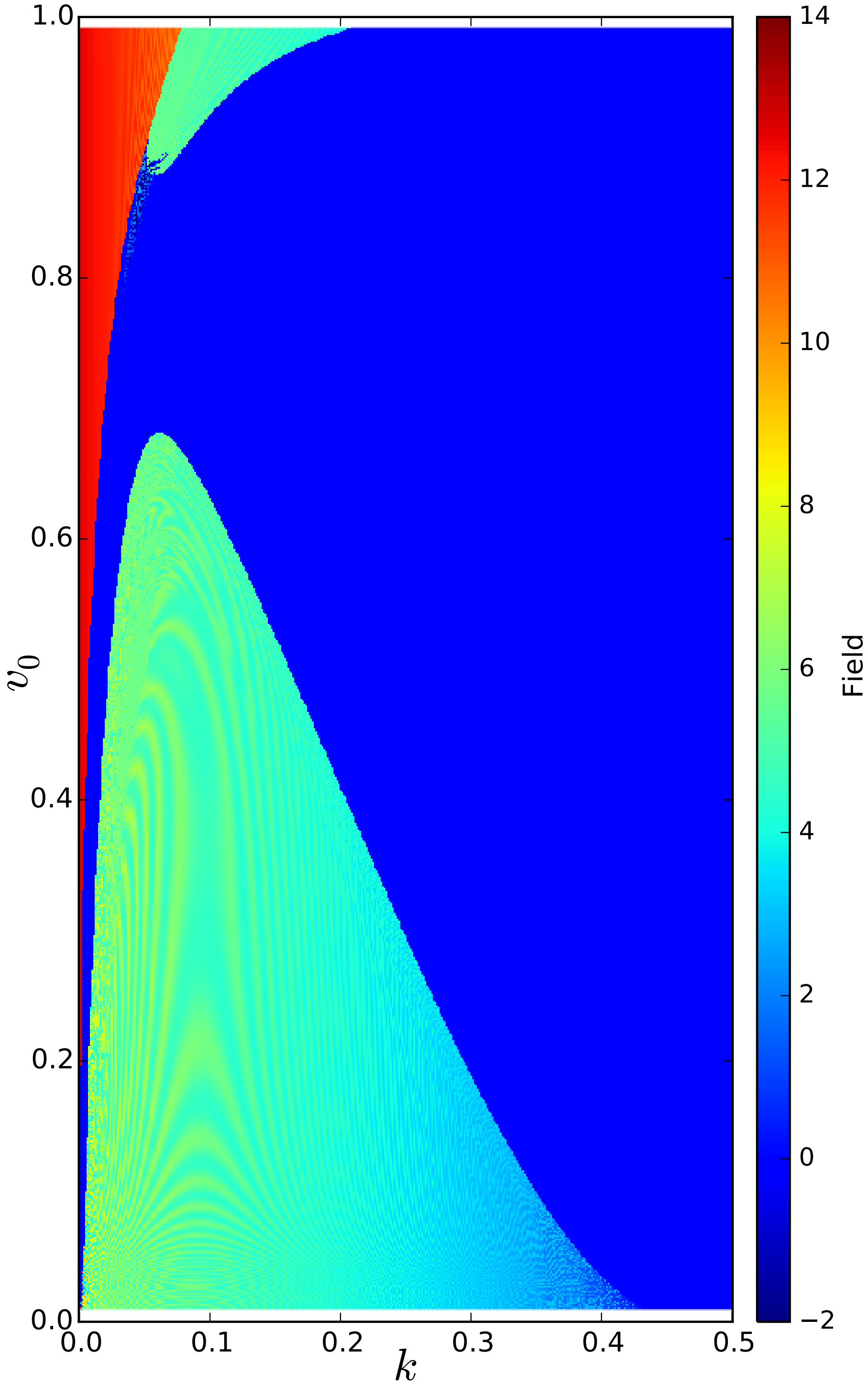

Figure 8: A snapshot of the field values at $x = 0$, $t = |x_0|/v_0 + 1000$ for the scattering of an initial antikink with velocity v_0, position x_0 and boundary parameter k. Fig. 14 below shows a zoomed-in view of the complicated structure near to $k = 0.06$, $v_0 = 0.89$.

Figure 14 from ref.1

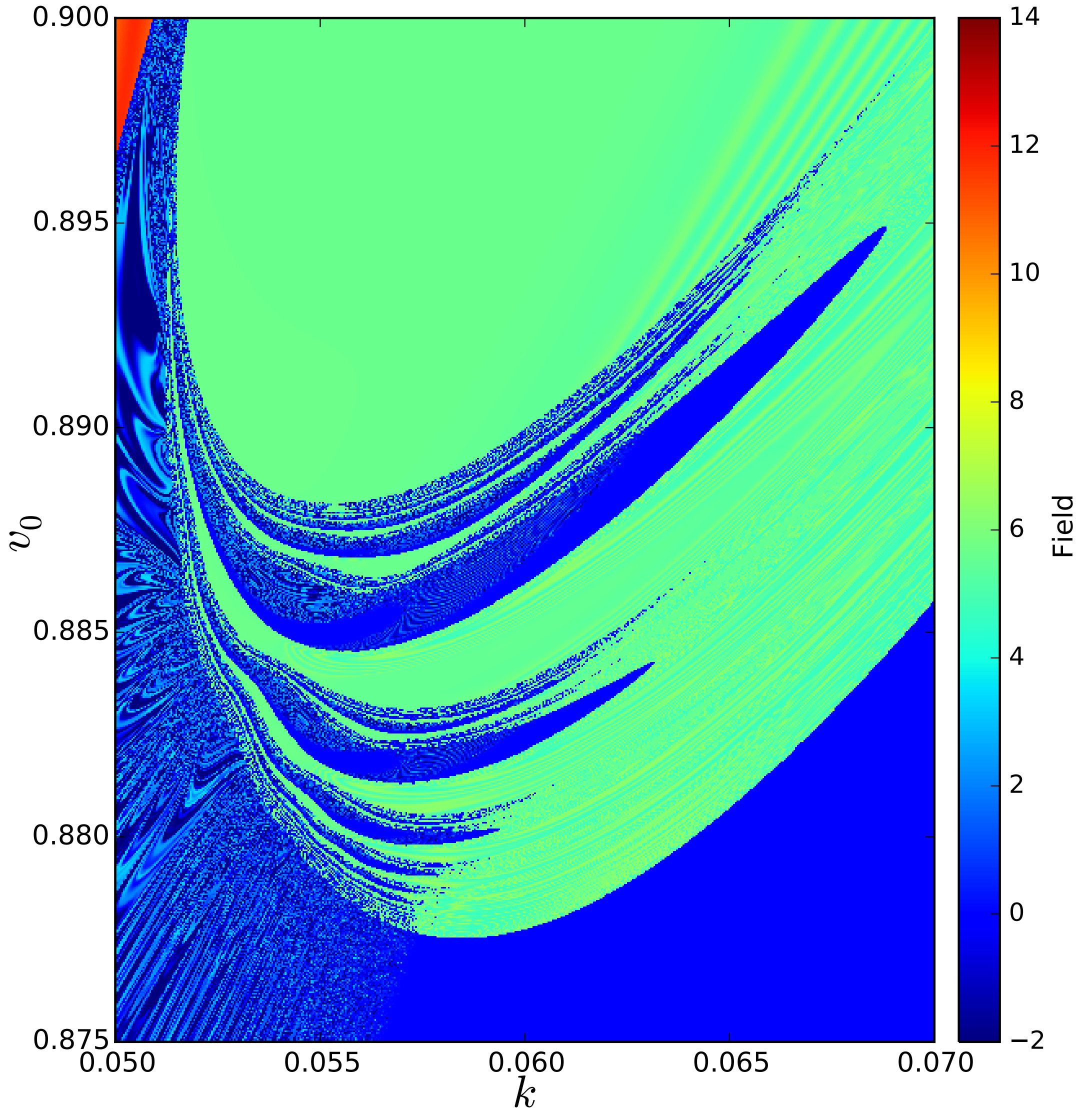

Figure 14: A zoomed-in plot of the shaded area in Fig. 9a, showing the value of the field at $x = 0$, $t = t_f = |x_0|/v_0 + 1000$ for an initial antikink with velocity v_0, position x_0, and boundary parameter k. The dark blue bands, where $u(0, t_f)$ is near zero, correspond to an antikink being emitted, while in the light green areas, where $u(0, t_f)$ is near 2π, only breathers are emitted. In between these areas are indeterminate regions where a very slight change in the initial parameters can cause an antikink to be produced or not. The oscillations in the boundary value of the field on the left of the plot are due to a breather becoming trapped at the boundary, only decaying very slowly there, in contrast to behaviour on the bottom right where this breather is able to escape and the field relaxes to zero much more quickly. The line separating these two regions, running from approximately $k = 0.0565$, $v_0 = 0.875$ to $k = 0.0574$, $v_0 = 0.8776$, is the top portion of the boundary between regions V_a and IV in Fig. 9a.