LONG TIME BEHAVIOR FOR THE FOCUSING NONLINEAR SCHROEDINGER EQUATION WITH REAL SPECTRAL SINGULARITIES

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ABSTRACT / SUMMARY

We consider the effect of real spectral singularities on the long time behavior of the solutions of the focusing Nonlinear Schrödinger equation. We find that for each spectral singularity $\lambda' \in \mathbb{R}$, such effect is limited to the region of the (x,t)-plane in which λ' is close to the point of stationary phase $\lambda_0 = \frac{-x}{4t}$ (the phase here being defined in a standard way by, say, the evolution of the Jost functions). As we approach that region, the solution performs decaying oscillations; the order of decay is $O((\frac{logt}{t})^{1/2})$. Inside that region the solution is asymptotically identified as a self-similar solution of NLS expressible in terms of a Painlevé 4 transcedent. We prove our result by using the Riemann-Hilbert factorization formulation of the inverse scattering problem. We recover our asymptotics by transforming our problem to one which is equivalent for large time. Around the shock front the limiting "model" Riemann-Hilbert problem can be interpreted as the one corresponding to the genus 0 algebro-geometric solution of the equation. Inside the shock region, the Riemann-Hilbert problem is related to the Painlevé 4 equaton.

1.INTRODUCTION

We consider the nonlinear Schrödinger equation (focusing case)

$$iq_t + q_{xx} + 2q|q|^2 = 0 \tag{1.1}$$

under initial data

$$q(x,0) = q_0(x), \tag{1.2}$$

belonging in the Schwartz class.

As is well known (see [NMPZ], [FT]), the problem (1.1)-(1.2) can be integrated through the method of inverse scattering. We will here present some of the results we will need without proof.

The associated linear system is

$$\psi_x = \begin{pmatrix} i\lambda & iq(x) \\ i\bar{q}(x) & -i\lambda \end{pmatrix} \psi, \tag{1.3}$$

where the bar denotes conjugation. Jost functions $\psi_1, \psi_2, \phi_1, \phi_2$ are defined on the real line as (column vector) solutions of (1.3) satisfying the asymptotic conditions

$$\begin{aligned} \psi_1(x,\lambda) &\sim \begin{pmatrix} e^{i\lambda x} \\ 0 \end{pmatrix}, \ as\ x \to +\infty, \\ \psi_2(x,\lambda) &\sim \begin{pmatrix} 0 \\ e^{-i\lambda x} \end{pmatrix}, \ as\ x \to +\infty, \\ \phi_1(x,\lambda) &\sim \begin{pmatrix} e^{i\lambda x} \\ 0 \end{pmatrix}, \ as\ x \to -\infty, \\ \phi_2(x,\lambda) &\sim \begin{pmatrix} 0 \\ e^{-i\lambda x} \end{pmatrix}, \ as\ x \to -\infty. \end{aligned} \tag{1.4}$$

Furthermore, ψ_1 and ϕ_2 can be meromorphically extended to the upper half-plane, while ψ_2 and ϕ_1 can be meromorphically extended to the lower half-plane. Indeed, one has

$$\begin{aligned} &\begin{pmatrix} \psi_1(x,\lambda)e^{-i\lambda x} & \phi_2(x,\lambda)e^{i\lambda x} \end{pmatrix} \to I, \ as\ \lambda \to \infty, \ Im\lambda \geq 0, \\ &\begin{pmatrix} \phi_1(x,\lambda)e^{-i\lambda x} & \psi_2(x,\lambda)e^{i\lambda x} \end{pmatrix} \to I, \ as\ \lambda \to \infty, \ Im\lambda < 0. \end{aligned} \tag{1.5}$$

I is here the identity matrix.

We point out the symmetry

$$\bar{\psi}(x,\bar{\lambda}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \psi(x,\lambda), \tag{1.6}$$

where $\psi = \begin{pmatrix} \psi_1 & \psi_2 \end{pmatrix}$; the same symmetry is satisfied by $\phi = \begin{pmatrix} \phi_1 & \phi_2 \end{pmatrix}$.

On the real line, ϕ and ψ are solutions of system (1.3). Hence there exist 'scattering coefficients' $a(\lambda), b(\lambda)$ such that

$$\begin{aligned} \phi_2(x,\lambda) &= a(\lambda)\psi_2(x,\lambda) + b(\lambda)\psi_1(x,\lambda) \\ \phi_1(x,\lambda) &= \bar{a}(\lambda)\psi_1(x,\lambda) - \bar{b}(\lambda)\psi_2(x,\lambda). \end{aligned} \tag{1.7}$$

Although $\bar{a}$ and $\bar{b}$ are a priori independent of a and b, one can see from the symmetries above that they are after all their complex conjugates. Furthermore one can show

$$|a(\lambda)|^2 + |b(\lambda)|^2 = 1. \tag{1.8}$$

The time evolution of the scattering coefficients is given by

$$\begin{aligned} a(\lambda, t) &= a(\lambda, 0), \\ b(\lambda, t) &= b(\lambda, 0)e^{4i\lambda^2 t}. \end{aligned} \tag{1.9}$$

It turns out that $a(\lambda)$ can be analytically extended to the upper half-plane, while $\bar{a}(\lambda)$ can be analytically extended to a function $a^*(\lambda) = \bar{a}(\bar{\lambda})$ in the lower half-plane. In general, this is not also true for b. Generically, a has only a finite number of zeros in the upper half-plane and no zero at all on the real line (cf. [BC]). However, there are cases where this is not true. One can have, for example, an infinity of non-real zeros with a limit point on the real line (see [Z] for an example).

It is well-known that in the generic case, non-real zeros of a correspond to solitons for long times (see [FT] for example). Our goal is to study the effect of *real* zeros of a. We will only consider the case of finitely many zeros of a. For simplicity we will state and prove our results in the case where no non-real zeros of a are present and only one real simple zero exists. However, it should eventually become clear that this is only a superficial constraint, and indeed we will indicate at the end of this work what happens in the more general case (of finitely many zeros).

A simple example of initial data producing exactly one real spectral singularity is the following ([CK]). Let $X > 0$ and

$$\begin{aligned} u_0(x) &= \frac{\pi}{2X},\ 0 < x < X, \\ &= 0,\ \textit{otherwise}. \end{aligned} \tag{1.10}$$

Then $a(\lambda)$ has exactly one (simple) zero at $\lambda = 0$ and $a'(0) = \frac{-2iX}{\pi}$. Even though u_0 is discontinuous, the standard direct and inverse scattering theory is applicable.

We define the 2×2-matrix-valued function Ψ as follows. Let

$$\begin{aligned} \Psi &= \begin{pmatrix} \psi_1 e^{-i\lambda x} & \frac{\phi_2 e^{i\lambda x}}{a(\lambda)} \end{pmatrix},\ for\ Im\lambda > 0, \\ &= \begin{pmatrix} \frac{\phi_1 e^{-i\lambda x}}{a^*(\lambda)} & \psi_2 e^{i\lambda x} \end{pmatrix},\ for\ Im\lambda < 0. \end{aligned} \tag{1.11}$$

Letting Ψ_+ and Ψ_- denote the limits of Ψ on the real line from above and below respectively, we have (after a few calculations)

$$\Psi_+(z) = \Psi_-(z) \begin{pmatrix} 1 & r(\lambda)e^{2i\lambda x + 4i\lambda^2 t} \\ \bar{r}(\lambda)e^{-2i\lambda x - 4i\lambda^2 t} & 1 + |r(\lambda)|^2 \end{pmatrix},\ Im\lambda = 0, \tag{1.12}$$

where $r(\lambda) = \frac{b(\lambda, t=0)}{a(\lambda, t=0)}$. Note that the jump matrix has determinant 1.

Finally, one can prove that $a(\lambda) = 1 + O(1/\lambda)$ as $\lambda \to \infty, Im\lambda \geq 0$. Hence

$$\Psi(\lambda = \infty) = I. \tag{1.13}$$

We thus end up with a Riemann-Hilbert factorization problem: Ψ is a matrix function, analytic in the complement of the real line, satisfying the jump condition (1.12) and the asymptotic condition (1.13).

Note that in the generic case of finitely many zeros off the real line and no real zero, Ψ is meromorphic and neither Ψ_+ or Ψ_- nor the jump matrix have any singularities. In the case we are interested in, however, both $\Psi_\pm$ and the jump matrix have singularities exactly at the zeros of $a(\lambda)$.

Conversely, the solution of the Riemann-Hilbert problem enables us to recover $q(x,t)$. Indeed (see e.g [NMPZ])

$$q(x,t) = -2lim_{\lambda\to\infty}\lambda\Psi_{12}. \tag{1.14}$$

Thus, the initial value problem (1.1)-(1.2) is reduced to the above Riemann-Hilbert on which we focus from now on.

The interest of the problem treated in this paper is twofold. On the first hand, as we will see later, the physical effect of the spectral singularity is a 'collisionless shock' type phenomenon; we thus have an interesting connection with the theory of 'dispersive shocks' for nonlinear wave equations (cf.also [DVZ], [AS]). On the other hand, ours is a first step towards the completion of the solution of the problem of long-time asymptotics of integrable equations in the case of Schwartz data, in the following sense: although the related direct and inverse scattering problems are now completely solved (see [Z], [DZ2]) for the most general cases of Lax operators and even in the non-generic exceptional cases of data for which there are either infinitely many spectral singularities off the associated Riemann-Hilbert contour or there are (possibly infinitely many) spectral singularities on the contour, the long-time asymptotics problem is still far from having a complete solution, even in cases as simple as the NLS equation. In fact, the present paper is the first of a series; in a recent article (see [K]) we treat the infinite-soliton case, while a treatment of the case of infinitely many real (i.e. on the Riemann-Hilbert contour) spectral singularities for the NLS equation will appear soon.

We now state our main result, to be proved in section 3.

THEOREM 1.1. Let q be the solution of (1.1) with initial data in the Schwartz class, and such that a (simple) spectral singularity exists at $\lambda = 0$ and nowhere else. Let $\lambda_0 = \frac{-x}{4t}$, $\tau = t\lambda_0^2$ and M be a given positive constant. Then the leading order asymptotics of q, as $t \to \infty$, is as follows.

In region I: $x < 0, \lambda_0 < -M, q$ is given by formulae (2.5) and (2.6).
In region II: $x < 0, \lambda_0 \to 0, \tau \to \infty$, q is given by formula (3.18).
In region III: $\lambda_0 \to 0, \tau < M$, q is given by (3.19).
In region IV: $x > 0, \lambda_0 \to 0, \tau \to \infty, q$ is given by (3.18).
In region V: $x > 0, \lambda_0 > M, q$ is given by (2.5)-(2.6).

The plan of the rest of the paper is as follows. In section 2, we solve the Riemann-Hilbert factorization problem in the case where the zeroes of a, are away from the stationary point $\lambda_0 = \frac{-x}{4t}$ of the phase $\Theta = \lambda x + 2\lambda^2 t$. Indeed, we show that no spectral singularity has any effect at all in the long time behavior of $q(x,t)$. In section 3, we consider the more interesting case where $\lambda_0 - \lambda' \to 0$ with time, for some real singularity λ'. In section 4, we discuss a generalization of our results.

The problem of a real spectral singularity was first considered by Ablowitz and Segur. In [AS] they study the collisionless shock phenomenon for the KdV equation with decaying initial data and they dedicate a small section to the focusing nonlinear Schroedinger equation, where they address the problem of a real spectral singularity by considering non-real ones, say κ_0, and taking the limit as $Im\kappa_0 \to 0$. Their treatment is heuristic and non-rigorous. It is satisfactory however that their estimate for the decay of the solution on the 'shock' front agrees with ours. For the sake of the reader, we present the result of Ablowitz and Segur below.

RESULT of [AS]. In the shock front region (corresponding to our regions II and IV), the solution of (1.1)-(1.2) has the following asymptotic expression

$$q(x,t) = t^{-1/2}R(x/t,t)e^{it\theta(x/t,t)}$$

where

$$\begin{aligned} \theta &\sim \frac{1}{4}\frac{x^2}{t^2} + 2\frac{logt}{t} f^2(\frac{x}{t}) + O(\frac{1}{t}), \\ R &\sim f(\frac{x}{t}) + 4f(\frac{x}{t})(3[f'(\frac{x}{t})]^2 + f(\frac{x}{t})f''(\frac{x}{t}))\frac{logt}{t}, \\ f^2(-4k) &\sim \frac{1}{4\pi}(logt - loglogt - 2logm), \\ &where\ k = \lambda' + m(\frac{logt}{t})^{1/2}. \end{aligned} \tag{1.15}$$

In particular (cf. (4.10) of[AS])

$$|q| \sim \frac{1}{2\pi^{1/2}}(\frac{logt}{t})^{1/2}.$$

Note that this formula agrees with our (3.18).

The method of this paper follows the spirit of the work of Deift and Zhou (see [DZ], [DVZ]), who invented a new (and for the first time rigorous) method for recovering long-time asymptotics of integrable 'soliton' equations, by using the fact that the inverse scattering problem for such equations can be stated as a Riemann-Hilbert factorization problem. We also make use of results of Deift, Its and Zhou for the defocusing nonlinear Schroedinger equation ([DIZ]). We note that the present work is the first in this spirit that deals with problems for which the jump matrix blows up at a point.

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2. AWAY FROM A SINGULARITY

As mentioned in the introduction, Deift, Its and Zhou have analyzed the long time behavior of the defocusing nonlinear Schroedinger equation. In that case, the Riemann-Hilbert problem agrees (modulo a minus sign) with (1.12)-(1.13) except that no spectral singularities exist at all (real or non-real). The question is how is the analysis of the problem affected when Ψ and the jump matrix in (1.12) have a singularity. In this section, we show that when for all singularities λ', $\lambda_0-\lambda' = O(1)$, they have no effect at all. We will only restrict ourselves to the case $\lambda_0 > \lambda'$, since obviously the case $\lambda_0 < \lambda'$ is similar (and easier).

1. We begin by considering an auxiliary scalar factorization problem. Let d be a function analytic in $\mathbb{C} \setminus (-\infty, \lambda_0]$ such that

$$\begin{aligned} d_+(\lambda) = d_-(\lambda)(1+|r(\lambda)|^2) \qquad &\text{for } -\infty < \lambda \leq \lambda_0, \\ d_+(\lambda) = d_-(\lambda) \qquad &\text{for } \lambda > \lambda_0 \\ d \to 1 \quad &\text{as } \lambda \to \infty. \end{aligned} \tag{2.1}$$

When there is no spectral singularity at 0, it can be easily checked that the solution of the above scalar problem is given by

$$d(\lambda) = exp(\frac{1}{2\pi i}\int_{-\infty}^{\lambda_0} \frac{log(1+|r(s)|^2)}{s-\lambda} ds).$$

PROPOSITION 2.1. Problem (2.1) has a unique solution which has no zeros and its only singularities are at zeros of a. More precisely, near a zero of a $d_+(\lambda)a(\lambda)$, $d(\lambda)a(\lambda)$, $\frac{d_-(\lambda)}{\bar{a}(\lambda)}$ and $\frac{d(\lambda)}{a^*(\lambda)}$ are bounded above and below.

PROOF: First note that

$$1+|r(\lambda)|^2 = \frac{1}{|a(\lambda)|^2} = \frac{1}{a(\lambda)a^*(\lambda)}, \; for \; \lambda \in \mathbb{R}.$$

Also recall that a is analytic in the upper half-plane, a^* is analytic in the lower half-plane and $a(\lambda) = 1 + O(\frac{1}{\lambda})$ as $\lambda \to \infty, \; Im \; \lambda \geq 0$.

Consider the contour depicted in figure 2.1 (the exact choice of lines l_2 and l_3 is irrelevant provided a, a^* have no zeros in regions A_2, A_3 or on lines l_2, l_3) and define

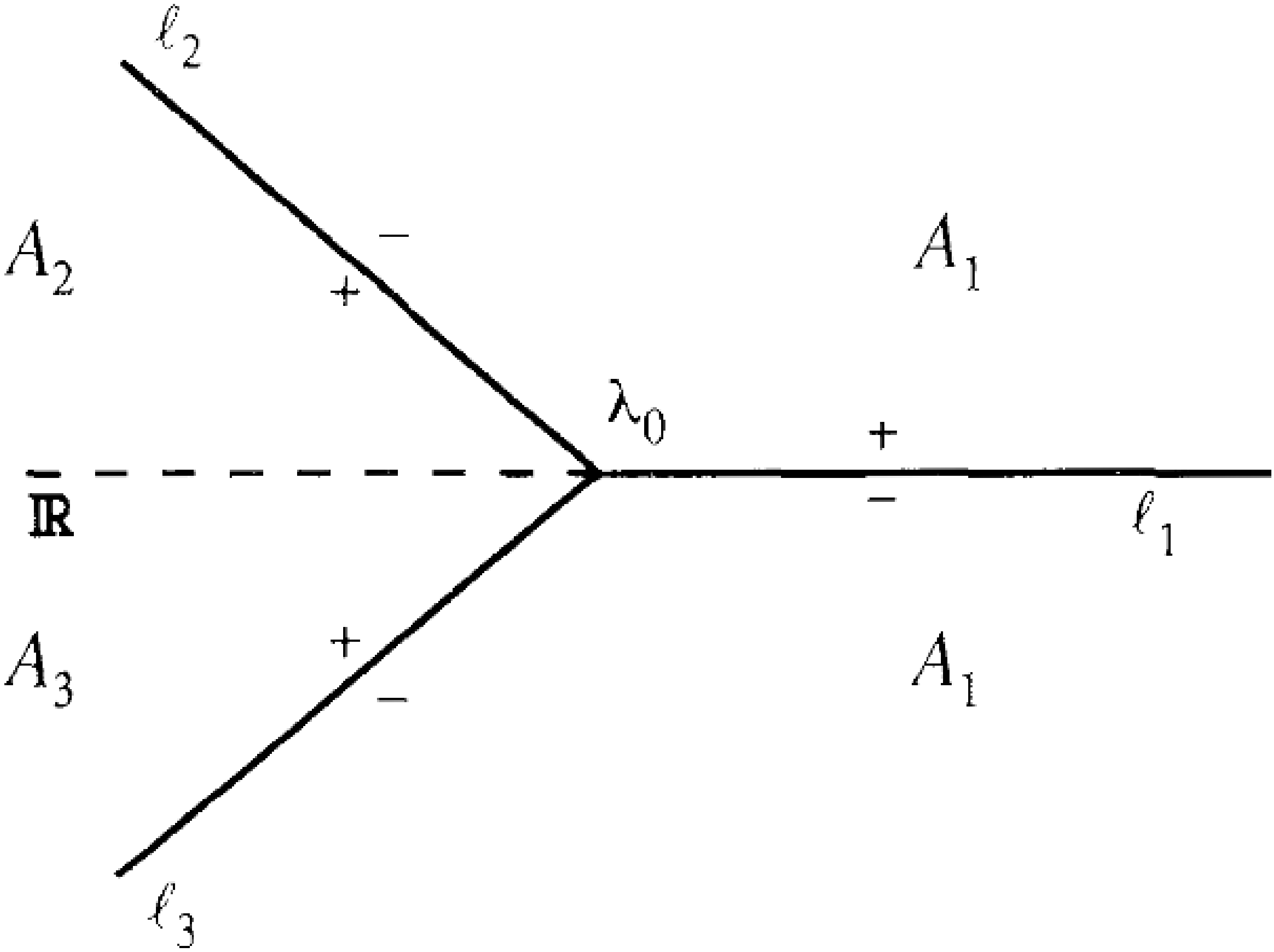
$$\delta(\lambda) = d(\lambda), \; \lambda \in A_1,$$
$$\delta(\lambda) = d(\lambda)a(\lambda), \; \lambda \in A_2,$$
$$\delta(\lambda) = d(\lambda)(a^*(\lambda))^{-1}, \; \lambda \in A_3.$$

δ satisfies the following scalar problem: it is analytic in $\mathbb{C} \setminus (l_2 \cup l_3)$ and,

$$\delta_+ = \delta_- a, \; on \; l_2,$$
$$\delta_+ = \delta_-(a^*)^{-1}, \; on \; l_3.$$

As a has neither zeros nor poles on $l_2 \cup l_3$ we see that this scalar factorization problem has a unique solution with neither zeros nor poles; even though a has a discontinuity at λ_0, δ is bounded near λ_0 (see e.g. [G], p.448). The result follows.

Figure 2.1



ℓ_2
A_2
$-$
$+$
A_1
λ_0
$+$
$-$
$\mathbb{R}$
ℓ_1
$+$
A_3
$-$
A_1
ℓ_3

It can be easily seen now that near λ_0 the behavior of d is

$$d(\lambda) \sim (\lambda-\lambda_0)^{-i\nu} exp(\frac{1}{2\pi i}\int_{-\infty}^{\lambda_0} |log(s-\lambda)| dlog(1+|r(\lambda(s))|^2)$$

where $\nu = \frac{1}{2\pi} log(1+|r(\lambda_0)|^2) > 0$.

2. We will next provide an appropriate contour deformation (following [DIZ]) which will be guided by an analysis of the signature of the phase $\Theta = \lambda x + 2\lambda^2 t$ appearing in the exponents of (1.12) (see figure 2.2).

A fundamental fact is that the jump matrix of (1.12) admits the following factorizations (hence justifies our construction of d).

$$\begin{pmatrix} d & 0 \\ \bar{r} d e^{-2i\Theta} & d^{-1} \end{pmatrix} \begin{pmatrix} d^{-1} & r d^{-1} e^{2i\Theta} \\ 0 & d \end{pmatrix} \; for \; \lambda > \lambda_0$$

$$and$$

$$\begin{pmatrix} d_- & \frac{r d_-^{-1} e^{2i\Theta}}{1+|r|^2} \\ 0 & d_-^{-1} \end{pmatrix} \begin{pmatrix} d_+^{-1} & 0 \\ \frac{\bar{r} d_+ e^{-2i\Theta}}{1+|r|^2} & d_+ \end{pmatrix}, \; for \; \lambda < \lambda_0.$$

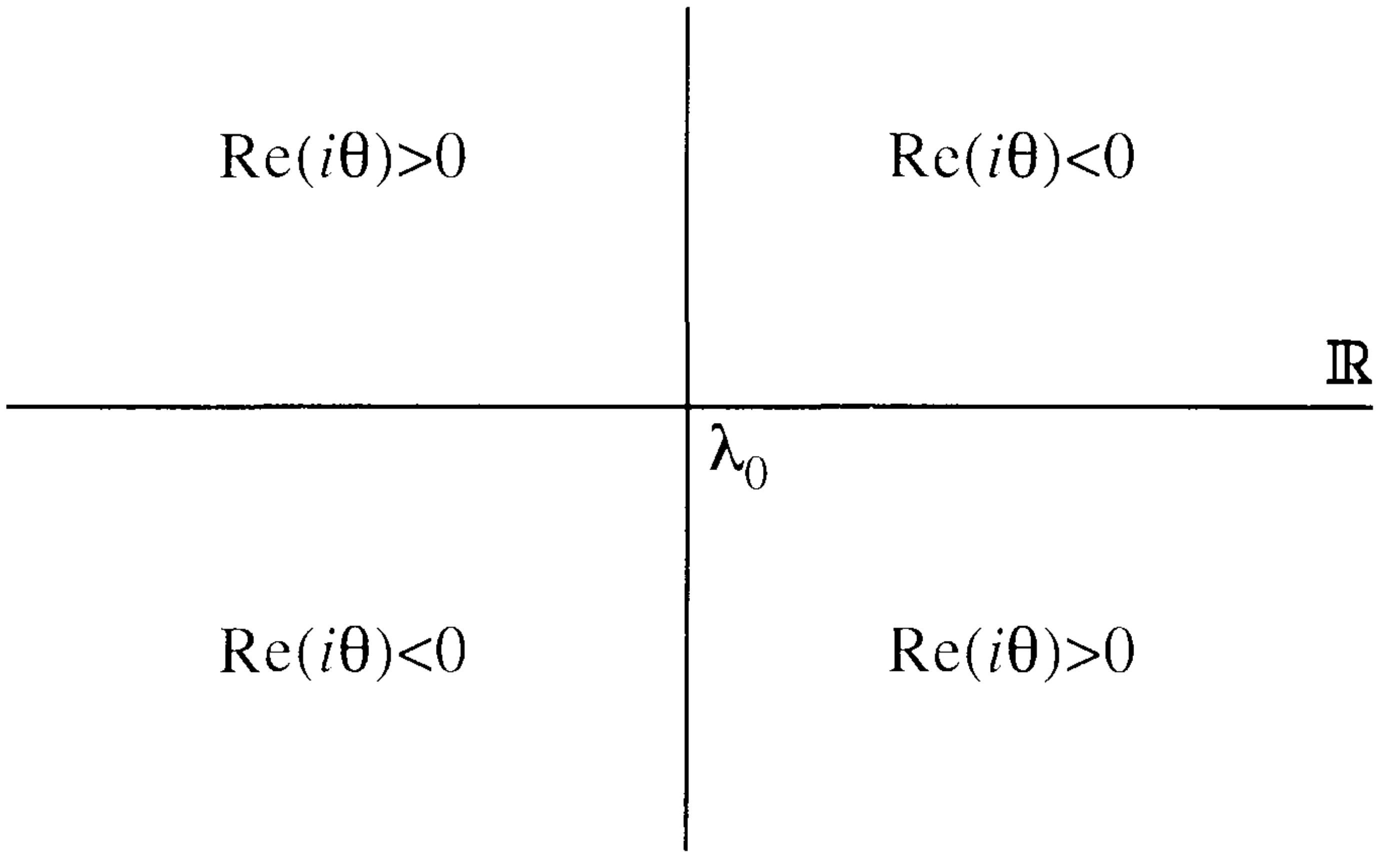
We deform our contour as in figure 2.3.

Guided by the above factorizations, we define

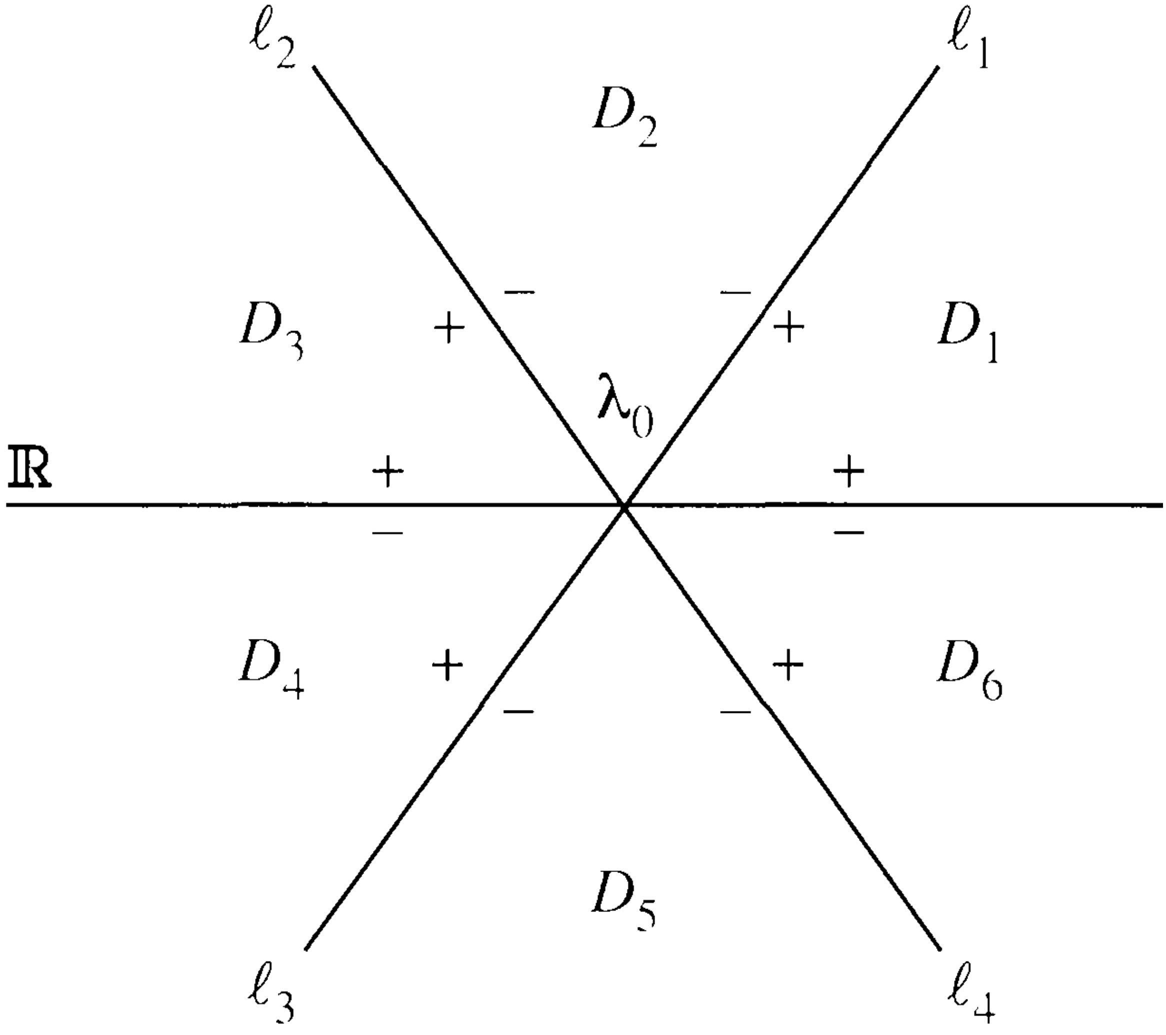
$$\begin{aligned}
\Psi^1 &= \Psi \begin{pmatrix} d & -r d^{-1} e^{2i\Theta} \\ 0 & d^{-1} \end{pmatrix}, \; \lambda \in D_1, \\
\Psi^1 &= \Psi \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix}, \; \lambda \in D_2, \\
\Psi^1 &= \Psi \begin{pmatrix} d & 0 \\ \frac{-\bar{r} d}{1+|r|^2} e^{-2i\Theta} & d^{-1} \end{pmatrix}, \; \lambda \in D_3, \\
\Psi^1 &= \Psi \begin{pmatrix} d & \frac{r d^{-1} e^{2i\Theta}}{1+|r|^2} \\ 0 & d^{-1} \end{pmatrix}, \; \lambda \in D_4, \\
\Psi^1 &= \Psi \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix}, \; \lambda \in D_5, \\
\Psi^1 &= \Psi \begin{pmatrix} d & 0 \\ \bar{r} d e^{-2i\Theta} & d^{-1} \end{pmatrix}, \; \lambda \in D_6.
\end{aligned} \tag{2.2}$$

REMARK. For such a deformation we need to assume that $b, \bar{b}$ can be analytically extended, at least in a small strip containing the real line (note that the actual choice of the curves l_j is not important as long as they are in the right quadrant). This would be indeed true under more restrictive data. However, such an assumption is not necessary. As shown in [DZ] (see also [DIZ]) b can be approximated by a rational function whose poles do not affect the analysis.

Figures 2.2 and 2.3



Re($i\theta$)>0
Re($i\theta$)<0
$\mathbb{R}$
λ_0
Re($i\theta$)<0
Re($i\theta$)>0



ℓ_2
ℓ_1
D_2
D_3
D_1
λ_0
$\mathbb{R}$
D_4
D_6
D_5
ℓ_3
ℓ_4

A straightforward calculation shows that there is no jump across the real line. We have

$$\Psi^1(\infty) = I. \tag{2.3}$$

The jump is

$$\Psi^1_+ = \Psi^1_- u^1_{x,t}, \ where$$

$$\begin{aligned} u^1_{x,t} &= \begin{pmatrix} 1 & -rd^{-2}e^{2i\Theta} \\ 0 & 1 \end{pmatrix}, \ on \ l_1, \\ &= \begin{pmatrix} 1 & 0 \\ \frac{-\bar{r}d^2e^{-2i\Theta}}{1+|r|^2} & 1 \end{pmatrix}, \ on \ l_2, \\ &= \begin{pmatrix} 1 & \frac{rd^{-2}e^{2i\Theta}}{1+|r|^2} \\ 0 & 1 \end{pmatrix}, \ on \ l_3, \\ &= \begin{pmatrix} 1 & 0 \\ \bar{r}d^2e^{-2i\Theta} & 1 \end{pmatrix} \ on \ l_4. \end{aligned} \tag{2.4}$$

3. The important observation is that this is a Riemann-Hilbert factorization problem without singularities at all. Indeed, the jump matrices have no poles as a has no zeros on $l_1 \cup l_2 \cup l_3 \cup l_4$. Also, Ψ_1 has no poles even at the points where $a(\lambda) = 0$. This follows easily from the scattering relations (1.7), the definition of Ψ (1.11), and proposition 2.1 which gives the behavior of $d_\pm$ near the zeros of a.

In other words, we end up with exactly the same problem as the one corresponding to initial data that produce no real spectral singularities at all. Hence, the analysis of Deift, Its and Zhou goes through completely unaltered. Note also that the solution of (1.1)-(1.2) is still recovered in the same way (see (1.14)) since the modifications above have no effect on Ψ up to order $\frac{1}{\lambda}$.

We will not provide the analysis of Deift, Its and Zhou in detail. We refer the reader to [DIZ] instead. We will only recall that as the jumps (2.4) are exponentially small away from the stationary phase point λ_0 the problem is reduced to one on a small cross near λ_0. The new problem can be solved explicitly (after some rescaling) in terms of parabolic cylinder functions. For the reader's convenience we provide the leading order asymptotics for q.

THEOREM 2.1. Let $M > 1, K > 0$ be fixed and assume that for any real zero of a, say λ', we have $\lambda - \lambda' \leq K$. We have

$$q(x,t) = t^{-1/2}\alpha(\lambda_0)e^{\frac{i|x|^2}{4t} - i\nu(\lambda_0)log(8t)} + E(x,t), \tag{2.5}$$

where, as $t \to \infty$,

$$E(x,t) = O(t^{-1}logt), \ for \ |\lambda_0| \leq M, \tag{2.6}$$

and for any j,

$$E(x,t) = O(|x|^{-j} + c_j(z_0)x^{-1}log|x|), \ for \ |\lambda_0| \geq M^{-1},$$

where

$$\nu(\lambda) = \frac{1}{2\pi}log(1 + |r(\lambda)|^2) > 0,$$

$$|\alpha(\lambda_0)|^2 = \frac{\nu(\lambda_0)}{2},$$

$$arg\alpha(\lambda_0) = arg\Gamma(i\nu(\lambda_0)) - argr(\lambda_0) + \frac{\pi}{4} + \frac{1}{\pi}\int_{-\infty}^{\lambda_0} log(\lambda_0 - \lambda)d(log(1 + |r(\lambda)|^2).$$

3. NEAR A SINGULARITY

In this section we consider the effect of a real spectral singularity which is close enough to the stationary phase point λ_0 . For simplicity, we will assume that there are no non-real singularities and that there is only one real simple singularity, i.e. a has a simple zero, say at zero (but see section 4 about these assumptions). We write $a(\lambda) = \lambda\, \tilde{a}(\lambda)$, with $\tilde{a}(\lambda) \neq 0$.

In this case, the analysis of [DIZ] breaks down, so the method of section 2 is no more useful. The behavior of d (the solution of problem (2.1)) is more complicated near λ_0; it is no more bounded there, and the relevant resolvent operators are also unbounded, so the standard method of [DIZ] cannot be applied directly. We will instead study this case by deforming the original problem in a different way.

1. Let

$$\tau = \frac{x^2}{16t} = t\lambda_0^2. \tag{3.1}$$

We consider the region defined by

$$\begin{aligned} \lambda_0 &\to 0, \\ \tau &\to \infty, \\ as\ t &\to \infty. \end{aligned} \tag{3.2}$$

Our first step will be to rescale appropriately so that the distance between the spectral singularity and the stationary phase point is O(1). We next introduce a contour deformation (different from section 2) which is still guided by the signature of the phase Θ. It now turns out that, in the region we are interested in, the factorization problem takes a very special shape. After a final conjugation involving an appropriate multi-valued function we end up with a problem on a vertical band, that can be solved in terms of the genus-0 algebro-geometric solution of equation (1.1).

REMARK. A comprehensive reference for scalar problems with singular jumps (like (2.1)) is the book by Gakhov ([G]).

2. Let $\Psi^{(1)}(\lambda) = \Psi(\lambda_0\lambda)$. Condition (1.13) becomes

$$\Psi_+^{(1)} = \Psi_-^{(1)} e^{i\tau(2\lambda^2-4\lambda)\sigma_3} \begin{pmatrix} 1 & r(\lambda_0\lambda) \\ \bar{r}(\lambda_0\lambda) & 1+|r(\lambda_0\lambda)|^2 \end{pmatrix} e^{-i\tau(2\lambda^2-4\lambda)\sigma_3}. \tag{3.3}$$

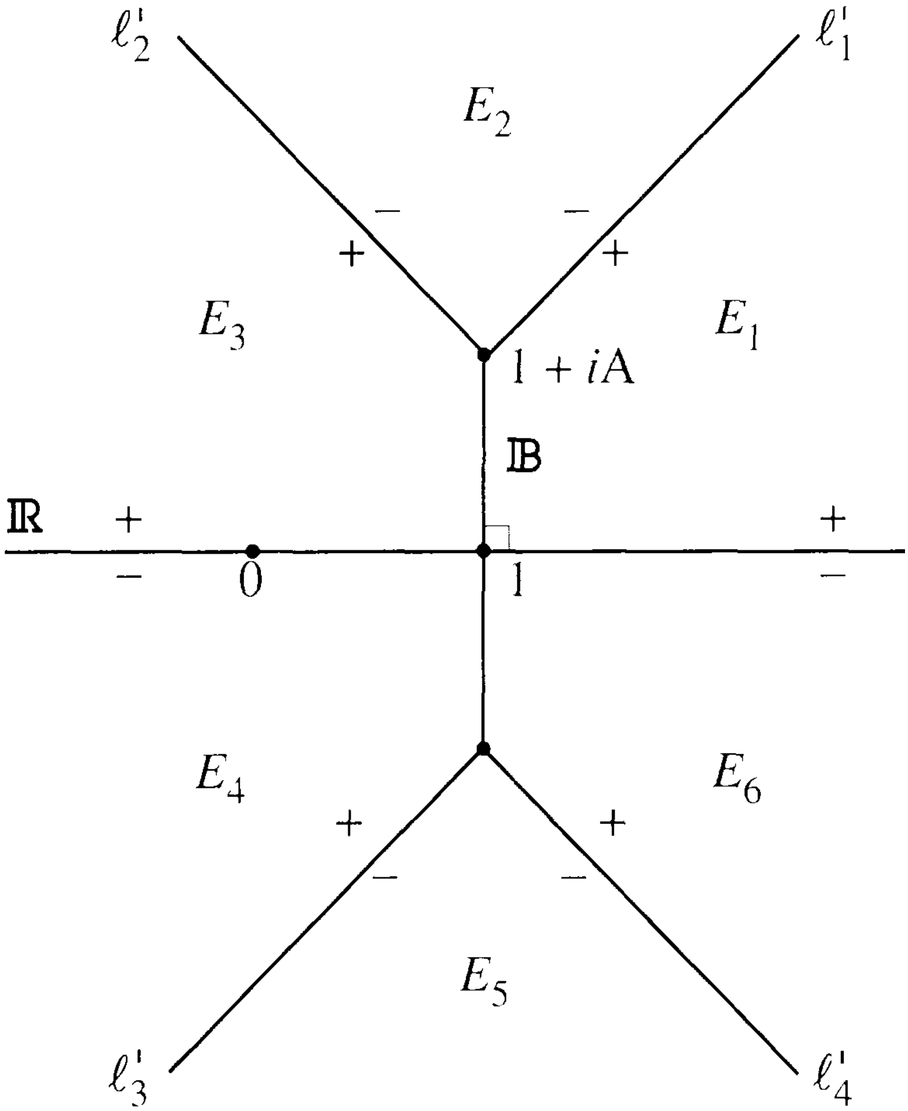
The rescaled phase is

$$\Phi = \tau(\lambda^2 - 2\lambda). \tag{3.4}$$

The singularity of the problem is now at $\lambda = 0$ while the stationary phase point is at $\lambda = 1$.

3. Let d be the solution of the scalar problem (2.1). We introduce a new contour Σ as shown in figure 3.1. We denote the vertical band joining points $1 + iA$ and $1 - iA$ by $\mathbb{B}$. A is a real positive parameter to be specified later.

Figure 3.1



ℓ_2'
ℓ_1'
E_2
$-$
$-$
$+$
$+$
E_3
E_1
$1 + iA$
$\mathbb{B}$
$\mathbb{R}$
$+$
$+$
$-$
0
1
$-$
E_4
E_6
$+$
$+$
$-$
$-$
E_5
ℓ_3'
ℓ_4'

Let

$$
\begin{aligned}
\Psi^{(2)} &= \Psi^{(1)} \begin{pmatrix} D & -RD^{-1}e^{2i\Phi} \\ 0 & D^{-1} \end{pmatrix}, \ \lambda \in E_1, \\
\Psi^{(2)} &= \Psi^{(1)} \begin{pmatrix} D & 0 \\ 0 & D^{-1} \end{pmatrix}, \ \lambda \in E_2, \\
\Psi^{(2)} &= \Psi^{(1)} \begin{pmatrix} D & 0 \\ \frac{-\bar{R}D}{1+|R|^2}e^{-2i\Phi} & D^{-1} \end{pmatrix}, \ \lambda \in E_3, \\
\Psi^{(2)} &= \Psi^{(1)} \begin{pmatrix} D & \frac{RD^{-1}e^{2i\Phi}}{1+|R|^2} \\ 0 & D^{-1} \end{pmatrix}, \ \lambda \in E_4, \\
\Psi^{(2)} &= \Psi^{(1)} \begin{pmatrix} D & 0 \\ 0 & D^{-1} \end{pmatrix}, \ \lambda \in E_5, \\
\Psi^{(2)} &= \Psi^{(1)} \begin{pmatrix} D & 0 \\ \bar{R}De^{-2i\Phi} & D^{-1} \end{pmatrix}, \ \lambda \in E_6.
\end{aligned} \tag{3.5}
$$

Here $R(\lambda) = r(\lambda_0\lambda)$ and $D(\lambda) = d(\lambda_0\lambda)$.

A straightforward calculation shows that there is no jump across the real line. We also have

$$\Psi^{(2)}(\infty) = I. \tag{3.6}$$

The jump conditions for $\Psi^{(2)}$ are

$$
\begin{aligned}
&\Psi^{(2)}_+ = \Psi^{(2)}_- u^{(2)}_{x,\tau}, \ where \\
u^{(2)}_{x,\tau} &= \begin{pmatrix} 1 & -RD^{-2}e^{2i\Phi} \\ 0 & 1 \end{pmatrix}, \ on \ l'_1, \\
&= \begin{pmatrix} 1 & 0 \\ \frac{-\bar{R}D^2e^{-2i\Phi}}{1+|R|^2} & 1 \end{pmatrix}, \ on \ l'_2, \\
&= \begin{pmatrix} 1 & \frac{RD^{-2}e^{2i\Phi}}{1+|R|^2} \\ 0 & 1 \end{pmatrix}, \ on \ l'_3, \\
&= \begin{pmatrix} 1 & 0 \\ \bar{R}D^2e^{-2i\Phi} & 1 \end{pmatrix}, \ on \ l'_4, \\
&= \begin{pmatrix} 1 & -RD^{-2}e^{2i\Phi} \\ \frac{\bar{R}D^2e^{-2i\Phi}}{1+|R|^2} & \frac{1}{1+|R|^2} \end{pmatrix}, \ on \ the \ top \ half \ of \ \mathbb{B}, \\
&= \begin{pmatrix} \frac{1}{1+|R|^2} & \frac{-RD^{-2}e^{2i\Phi}}{1+|R|^2} \\ \bar{R}D^2e^{-2i\Phi} & 1 \end{pmatrix}, \ on \ the \ bottom \ half \ of \ \mathbb{B}.
\end{aligned} \tag{3.7}
$$

Once more this is a Riemann-Hilbert factorization problem without singularities at all. The proof of this is the same as in section 2.

4. We observe that, as in section 2, because of the structure sign of Φ the jumps across $\Sigma \setminus \mathbb{B}$ are exponentially close to I as $\tau \to +\infty$. We thus end up with a Riemann-Hilbert factorization problem across $\mathbb{B}$.

THEOREM 3.1. Let $\Psi^{(3)}$ be analytic in $\mathbb{C} \setminus \mathbb{B}$, such that

$$\Psi^{(3)}_+ = \Psi^{(3)}_- u^{(3)}_{x,\tau} \ on \ , \ where$$

$$
\begin{aligned}
u^{(3)}_{x,\tau} &= \begin{pmatrix} 1 & -RD^{-2}e^{2i\Phi} \\ \frac{\bar{R}D^2e^{-2i\Phi}}{1+|R|^2} & \frac{1}{1+|R|^2} \end{pmatrix}, \ on \ the \ top \ half \ of \ \mathbb{B}, \\
&= \begin{pmatrix} \frac{1}{1+|R|^2} & \frac{-RD^{-2}e^{2i\Phi}}{1+|R|^2} \\ \bar{R}D^2e^{-2i\Phi} & 1 \end{pmatrix}, \ on \ the \ bottom \ half \ of \ \mathbb{B}.
\end{aligned} \tag{3.8}
$$

and

$$\Psi^{(3)}(\infty) = I. \tag{3.9}$$

Then $\Psi^{(2)} - \Psi^{(3)} = O(\tau^{-l})$, for any positive l, as $\tau \to \infty$, uniformly in x, in compact subsets of the $\lambda-$Riemann-sphere.

PROOF: The proof is omitted (see [DZ]). The important observation here is that RD^{-2} and $\frac{\bar{R}D^2}{1+|R|^2}$ are actually under control on the top half of $\mathbb{B}$ while $\frac{-RD^{-2}}{1+|R|^2}$ and $\bar{R}D^2$ are under control on the bottom half. For example (again see [G], p.448) D behaves like $\lambda_0^{-1/2}(\lambda-1)^{-1/2}$ and RD^{-2} is bounded. Note here that it is crucial to chose the segment $\mathbb{B}$ at right angles with the real line; the behavior of D depends essentially on the direction in which we approach $\lambda = 1$.

5. We introduce the multi-valued function

$$\Omega = 2(\lambda-1)((\lambda-1)^2 + A^2)^{1/2} - A^2 - 2. \tag{3.10}$$

We consider this as a function on $\mathbb{C}$, chosing the branch consistently with the condition

$$\Omega = 2\lambda^2 - 4\lambda + O(\frac{1}{\lambda^2}),\ as\ \lambda \to \infty. \tag{3.11}$$

Naturally, Ω has a jump across $\mathbb{B}$. Also,

$$\begin{aligned} \Omega_+ + \Omega_- &= -2(A^2+2), \\ i(\Omega_+ - \Omega_-) &< 0,\ on\ the\ top\ half\ of\ \mathbb{B}, \\ i(\Omega_+ - \Omega_-) &> 0,\ on\ the\ bottom\ half\ of\ \mathbb{B}. \end{aligned} \tag{3.12}$$

Defining

$$\Psi^{(4)} = \Psi^{(3)} e^{i(2\Phi - \tau\Omega)\sigma_3}, \tag{3.13}$$

with $\sigma_3 = diag(1,-1)$ (a Pauli matrix), we end up with the following Riemann-Hilbert problem.

(3.14)

$$\Psi^{(4)}_+ = \Psi^{(4)}_- u^{(4)}_{x,\tau},\ where$$

$$u^{(4)}_{x,\tau} = \begin{pmatrix} e^{i\tau(\Omega_+ - \Omega_-)} & -RD^{-2}e^{i\tau(\Omega_+ + \Omega_-)} \\ \frac{\bar{R}D^2 e^{-i\tau(\Omega_+ + \Omega_-)}}{1+|R|^2} & \frac{1}{1+|R|^2}e^{-i\tau(\Omega_+ - \Omega_-)} \end{pmatrix},\ on\ the\ top\ half\ of\ \mathbb{B},$$

$$= \begin{pmatrix} \frac{e^{i\tau(\Omega_+ - \Omega_-)}}{1+|R|^2} & \frac{-RD^{-2}e^{i\tau(\Omega_+ + \Omega_-)}}{1+|R|^2} \\ \bar{R}D^2 e^{-i\tau(\Omega_+ + \Omega_-)} & e^{-i\tau(\Omega_+ - \Omega_-)} \end{pmatrix},\ on\ the\ bottom\ half\ of\ \mathbb{B},$$

$$\Psi^{(4)}(\infty) = I.$$

REMARK. The choice of Ω is inspired by the theory of algebro-geometric solutions of the NLS equation (cf. [BBEIM]). We are seeking a generalized differential $\tau d\Omega$ on a Riemann surface, whose integral behaves like the phase 2Φ as $\lambda \to \infty$. The actual Riemann surface (of genus 0 in our case) is dictated by the Riemann-Hilbert contour $\mathbb{B}$ in (3.8).

REMARK. The top diagonal term in the first matrix of (3.14) is small. To make the second diagonal term small we need to specify A. This will be done below. Similarly, the top diagonal term in the second matrix will be small when A is appropriately specified.

6. As it stands, (3.14) does not look like an improvement over the original Riemann-Hilbert problem. However, we shall now show that near the 'shock' region, it can be much simplified. In this subsection, we consider the 'shock front', i.e. the region II of theorem 1.1.

We choose A such that $|\lambda_0| = e^{-\tau A^2}$, i.e.

$$A = (\frac{-log\ |\lambda_0|}{\tau})^{1/2}. \tag{3.15}$$

A and τ depend on x and t and should be thought of as a 'slow' and a 'fast' varaible, respectively.

Because of (3.15), the diagonal terms of $u^{(4)}_{x,\tau}$ tend rapidly to 0, as $\tau \to \infty, \lambda_0 \to 0$. As in subsection 4, we end up with

$$\begin{aligned} \Psi^{(5)}_+ = \Psi^{(5)}_- u^{(5)}_{x,\tau},\ where \\ u^{(5)}_{x,\tau} = \begin{pmatrix} 0 & C(\lambda_0)\lambda_0 e^{-i\tau(2A^2+4)} \\ -\frac{1}{C(\lambda_0)\lambda_0} e^{i\tau(2A^2+4)} & 0 \end{pmatrix},\ \lambda \in \mathbb{B}, \\ \Psi^{(5)}(\infty) = I, \end{aligned} \tag{3.16}$$

where

(3.17)

$$C(\lambda_0) \sim -\frac{b(0)}{exp(iarg(a'(0)))}(\lambda_0)^{2i\nu} exp(\frac{1}{2\pi i}\int_{-\infty}^0 log(-s)dlog(|\tilde{a}(s)|^2)\ ),\ as\ \lambda_0 \to 0.$$

Note that there is no discontinuity at $\lambda = 1$. Here one uses the fact that when $a(\lambda) = 0, |b(\lambda)| = 1$ (see (1.8)), hence $\bar{b}(0) = \frac{1}{b(0)}$.

Problem (3.16) can be solved explicitly. It can be interpreted as a form of the Riemann-Hilbert problem for the genus-0 algebro-geometric solution of (1.1) (see e.g. [MA], [LM], [BBEIM] ch.4). We analyze and solve (3.16) in the appendix.

Keeping track of the different transformations of Ψ and recalling (1.14), we have:

THEOREM 3.2. The effect of a real spectral singularity at λ' is only felt in the region $\lambda' - \lambda_0 \to 0$. In the case, say, $\lambda' = 0$ we have, as $\lambda_0 \to 0$ *and* $\tau \to \infty$,

$$\begin{aligned} q(x,t) \sim \frac{(-log(|\lambda_0|))^{1/2}}{t^{1/2}}\cdot \\ exp(-4i\tau - \frac{i}{\pi}log(|\lambda_0|)log(|a'(0)\lambda_0|) - \frac{i}{2\pi}\int_{-\infty}^0 log(-s)dlog|\tilde{a}(s)|^2), \end{aligned} \tag{3.18}$$

where $\lambda_0 = \frac{-x}{4t}$, a, b are the scattering coefficients defined in (1.7) and $\tilde{a}(s) = \frac{a(s)}{s}$.

In particular, in the region $\frac{-x}{4t} \sim (\frac{logt}{t})^{1/2}$ the amplitude of the solution decays like $O((\frac{log\ t}{t})^{1/2})$.

7. The region $\tau \to \infty, \lambda_0 \to 0$ corresponds to the 'front' of the region in which the effect of the real singularity is felt. We conclude this section by considering the region where τ is bounded.

In this region (especially as $\tau \to 0$) no simple rescaling works with the same geometry. One needs a new kind of Riemann-Hilbert deformation.

We begin with the problem (3.7) which is still exactly equivalent to the original Riemann-Hilbert problem (1.12). But we do not aim to "erase" the half-lines l_i this time. Instead we see that the problem (3.7) is asymptotically equivalent to the Riemann-Hilbert problem for the Painlevé 4 equation ([FMA],[FZ])

$$\frac{d^2u}{dx^2} = \frac{1}{2u}(\frac{du}{dx})^2 + \frac{3}{2}u^3 + 4xu^2 + 2x^2u - 2u,$$

by means of the rescaling $\lambda \to \lambda(4it)^{1/2}, x \to x(4it)^{-1/2}$.

Choosing $A \sim \tau$ and rotating the Painlevé 4 problem appropriately, we see that the two Riemann-Hilbert problems are identified outside the circle of radius A. Inside the circle there is no identification, but we can show, using general symmetry facts and a vanishing theorem of Zhou ([Z]), that there is a matching transformation taking one Riemann-Hilbert problem to another. We thus have

THEOREM 3.3. When $\tau < M$, as $t \to \infty$, the asymptotics for the solution of (1.1),(1.2) is

$$q(x,t) \sim \frac{1}{t^{1/2}} u(xt^{-1/2}), \tag{3.19}$$

where u can be expressed in terms of a solution of Painlevé 4 such that $\frac{1}{t^{1/2}}u(xt^{-1/2})$ is actually a self-similar exact solution of (1.1) (see e.g. [BP], [C], [BCH]). The actual solution is defined in terms of 4 constants ([FMA]) a, b, c, d which can be explicitly expressed in terms of the scattering data. More precisely

$$a = \frac{\bar{b}(0)}{a'(0)} exp(\frac{1}{\pi i} \int_{-\infty}^{0} log(-s) dlog(|\tilde{a}(s)|^2)),$$

and $b = -\bar{a},\ c = -\frac{a}{|\tilde{a}(0)|^2},\ d = -\bar{c}.$

MATCHING. It can be shown that the formulae of region I match the formulae for region II by extending the [DIZ] analysis slightly into the region where $\tau = O(logt)$. Using known results on Painlevé 4 asymptotics by Kitaev [Ki], one can also show that the formulae of region II match the formulae for region III. Similarly, the formulae of region IV match the formulae for region II. and the the formulae of region V match the formulae for region IV.

4. HIGHER ORDER ZEROS AND FINITELY MANY SINGULARITIES

It should be clear by the discussion of sections 2 and 3 that there is nothing special about a singularity at zero. In the general case of finitely many singularities the following is true: for each real singularity λ_i, there is a region where $\lambda_i - \lambda_0 \to 0,\ as\ t \to \infty$, in which the solution decays with leading asymptotics similar to (3.18) and (3.19). Non-real singularities correspond to solitons as usual (see e.g [FT], [K] for the infinite case).

On the other hand, it is clear from section 3 that the order of the zero of a is not important; the order of the solution remains the same but there is a minor change in the phase. We leave the details to the reader.

As mentioned in section 1, a study of the interesting generalization of our result in the case of infinitely many real singularities is under way (for examples of such initial data see [Z]).

APPENDIX. THE RIEMANN-HILBERT PROBLEM FOR THE GENUS-0 SOLUTION

In this appendix we solve the Riemann-Hilbert problem (3.16). We do this by diagonalizing the jump matrix and thus reducing the problem to a scalar one which can then be solved explicitly.

Let $G = Cexp(-i\tau(2A^2+4))$ and

$$S = \begin{pmatrix} -iG & 1 \\ 1 & -\frac{i}{G} \end{pmatrix}. \tag{A.1}$$

The eigenvalues of S are i and $-i$; we have

$$\begin{pmatrix} 0 & G \\ -\frac{1}{G} & 0 \end{pmatrix} = S \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} S^{-1}. \tag{A.2}$$

Define

$$\Lambda = \left(\frac{\lambda - 1 + iA}{\lambda - 1 - iA}\right)^{1/4}. \tag{A.3}$$

Then Λ solves the scalar problem

$$\begin{aligned} \Lambda_+ &= \Lambda_- i,\ on\ \mathbb{B}, \\ \Lambda(\infty) &= 1. \end{aligned} \tag{A.4}$$

Let the matrix μ be defined by

$$\mu = S \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1} \end{pmatrix} S^{-1}. \tag{A.5}$$

Then

$$\begin{aligned} \mu_+ &= \mu_- \begin{pmatrix} 0 & G \\ -\frac{1}{G} & 0 \end{pmatrix},\ on\ \mathbb{B}, \\ \mu(\infty) &= I. \end{aligned} \tag{A.6}$$

Thus μ is the solution of (3.16). Furthermore, near infinity,

$$\mu(\lambda) = I + \begin{pmatrix} 0 & -\frac{GA}{2(\lambda-1-iA)} \\ -\frac{2}{GA(\lambda-1-iA)} & 0 \end{pmatrix}. \tag{A.7}$$

In particular $lim_{\lambda\to\infty}(2\lambda\mu_{12}) = -GA$. The asymptotics of q is now immediately recovered through (1.14).

REMARK: One can interpret problem (3.16) as the Riemann-Hilbert problem for the genus-0 solution of the NLS equation. Although the use of the term 'algebro-geometric solution' may sound pretentious in the very simple special case of genus 0, it is worth pointing out the fact that what we encounter here is an instance of a very general phenomenon, where the long-time behavior of the solution of a soliton equation in a particular region is related to a different type of solution, which is indeed connected with Riemann surfaces, and the associated theory of theta functions connected with the Abel map. For a discussion of that theory in the context of the nonlinear Schroedinger equation (focusing and defocusing) see [MA], [LM] and [BBEIM] (chapter 4).

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