JOURNAL OF MATHEMATICAL PHYSICS **48**, 065401 (2007)

Point vortex dynamics: A classical mathematics playground

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(Received 11 October 2006; accepted 30 November 2006; published online 12 June 2007)

The idealization of a two-dimensional, ideal flow as a collection of point vortices embedded in otherwise irrotational flow yields a surprisingly large number of mathematical insights and connects to a large number of areas of classical mathematics. Several examples are given including the integrability of the three-vortex problem, the interplay of relative equilibria of identical vortices and the roots of certain polynomials, addition formulas for the cotangent and the Weierstraß ζ function, projective geometry, and other topics. The hope and intent of the article is to garner further participation in the exploration of this intriguing dynamical system from the mathematical physics community. © *2007 American Institute of Physics.*
[DOI: 10.1063/1.2425103]

I. INTRODUCTION

Some years ago, after a talk at the Newton Institute for Mathematical Sciences in Cambridge about point vortex dynamics, a member of the audience, a Russian mathematician and mathematical physicist, commented that this was an example of a "classical mathematics playground." The characterization, which has stuck in my mind, contains more than a germ of truth as I hope to illustrate in this article. Somehow, in the problem of point vortex dynamics many strands of classical mathematical physics come together. Of course, one encounters the theory of dynamical systems, of systems of ordinary differential equations (ODEs), Hamiltonian dynamics, potential theory, and several other topics that one might think of as "expected." But there are also unexpected—or less expected—connections to subjects such as projective geometry, to aspects of the theory of polynomials, to elliptic functions (when the vortices are in periodic or bounded domains), and to pole decompositions of some of the integrable partial differential equations [such as Burgers' equation and the Korteweg–de Vries (KdV) equation]. Applications of even more exotic objects such as the Schwarz function[19] and the Schottky-Klein prime function[20] have appeared. In recent years we have understood how to apply Thurston-Nielsen theory to the motion of point vortices[15] and to the mixing that they induce of the surrounding fluid.[14] I am sure there will be many more such connections that serve, on one hand, to introduce new bodies of mathematical knowledge into applications (to the extent point vortex dynamics can rightfully be called "applications") and, on the other hand, to give physical motivation to certain mathematical concepts and theorems. Some of us, including I suspect many readers of this journal, find such physical motivation to be both useful and helpful. In this article my goal is to illustrate some of these avenues of investigation. I cannot possibly illustrate them all to any degree of completeness without turning the article into a monograph, but the examples and the bibliography will, hopefully, whet the reader's appetite sufficiently to continue the explorations. I submit that there is much beautiful, interesting, and useful mathematical physics still to be uncovered.

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II. THE BASIC DYNAMICAL EQUATIONS

The equations of motion of interacting point vortices were introduced by Helmholtz[27] in a seminal paper published in 1858, squeezed in between his equally seminal—and maybe in the long run more important—work on the physics of sound and the physiology of hearing. The paper on vorticity and vortex dynamics was entitled "Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen" ("On integrals of the hydrodynamical equations, which express vortex-motion" in Tait's 1867 translation). In this paper[27] Helmholtz was the first to elucidate key properties of those portions of a fluid in which vorticity occurs. Although the investigation was motivated, at least in part, by an interest in the effect of friction within a fluid, the theory developed is restricted to the dynamics of an ideal, incompressible fluid with "embedded vorticity"—as we would phrase it today. Helmholtz showed that in such a substance vortex motion could neither be produced from irrotational flow nor be destroyed entirely by any natural forces that themselves have a potential. If vorticity exists within a group of fluid particles, in the inviscid limit they are incapable of transmitting it to particles that have none. Conversely, the particles that have vorticity cannot be entirely deprived of it, although the vorticity of any individual particle may change in three-dimensional (3D) flow by the all-important mechanism of vortex stretching. In two-dimensional (2D) flow of an inviscid fluid, on the other hand, the vorticity of each particle is a constant of the motion. For an ideal fluid, then, the laws of vortex motion establish a curious, invariable linkage between fluid particles and their state of rotation.

Towards the end of his paper, in Sec. V entitled "Straight parallel vortex-filaments" (in Tait's transation), Helmholtz introduces the point vortex model and considers some of its most immediate consequences. In this model one envisions that the vorticity is confined to a set of infinitely thin, straight, parallel vortex filaments each of which carries an invariant amount of *circulation*. Equivalently, one can trace the points of intersection of this family of filaments with a plane perpendicular to all of them. These points of intersection are known as *point vortices*. One may think of them as playing a role in ideal hydrodynamics similar to that played by point masses in celestial mechanics. Point vortices only exist in 2D flow. Since the 3D vorticity field is divergence-free, such vorticity "monopoles" cannot exist in 3D flow. Attempts to come up with suitable generalizations, e.g., Novikov's "vortons,"[40] which may be thought of as infinitesimal vortex rings, have not met with broad acceptance.

The equations of motion for N interacting point vortices on the unbounded xy plane, with vortex $\alpha=1,\ldots,N$ having circulation or *strength* Γ_α and position (x_α,y_α), consist of the following $2N$ first-order, nonlinear, ODEs:

$$\frac{dx_\alpha}{dt}=-\frac{1}{2\pi}\sum_{\beta=1}^{N}{}'\frac{\Gamma_\beta(y_\alpha-y_\beta)}{l_{\alpha\beta}^2},\quad \frac{dy_\alpha}{dt}=\frac{1}{2\pi}\sum_{\beta=1}^{N}{}'\frac{\Gamma_\beta(x_\alpha-x_\beta)}{l_{\alpha\beta}^2},\quad \alpha=1,2,\ldots,N, \tag{1}$$

where $l_{\alpha\beta}^2=(x_\alpha-x_\beta)^2+(y_\alpha-y_\beta)^2$, and the prime on the summation indicates omission of the singular term $\beta=\alpha$. Typically, an initial value problem is addressed with the initial positions of the vortices and their strengths given so as to represent some flow situation of interest, where for classical fluids "represent" is to be understood in the sense of providing an approximate model. In superfluids vortices have cross sections of atomic dimensions—and quantized circulations—so Eqs. (1) provide an excellent basis for macroscopic flow considerations and give a quantitatively accurate representation of 2D flow of such a fluid. The physical content of Eqs. (1) is that each vortex sets up about itself a circumferential velocity field of magnitude $\Gamma_\alpha/2\pi r$, where r is the distance from that vortex, and that any one vortex is advected by the combined velocity produced by all the other vortices, i.e., its velocity at any instant is equal to the vector sum of the velocities produced at its position by all the other vortices.

If the plane in which the motion takes place is thought of as the complex plane, system (1) may be also written elegantly as N ODEs for N complex positions of the vortices $z_\alpha=x_\alpha+iy_\alpha$:

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$$\overline{\frac{dz_\alpha}{dt}} = \frac{1}{2\pi i}\sum_{\beta=1}^{N}{}' \frac{\Gamma_\beta}{z_\alpha - z_\beta}, \quad \alpha = 1,2,\ldots,N, \tag{2}$$

where the overline denotes complex conjugation.

The point vortex equations [Eqs. (1)] may be thought of as a discretization of the continuum equations for 2D inviscid flow—commonly referred to as the *Euler equations*—that is useful both for analytical and numerical approximation purposes. The relevance and adequacy of this correspondence is itself the subject of an extended literature dealing with issues of convergence and accuracy over time. The reader interested in such issues may wish to consult the book by Marchioro and Pulvirenti.[35] Our viewpoint in this article will be to treat Eqs. (1) or (2) as an interesting dynamical system in its own right and proceed from there. This also implies that we shall not pursue extensions of Eqs. (1) to bounded domains or to surfaces other than the plane (the sphere being the surface of particular interest), although we shall briefly consider vortices in periodic domains. The extension of the basic dynamical equations of point vortices to bounded domains was considered in two short papers by Lin,[32,33] later published as a monograph.[34] This theory has recently been revisited by Crowdy and Marshall[20,21] who show that the Hamiltonian may be given explicitly for a multiconnected domain in terms of the Riemann mapping function of that domain onto a topologically equivalent domain with all boundaries being circles and the Schottky-Klein prime function associated with this set of circles.

Several investigators seized upon Helmholtz's new model for 2D flow. In his influential lectures on mathematical physics (see Ref. 29, Lecture 20) Kirchhoff demonstrated that system (1) may be restated in Hamilton's canonical form:

$$\Gamma_\alpha \frac{dx_\alpha}{dt} = \frac{\partial H}{\partial y_\alpha}, \quad \Gamma_\alpha \frac{dy_\alpha}{dt} = -\frac{\partial H}{\partial x_\alpha}, \quad \alpha = 1,2,\ldots,N. \tag{3}$$

A complete correspondence results by setting the "generalized coordinates" $q_\alpha = x_\alpha$ and the "generalized momenta" $p_\alpha = \Gamma_\alpha y_\alpha$. This results in the remarkable insight that the "phase space"—in the sense of Hamiltonian dynamics—for the point vortex system is, in essence, its configuration space. The autonomous Hamiltonian,

$$H = -\frac{1}{4\pi}\sum_{\alpha,\beta=1}^{N}{}' \Gamma_\alpha \Gamma_\beta \ln l_{\alpha\beta}, \tag{4}$$

is conserved during the motion of the point vortices (by the usual arguments of Hamiltonian dynamics). Although H in Eq. (4) looks like a potential energy, it is in fact the kinetic energy of the fluid motion surrounding the vortices.

In addition to H system (3) has three independent first integrals:

$$Q = \sum_{\alpha=1}^{N} \Gamma_\alpha x_\alpha, \quad P = \sum_{\alpha=1}^{N} \Gamma_\alpha y_\alpha, \quad I = \sum_{\alpha=1}^{N} \Gamma_\alpha (x_\alpha^2 + y_\alpha^2). \tag{5}$$

There are at least three ways to see this. First, one may verify these integrals by direct calculation either from Eqs. (1) or (2). Second, following Kirchhoff, one may observe the invariance of the Hamiltonian [Eq. (4)] to translation or rotation of the coordinates and apply what today would be called *Noether's theorem*. The two components of the *linear impulse*, Q and P, result from the translational invariance. The *angular impulse* I is a constant of the motion because of the rotational invariance of H, Eq. (4).

Third, one may develop the Hamiltonian formalism a bit further by introducing the notion of a *Poisson bracket*, i.e., set

$$[f,g] = \sum_{\alpha=1}^{N} \frac{1}{\Gamma_\alpha}\left(\frac{\partial f}{\partial x_\alpha}\frac{\partial g}{\partial y_\alpha} - \frac{\partial f}{\partial y_\alpha}\frac{\partial g}{\partial x_\alpha}\right). \tag{6}$$

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The so-called *fundamental Poisson brackets* are

$$[x_1,\Gamma_1 y_1]=[x_2,\Gamma_2 y_2]=\cdots=1, \tag{7}$$

$$[x_1,y_2]=[x_2,y_1]=[x_1,x_2]=[y_1,y_2]=\cdots=0, \tag{8}$$

or, when written in terms of the complex coordinates,

$$[z_\alpha,z_\beta]=0, \quad [z_\alpha,\overline{z_\beta}]=-2i\delta_{\alpha\beta}\Gamma_\alpha. \tag{9}$$

This formalism was developed by Poincaré[44] and Laura.[30] (For a modern account see Newton's monograph.[38])

It now follows by direct calculation from Eqs. (7) and (8) that

$$[Q,H]=[P,H]=0, \quad [I,H]=0, \tag{10}$$

and since the evolution equation for any function f of the generalized coordinates is

$$\frac{\mathrm{d}f}{\mathrm{d}t}=\frac{\partial f}{\partial t}+[f,H] \tag{11}$$

and Q, P, and I do not depend explicitly on time, it follows once again that Q, P, and I are integrals of the motion.

III. INTEGRABILITY OF THE THREE-VORTEX PROBLEM

Exploring the Poisson brackets of Q, P, and I with one another, we find

$$[Q,P]=\sum_{\alpha=1}^{N}\Gamma_\alpha, \quad [Q,I]=2P, \quad [P,I]=-2Q. \tag{12}$$

These relations show (a) that no new integrals arise by taking Poisson brackets between the three known general integrals [Eqs. (5)] and (b) that

$$[Q^2+P^2,I]=2Q[Q,I]+2P[P,I]=0 \tag{13}$$

regardless of the values of the vortex strengths. Since we then always have three independent integrals *in involution*, viz., H, I, and Q^2+P^2, Liouville's theorem of classical mechanics assures integrability of the N-vortex problem for $N\leqslant 3$ and any values of the vortex strengths. Indeed, the one-vortex problem is trivial: A single vortex on the unbounded plane remains stationary. The simple solution of the two-vortex problem was, in essence, already included in Helmholtz's paper.[27]

Poincaré's treatment[44] stops after the formal statement of having three independent integrals in involution, whereas the earlier work of Gröbli[24] expounds in elaborate and ingenious detail on how to determine the trajectories of the three vortices. For additional background on the history of the three-vortex problem the reader may enjoy the account given in Ref. 7.

We should also note in passing that for four vortices the relations Eqs. (12) show that if (i) the sum of the circulations vanishes and if (ii) we consider states such that $P=Q=0$, then the four-vortex problem has four independent integrals in involution, viz., P, Q, I, and H. Thus, the problem of a "neutral" four-vortex system with vanishing linear impulse is integrable. Details for this case are presented in Refs. 22 and 9.

The actual "reduction to quadratures" of the three-vortex problem generally makes use of the integrals H, Eq. (4), and

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$$L = \sum_{1 \leqslant \alpha < \beta \leqslant N} \Gamma_\alpha \Gamma_\beta l_{\alpha\beta}^2 = \left(\sum_{\alpha=1}^{N} \Gamma_\alpha \right) I - (P^2 + Q^2). \tag{14}$$

As a combination of known integrals L is clearly an integral of the point vortex equations, and a few easy calculations give

$$[L,P] = [L,Q] = [L,I] = 0, \tag{15}$$

whence also

$$[L, P^2 + Q^2] = 0. \tag{16}$$

Both L and H are expressed in terms of the distances between the vortices. For three vortices let $s_1 = l_{23}$, $s_2 = l_{31}$, and $s_3 = l_{12}$. Then

$$L = \Gamma_2 \Gamma_3 s_1^2 + \Gamma_3 \Gamma_1 s_2^2 + \Gamma_1 \Gamma_2 s_3^2,$$

$$H = -\frac{1}{4\pi}[\Gamma_2 \Gamma_3 \log s_1^2 + \Gamma_3 \Gamma_1 \log s_2^2 + \Gamma_1 \Gamma_2 \log s_3^2]. \tag{17}$$

The fundamental Poisson brackets [Eqs. (7) and (8)] yield

$$\begin{aligned}[s_1^2, s_2^2] &= [(x_2 - x_3)^2 + (y_2 - y_3)^2, (x_3 - x_1)^2 + (y_3 - y_1)^2] = 4\{(x_2 - x_3)(y_3 - y_1)[x_2 - x_3, y_3 - y_1] \\ &\quad + (y_2 - y_3)(x_3 - x_1)[y_2 - y_3, x_3 - x_1]\} = -\frac{4}{\Gamma_3}\{(x_2 - x_3)(y_3 - y_1) - (y_2 - y_3)(x_3 - x_1)\} \\ &= -\frac{4}{\Gamma_3}(x_1 y_2 + x_2 y_3 + x_3 y_1 - x_1 y_3 - x_3 y_2 - x_2 y_1) = -\frac{8\Delta}{\Gamma_3},\end{aligned} \tag{18}$$

where

$$\Delta = \frac{1}{2}(x_1 y_2 + x_2 y_3 + x_3 y_1 - x_1 y_3 - x_3 y_2 - x_2 y_1) \tag{19}$$

is the area of triangle 123 with orientation, i.e., reckoned positive if 123 appear counterclockwise and negative if 123 appear clockwise. By permutation of indices we obtain

$$[s_1^2, s_2^2] = -\frac{8\Delta}{\Gamma_3}, \quad [s_2^2, s_3^2] = -\frac{8\Delta}{\Gamma_1}, \quad [s_3^2, s_1^2] = -\frac{8\Delta}{\Gamma_2}. \tag{20}$$

The equations of motion for the squares of the sides of the triangle may now be found as follows:

$$\frac{\mathrm{d}(s_1^2)}{\mathrm{d}t} = [s_1^2, H] = -\frac{1}{4\pi}\left(\frac{\Gamma_3 \Gamma_1}{s_2^2}[s_1^2, s_2^2] + \frac{\Gamma_1 \Gamma_2}{s_3^2}[s_1^2, s_3^2] \right) = \frac{2}{\pi}\Gamma_1 \Delta \left(\frac{1}{s_2^2} - \frac{1}{s_3^2} \right). \tag{21}$$

By permutation of indices we get the following system of ODEs:

$$\frac{\mathrm{d}(s_1^2)}{\mathrm{d}t} = \frac{2}{\pi}\Gamma_1 \Delta \left(\frac{1}{s_2^2} - \frac{1}{s_3^2} \right),$$

$$\frac{\mathrm{d}(s_2^2)}{\mathrm{d}t} = \frac{2}{\pi}\Gamma_2 \Delta \left(\frac{1}{s_3^2} - \frac{1}{s_1^2} \right),$$

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$$\frac{\mathrm{d}(s_3^2)}{\mathrm{d}t} = \frac{2}{\pi}\Gamma_3\Delta\left(\frac{1}{s_1^2} - \frac{1}{s_2^2}\right). \tag{22}$$

These equations, which are in Gröbli's 1877 thesis,[24] are usually called the *equations of relative motion*. (The present derivation is in the spirit of the work by Laura,[30] see also Ref. 13.) They may be generalized to N vortices quite straightforwardly. Note that by Hero's formula the absolute value of Δ, Eq. (19), may be expressed in terms of the three sides s_1, s_2, and s_3 so that Eqs. (22) form a closed dynamical system for the evolution of the sides in the vortex triangle except for instants at which the orientation of the vortex triangle is indeterminate, i.e., instants at which the three vortices become collinear. Modulo a treatment of collinear configurations, which we provide below, the three-degree-of-freedom dynamical system [Eqs. (22)] has the two integrals [Eqs. (17)] and is thus integrable.

We note in passing the remark by Borisov and Palmov[13] that if time is rescaled in Eqs. (22) by setting $2\Delta\ \mathrm{d}t = \pi s_1^2 s_2^2 s_3^2 \mathrm{d}\tau$, one obtains

$$\frac{\mathrm{d}(s_1^2)}{\mathrm{d}\tau} = \Gamma_1 s_1^2(s_3^2 - s_2^2), \quad \frac{\mathrm{d}(s_2^2)}{\mathrm{d}\tau} = \Gamma_2 s_2^2(s_1^2 - s_3^2), \quad \frac{\mathrm{d}(s_3^2)}{\mathrm{d}\tau} = \Gamma_3 s_3^2(s_2^2 - s_1^2). \tag{23}$$

These equations are similar in form to the celebrated *Lotka-Volterra equations* used in modeling predator-prey interactions in ecology. It is, of course, quite unlikely that there are any deep philosophical implications to this formal observation.

Continuing the general arguments, we reiterate that Eqs. (22) describe the evolution of the vortex triangle except for instants at which $\Delta = 0$, i.e., it is emphatically not true that any configuration of three collinear vortices remains collinear. To understand what happens at an instant when the vortices become collinear we require the equation of motion for Δ.

Using expression (19) for Δ and the Poisson brackets given previously, one finds after a straightforward albeit somewhat tedious calculation that

$$[s_1^2, 2\Delta] = \left(\frac{1}{\Gamma_2} - \frac{1}{\Gamma_3}\right)s_1^2 + \left(\frac{1}{\Gamma_2} + \frac{1}{\Gamma_3}\right)(s_2^2 - s_3^2),$$

$$[s_2^2, 2\Delta] = \left(\frac{1}{\Gamma_3} - \frac{1}{\Gamma_1}\right)s_2^2 + \left(\frac{1}{\Gamma_3} + \frac{1}{\Gamma_1}\right)(s_3^2 - s_1^2),$$

$$[s_3^2, 2\Delta] = \left(\frac{1}{\Gamma_1} - \frac{1}{\Gamma_2}\right)s_3^2 + \left(\frac{1}{\Gamma_1} + \frac{1}{\Gamma_2}\right)(s_1^2 - s_2^2). \tag{24}$$

(The last two of these follow by permutation of indices from the first.) Thus,

$$\begin{aligned}\frac{\mathrm{d}\Delta}{\mathrm{d}t} = [\Delta, H] &= -\frac{1}{4\pi}\left(\frac{\Gamma_2\Gamma_3}{s_1^2}[\Delta, s_1^2] + \frac{\Gamma_3\Gamma_1}{s_2^2}[\Delta, s_2^2] + \frac{\Gamma_1\Gamma_2}{s_3^2}[\Delta, s_3^2]\right) \\ &= \frac{1}{8\pi}\left[\frac{\Gamma_2 + \Gamma_3}{s_1^2}(s_2^2 - s_3^2) + \frac{\Gamma_3 + \Gamma_1}{s_2^2}(s_3^2 - s_1^2) + \frac{\Gamma_1 + \Gamma_2}{s_3^2}(s_1^2 - s_2^2)\right].\end{aligned} \tag{25}$$

This equation shows that even when the vortices are collinear the configuration is not a relative equilibrium unless the term in square brackets in Eq. (25) vanishes. That is, even if the vortices become collinear and Δ vanishes momentarily, the rate of change of Δ will not vanish in general, and the vortex triangle will "rebound" in the next instant, usually accompanied by a change in orientation.

The condition that a collinear configuration is a relative equilibrium of the three vortex system, then, is that

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$$\frac{\Gamma_2+\Gamma_3}{s_1^2}(s_2^2-s_3^2)+\frac{\Gamma_3+\Gamma_1}{s_2^2}(s_3^2-s_1^2)+\frac{\Gamma_1+\Gamma_2}{s_3^2}(s_1^2-s_2^2)=0. \tag{26}$$

In order to discuss the solution in detail and its dependence on the values of the circulations, one can use the relation $s_1=s_2+s_3$ (or one of its permutations), which follows from the collinearity of the vortices, to turn Eq. (26) into a cubic for the quantity $z=s_2/s_3$. Setting $s_2=zs_3$, $s_1=(1+z)s_3$ in Eq. (26) produces this cubic that is (except for the sign of z) Eq. (19) of the paper by Tavantzis and Ting.[50] The discussion proceeds by elementary means.

There is a neat geometric interpretation to all this: Think of the three quantities

$$b_1=\lambda\frac{s_1^2}{\Gamma_1},\quad b_2=\lambda\frac{s_2^2}{\Gamma_2},\quad b_3=\lambda\frac{s_3^2}{\Gamma_3},\quad \text{where } \lambda=\frac{\Gamma_1\Gamma_2\Gamma_3}{L}, \tag{27}$$

derived from the side lengths in the vortex triangle as trilinear coordinates in the plane. (This definition excludes the important case $L=0$ to which we must return separately.) Assume units of length and time chosen such that $|\lambda|=1$. Now, according to Eqs. (17), as the vortices move about we always have

$$b_1+b_2+b_3=1, \tag{28}$$

i.e., by chosing b_1,b_2,b_3 as trilinear coordinates we can "build in" one of the conservation laws of the motion. The second conservation law, $H=$const., is then represented as a set of level curves in the (b_1,b_2,b_3) trilinear coordinate plane.

There is an interesting geometric result for this representation, which must be well known but that I would be hard pressed to find in the literature: Consider the constraint imposed on the b by their derivation from side lengths in a triangle due to the *triangle inequalities* that pertain to said triangle. These may be expressed by demanding that the argument of the square root in Hero's formula be greater than or equal to zero, viz.,

$$(s_1+s_2+s_3)(s_1+s_2-s_3)(s_1-s_2+s_3)(-s_1+s_2+s_3)\geqslant 0. \tag{29}$$

(We have retained the first factor, even though it is always positive, because the b are defined in terms of squares of the side lengths.) Inequality (29) may also be written as

$$2(s_2^2s_3^2+s_3^2s_1^2+s_1^2s_2^2)\geqslant s_1^4+s_2^4+s_3^4. \tag{30}$$

When written out in term of the b, Eq. (30) becomes the quadratic form

$$2(\Gamma_1\Gamma_2b_1b_2+\Gamma_2\Gamma_3b_2b_3+\Gamma_3\Gamma_1b_3b_1)\geqslant(\Gamma_1b_1)^2+(\Gamma_2b_2)^2+(\Gamma_3b_3)^2. \tag{31}$$

The geometric representation in the trilinear coordinate plane of this quadratic expression—i.e., Eq. (31) with equality rather than inequality—is a conic section. Points "inside" the conic represent realizable vortex triangles, and we refer to this portion of the $b_1b_2b_3$ plane as the *physical region*. The boundary of the physical region corresponds to collinear states of the three vortices.

The classification of the kind of conic (ellipse, parabola, hyperbola) proceeds by elementary means. On physical grounds we may always assume the three vortices numbered such that $\Gamma_1\geqslant\Gamma_2>0$ (two of the vortices must have the same sign and this sign may be chosen to be positive; sign reversal of all vortex strengths corresponds to time reversal of the motion). If γ denotes the sum of the three strengths, i.e., $\gamma=\Gamma_1+\Gamma_2+\Gamma_3$, the results of this classification may be stated as follows:

(i) If Γ_3 and γ are both >0 or both <0, the boundary is an ellipse located in region I for $\Gamma_3>0$ and in region II for $\Gamma_3<0$.
(ii) If $\gamma=0$, the boundary is a parabola located in region II.
(iii) If $\Gamma_3<0$ but $\gamma>0$, the boundary is a hyperbola with branches in regions I and II.

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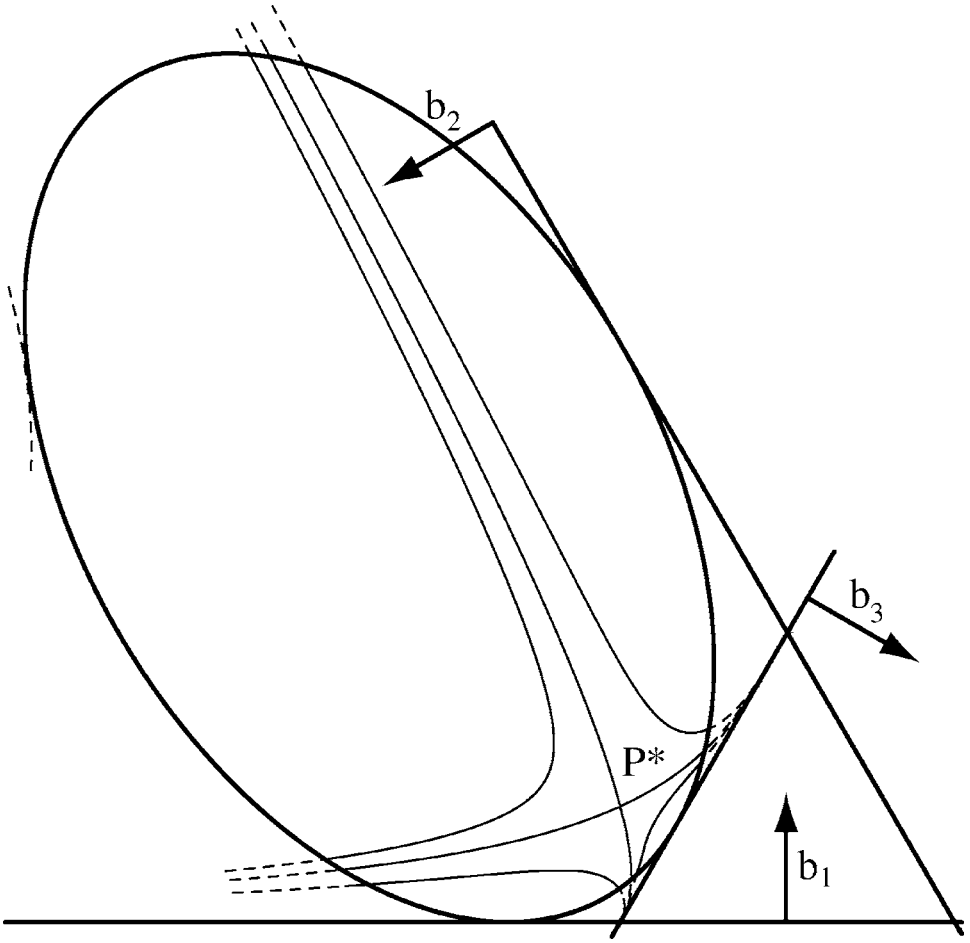


FIG. 1. An example of the trilinear coordinate representation of relative motion of three vortices of strengths 2, 1, and −4 (see the text for details).

This result already contains the nontrivial information that in case (i) all relative motion of the three vortices is bounded. While this is easy to see when $\Gamma_3 > 0$, it is trickier to prove when $\Gamma_3 < 0$. An example of a physical region boundary when Γ_3 and γ are both negative is given in Fig. 1.

Because the quadratic form [Eq. (31)] is homogeneous, the three trilinear axes are tangent to the physical region boundary. The points of tangency are easily found. For example, setting $b_1 = 0$ in Eq. (31) we find $\Gamma_2 b_2 = \Gamma_3 b_3$. From this and Eq. (28) we get the point of tangency between the physical region boundary and the b_1 axis as

$$(b_1, b_2, b_3) = \left(0, \frac{\Gamma_3}{\Gamma_2 + \Gamma_3}, \frac{\Gamma_2}{\Gamma_2 + \Gamma_3}\right). \tag{32}$$

The other points of tangency follow by permutation of indices. Physically the limit $b_1 = 0$ means that vortices 2 and 3 coincide. That is, the three-vortex problem has really "degenerated" to a two-vortex problem. Since two vortices are always collinear, the points of tangency, i.e., Eq. (32) and its index permutations, satisfy Eq. (26). Said differently, if we view Eq. (26) as defining a curve of degree 3 in the trilinear coordinate plane, it will pass through the points of tangency of the physical region boundary and the trilinear axes.

Since the physical region boundary is of the second degree, and curve (26) is of the third

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degree, we expect by *Bezout's theorem* that there will, in general, be six points of intersection. Three of these are the points of tangency of the physical region boundary with the trilinear axes. The other three are the collinear equilibria. This is the generic case, as one also deduces from writing Eq. (26) as a cubic in s_2/s_3. But there are degeneracies, and the overall picture is more complicated.[50]

A level curve of the Hamiltonian [Eqs. (17)] in the trilinear coordinate plot may be wholly situated within the physical region or it may intersect the physical region boundary at one or more points. The significance of Eq. (26) is that it singles out points where the physical region boundary and a level curve of the Hamiltonian have a common tangent. To verify this last statement let $f(x,y)=0$, $g(x,y)=0$ be the equations of two plane curves in Cartesian coordinates. Assume that the curves intersect at some point. Then it is clear that the curves will have a common tangent at the point of intersection if

$$\frac{\partial f}{\partial x}\frac{\partial g}{\partial y}-\frac{\partial f}{\partial y}\frac{\partial g}{\partial x}=0 \tag{33}$$

at that point. Assume now that f and g are given as functions of trilinear coordinates b_1, b_2, and b_3 rather than as functions of x and y. The condition for a common tangent then becomes

$$f_1g_2+f_2g_3+f_3g_1-f_2g_1-f_3g_2-f_1g_3=0 \tag{34}$$

(where f_1 means $\partial f/\partial b_1$, etc.). We use this relation when $f(b_1,b_2,b_3)=0$ is the equation representing conservation of the Hamiltonian, and $g(b_1,b_2,b_3)=0$ is the equation of the physical region boundary, i.e., Eq. (31) with the inequality sign replaced by an equal sign. The condition that these two curves have a common tangent at their point of intersection then becomes precisely the condition that the right hand side of Eq. (26) vanishes.

Figure 1 provides an example of the preceding. The vortex strengths are $(2,1,-4)$ (only the relative strengths matter). According to the classification given previously this is case (i) and one can only have $L\leqslant 0$, i.e., $b_1,b_2\geqslant 0$, $b_3\leqslant 0$ in the trilinear coordinate plane. All relative motion of the three vortices is bounded. There is a collinear state (up to rotation of the three vortices) that is represented in Fig. 1 by a level curve of the Hamiltonian touching the physical boundary ellipse from the outside (to the left in the figure). The diagram shows that this configuration is stable in the sense that a small perturbation of it will lead to configurations that pulsate about the collinear state while rotating as a whole. There are no other points where a level curve of the Hamiltonian and the physical region boundary have a common tangent. Hence, there is only this one collinear relative equilibrium for this choice of vortex strengths. This result is, of course, consistent with what one finds from an examination of the solutions to Eq. (26).

The point labeled P^* in Fig. 1 corresponds to an equilateral triangle of vortices. An equilateral triangle configuration is always a relative equilibrium regardless of the strengths of the vortices placed at the vertices. For the particular choice of vortex strengths leading to Fig. 1 it is seen that this equilateral triangle configuration is unstable to infinitesimal perturbations since P^* is a saddle point of the Hamiltonian.

For further discussion of the trilinear representation of the solution to the three-vortex problem on the unbounded plane see Refs. 1, 7, 49, and 50 and references therein. Gröbli's thesis of 1877 provided a blueprint for how to solve any given three-vortex problem.[24] For special cases of the vortex strengths the procedure that he outlined can be carried out and explicit formulas, which typically involve elliptic or hyperelliptic integrals, can be obtained. One might here draw an analogy to another famous problem in dynamics, the asymmetrical top. In that problem the solution of Euler's equations can be obtained in terms of elliptic functions and the problem may be thought of as solved once these expressions are established. However, to gain an overview of the motion and how it depends on the various parameters of the problem, which in the case of the top are the principal moments of inertia, one inevitably turns to the beautiful geometric construction of Poinsot. Soon the qualitative insights provided by "polhode" and "herpolhode" rival the quantitative accuracy of the elliptic function formulas. In the introduction to their classic four-volume

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work on the motion of the top, *Über die Theorie des Kreisels*, published over the span of a dozen years, 1897–1910, Klein and Sommerfeld lament the British and German insistence on quantitative formulas and extoll the virtues of the French qualitative, geometrical approach. In the case of three vortices the qualitative approach was Irish-American.[1,49] Actually, for the symmetric case of three identical vortices Gröbli had already found such a representation. This analysis was rediscovered by Novikov[39] almost a century later in a paper that provided much of the modern impetus to research on few-vortex dynamics.

Before leaving this topic, we note that the integrable case of four-vortex motion can also be analyzed by using the trilinear coordinates mentioned. The value of the integral L, Eq. (14), is clearly zero for this case. However, if we calculate

$$L_1 = \Gamma_2\Gamma_4 l_{24}^2 + \Gamma_3\Gamma_4 l_{34}^2 + \Gamma_2\Gamma_3 l_{23}^2, \quad I_1 = \Gamma_2|z_2|^2 + \Gamma_3|z_3|^2 + \Gamma_4|z_4|^2,$$

i.e., an L-like quantity and an I-like quantity that both omit vortex 1, we find precisely as in Eq. (14) that

$$L_1 = (\Gamma_2 + \Gamma_3 + \Gamma_4)I_1 - |\Gamma_2 z_2 + \Gamma_3 z_3 + \Gamma_4 z_4|^2 = -\Gamma_1 I_1 - \Gamma_1^2|z_1|^2 = -\Gamma_1 I.$$

In the second equality we have used that the sum of the strengths is zero and that $Q=P=0$. Thus L_1 is an integral of this particular four-vortex problem, and the same goes for the three analogous quantities L_2, L_3, and L_4 constructed by permutation of indices. The last of these, L_4, is of course just the quantity L for the three vortices 1, 2, 3, cf. Eqs. (17). We can now eliminate l_{14}, l_{24}, l_{34} in favor of l_{12}, l_{23}, l_{31}. The resulting Hamiltonian is somewhat more complicated than in the three-vortex case, but its level curves still carry the same significance in the phase plane diagrams. The singularities $l_{14}=0$, $l_{24}=0$, and $l_{34}=0$ define three lines in the trilinear plot that form an equilateral triangle. These must again be tangent to the conic bounding the physical region (just as the trilinear axes, which correspond to $l_{23}=0$, $l_{31}=0$, and $l_{12}=0$, respectively, are tangent to this conic). Several examples and a complete analysis may be found in the paper by Aref and Stremler.[9] The earlier analysis by Eckhardt[22] uses a different canonical reduction of the equations of motion that is again elucidated using various phase diagrams. However, the variables are different from the intervortex separations.

While the availability of an integrable three-body problem is fascinating, this really results from the relative simplicity of the one- and two-vortex problems, as briefly indicated. The three-vortex problem is the first for which the distances between the vortices change in time, but the problem of relative motion of the three vortices still contains just one frequency (when it is periodic). As one crosses into the four-vortex problem, integrability is, in general, lost. The transition from integrability to nonintegrability is analogous to what one finds in the N-body problem of celestial mechanics: The motion of a "vortex of zero strength" in the field of three interacting vortices—the "restricted four-vortex problem," as we call it—is nonintegrable, just as the restricted three-body problem of celestial mechanics is nonintegrable. The interpretation of the restricted four-vortex problem is the advection of a "speck of dust" in the unsteady flow field produced by the integrable motion of three vortices. The speck of dust formally enters the equations of motion as would an extra vortex of vanishing circulation. The notion that the advection, and by extension the stirring, of passive particles by a simple time-dependent flow can be chaotic has spawned a new subfield of fluid mechanics known as *chaotic advection*, a term that is today used as a classifier and keyword for journals and conferences.[2]

One final comment on the dynamics of three vortices: When $\Gamma_1\Gamma_2+\Gamma_2\Gamma_3+\Gamma_3\Gamma_1=0$ and $L=0$, there exist motions where the triangle of vortices collapses self-similarly (i.e., without change of shape) to a point in a finite time. The vortices move according to the similarity solution

$$z_\alpha(t) = z_\alpha(0)\left(1 - \frac{t}{\tau}\right)^{1/2 - i\omega\tau}, \tag{35}$$

where τ and ω are real parameters dependent on the initial condition (which must satisfy $L=0$). Positive τ has the physical meaning of a *collapse time*. When τ is negative the vortex triangle

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expands self-similarly for all times. In these motions all the vortices move on similar logarithmic spirals, as one easily sees from Eq. (35). For three vortices there is an explicit construction of initial states that will lead to this phenomenon of *vortex collapse*.[1] Collapsing states may also be found analytically for four and five vortices.[41] We believe such configurations exist for an arbitrary number of vortices subject to the necessary constraint on circulations that

$$\sum_{1\leqslant\alpha<\beta\leqslant N}\Gamma_\alpha\Gamma_\beta=0 \quad \text{or} \quad \left(\sum_{\alpha=1}^{N}\Gamma_\alpha\right)^2=\sum_{\alpha=1}^{N}\Gamma_\alpha^2. \tag{36}$$

The existence of vortex collapse states raises a number of intriguing mathematical questions: Initially smooth vorticity evolving under the 2D Euler equation is known to remain smooth for all times. Thus, arbitrarily close to certain point vortex configurations that collapse in finite time, there are smooth 2D flows that will not lead to a singularity in finite time but that must lead to something very close to a singularity, cf. Ref. 51. What is the nature of this "singularity avoidance?" Can one exploit the singular solutions for the point vortices to construct singular solutions of smooth vorticity distributions in 3D? For example, what if one looks at a weakly 3D flow consisting of vortex filaments that are almost parallel but have a slight 3D variation and where the two-dimensional configuration in a dense set of cuts through these filaments (essentially perpendicular to them) is a collapse configuration of point vortices? Can such initial states be constructed that will display a singularity after a finite time when evolved according to the 3D Euler equation? (This intriguing suggestion was made by Zakharov.[52]) Is this a productive route to demonstrating by explicit construction that the 3D Euler equations have a singularity after a finite time for certain initial conditions? Other ideas that have been pursued, in particular, by the late Pelz, are to construct fully 3D initial states from collapse configurations in three mutually perpendicular planes, and then let such filament configurations evolve under 3D vortex dynamics, hopefully ending in a collapse singularity of the 3D Euler equations. We must leave these speculations to tantalize the reader.

IV. POINT VORTICES AND POLYNOMIALS

We have introduced the point vortex equations in their "complex coordinate" form, Eq. (2). It is tempting to consider the polynomial that has roots at the vortex locations, with each root weighted according to the corresponding vortex strength, and to attempt to find properties of this polynomial from features of the motion of the vortices. Two cases have received the preponderance of attention: the case of identical vortices and the case where all the vortices have the same absolute strength but are either positive or negative. In the former case one considers the polynomial

$$P(z)=(z-z_1)(z-z_2)\cdots(z-z_N), \tag{37}$$

where $z_1,\ldots,z_N$ are the vortex positions. In the latter case, with N positive and M negative vortices, one considers Eq. (37) for the positive vortices and

$$Q(z)=(z-\zeta_1)(z-\zeta_2)\cdots(z-\zeta_M), \tag{38}$$

where $\zeta_1,\ldots,\zeta_M$ are the positions of the M negative vortices.

This approach of embedding the vortex positions as roots of polynomials has thus far primarily yielded results for *relative equilibria* of the vortex assembly, i.e., for cases where the vortex motion is simply a uniform rotation about the center of vorticity, a uniform translation or, in the important special case [Eq. (36)], a stationary equilibrium. The following results have been established:[6]

1. N collinear, identical vortices are in relative equilibrium if, and only if, they are situated at the roots of the Nth *Hermite polynomial*;[46]
2. N identical vortices on a circle (with or without a vortex at the center) are in relative equilibrium if, and only if, they are situated at the vertices of a regular N-gon;[31]

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3. relative equilibria of n nested, regular p-gons may be found, and have been calculated in detail for $n \leqslant 3$;[5]
4. $n(n+1)/2$ positive and $n(n-1)/2$ negative vortices form a stationary equilibrium if, and only if, the vortices are situated at the roots of successive *Adler-Moser polynomials*.[11,16] (It is easily seen that when the vortices have strengths either $+1$ or -1, the numbers of minority and majority populations are successive triangular numbers.)
5. N positive and N negative vortices form a uniformly translating equilibrium if, and only if, the vortices of one sign are situated at the roots of an Adler-Moser polynomial.[11,16] The vortices of opposite sign are then at the roots of a polynomial of the same degree derived from the Adler-Moser polynomial. In particular, such equilibria are only possible when N is a triangular number.

Particularly the theory of stationary equilibria and its relation to rational solutions of the *KdV equation*—the context in which the Adler-Moser polynomials were first discussed—is quite unexpected and very beautiful. We do not pause to write it out here but refer the reader to other expositions in which more details and further references are given.[11,6,3]

The problem of relative equilibria of point vortices is very rich. Although these equilibria are all 2D patterns of points, the possibility of both positive and negative strengths adds considerable variety to the possibilities. A number of methods have now been tried in seeking relative equilibria for point vortices. From a physical point of view the most important problem is probably the determination of vortex positions given a set of strengths although the problem of finding vortex strengths that make a given set of points a relative equilibrium is formally simpler since it is linear in the vortex strengths, cf. Ref. 42. Very intriguing are point configurations with the property that they are relative equilibria of point vortices regardless of the strengths of the vortices used to populate them. The equilateral triangle is one such configuration. Are there others (possibly for a restricted class of vortex strengths)? This class of problems is not well explored.

Of particular interest are relative equilibria of identical vortices. This problem arises in the application of point vortex dynamics to superfluids, for example, where quantum mechanics dictates that the vortex circulations are quantized. One might feel that this most symmetrical problem would be amenable to a complete solution given that some of the other problems with vortices of mixed strengths have yielded to a solution. Thus far, however, this has not been the case. After simple rescaling the problem of relative equilibria of N identical vortices is, in essence, the problem of determining all solutions of the system of N algebraic equations,

$$\overline{z}_\alpha = \sum_{\beta=1}^{N}{}' \frac{1}{z_\alpha - z_\beta}, \tag{39}$$

in the N complex variables $z_1, \ldots, z_N$. This problem is currently unsolved although many solutions are known both analytically and numerically to high precision. Some examples are given in Fig. 2.

One approach to this problem is to consider the iterative scheme

$$\overline{\hat{z}_\alpha} = \frac{\sum_{\beta=1}^{N}{}' [1/(z_\alpha - z_\beta)]}{\sqrt{\sum_{\lambda=1}^{N} \left| \sum_{\mu=1}^{N}{}' [1/(z_\lambda - z_\mu)] \right|^2}}. \tag{40}$$

Note that the denominator cannot vanish: If it were assumed to vanish, we would have

$$\sum_{\mu=1}^{N}{}' \frac{1}{z_\lambda - z_\mu} = 0 \quad \text{for } \lambda = 1, \ldots, N. \tag{41}$$

But

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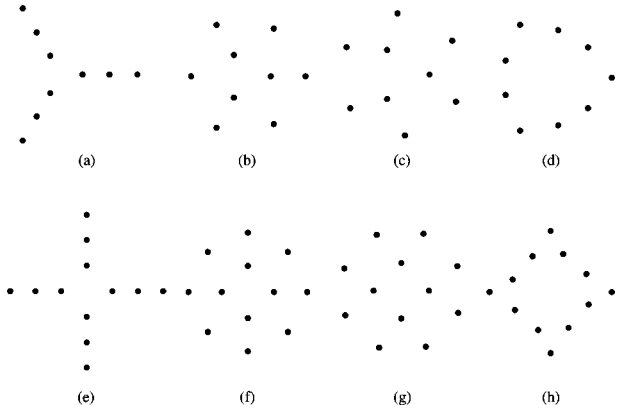


FIG. 2. Examples of relative equilibria of N identical point vortices that are understood analytically: [(a)–(d)] $N=9$; [(e)–(h)] $N=12$ (cf. Ref. 5).

$$\sum_{\lambda=1}^{N}\sum_{\mu=1}^{N}{}'\frac{z_\lambda}{z_\lambda - z_\mu} = N(N-1) + \sum_{\lambda=1}^{N}\sum_{\mu=1}^{N}{}'\frac{z_\mu}{z_\lambda - z_\mu},$$

so that

$$\sum_{\lambda=1}^{N}\sum_{\mu=1}^{N}{}'\frac{z_\lambda}{z_\lambda - z_\mu} = \frac{N(N-1)}{2}. \tag{42}$$

This contradicts Eq. (41) since according to that equation the left hand side of Eq. (42) should vanish, which clearly it does not (except in the trivial case $N=1$).

The setup in Eq. (40), particularly the denominator on the right hand side, assures that

$$\sum_{\alpha=1}^{N}\hat{z}_\alpha = 0, \tag{43}$$

$$\sum_{\alpha=1}^{N}|\hat{z}_\alpha|^2 = \sum_{\alpha=1}^{N}\hat{z}_\alpha\overline{\hat{z}_\alpha} = \frac{\displaystyle\sum_{\alpha=1}^{N}\left|\sum_{\beta=1}^{N}{}'[1/(z_\alpha - z_\beta)]\right|^2}{\displaystyle\sum_{\lambda=1}^{N}\left|\sum_{\mu=1}^{N}{}'[1/(z_\lambda - z_\mu)]\right|^2} = 1. \tag{44}$$

Thus, iteration (40) becomes a map of the unit sphere in $2N$-dimensional space (N-dimensional complex space) onto itself. By *Brouwer's fixed point theorem* it follows that the iteration has a fixed point. For such a fixed point we have

$$\Omega\bar{z}_\alpha = \sum_{\beta=1}^{N}{}'\frac{1}{z_\alpha - z_\beta}, \quad \Omega = \sqrt{\sum_{\lambda=1}^{N}\left|\sum_{\mu=1}^{N}{}'\frac{1}{z_\lambda - z_\mu}\right|^2}, \tag{45}$$

and it is trivial to rescale to obtain a solution of Eq. (39). Several of the equilibria shown in Fig. 3 were obtained numerically by iteration of eq. (40) or variations thereof. The issue of determining these configurations—even of counting them accurately—is still largely unresolved.

It may add to the intrinsic interest of these solutions if we remark that they provide extrema of

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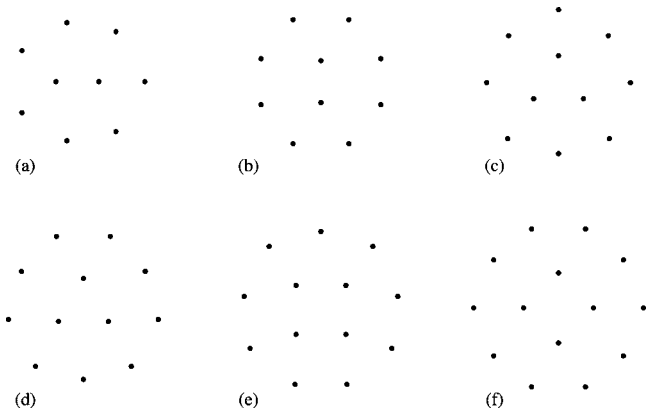


FIG. 3. Examples of relative equilibria of N identical point vortices with an axis of symmetry, $N=9,\ldots,14$. These are determined to high precision numerically but poorly/not understood analytically. These configurations first appeared in the *Los Alamos Catalog* (Refs. 17 and 18).

$$\prod_{\alpha,\beta=1}^{N}{}' l_{\alpha\beta}^2, \tag{46}$$

under the subsidiary condition

$$\sum_{\alpha,\beta=1}^{N}{}' l_{\alpha\beta}^2 = \text{const}. \tag{47}$$

This is basically the content of *Kelvin's variational principle* for relative equilibria of point vortices.

Considering Eq. (39) it is easy to see that it has the following two symmetries: (a) we may rotate each vortex position by the same angle (this amounts to multiplying each z_α by the same factor $e^{i\varphi}$) and the rotated configuration is also a relative equilibrium, or (b) we may reflect all vortex positions in the real axis (this corresponds to replacing each z_α by its complex conjugate). The first symmetry is not surprising since the relative equilibria rotate uniformly about the origin. The second symmetry is less intuitive. Taken together these two symmetries might suggest that all solutions of Eq. (39) have at least an axis of symmetry (which by rotation can be taken as the real axis).

The "classical" relative equilibria for N identical vortices, the regular N-gon and regular, centered $(N-1)$-gon, do of course have a discrete rotational symmetry by angles $2\pi/N$ and $2\pi/(N-1)$, respectively. There are "nested" polygon relative equilibria, where two or more n-gons are arranged within one another, that may be calculated[26,5] in detail. These configurations also exist in a "centered form," i.e., with a number of nested, regular n-gons and a vortex at the center. For up to three nested regular polygons it may be established that the only possibilities are for all polygons to have the same number of vertices. (This may be true also for a larger number of nested regular polygons but that is unproven.) There are also "degenerate cases" where two of the n-gons have the same radius, but the vortices on these two do not form a regular $2n$-gon. A complete picture is available for two or three nested, regular polygons in the references just cited. All these configurations have a symmetry group that is a discrete subgroup of the group of rotations about the centroid of the configuration. Early on, the "folklore" about relative equilibria of identical vortices was that they would all have a symmetry group of this kind. Figure 2 gives a sampling of these analytically understood relative equilibria. There are, however, numerous additional relative equilibria, found by iteration of Eq. (40) or a numerical scheme that is some variant thereof, with no complete analytical understanding but calculated to considerable numerical accuracy, that have the appearance of a system of nested, regular polygons but really are not. This class

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of equilibria has an axis of symmetry, a feature that may implicitly be picked out by the numerical methods used to compute it. The existence of these configurations led to the conjecture, viable for a couple of decades, that all relative equilibria of identical point vortices had an axis of symmetry.

For small N we know all. For $N=1,2$ all configurations are relative equilibria. For $N=3$ we have the equilateral triangle and three vortices on a line with one at the midpoint of the line joining the other two. For $N=4$ we have the collinear configurations (the vortices are at the roots of the Hermite polynomial H_4), the centered equilateral triangle, and the square. This much is firmly established analytically. For $N=5$ we have the collinear configuration (vortices at the roots of H_5), the centered square, and the regular pentagon. In all likelihood this is the complete list but a proof has not been given (so far as I am aware). Already for $N=6$ the space of solutions is richer. It includes, of course, the collinear configuration, the centered regular pentagon, and the regular hexagon (two nested equilateral triangles, one turned $\pi/3$ with respect to the other, which happen to have the same radius). But there is also a symmetric configuration of two nested equilateral triangles with the vortices situated on the same three "spokes" from the center [cf. Fig. 2(a) for the related configuration with $N=9$]. Furthermore, a state consisting of three nested digons exists which has four vortices on a line and two on a line perpendicular to it. We believe this is the complete list of relative equilibria for $N=6$ but a rigorous proof is not available. For $N\geqslant 7$ all our knowledge is based on numerical explorations.

There are, of course, other ways than iterating Eq. (40) to obtain relative equilibria of N vortices numerically. The following has proven particularly productive: Imagine a relative equilibrium of $N-1$ vortices, where N is beyond the "trivial range" outlined above. Associated with any such equilibrium are a number of points in the fluid that are at rest relative to the vortices and so rotate with them at constant angular velocity. We call these the "corotating points." In terms of Eq. (39) such points are solutions of

$$\bar{z}=\sum_{\alpha=1}^{N}\frac{1}{z-z_\alpha}, \tag{48}$$

where the z_α solve Eq. (39). I call this *Morton's equation* since Morton studied problems of this kind in his 1933 paper.[37] Multiplying through by the product of the denominators, Eq. (48) may be written in the form $P(z,\bar{z})=0$, where P is a polynomial of degree N in z and of degree 1 in $\bar{z}$. If the vortex configuration is symmetrical with respect to the real axis, then the solutions of Eq. (48) are also either real or come in complex conjugate pairs. One might think that there would just be $N+1$ such corotating points since in Eq. (48) that is the overall degree of the polynomial P. However, this would be an underestimate. Indeed, it is possible for a polynomial in z and $\bar{z}$ to have infinitely many roots, cf. $z\bar{z}=1$. It turns out that Eq. (48) "retains" the property of polynomials in a single variable of having finitely many roots. For simple, analytically known configurations, e.g., the regular N-gon, the roots may be found directly. There are $3N+1$ corotating points.

Once the corotating points have been found, one may think of them as "embryonic" or "ghost" vortices of strength zero. One may then consider an algorithm in which the vortex strength associated with a corotating point is gradually increased, i.e., one imagines solving the system

$$\bar{z}_\alpha(\epsilon)=\sum_{\beta=1}^{N}{}'\frac{1}{z_\alpha(\epsilon)-z_\beta(\epsilon)}+\frac{\epsilon}{z_\alpha(\epsilon)-z(\epsilon)},\quad \bar{z}(\epsilon)=\sum_{\alpha=1}^{N}\frac{1}{z(\epsilon)-z_\alpha(\epsilon)} \tag{49}$$

step by step as ϵ is incrementally increased from 0 to 1. The "initial conditions" for this procedure, i.e., the values of z and z_α for $\epsilon=0$, are of course the N-vortex equilibrium and one of its corotating points.

This approach was used by Aref and Vainchtein[10] and resulted in a multitude of relative equilibrium configurations quite different from any that had been found previously. In particular, none of these configurations seemed to have the concentric polygon appearance that was prevalent for the solutions of the iteration [Eq. (40)]. Remarkably, this method also produced relative equilibria without any apparent symmetry whatsoever (and certainly without an axis of symme-

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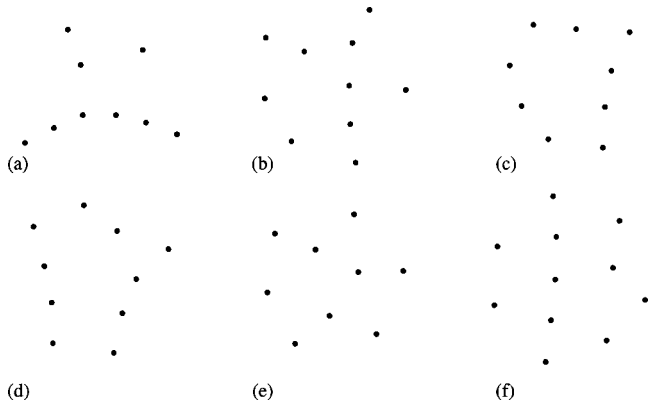


FIG. 4. Asymmetric relative equilibria of N identical point vortices, $N=9,10,11$, determined by the method of "ghost" vortices, Eqs. (49). From an analytical point of view these configurations remain a mystery.

try). These equilibria appeared for $N\geqslant 8$ and several "families" of them emerged as N was increased further. Examples are shown in Fig. 4. Note that each panel of this figure may be thought of as representing two relative equilibria, viz., the one shown and its reflection in an axis through the origin.

In terms of the notion of a generating polynomial, the discovery of asymmetric relative equilibria is somewhat unpleasant. At least, if the relative equilibrium has an axis of symmetry, which we take to be the real axis as before, a generating polynomial will have real coefficients and there is a chance that it is "recognizable" in terms of the known families of polynomials of mathematical physics. If the polynomial does not have real coefficients, it is unlikely that it is a member of any of the well known families. One may, of course, simply turn the problem around and consider polynomials the roots of which satisfy Eq. (39). Developing a theory of polynomials based solely on the knowledge that their roots satisfy Eq. (39) remains an open challenge.

V. A TRIGONOMETRIC IDENTITY

In this section I want to give a "proof by vortex dynamics" of the identity

$$\frac{1}{p}\sum_{n=0}^{p-1}\cot\left(\frac{z-n\pi}{p}\right)=\cot z. \tag{50}$$

Imagine enclosing the N vortices we have been considering in a strip of width L. Now duplicate the strip to the left and to the right such that each vortex has an infinite number of periodic images on either side, i.e., each z_α, $\alpha=1,\ldots,N$, now represents a row of vortices located at $z_\alpha\pm nL$, where $n=0,1,2,\ldots$. runs through the integers. If this representation is substituted into the point vortex equations, we get, formally,

$$\overline{\frac{dz_\alpha}{dt}}=\frac{1}{2\pi i}{\sum_{\beta=1}^{N}}'\left\{\frac{\Gamma_\beta}{z_\alpha-z_\beta}+\sum_{n=1}^{\infty}\left[\frac{\Gamma_\beta}{z_\alpha-(z_\beta+nL)}+\frac{\Gamma_\beta}{z_\alpha-(z_\beta-nL)}\right]\right\}.$$

Two points are important to note in this expression. First, we have indicated how we plan to perform the infinite sum, which in the absence of such a prescription is not convergent. We do the sum pairwise, adding the contributions from images on the left, $z_\beta-nL$, and the right, $z_\beta+nL$, at equal distances from the "base vortex" at z_β. With such a prescription the sum is convergent. Second, one might have expected to see contributions from the periodic images of vortex α itself. However, these contributions, when added according to the prescription just mentioned, cancel pairwise and so contribute nothing to the sum.

We now transcribe the sum just written as follows:

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$$\overline{\frac{dz_\alpha}{dt}} = \frac{1}{2\pi i}\sum_{\beta=1}^{N}{}'\Gamma_\beta\left[\frac{1}{z_\alpha - z_\beta} + 2\sum_{n=1}^{\infty}\frac{z_\alpha - z_\beta}{(z_\alpha - z_\beta)^2 - (nL)^2}\right].$$

The function in square brackets,

$$\frac{1}{z} + 2\sum_{n=1}^{\infty}\frac{z}{z^2 - (nL)^2} = \frac{\pi}{L}\left[\frac{L}{\pi z} + 2\sum_{n=1}^{\infty}\frac{\pi z/L}{(\pi z/L)^2 - (n\pi)^2}\right] = \frac{\pi}{L}\cot\left(\frac{\pi z}{L}\right)$$

by the *partial fraction decomposition* of cot.

Thus we have, finally,

$$\overline{\frac{dz_\alpha}{dt}} = \frac{1}{2Li}\sum_{\beta=1}^{N}{}'\Gamma_\beta \cot\left[\frac{\pi(z_\alpha - z_\beta)}{L}\right]. \tag{51}$$

These are the equations of motion for N vortices in a periodic strip of width L or, equivalently, for N vortex rows within each of which the spacing between the identical vortices in the row is L. Equation (51) appears to have been first published in 1928 by Friedmann and Poloubarinova.[23]

Equation (51) shares many properties of the equations of motion on the infinite plane. The periodic strip system may be cast in Hamiltonian form. The Hamiltonian, as may be easily seen from Eq. (51), is

$$H = -\frac{1}{4\pi}\sum_{\alpha,\beta=1}^{N}{}'\Gamma_\alpha\Gamma_\beta \log\left|\sin\left[\frac{\pi(z_\alpha - z_\beta)}{L}\right]\right|. \tag{52}$$

Two of the integrals familiar from the infinite plane case, the components of the linear impulse, P and Q,

$$Q + iP = \sum_{\alpha=1}^{N}\Gamma_\alpha z_\alpha, \tag{53}$$

remain integrals for the periodic strip system, as was emphasized in 1959 by Birkhoff and Fisher[12] in a thought-provoking paper. However, the integral I is lost. The invariance of the Hamiltonian to translations in the y direction is clear enough, since this is the direction of the strips. The invariance to translations in the periodic direction is more mysterious. Rigorously speaking there is a subtlety involved since each x_α appearing in P is only defined modulo the strip width L, i.e., if in the course of its motion a vortex "leaves" the basic strip at $x=L$, it "reappears" at $x=0$. Hence, Q is not solely determined by the instantaneous configuration of the vortices in the basic strip—in order to verify that Q is an integral of the motion, one also needs to know or keep track of how often each vortex has gone through the basic strip. Nevertheless, Q is well defined over time intervals short enough for the same vortices to remain in the basic strip, and we have the usual linkage between the invariance of H to translations and the invariance of Q and P in time. The subtlety deepens a bit when we realize that Q is an integral because of the invariance of H to translations in the "trivial" y direction, whereas P, which is well defined without recourse to vortex trajectory histories, corresponds to the translational invariance along the x axis.

Again, the "one-vortex problem"—which is now really a "one-row-of-vortices problem"—for the dynamics [Eq. (51)] is trivial. The vortex (and the entire row) remains stationary. A single row of identical vortices, in the limit where the vortices are very close together, has often been used to model a *vortex sheet*, and the basic integrodifferential equation of motion for such a sheet, the so-called *Birkhoff-Rott equation*, may be obtained as a limit of the point vortex equations.

The result [Eq. (50)] is obtained by considering the flow field produced at $z=x+iy$ by a periodic row of identical vortices. Assuming the vortices all to have circulation Γ, this flow field is given by

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$$u - iv = \frac{1}{2Li} \sum_{\alpha=1}^{N} \Gamma_\alpha \cot\left[\frac{\pi(z - z_\alpha)}{L} \right]. \tag{54}$$

Equation (50) is nothing but the statement that the induced velocity from a periodic row is invariant to our decision as to how we partition this row into periodic segments.

Thus, consider vortices of circulation Γ placed at the origin and along the x axis at $\pm nL$, n being integer. Viewing this as a periodic system with one vortex per strip of width L, i.e., using $N=1$ in Eq. (54), we find the velocity at an arbitrarily chosen field point, z, to be given as

$$u - iv = \frac{1}{2Li} \Gamma \cot\left(\frac{\pi z}{L} \right). \tag{55}$$

Now, consider this exact same vortex configuration but use a strip of width pL, where p is an integer $\geqslant 2$. In this view of the row we have p vortices in the basic strip at $x=0, L, \ldots, (p-1)L$. The velocity at the arbitrary field point z must, of course, be exactly the same on physical grounds, since nothing but our viewpoint concerning the width of the periodic strip has changed. By Eq. (54) the velocity calculated from the second point of view is given as

$$u - iv = \frac{\Gamma}{2pLi} \sum_{n=0}^{p-1} \cot\left[\frac{\pi(z - nL)}{pL} \right]. \tag{56}$$

Equating the two expressions for the velocity [Eqs. (55) and (56)] we have a proof by vortex dynamics of the trigonometric identity [Eq. (50)].

Taking the limit $z \to 0$ of Eq. (50) we have

$$\frac{1}{p} \sum_{n=1}^{p-1} \cot\left(\frac{n\pi}{p} \right) = 0. \tag{57}$$

Identity (50) leads to many further identities by taking derivatives of it and/or substituting particular values for z. For example, taking the z derivative one obtains

$$\frac{1}{p^2} \sum_{n=0}^{p-1} \frac{1}{\sin^2((z - n\pi)/p)} = \frac{1}{\sin^2 z},$$

which for $z=-\pi/2$ yields

$$\sum_{n=0}^{p-1} \frac{1}{\sin^2(\pi[(2n+1)/2p])} = p^2,$$

a formula that can, in turn, be used in proofs of further well known identities, cf. Ref. 28 and references therein.

For vortices in a periodic parallelogram the cot interaction in Eq. (51) is replaced by

$$\frac{d\bar{z}_\alpha}{dt} = \frac{1}{2\pi i} {\sum_{\beta=1}^{N}}' \Gamma_\beta \zeta(z_\alpha - z_\beta; \omega_1, \omega_2) + \frac{\eta_1}{2\pi i \omega_1}(Q + iP) - \frac{P}{\Delta}. \tag{58}$$

In this formula ζ is the *Weierstraß* ζ *function* for the parallelogram in question. The quantities ω_1 and ω_2 are the half-periods of the parallelogram, $\eta_1 = \zeta(\omega_1)$, and the particular form of Eq. (58) arises when ω_1 is real. The quantities Q and P are the components of the linear impulse as before. In this case the periodicity of the flow implies (by Stokes theorem) that the sum of the vortex strengths in the basic parallelogram must vanish. Again, this is a Hamiltonian system that retains the integrals Q and P [Eq. (53)] for similar reasons and with similar caveats about tracking vortices as they cross in and out of the basic periodic parallelogram as we discussed in the case of the periodic strip. The details may be found in Ref. 48.

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There is, again, an addition formula for the Weierstraß ζ function which expresses that the velocity at a field point in such a doubly periodic vortex array must be independent of how we choose to "carve up" the array into periodic boxes, i.e., whether we choose to compute it using one box, as in Eq. (58), or whether we choose to lump together several boxes to form a larger "basic cell" (with more vortices). This addition formula, the counterpart to Eq. (50), is

$$\zeta(z;\omega_1,\omega_2) = \sum_{m,n=0}^{p-1} \zeta(z - m\omega_1 - n\omega_2; p\omega_1, p\omega_2) + (p-1)\zeta(\omega_1 + \omega_2; \omega_1, \omega_2). \tag{59}$$

There are other interesting connections that emerge from considering these systems. For example, Stremler[47] shows how consideration of the Hamiltonian for the vortices, which is the kinetic energy of the flow field less the infinite "self-energy" of the singular vortices, leads to an identity for a class of infinite lattice sums. Essentially, this identity arises by comparing the expression for the energy based on contributions from the vortices to the expression for the energy based on an integral of the velocity field expanded in Fourier modes over the basic cell.

I submit that there is something of a "gold mine" of results here, many of them probably well known to specialists in elliptic function theory, but maybe not connected in the way they could become when viewed as expressions for velocities, linear and angular momentum, or kinetic energy of a system of vortices. Possibly, instead of speaking of a classical mathematical playground, we ought to think of point vortex dynamics as a "smorgasbord."

VI. PROJECTIVE GEOMETRY OF STAGNATION POINTS

There is an intriguing connection between certain characteristics of the flow field surrounding an assembly of vortices and some classical results in projective geometry. This connection arises principally from the notion that the vortex positions are the roots of some polynomial or, in the case of mixed signs of the vortex strengths, are the zeros and poles of some rational function. We again consider N vortices with strengths $\Gamma_1,\ldots,\Gamma_N$ situated in the complex plane. For simplicity we shall assume all the Γ to be rational. Although, in principle, the vortex strengths are real numbers, one would assume that approximating them closely by a set of rationals should be entirely adequate. (This point is hardly worth commenting on were it not for the observation that in solving the three-vortex problem in a periodic strip[8] or a periodic parallelogram[48] it turns out not to be true.) Assuming this, we may rescale all Γ by a common denominator for the entire set of rational numbers. Physically, this corresponds to changing the scales of length and/or time. Ultimately, then, we may assume the Γ to be a set of integers.

The 2D flow field, (u,v), produced by these vortices is given by

$$u - iv = \frac{1}{2\pi i} \sum_{\alpha=1}^{N} \frac{\Gamma_\alpha}{z - z_\alpha}. \tag{60}$$

Now, consider the rational function

$$R(z) = (z - z_1)^{\Gamma_1}(z - z_2)^{\Gamma_2} \cdots (z - z_N)^{\Gamma_N}. \tag{61}$$

This is a rational function because of our convention that the Γ are integers. If all the Γ are positive, $R(z)$ is a polynomial.

At issue is the location of the zeros of the derivative of $R(z)$. These zeros are the instantaneous stagnation points of the flow [Eq. (60)]. We have, thus, a classical problem of analysis, viz., to characterize the location of the zeros of the derivative of a polynomial (or rational function) in terms of the zeros (and/or poles) of the polynomial (rational function) itself. The general, and very beautiful, result due to Siebeck[45] is this: The stagnation points, i.e., zeros of Eq. (60), are the foci of an explicitly given curve of class $N-1$ that touches each line segment connecting a z_α to a z_β

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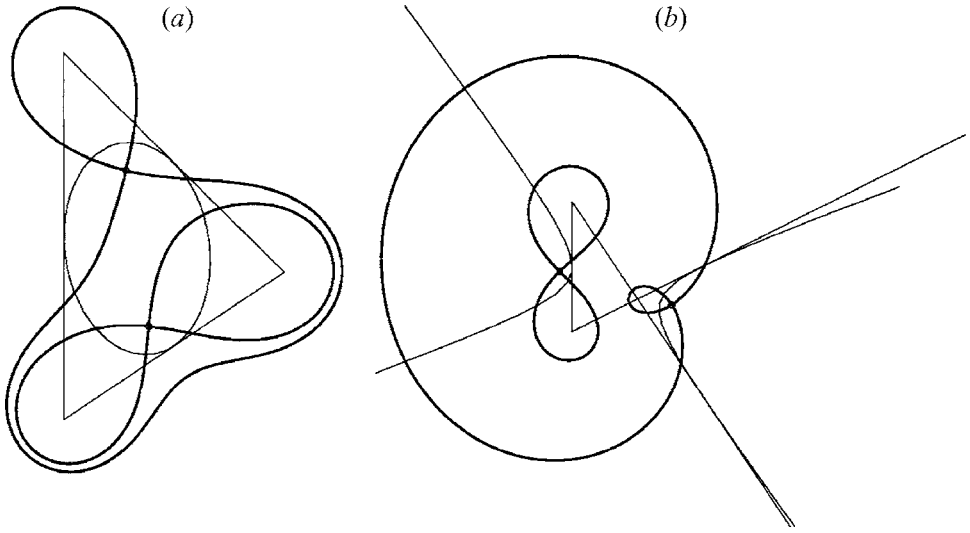


FIG. 5. Stagnation points for three-vortex flow as foci of the Siebeck conic. (a) Identical vortices; (b) vortices with strength ratio $1:1:-1$.

in the point that divides that line segment in the ratio $\Gamma_\alpha:\Gamma_\beta$. This result is proven early on in the book of Marden.[36] It has been elaborated and proven in a manner more accessible to fluid mechanicians in Ref. 4.

We illustrate the result by two examples reproduced from Ref. 4. In Fig. 5(a) we are considering the symmetrical case of three identical vortices. The vortex triangle is shown, with a vortex at each vertex. In this case the Siebeck conic is an ellipse. It touches the sides in their midpoints, the points that divide the side in the ratio 1:1. The two foci of the ellipse are the instantaneous stagnation points. To amplify this we have drawn also the instantaneous dividing streamline which bifurcates at the two stagnation points.

The problem discussed here is related to the general problem of finding the locations of the derivative of a polynomial given the location of its roots. The well known *Gauss-Lucas theorem* states that the roots of the derivative of a polynomial are situated within the convex hull spanned by the roots of the polynomial. Siebeck's theorem provides a different restriction on the positions of the zeros (stagnation points). In the case of three identical vortices the stagnation points are the foci of an ellipse inscribed in the vortex triangle. This is a sharper characterization of their location than just knowing the stagnation points to be inside the vortex triangle. Considering vortices of different strengths (still all positive) gives the location of the roots of the derivative of a polynomial with multiple roots in terms of the location of those roots.

In Fig. 5(b) we illustrate the case of three vortices with strengths $(1,1,-1)$. Again, the vortex triangle is shown. This time the Siebeck conic is a hyperbola. It touches the side connecting the two identical vortices at its midpoint. It touches the two other sides in the points that divide those sides in the ratio $1:(-1)$, i.e., in points located a side length further along on the extension of the side beyond the negative vortex. The foci of this hyperbola are the instantaneous stagnation points. To emphasize this, the instantaneous dividing streamline is shown. It has points of bifurcation at the stagnation points. As the vortices trace out their trajectories, it is clear that a diagram similar to Fig. 5(b) will prevail so long as the vortices do not become collinear. In particular, the two stagnation points will be situated one in each part of the sector between the sides in the vortex triangle that connect the negative vortex to the two positive vortices, outside the vortex triangle itself. Even a simple conclusion such as this is not so easy to derive directly from Eq. (60), with $u=v=0$, without recourse to the geometrical interpretation.

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VII. MORE, EVER MORE

I hope the reader is convinced that point vortex dynamics touches on a multitude of areas of mathematics and that it can provide interesting physical interpretations to a number of mathematical results as well as suggest new ones for investigation. There are many further examples that could have been given. There are extensions of the problems discussed to manifolds such as the sphere and the torus, to bounded domains, and to multiply connected domains.

If I were to single out an area of investigation that is largely unexplored and seems to hold considerable promise, it would be the application of probabilistic ideas to point vortex motion as an alternative approach to problems that are—at least superficially—related to turbulence. This approach is exemplified by the seminal paper of Onsager[43] who used ideas from equilibrium statistical physics to try to elucidate the formation of large coherent vortices in predominantly 2D flows. Onsager may have had in mind the spontaneous emergence of long-lived vortices in the planetary atmospheres of the giant gaseous planets, Jupiter in particular, and in our own oceans and atmosphere. However, the statistical approach to point vortex dynamics still has many unexplored aspects. Theories beyond equilibrium statistical mechanics ought to be applied to this system.

Another intriguing area that is just beginning to be explored is the cross section between the celestial mechanics of point masses and the dynamics of point vortices. The theory of relative equilibria of point masses and many aspects of point mass dynamics are in a much more mature state than their point vortex counterparts, having been studied for a longer period of time and also having been of more immediate concern, e.g., in connection with space exploration, the trajectories of satellites, planetary and galactic dynamics, and so on. Thus, Hampton and Moeckel[25] have just announced proofs "that the number of relative equilibria, equilibria, and rigidly translating configurations in the problem of four point vortices is finite." Their proofs are "based on symbolic and exact integer computations which are carried out by computer." The main result of their paper is the following.

Theorem: If the vortex strengths Γ_α are nonzero then the four-vortex problem has

(i) exactly 2 equilibria when the necessary condition $L=0$ holds;
(ii) at most 6 rigidly translating configurations when the necessary condition $\Gamma_1+\Gamma_2+\Gamma_3+\Gamma_4=0$ holds;
(iii) at most 12 collinear relative equilibria;
(iv) at most 14 strictly planar relative equilibria when $\Gamma_1+\Gamma_2+\Gamma_3+\Gamma_4=0$; and
(v) at most 74 strictly planar relative equilibria when $\Gamma_1+\Gamma_2+\Gamma_3+\Gamma_4\neq 0$ provided no two vortex strengths or no three vortex strengths sum to zero.

Here "strictly planar" means planar (as all these equilibria are) but not collinear. Relative equilibria are equivalent if they arise from one another by rotation, translation, or dilatation. However, relative equilibria that arise by reflection are not considered equivalent in this classification and enumeration. While many fluid mechanicians may consider the four-vortex problem to be too simple to merit much attention, the necessary mathematical tools to prove even this theorem are quite impressive. Furthermore, many of these mathematical tools and concepts are entirely new to vortex dynamics and by extension to fluid mechanics. Again, I see results of this type as vindicating the theme of this article that point vortex dynamics is, indeed, a classical mathematics playground.

ACKNOWLEDGMENTS

I am indebted to many coauthors and collaborators over the years for improving my understanding of the dynamics of point vortices. Thanks are due to (in alphabetical order) Phil Boyland, Morten Brøns, Martin van Buren, Bruno Eckhardt, Yoshi Kimura, Slava Meleshko, Paul Newton, Neil Pomphrey, Nicholas Rott, Mark Stremler, Tadashi Tokieda, Dmitri Vainchtein, and Irek Zawadzki. This work was supported by a Niels Bohr Visiting Professorship at the Technical University of Denmark provided by the Danish National Research Foundation.

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